Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 85–126.
DOI: 10.3934/math.2026005
Received: 26 October 2025
Revised: 24 November 2025
Accepted: 03 December 2025
Published: 04 January 2026

Research article

A new almost unbiased estimator for beta regression model under multicollinearity

Randa Alharbi[1] **and Abdulaziz S. Alghamdi**[2,*]

[1] Department of Statistics, Faculty of Science, University of Tabuk, Tabuk, Saudi Arabia; ralharbi@ut.edu.sa

[2] Department of Mathematics, College of Science & Arts, King Abdulaziz University, P. O. Box 344, Rabigh 21911, Saudi Arabia

* **Correspondence:** Email: ashalghamedi@kau.edu.sa.

Abstract: The beta regression model (BRM) is designed to model continuous response variables constrained to the open interval (0, 1). It is particularly suitable for analyzing proportions, rates, and other fractional data. Maximum likelihood estimation (MLE) is the most common way to estimate parameters in the BRM. However, when multicollinearity is present, i.e., when explanatory variables exhibit high intercorrelation, the MLE may produce unstable and biased parameter estimates, inflated variance, and increased scalar mean squared error (MSE), ultimately undermining the model's statistical reliability. To address the effects of multicollinearity, some biased estimators have been proposed for the BRM. In this study, we introduce a new almost unbiased estimator for the BRM. A theoretical comparison of the proposed estimator with existing estimators is derived using the matrix mean squared error (MMSE) and MSE. The performance of the proposed estimator is subsequently assessed and contrasted with existing estimators via a comprehensive Monte Carlo simulation study and applied to two real-world datasets. The simulation and applications consistently show that the proposed estimator is better than other existing estimators, providing more accurate and stable parameter estimates for the BRM under multicollinearity.

Keywords: beta regression model; biased estimator; almost unbiased estimator; modified ridge-type estimator; multicollinearity
Mathematics Subject Classification: 62H12, 62J07, 62J10, 62J12, 62P10

1. Introduction

The beta regression model (BRM) constitutes a specialized subclass of generalized linear models explicitly to model a continuous response variable within the open unit interval (0, 1) with one or more explanatory covariates. The BRM is widely used for model proportions, rates, and percentages. The BRM has been widely utilized across multiple disciplines, including economics, medicine, environmental studies, and the social sciences. Traditional modeling strategies, such as Gaussian linear regression, binary logistic regression, and Poisson or negative-binomial count regression, exhibit limitations when applied to bounded data within the interval (0, 1). To handle these methodological issues, Ferrari and Cribari-Neto [1] formulated the BRM as a specialized model for continuously distributed outcomes constrained to the unit interval. By assuming the response follows a beta distribution, the BRM makes the regression coefficients straightforward to interpret and naturally handles skewed data and heteroskedasticity [2, 3].

Recently, the BRM has been widely used, such as in the following papers: Stein et al. [4] used the BRM for network analysis, Amin et al. [5] developed the BRM to model the thermal power plants data, and Tannous et al. [6] applied the BRM to predictive modelling of medication adherence in post-myocardial infarction patients. Although the standard assumption assumes that predictors are not linearly correlated, empirical data sets routinely violate this requirement, a phenomenon known as multicollinearity [7, 8]. This problem was first introduced by Frisch [9]. When multicollinearity is present, this makes the beta maximum likelihood estimator inefficient, as it affects the accuracy and stability of parameter estimates, inflated variance, and increased scalar mean squared error (MSE), ultimately undermining the model's statistical reliability. Uncommon strategies for addressing multicollinearity involve increasing the sample size, altering the model specification, or excluding highly correlated predictors. Nevertheless, these approaches are frequently impractical or offer limited effectiveness in applied settings.

To overcome this problem, biased estimation techniques have become popular alternatives. One of the most widely used methods is the ridge regression estimator, introduced by Hoerl and Kennard [10]. This method adds a biasing parameter k to the variance-covariance matrix to stabilize the estimates and reduce variance. Although ridge regression is effective, its performance depends on choosing an appropriate k value. Additionally, Liu [11] introduced the Liu estimator, which combines features of ridge regression and the Stein estimator. The Liu estimator applies a linear shrinkage technique defined as a linear function of the Liu parameter d to handle problems of multicollinearity, offering better performance in models with both independent and correlated predictors. In addition to linear regression models, several biased estimation methods have been proposed in the literature. These include the modified ridge-type estimator [12], the new ridge estimator [13], the new two-parameter ridge estimator [14], the modified two-parameter Liu estimator [15], the almost unbiased modified ridge-type estimator [16], the almost unbiased general ridge-type estimator [17], and the almost unbiased Liu-type estimator [18], among others.

For BRM, various estimators have been introduced and extended by several researchers. Abonazel and Taha [19] and Qasim et al. [20] proposed the ridge estimator, while Karlsson et al. [21] extended the Liu estimator. Amin et al. [22] developed the James-Stein estimator, Hammad et al. [8] proposed modified Liu estimators, and Erkoç et al. [23] proposed the Özkale-Kaciranlar estimator. Additionally, Lukman et al. [24] introduced the modified ridge-type estimator, Koç and Dünder [25] developed the Kibria-Lukman estimator and its jackknifed version, and Farghali [26] defined the jackknifed beta ridge regression estimator.

Although many works have presented biased estimators for addressing multicollinearity in a BRM, there is no limit to the value of bias, and large shrinkage parameter values can lead to large biases.

Therefore, almost unbiased estimators have recently emerged, offering a greater balance between bias and variance in estimation under multicollinearity. Due to the limited application and study of this type of estimator in BRM. This study proposes a new effective estimator called the almost unbiased modified ridge-type regression estimator to address multicollinearity in the BRM. This type of estimator provides a greater balance between bias and variance than existing biased estimators. The effectiveness of this estimator is evaluated through a comprehensive study of the biased estimators in the literature and by comparing our proposed estimator with existing estimators through theoretical comparisons and simulation studies. The simulation study confirms the superiority of the proposed estimator under different scenarios. The applications also support the simulation results using real data and confirm the superiority of the proposed estimator in reducing MSE and providing more stable coefficients, which result in more robust solutions than current estimators in the presence of multicollinearity.

The remainder of this article is organized as follows: Section 2 provides a literature review on the BRM and existing estimators. In Section 3, we introduce the proposed estimator, followed by a discussion of its statistical properties, a comparison with existing estimators, and shrinkage parameter selection for proposed and existing estimators. Section 4 details the simulation study, considering various factors. Section 5 evaluates the proposed estimators using real-world datasets. Finally, Section 6 presents the concluding and future works.

2. Methodology

Let Y be a continuous random variable following a beta distribution with parameters a and b. The probability density function of the variable Y is given by

$$f(y;a,b) = \frac{\Gamma(a+b)y^{a-1}(1-y)^{b-1}}{\Gamma(a)\Gamma(b)}, \quad 0 < y < 1, \tag{2.1}$$

where $\Gamma(.)$ denotes the gamma function and $a, b > 0$. The expected value and variance of the variable Y are

$$E(Y) = \frac{a}{a+b}, \quad \text{Var}(Y) = \frac{ab}{(a+b)^2(a+b+1)}.$$

Ferrari and Cribari-Neto [1] proposed a reparameterization of the beta distribution by setting

$$\mu = \frac{a}{a+b}$$

and

$$\theta = a + b$$

in Eq (2.1). This leads to

$$a = \mu\theta$$

and

$$b = \theta(1-\mu).$$

The reparameterized probability density function is as follows:

$$f(y;\mu,\theta) = \frac{\Gamma(\theta) y^{\mu\theta-1}(1-y)^{\theta(1-\mu)-1}}{\Gamma(\mu\theta)\Gamma(\theta(1-\mu))}, \quad 0 < y < 1, \tag{2.2}$$

where μ is the mean of Y, and θ is the precision parameter. The expected value and variance under this parameterization are

$$E(Y) = \mu, \quad \mathrm{Var}(Y) = \frac{\mathrm{Var}(\mu)}{1+\theta}.$$

Ferrari and Cribari-Neto [1] introduced the BRM by the linear predictor

$$g(\mu_i) = x_i^T \beta = \eta_i,$$

where

$$x_i^T = (1, x_{i1}, x_{i2}, \ldots, x_{ip})$$

is the vector of covariates for the i-th observation from the design matrix

$$X = [1, x_1^T, ..., x_n^T]^T,$$

and

$$\beta = (\beta_0, \beta_1, ..., \beta_p)^T$$

is the vector of regression coefficients, and $g(.)$ is called the link function. The BRM commonly employs the logit link defined as follows:

$$\eta_i = g(\mu_i) = \log \frac{\mu_i}{1-\mu_i}, \quad \mu_i = \frac{\exp(x_i^T\beta)}{1+\exp(x_i^T\beta)}.$$

Then, the log-likelihood function for Eq (2.2) using the link function to estimate the parameters via the beta maximum likelihood estimator (BMLE) is

$$\begin{aligned} l(\beta) = \sum_{i=1}^{n} \Big[& \log\Gamma(\theta) - \log\Gamma(\mu_i\theta) - \log\Gamma((1-\mu_i)\theta) \\ & + (\mu_i\theta - 1)\log y_i + (\theta(1-\mu_i) - 1)\log(1-y_i) \Big]. \end{aligned} \tag{2.3}$$

Differentiating the log-likelihood in Eq (2.3) for β yields the score function for β, expressed as:

$$U(\beta) = \theta X^T T (y^* - \mu^*), \tag{2.4}$$

where

$$\begin{aligned} T &= \mathrm{diag}\left(\frac{1}{g'(\mu_1)}, \ldots, \frac{1}{g'(\mu_n)}\right), \\ y^* &= (y_1^*, \ldots, y_n^*)^T, \ \ \mu^* = (\mu_1^*, \ldots, \mu_n^*)^T, \\ y_i^* &= \log\left(\frac{y_i}{1-y_i}\right), \ \ \mu_i^* = \psi(\mu_i\theta) - \psi((1-\mu_i)\theta), \end{aligned}$$

and $\psi(.)$ represent the digamma function. The iterative reweighted least-squares algorithm, also known as the Fisher scoring algorithm, is used to estimate β [27, 28]. This algorithm can be written as:

$$\beta^{(l+1)} = \beta^{(l)} + \left(\mathbb{I}_{\beta\beta}^{(l)}\right)^{-1} U_{\beta}^{(l)}(\beta), \tag{2.5}$$

where $U_{\beta}^{(l)}$ is the score function as defined in Eq (2.4), and $\mathbb{I}_{\beta\beta}^{(l)}$ is the information matrix for β. For further details, refer to [27]. The iterations, indexed by $l = 0, 1, 2, \ldots$, continue until the difference between successive estimates is smaller than a specified threshold. At convergence, the BMLE of β is

$$\hat{\beta}_{\text{BMLE}} = A^{-1} X^T \hat{W} \hat{z}, \tag{2.6}$$

where

$$\begin{aligned}
A &= X^T \hat{W} X, \quad \hat{W} = \operatorname{diag}(\hat{w}_1, \ldots, \hat{w}_n), \\
\hat{w}_i &= \hat{\theta} \left\{\psi^T \left(\hat{\mu}_i \hat{\theta}\right) + \psi^T \left((1 - \hat{\mu}_i)\hat{\theta}\right)\right\} \frac{1}{\{g^T(\hat{\mu}_i)\}^2}, \\
\hat{z} &= \hat{\eta} + \hat{W}^{-1} \hat{T}(y^* - \mu^*).
\end{aligned}$$

Here, $\hat{W}$ and $\hat{T}$ are the matrices W and T, respectively, evaluated at the BMLE. The covariance, matrix mean squared error (MMSE), and MSE of $\hat{\beta}_{\text{BMLE}}$ are

$$\operatorname{Cov}(\hat{\beta}_{\text{BMLE}}) = \varphi A^{-1}, \tag{2.7}$$

$$\operatorname{MMSE}(\hat{\beta}_{\text{BMLE}}) = \varphi \xi \Lambda^{-1} \xi^T, \tag{2.8}$$

$$\operatorname{MSE}(\hat{\beta}_{\text{BMLE}}) = \varphi \sum_{j=1}^{r} \frac{1}{\lambda_j}, \tag{2.9}$$

where

$$\varphi = \theta^{-1}, \quad \alpha = \xi^t \hat{\beta}_{\text{BMLE}}$$

and

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r),$$

which correspond to $\xi A \xi^t$. Here, ξ represents the orthogonal matrix whose columns contain the eigenvectors of A; that is,

$$\xi = \xi_1, \ldots, \xi_r.$$

The values

$$\lambda_1 > \lambda_2 > \ldots > \lambda_r > 0$$

are the eigenvalues of the matrix A.

The design matrix becomes ill-conditioned when the highly correlated explanatory variables lead to very small eigenvalues. As a result, the BMLE for the BRM becomes unstable and inflated. Multicollinearity affects the reliability of the results by increasing the variances and confidence intervals of the parameter estimates, which may lead to incorrect inferences. To address this issue, several biased estimators have been introduced in the literature.

Qasim et al. [20] proposed the beta ridge regression estimator (BRRE), defined as

$$\hat{\beta}_{\mathrm{BRRE}} = (A + kI_r)^{-1}A\hat{\beta}_{\mathrm{BMLE}}, \quad k > 0, \tag{2.10}$$

where k is the ridge parameter, and I_r is the identity matrix of order $r \times r$. If $k = 0$, then $\hat{\beta}_{\mathrm{BRRE}}$ reduces to $\hat{\beta}_{\mathrm{BMLE}}$.

Karlsson et al. [21] introduced the beta Liu estimator (BLE), given by

$$\hat{\beta}_{\mathrm{BLE}} = (A + I_r)^{-1}(A + dI_r)\hat{\beta}_{\mathrm{BMLE}}, \quad 0 < d < 1, \tag{2.11}$$

where d is a Liu shrinkage parameter in the range $(0, 1)$. When $d = 1$, the estimator simplifies to $\hat{\beta}_{\mathrm{BMLE}}$.

Amin et al. [22] developed a James-Stein estimator, known as the beta James-Stein estimator (BJSE), which is defined as follows:

$$\hat{\beta}_{\mathrm{BJSE}} = c\hat{\beta}_{\mathrm{BMLE}}, \quad 0 < c < 1, \tag{2.12}$$

where

$$c = \frac{\left(\hat{\beta}^T_{\mathrm{BMLE}}\hat{\beta}_{\mathrm{BMLE}}\right)}{\left(\hat{\beta}^T_{\mathrm{BMLE}}\hat{\beta}_{\mathrm{BMLE}} + \mathrm{trace}(A)^{-1}\right)}.$$

If $c = 1$, the estimator reduces to $\hat{\beta}_{\mathrm{BMLE}}$.

Algamal and Abonazel [29] proposed the beta Liu-type estimator (BLTE), which is given by

$$\hat{\beta}_{\mathrm{BLTE}} = (A + kI_r)^{-1}(A - dI_r)\hat{\beta}_{\mathrm{BMLE}}, \quad -\infty < d < \infty, \;\; k > 0, \tag{2.13}$$

where d and k are Liu-type shrinkage parameters. When $d = 0$, the estimator simplifies to $\hat{\beta}_{\mathrm{BRRE}}$, and when both $d = k = 0$, it reduces to $\hat{\beta}_{\mathrm{BMLE}}$.

Abonazel et al. [30] introduced the beta Özkale-Kaciranlar estimator (BOKE) as follows:

$$\hat{\beta}_{\mathrm{BOKE}} = (A + kI_r)^{-1}(A + kdI_r)\hat{\beta}_{\mathrm{BMLE}}, \quad 0 < d < 1, \;\; k > 0, \tag{2.14}$$

where d and k are shrinkage parameters. If $k = 1$, then

$$\hat{\beta}_{\mathrm{BOKE}} = \hat{\beta}_{\mathrm{BLE}};$$

if $d = 0$, then

$$\hat{\beta}_{\mathrm{BOKE}} = \hat{\beta}_{\mathrm{BRRE}};$$

and if both $d = k = 0$, then

$$\hat{\beta}_{\mathrm{BOKE}} = \hat{\beta}_{\mathrm{BMLE}}.$$

Abonazel et al. [31] proposed the beta Dawoud-Kibria estimator (BDKE) to address severe multicollinearity, defined as

$$\hat{\beta}_{\mathrm{BDKE}} = (A + k(1 + d)I_r)^{-1}(A - k(1 + d)I_r)\hat{\beta}_{\mathrm{BMLE}}, \quad 0 < d < 1, \;\; k > 0, \tag{2.15}$$

where d and k are the Dawoud-Kibria shrinkage parameters. If $k = 0$, then

$$\hat{\beta}_{\mathrm{BDKE}} = \hat{\beta}_{\mathrm{BMLE}}.$$

Akram et al. [32] introduced the beta-modified ridge-type estimator (BMRTE), given by

$$\hat{\beta}_{\mathrm{BMRTE}} = (A + k(1 + d)I_r)^{-1}A\hat{\beta}_{\mathrm{BMLE}}, \quad 0 < d < 1,\ k > 0, \tag{2.16}$$

where d and k are modified ridge-type shrinkage parameters. If $k = 0$, then

$$\hat{\beta}_{\mathrm{BMRTE}} = \hat{\beta}_{\mathrm{BMLE}},$$

and if $d = 0$, then

$$\hat{\beta}_{\mathrm{BMRTE}} = \hat{\beta}_{\mathrm{BRRE}}.$$

Koç and Dünder [25] proposed the beta Kibria-Lukman estimator (BKLE), which is given by

$$\hat{\beta}_{\mathrm{BKLE}} = (A + kI_r)^{-1}(A - kI_r)\hat{\beta}_{\mathrm{BMLE}}, \tag{2.17}$$

where k is the Kibria-Lukman shrinkage parameter. If $k = 0$, then

$$\hat{\beta}_{\mathrm{BKLE}} = \hat{\beta}_{\mathrm{BMLE}}.$$

Farghali [26] presented the jackknifed beta ridge regression estimator (BJRRE) as follows:

$$\hat{\beta}_{\mathrm{BJRRE}} = (I_r - k^2(A + kI_r)^{-2})\hat{\beta}_{\mathrm{BMLE}}, \qquad k > 0, \tag{2.18}$$

where k is the jackknifed ridge parameter. If $k = 0$, then $\hat{\beta}_{\mathrm{BJRRE}}$ reduces to $\hat{\beta}_{\mathrm{BMLE}}$.

Koç and Dünder [25] developed a beta jackknife Kibria-Lukman estimator (BJKLE) to handle multicollinearity, defined as

$$\hat{\beta}_{\mathrm{BJKLE}} = (I_r + 2k(A + kI_r)^{-1})(I_r - (2k(A + kI_r)^{-1})^2)\hat{\beta}_{\mathrm{BMLE}}, \qquad k > 0, \tag{2.19}$$

where k is the Jackknife Kibria-Lukman parameter. If $k = 0$, then $\hat{\beta}_{\mathrm{BJKLE}}$ reduces to $\hat{\beta}_{\mathrm{BMLE}}$.

The MMSE and MSE for the previously mentioned estimators are defined as follows:

$$\mathrm{MMSE}(\hat{\beta}_{\mathrm{BRRE}}) = \varphi\xi\Lambda_k^{-1}\Lambda\Lambda_k^{-1}\xi^T + k^2\xi\Lambda_k^{-1}\alpha\alpha^T\Lambda_k^{-1}\xi^T, \tag{2.20}$$

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BRRE}}) = \varphi\sum_{j=1}^{r}\frac{\lambda_j}{(\lambda_j + k)^2} + k^2\sum_{j=1}^{r}\frac{\alpha_j^2}{(\lambda_j + k)^2}, \tag{2.21}$$

$$\mathrm{MMSE}(\hat{\beta}_{\mathrm{BLE}}) = \varphi\xi\Lambda_1^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_1^{-1}\xi^T + (d-1)^2\xi\Lambda_1^{-1}\alpha\alpha^T\Lambda_1^{-1}\xi^T, \tag{2.22}$$

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BLE}}) = \varphi\sum_{j=1}^{r}\frac{(\lambda_j + d)^2}{\lambda_j(\lambda_j + 1)^2} + (d-1)^2\sum_{j=1}^{r}\frac{\alpha_j^2}{(\lambda_j + 1)^2}, \tag{2.23}$$

$$\mathrm{MMSE}(\hat{\beta}_{\mathrm{BJSE}}) = \varphi c\Lambda^{-1}c^T + \left(-\frac{\Lambda^{-1}}{\xi^T\alpha^T\alpha\xi + \Lambda^{-1}}\xi\alpha\right)^T\left(-\frac{\Lambda^{-1}}{\xi^T\alpha^T\alpha\xi + \Lambda^{-1}}\xi\alpha\right), \tag{2.24}$$

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BJSE}}) = \varphi\sum_{j=1}^{r}\frac{\alpha_j^4\lambda_j}{(\alpha_j^2\lambda_j + 1)^2} + \sum_{j=1}^{r}\frac{\alpha_j^2}{(\alpha_j^2\lambda_j + 1)^2}, \tag{2.25}$$

$$\mathrm{MMSE}(\hat{\beta}_{\mathrm{BLTE}}) = \varphi\xi\Lambda_k^{-1}\Lambda_{d_0}\Lambda^{-1}\Lambda_{d_0}\Lambda_k^{-1}\xi^T + (d+k)^2\xi\Lambda_k^{-1}\alpha\alpha^T\Lambda_k^{-1}\xi^T, \tag{2.26}$$

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BLTE}}) = \varphi \sum_{j=1}^{r} \frac{(\lambda_j - d)^2}{\lambda_j(\lambda_j + k)^2} + (d + k)^2 \sum_{j=1}^{r} \frac{\alpha_j^2}{(\lambda_j + k)^2}, \tag{2.27}$$

$$\mathrm{MMSE}(\hat{\beta}_{\mathrm{BOKE}}) = \varphi\xi\Lambda_k^{-1}\Lambda_{kd}\Lambda^{-1}\Lambda_{kd}\Lambda_k^{-1}\xi^T + k^2(d-1)^2\xi\Lambda_k^{-1}\alpha\alpha^T\Lambda_k^{-1}\xi^T, \tag{2.28}$$

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BOKE}}) = \varphi \sum_{j=1}^{r} \frac{(\lambda_j + kd)^2}{\lambda_j(\lambda_j + k)^2} + k^2(d-1)^2 \sum_{j=1}^{r} \frac{\alpha_j^2}{(\lambda_j + k)^2}, \tag{2.29}$$

$$\mathrm{MMSE}(\hat{\beta}_{\mathrm{BDKE}}) = \varphi\xi\Lambda_F^{-1}\Lambda_T\Lambda^{-1}\Lambda_T\Lambda_F^{-1}\xi^T + 4k^2(d+1)^2\xi\Lambda_F^{-1}\alpha\alpha^T\Lambda_F^{-1}\xi^T, \tag{2.30}$$

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BDKE}}) = \varphi \sum_{j=1}^{r} \frac{(\lambda_j - k(d+1))^2}{\lambda_j(\lambda_j + k(d+1))^2} + \sum_{j=1}^{r} \frac{4k^2(d+1)^2\alpha_j^2}{(\lambda_j + k(d+1))^2}, \tag{2.31}$$

$$\mathrm{MMSE}(\hat{\beta}_{\mathrm{BMRTE}}) = \varphi\xi\Lambda_F^{-1}\Lambda\Lambda_F^{-1}\xi^T + k^2(d+1)^2\xi\Lambda_F^{-1}\alpha\alpha^T\Lambda_F^{-1}\xi^T, \tag{2.32}$$

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BMRTE}}) = \varphi \sum_{j=1}^{r} \frac{\lambda_j}{(\lambda_j + k(d+1))^2} + \sum_{j=1}^{r} \frac{k^2(d+1)^2\alpha_j^2}{(\lambda_j + k(d+1))^2}, \tag{2.33}$$

$$\mathrm{MMSE}(\hat{\beta}_{\mathrm{BKLE}}) = \varphi\xi\Lambda_k^{-1}\Lambda_{k_0}\Lambda^{-1}\Lambda_{k_0}\Lambda_k^{-1}\xi^T + 4k^2\xi\Lambda_k^{-1}\alpha\alpha^T\Lambda_k^{-1}\xi^T, \tag{2.34}$$

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BKLE}}) = \varphi \sum_{j=1}^{r} \frac{(\lambda_j - k)^2}{\lambda_j(\lambda_j + k)^2} + 4k^2 \sum_{j=1}^{r} \frac{\alpha_j^2}{(\lambda_j + k)^2}, \tag{2.35}$$

$$\begin{aligned}\mathrm{MMSE}\left(\hat{\beta}_{\mathrm{BJRRE}}\right) &= \varphi\xi\Lambda^{-1}(I_r - (\Lambda + kI_r)^{-2}k^2)^2\xi^T + k^4\xi(\Lambda + kI_r)^{-2} \\ &\quad \times \alpha\alpha^T((\Lambda + kI_r)^{-2})^T\xi^T,\end{aligned} \tag{2.36}$$

$$\mathrm{MSE}\left(\hat{\beta}_{\mathrm{BJRRE}}\right) = \varphi \sum_{j=1}^{r} \frac{\left(\lambda_j^2 + 2\lambda_j k\right)^2}{(\lambda_j + k)^4\lambda_j} + k^4 \sum_{j=1}^{r} \frac{\alpha_j^2}{(\lambda_j + k)^4}, \tag{2.37}$$

$$\begin{aligned}\mathrm{MMSE}\left(\hat{\beta}_{\mathrm{BJKLE}}\right) &= \varphi\xi(I_r - (2k(\Lambda + kI_r)^{-1})^2)^2\Lambda(I_r - (2k(\Lambda + kI_r)^{-1})^2)\xi^T \\ &\quad + \Big[(I_r + 2k(\Lambda + kI_r)^{-1})(I_r - (2k(\Lambda + kI_r)^{-1})^2) - I\Big]\beta\beta^T\Big[(I_r + 2k(\Lambda + kI_r)^{-1}) \\ &\quad \times (I_r - (2k(\Lambda + kI_r)^{-1})^2) - I\Big],\end{aligned} \tag{2.38}$$

$$\begin{aligned}\mathrm{MSE}\left(\hat{\beta}_{\mathrm{BJKLE}}\right) &= \varphi \sum_{j=1}^{r} \frac{\left(\left(\lambda_j + k\right)^2 - 4k^2\right)^2(\lambda_j - k)^2}{(\lambda_j + k)^6\lambda_j} \\ &\quad + \sum_{j=1}^{r} \frac{\left((\lambda_j - k)^2(\lambda_j + 3k) - (\lambda_j + k)^3\right)^2\alpha_j^2}{(\lambda_j + k)^6},\end{aligned} \tag{2.39}$$

where

$$\begin{aligned}\Lambda_k &= \mathrm{diag}(\lambda_1 + k, \lambda_2 + k, ..., \lambda_p + k), \\ \Lambda_d &= \mathrm{diag}(\lambda_1 + d, \lambda_2 + d, ..., \lambda_p + d), \\ \Lambda_1 &= \mathrm{diag}(\lambda_1 + 1, \lambda_2 + 1, ..., \lambda_p + 1), \\ \Lambda_{kd} &= \mathrm{diag}(\lambda_1 + kd, \lambda_2 + kd, ..., \lambda_p + kd), \\ \Lambda_{k_0} &= \mathrm{diag}(\lambda_1 - k, \lambda_2 - k, ..., \lambda_p - k),\end{aligned}$$

$$\begin{aligned}
\Lambda_F &= \text{diag}(\lambda_1 + k(1+d), \lambda_2 + k(1+d), ..., \lambda_p + k(1+d)),\\
\Lambda_T &= \text{diag}(\lambda_1 - k(1+d), \lambda_2 - k(1+d), ..., \lambda_p - k(1+d)),\\
\Lambda_{d_0} &= \text{diag}(\lambda_1 - d, \lambda_2 - d, ..., \lambda_p - d).
\end{aligned}$$

3. Proposed estimators

Although biased estimators have made significant progress in addressing multicollinearity in BRMs, they introduce bias in exchange for reducing variance through shrinkage parameters. However, large values of these parameters lead to considerable bias, which can negatively impact their stability and reliability. Therefore, researchers have proposed an alternative estimator known as the almost-unbiased estimator. Several researchers, including Chang [33], Wu et al. [34], and Jegede et al. [16], have studied these estimators. In this study, we introduce the almost unbiased modified ridge-type estimator for the BRM, using the following definition.

Definition 1. [35] Let $\hat{\beta}$ be a biased estimator of the parameter β, with bias given by

$$Bias(\hat{\beta}) = E(\hat{\beta}) - \beta = C\beta,$$

which implies that

$$E(\hat{\beta} - C\beta) = \beta.$$

Then, the estimator

$$\hat{\beta} = \hat{\beta} - C\beta = (I_r - C)\hat{\beta}$$

is called the almost unbiased estimator derived from the biased estimator $\hat{\beta}$.

Following the work of Omara [36], we introduce the beta almost unbiased modified ridge-type estimator (BAUMRTE) as follows:

$$\begin{aligned}
\hat{\beta}_{\text{BAUMRTE}} &= [I_r - \left((A + k(1+d)I_r)^{-1}A - I_r\right)]\hat{\beta}_{\text{BMRTE}}\\
&= [I_r - \left((A + k(1+d)I_r)^{-1}A - I_r\right)(A + k(1+d)I_r)^{-1}A\hat{\beta}_{\text{BMLE}}\\
&= \left[2I_r - (A + k(1+d)I_r)^{-1}A\right](A + k(1+d)I_r)^{-1}A\hat{\beta}_{\text{BMLE}}\\
&= [I_r - \left((A + k(1+d)I_r)^{-2}k^2(1+d)^2\right)]\hat{\beta}_{\text{BMLE}} = H\hat{\beta}_{\text{BMLE}},
\end{aligned} \tag{3.1}$$

where

$$H = [I_r - \left((A + k(1+d)I_r)^{-2}k^2(1+d)^2\right)],$$

k ($k > 0$), and d ($0 < d < 1$) are the almost unbiased modified ridge-type parameters. The BAUMRTE(0,0) reduces to BMLE and the BAUMRTE$(1, k/2)$ reduces to BJRRE. The bias and the variance of the BAUMRTE can be found as

$$Bias\left(\hat{\beta}_{\text{BAUMRTE}}\right) = E\left(\hat{\beta}_{\text{BAUMRTE}}\right) - \beta = E\left(H\hat{\beta}_{\text{BMLE}}\right) - \beta = [H - I_r]\beta \tag{3.2}$$

and

$$\begin{aligned}
\text{Cov}\left(\hat{\beta}_{\text{BAUMRTE}}\right) &= E\left(\left[\hat{\beta}_{\text{BAUMRTE}} - E(\hat{\beta}_{\text{BAUMRTE}})\right]\left[\hat{\beta}_{\text{BAUMRTE}} - E(\hat{\beta}_{\text{BAUMRTE}})\right]^T\right)\\
&= [H - I_r]\text{Cov}(\beta_{\text{BMLE}})[H - I_r]^T = [H - I_r]\varphi(\Lambda)^{-1}[H - I_r]^T.
\end{aligned} \tag{3.3}$$

Using Eqs (3.2) and (3.3), the MMSE and MSE of BAUMRTE can be simplified as

$$\begin{aligned} \mathrm{MMSE}\Big(\hat{\beta}_{\mathrm{BAUMRTE}}\Big) &= \mathrm{Cov}\Big(\hat{\beta}_{\mathrm{BAUMRTE}}\Big) + \mathrm{Bias}\Big(\hat{\beta}_{\mathrm{BAUMRTE}}\Big)\mathrm{Bias}\Big(\hat{\beta}_{\mathrm{BAUMRTE}}\Big)^T \\ &= \varphi[H - I_r](\Lambda)^{-1}[H - I_r]^T + [H - I_r]\beta\beta^T[H - I_r]^T \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \mathrm{MSE}\Big(\hat{\beta}_{\mathrm{BAUMRTE}}\Big) &= \mathrm{Tr}\left(\mathrm{MMSE}\Big(\hat{\beta}_{\mathrm{BAUMRTE}}\Big)\right) \\ &= \varphi \sum_{j=1}^{r} \frac{1}{\lambda_j}\left(1 - \frac{k^2(1+d)^2}{(\lambda_j + k(1+d)^2}\right)^2 + \sum_{j=1}^{r}\left(\frac{k^2(1+d)^2}{(\lambda_j + k(1+d))^2}\right)^2 \alpha_j^2. \end{aligned} \tag{3.5}$$

The MSE of the BAUMRTE depends on the choice of k and d. Selecting appropriate values of k and d helps achieve the minimum MSE for the BAUMRTE. For more details, see [36]. Therefore, in Subsection 3.2, we propose new methods for estimating parameters.

3.1. Superiority of the proposed estimators

The following lemma and theorem will be used to prove some theorems in this section.

Lemma 3.1. *Assuming B is a positive definite (pd) matrix, c is a positive constant, and γ is a vector of nonzero constants, the expression*

$$cB - \gamma\gamma^T > 0$$

holds true iff $\gamma B \gamma^T < c$ [37].

Theorem 3.1. *Let $k > 0$, $0 < d < 1$, and*

$$(\lambda_j + k(1+d))^4 > \left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2,$$

then $\hat{\beta}_{BAUMRTE}$ is superior to $\hat{\beta}_{BMLE}$ if and only if (iff)

$$MMSE(\hat{\beta}_{BMLE}) - MMSE(\hat{\beta}_{BAUMRTE}) > 0,$$

where

$$R = Bias(\hat{\beta}_{BAUMRTE}).$$

Proof. The difference between $\mathrm{MMSE}(\hat{\beta}_{\mathrm{BMLE}})$ and $\mathrm{MMSE}(\hat{\beta}_{\mathrm{BAUMRTE}})$, as computed using Eq (2.8) and Eq (3.4), is

$$\mathrm{MMSE}(\hat{\beta}_{\mathrm{BMLE}}) - \mathrm{MMSE}(\hat{\beta}_{\mathrm{BAUMRTE}}) = \varphi\xi\Big[\Lambda^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\Big]\xi^T - RR^T. \tag{3.6}$$

Equation (3.6) can be described using the MSE as:

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BMLE}}) - \mathrm{MSE}(\hat{\beta}_{\mathrm{BAUMRTE}}) = \varphi\xi\mathrm{diag}\left(\frac{1}{\lambda_j} - \frac{(\lambda_j^2 + 2\lambda_j k(1+d))^2}{(\lambda_j + k(1+d))^4\lambda_j}\right)\xi^T - RR^T.$$

The matrix $\left[\Lambda^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd iff

$$(\lambda_j + k(1+d))^4 - \left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 > 0,$$

which is equivalent to $(\lambda_j + k(1+d))^4 > \left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2$ being non-negative. Therefore the matrix $\left[\Lambda^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd, and the proof is completed by Lemma 3.1. □

Theorem 3.2. *Let $k > 0$, $0 < d < 1$, and*

$$\lambda_j^2(\lambda_j + k(1+d))^4 > \left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 (\lambda_j + k)^2,$$

then $\hat{\beta}_{BAUMRTE}$ is superior to $\hat{\beta}_{BRRE}$ iff

$$MMSE(\hat{\beta}_{BRRE}) - MMSE(\hat{\beta}_{BAUMRTE}) > 0,$$

where

$$R_{Ridge} = Bias(\hat{\beta}_{BRRE})$$

and

$$R = Bias(\hat{\beta}_{BAUMRTE}).$$

Proof. The difference between $\text{MMSE}(\hat{\beta}_{\text{BRRE}})$ and $\text{MMSE}(\hat{\beta}_{\text{BAUMRTE}})$, as computed using Eqs (2.20) and (3.4), is

$$\begin{aligned}\text{MMSE}(\hat{\beta}_{\text{BRRE}}) - \text{MMSE}(\hat{\beta}_{\text{BAUMRTE}}) =& \varphi\xi\left[\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]\xi^T \\ &+ R_{Ridge}R_{Ridge}^T - RR^T.\end{aligned} \tag{3.7}$$

Equation (3.7) can be described using the MSE as:

$$\text{MSE}(\hat{\beta}_{\text{BRRE}}) - \text{MSE}(\hat{\beta}_{\text{BAUMRTE}}) = \varphi\xi\text{diag}\left(\frac{\lambda_j}{(\lambda_j + k)^2} - \frac{(\lambda_j^2 + 2\lambda_j k(1+d))^2}{(\lambda_j + k(1+d))^4\lambda_j}\right)\xi^T + R_{Ridge}R_{Ridge}^T - RR^T.$$

The matrix $\left[\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd iff

$$\lambda_j^2(\lambda_j + k(1+d))^4 - \left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 (\lambda_j + k)^2 > 0,$$

which is equivalent to

$$\lambda_j^2(\lambda_j + k(1+d))^4 > \left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 (\lambda_j + k)^2$$

being non-negative. Therefore the matrix $\left[\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd, and the proof is completed by Lemma 3.1. □

Theorem 3.3. *Let $k > 0$, $0 < d < 1$, and*

$$(\lambda_j + d)^2(\lambda_j + k(1+d))^4 > \left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 (\lambda_j + 1)^2,$$

then $\hat{\beta}_{BAUMRTE}$ is superior to $\hat{\beta}_{BLE}$ iff

$$MMSE(\hat{\beta}_{BLE}) - MMSE(\hat{\beta}_{BAUMRTE}) > 0,$$

where

$$R_{Liu} = Bias(\hat{\beta}_{BLE})$$

and

$$R = Bias(\hat{\beta}_{BAUMRTE}).$$

Proof. The difference between MMSE($\hat{\beta}_{\mathrm{BLE}}$) and MMSE($\hat{\beta}_{\mathrm{BAUMRTE}}$), as computed using Eqs (2.22) and (3.4), is

$$\begin{aligned}\mathrm{MMSE}(\hat{\beta}_{\mathrm{BLE}}) - \mathrm{MMSE}(\hat{\beta}_{\mathrm{BAUMRTE}}) = \varphi\xi\Big[&\Lambda_1^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_1^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\Big]\xi^T \\ &+ R_{Liu}R_{Liu}^T - RR^T.\end{aligned} \tag{3.8}$$

Equation (3.8) can be described using the MSE as:

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BLE}}) - \mathrm{MSE}(\hat{\beta}_{\mathrm{BAUMRTE}}) = \varphi\xi\mathrm{diag}\left(\frac{(\lambda_j + d)^2}{\lambda_j(\lambda_j + 1)^2} - \frac{(\lambda_j^2 + 2\lambda_j k(1+d))^2}{(\lambda_j + k(1+d))^4\lambda_j}\right)\xi^T + R_{Liu}R_{Liu}^T - RR^T.$$

The matrix $\left[\Lambda_1^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_1^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd iff

$$(\lambda_j + d)^2(\lambda_j + k(1+d))^4 - \left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 (\lambda_j + 1)^2 > 0,$$

which is equivalent to

$$(\lambda_j + d)^2(\lambda_j + k(1+d))^4 > \left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 (\lambda_j + 1)^2$$

being non-negative. Therefore, the matrix $\left[\Lambda_1^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_1^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd, and the proof is completed by Lemma 3.1. □

Theorem 3.4. *Let $k > 0$, $0 < d < 1$, and*

$$\alpha_j^4\lambda_j^2(\lambda_j + k(1+d))^4 > \left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 (\alpha_j^2\lambda_j + 1)^2,$$

then $\hat{\beta}_{BAUMRTE}$ is superior to $\hat{\beta}_{BJSE}$ iff

$$MMSE(\hat{\beta}_{BJSE}) - MMSE(\hat{\beta}_{BAUMRTE}) > 0,$$

where

$$R_{JSE} = Bias(\hat{\beta}_{BJSE})$$

and

$$R = Bias(\hat{\beta}_{BAUMRTE}).$$

Proof. The difference between MMSE($\hat{\beta}_{\text{BJSE}}$) and MMSE($\hat{\beta}_{\text{BAUMRTE}}$), as computed using Eqs (2.24) and (3.4), is

$$\begin{aligned}\text{MMSE}(\hat{\beta}_{\text{BJSE}}) - \text{MMSE}(\hat{\beta}_{\text{BAUMRTE}}) = \varphi\xi\Big[c\Lambda^{-1}c^T - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\Big]\xi^T \\ + R_{JSE}R_{JSE}^T - RR^T.\end{aligned} \tag{3.9}$$

Equation (3.9) can be described using the MSE as:

$$\text{MSE}(\hat{\beta}_{\text{BJSE}}) - \text{MSE}(\hat{\beta}_{\text{BAUMRTE}}) = \varphi\xi\text{diag}\left(\frac{\alpha_j^4\lambda_j}{(\alpha_j^2\lambda_j+1)^2} - \frac{(\lambda_j^2 + 2\lambda_j k(1+d))^2}{(\lambda_j + k(1+d))^4\lambda_j}\right)\xi^T + R_{JSE}R_{JSE}^T - RR^T.$$

The matrix $\left[c\Lambda^{-1}c^T - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd iff

$$\alpha_j^4\lambda_j^2(\lambda_j + k(1+d))^4 - \left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2(\alpha_j^2\lambda_j+1)^2 > 0,$$

which is equivalent to

$$\alpha_j^4\lambda_j^2(\lambda_j + k(1+d))^4 > \left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2(\alpha_j^2\lambda_j+1)^2$$

being non-negative. Therefore the matrix $\left[c\Lambda^{-1}c^T - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd , and the proof is completed by Lemma 3.1. □

Theorem 3.5. *Let $k > 0$, $0 < d < 1$, and*

$$(\lambda_j - d)^2(\lambda_j + k(1+d))^4 > (\lambda_j + k)^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2,$$

then $\hat{\beta}_{BAUMRTE}$ is superior to $\hat{\beta}_{BLTE}$ iff

$$MMSE(\hat{\beta}_{BLTE}) - MMSE(\hat{\beta}_{BAUMRTE}) > 0,$$

where

$$R_{LT} = Bias(\hat{\beta}_{BLTE})$$

and

$$R = Bias(\hat{\beta}_{BAUMRTE}).$$

Proof. The difference between MMSE($\hat{\beta}_{\text{BLTE}}$) and MMSE($\hat{\beta}_{\text{BAUMRTE}}$), as computed using Eqs (2.26) and (3.4), is

$$\begin{aligned}\text{MMSE}(\hat{\beta}_{\text{BLTE}}) - \text{MMSE}(\hat{\beta}_{\text{BAUMRTE}}) = \varphi\xi\Big[\Lambda_k^{-1}\Lambda_{d_0}\Lambda^{-1}\Lambda_{d_0}\Lambda_k^{-1} - \Lambda^{-1}\times(I_r \\ - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\Big]\xi^T + R_{LT}R_{LT}^T - RR^T.\end{aligned} \tag{3.10}$$

Equation (3.10) can be described using the MSE as:

$$\text{MSE}(\hat{\beta}_{\text{BLTE}}) - \text{MSE}(\hat{\beta}_{\text{BAUMRTE}}) = \varphi\xi\text{diag}\left(\frac{(\lambda_j - d)^2}{\lambda_j(\lambda_j+k)^2} - \frac{(\lambda_j^2 + 2\lambda_j k(1+d))^2}{(\lambda_j + k(1+d))^4\lambda_j}\right)\xi^T + R_{LT}R_{LT}^T - RR^T.$$

The matrix $\left[\Lambda_k^{-1}\Lambda_{d_0}\Lambda^{-1}\Lambda_{d_0}\Lambda_k^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd iff

$$(\lambda_j - d)^2(\lambda_j + k(1+d))^4 - (\lambda_j + k)^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 > 0,$$

which is equivalent to

$$(\lambda_j - d)^2(\lambda_j + k(1+d))^4 > (\lambda_j + k)^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2$$

being non-negative. Therefore the matrix $\left[\Lambda_k^{-1}\Lambda_{d_0}\Lambda^{-1}\Lambda_{d_0}\Lambda_k^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd, and the proof is completed by Lemma 3.1. □

Theorem 3.6. *Let $k > 0$, $0 < d < 1$, and*

$$(\lambda_j + kd)^2(\lambda_j + k(1+d))^4 > (\lambda_j + k)^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2,$$

then $\hat{\beta}_{BAUMRTE}$ is superior to $\hat{\beta}_{BOKE}$ iff

$$MMSE(\hat{\beta}_{BOKE}) - MMSE(\hat{\beta}_{BAUMRTE}) > 0,$$

where

$$R_{OK} = Bias(\hat{\beta}_{BOKE})$$

and

$$R = Bias(\hat{\beta}_{BAUMRTE}).$$

Proof. The difference between MMSE($\hat{\beta}_{\text{BOKE}}$) and MMSE($\hat{\beta}_{\text{BAUMRTE}}$), as computed using Eqs (2.28) and (3.4), is

$$\begin{aligned}\text{MMSE}(\hat{\beta}_{\text{BOKE}}) - \text{MMSE}(\hat{\beta}_{\text{BAUMRTE}}) = \varphi\xi\Big[&\Lambda_k^{-1}\Lambda_{kd}\Lambda^{-1}\Lambda_{kd}\Lambda_k^{-1} - \times\Lambda^{-1}(I_r \\ &- (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\Big]\xi^T + R_{OK}R_{OK}^T - RR^T.\end{aligned} \tag{3.11}$$

Equation (3.11) can be described using the MSE as

$$\text{MSE}(\hat{\beta}_{\text{BOKE}}) - \text{MSE}(\hat{\beta}_{\text{BAUMRTE}}) = \varphi\xi\text{diag}\left(\frac{(\lambda_j + kd)^2}{\lambda_j(\lambda_j + k)^2} - \frac{(\lambda_j^2 + 2\lambda_j k(1+d))^2}{(\lambda_j + k(1+d))^4\lambda_j}\right)\xi^T + R_{OK}R_{OK}^T - RR^T.$$

The matrix $\left[\Lambda_k^{-1}\Lambda_{kd}\Lambda^{-1}\Lambda_{kd}\Lambda_k^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd iff

$$(\lambda_j + kd)^2(\lambda_j + k(1+d))^4 - (\lambda_j + k)^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 > 0,$$

which is equivalent to

$$(\lambda_j + kd)^2(\lambda_j + k(1+d))^4 > (\lambda_j + k)^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2$$

being non-negative. Therefore the matrix $\left[\Lambda_k^{-1}\Lambda_{kd}\Lambda^{-1}\Lambda_{kd}\Lambda_k^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd, and the proof is completed by Lemma 3.1. □

Theorem 3.7. *Let $k > 0$, $0 < d < 1$, and*

$$(\lambda_j - k(d+1))^2(\lambda_j + k(1+d))^4 > (\lambda_j + k(d+1))^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2,$$

then $\hat{\beta}_{BAUMRTE}$ is superior to $\hat{\beta}_{BDKE}$ iff

$$MMSE(\hat{\beta}_{BDKE}) - MMSE(\hat{\beta}_{BAUMRTE}) > 0,$$

where

$$R_{DK} = Bias(\hat{\beta}_{BDKE})$$

and

$$R = Bias(\hat{\beta}_{BAUMRTE}).$$

Proof. The difference between MMSE($\hat{\beta}_{\mathrm{BDKE}}$) and MMSE($\hat{\beta}_{\mathrm{BAUMRTE}}$), as computed using Eqs (2.30) and (3.4), is

$$\begin{aligned}\mathrm{MMSE}(\hat{\beta}_{\mathrm{BDKE}}) - \mathrm{MMSE}(\hat{\beta}_{\mathrm{BAUMRTE}}) = \varphi\xi\Big[&\Lambda_F^{-1}\Lambda_T\Lambda^{-1}\Lambda_T\Lambda_F^{-1} - \Lambda^{-1}(I_r \\ &- (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\Big]\xi^T + R_{DK}R_{DK}^T - RR^T.\end{aligned} \tag{3.12}$$

Equation (3.12) can be described using the MSE as

$$\begin{aligned}\mathrm{MSE}(\hat{\beta}_{\mathrm{BDKE}}) - \mathrm{MSE}(\hat{\beta}_{\mathrm{BAUMRTE}}) =&\varphi\xi\mathrm{diag}\left(\frac{(\lambda_j - k(d+1))^2}{\lambda_j(\lambda_j + k(d+1))^2} - \frac{(\lambda_j^2 + 2\lambda_j k(1+d))^2}{(\lambda_j + k(1+d))^4\lambda_j}\right)\xi^T \\ &+ R_{DK}R_{DK}^T - RR^T.\end{aligned}$$

The matrix $\left[\Lambda_F^{-1}\Lambda_T\Lambda^{-1}\Lambda_T\Lambda_F^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd iff

$$(\lambda_j - k(d+1))^2(\lambda_j + k(1+d))^4 - (\lambda_j + k(d+1))^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 > 0,$$

which is equivalent to

$$(\lambda_j - k(d+1))^2(\lambda_j + k(1+d))^4 > (\lambda_j + k(d+1))^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2$$

being non-negative. Therefore the matrix $\left[\Lambda_F^{-1}\Lambda_T\Lambda^{-1}\Lambda_T\Lambda_F^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd , and the proof is completed by Lemma 3.1. □

Theorem 3.8. *Let $k > 0$, $0 < d < 1$, and*

$$\lambda_j^2(\lambda_j + k(1+d))^4 > (\lambda_j + k(d+1))^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2,$$

then $\hat{\beta}_{BAUMRTE}$ is superior to $\hat{\beta}_{BMRTE}$ iff

$$MMSE(\hat{\beta}_{BMRTE}) - MMSE(\hat{\beta}_{BAUMRTE}) > 0,$$

where

$$R_{MRT} = Bias(\hat{\beta}_{BMRTE})$$

and

$$R = Bias(\hat{\beta}_{BAUMRTE}).$$

Proof. The difference between MMSE($\hat{\beta}_{\text{BMRTE}}$) and MMSE($\hat{\beta}_{\text{BAUMRTE}}$), as computed using Eqs (2.32) and (3.4), is

$$\begin{aligned}\text{MMSE}(\hat{\beta}_{\text{BMRTE}}) - \text{MMSE}(\hat{\beta}_{\text{BAUMRTE}}) =&\varphi\xi\Big[\Lambda_F^{-1}\Lambda\Lambda_F^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2} \\ &(k(1+d))^2)^2\Big]\xi^T + R_{MRT}R_{MRT}^T - RR^T.\end{aligned} \tag{3.13}$$

Equation (3.13) can be described using the MSE as:

$$\begin{aligned}\text{MSE}(\hat{\beta}_{\text{BMRTE}}) - \text{MSE}(\hat{\beta}_{\text{BAUMRTE}}) =&\varphi\xi\text{diag}\left(\frac{\lambda_j}{(\lambda_j + k(d+1))^2} - \frac{(\lambda_j^2 + 2\lambda_j k(1+d))^2}{(\lambda_j + k(1+d))^4\lambda_j}\right)\xi^T \\ &+ R_{MRT}R_{MRT}^T - RR^T.\end{aligned} \tag{3.14}$$

The matrix $\left[\Lambda_F^{-1}\Lambda\Lambda_F^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd iff

$$\lambda_j^2(\lambda_j + k(1+d))^4 - (\lambda_j + k(d+1))^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 > 0,$$

which is equivalent to

$$\lambda_j^2(\lambda_j + k(1+d))^4 > (\lambda_j + k(d+1))^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2$$

being non-negative. Therefore the matrix $\left[\Lambda_F^{-1}\Lambda\Lambda_F^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd, and the proof is completed by Lemma 3.1. □

Theorem 3.9. *Let $k > 0$, $0 < d < 1$, and*

$$(\lambda_j - k)^2(\lambda_j + k(1+d))^4 > (\lambda_j + k)^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2,$$

then $\hat{\beta}_{BAUMRTE}$ is superior to $\hat{\beta}_{BKLE}$ iff

$$MMSE(\hat{\beta}_{BKLE}) - MMSE(\hat{\beta}_{BAUMRTE}) > 0, \quad R_{KL} = Bias(\hat{\beta}_{BKLE})$$

and

$$R = Bias(\hat{\beta}_{BAUMRTE}).$$

Proof. The difference between MMSE($\hat{\beta}_{\text{BKLE}}$) and MMSE($\hat{\beta}_{\text{BAUMRTE}}$), as computed using Eqs (2.34) and (3.4), is

$$\begin{aligned}\text{MMSE}(\hat{\beta}_{\text{BKLE}}) - \text{MMSE}(\hat{\beta}_{\text{BAUMRTE}}) =& \varphi\xi\Big[\Lambda_k^{-1}\Lambda_{k_0}\Lambda^{-1}\Lambda_{k_0}\Lambda_k^{-1} - \Lambda^{-1}(I_r \\ &- (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\Big]\xi^T + R_{KL}R_{KL}^T - RR^T.\end{aligned} \tag{3.15}$$

Equation (3.15) can be described using the MSE as:

$$\text{MSE}(\hat{\beta}_{\text{BKLE}}) - \text{MSE}(\hat{\beta}_{\text{BAUMRTE}}) =\varphi\xi\text{diag}\left(\frac{(\lambda_j - k)^2}{\lambda_j(\lambda_j + k)^2} - \frac{(\lambda_j^2 + 2\lambda_j k(1+d))^2}{(\lambda_j + k(1+d))^4\lambda_j}\right)\xi^T + R_{KL}R_{KL}^T - RR^T.$$

The matrix $\left[\Lambda_k^{-1}\Lambda_{k_0}\Lambda^{-1}\Lambda_{k_0}\Lambda_k^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd iff

$$(\lambda_j - k)^2(\lambda_j + k(1+d))^4 - (\lambda_j + k)^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 > 0,$$

which is equivalent to

$$(\lambda_j - k)^2(\lambda_j + k(1+d))^4 > (\lambda_j + k)^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2$$

being non-negative. Therefore the matrix $\left[\Lambda_k^{-1}\Lambda_{k_0}\Lambda^{-1}\Lambda_{k_0}\Lambda_k^{-1} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd, and the proof is completed by Lemma 3.1. □

Theorem 3.10. *Let $k > 0$, $0 < d < 1$, and*

$$(\lambda_j - k)^2(\lambda_j + k(1+d))^4 > (\lambda_j + k)^2\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2,$$

then $\hat{\beta}_{BAUMRTE}$ is superior to $\hat{\beta}_{BJRRE}$ iff

$$MMSE(\hat{\beta}_{BJRRE}) - MMSE(\hat{\beta}_{BAUMRTE}) > 0, \quad R_{JR} = Bias(\hat{\beta}_{BJRRE})$$

and

$$R = Bias(\hat{\beta}_{BAUMRTE}).$$

Proof. The difference between $\mathrm{MMSE}(\hat{\beta}_{\mathrm{BJRRE}})$ and $\mathrm{MMSE}(\hat{\beta}_{\mathrm{BAUMRTE}})$, as computed using Eqs (2.36) and (3.4), is

$$\begin{aligned}\mathrm{MMSE}(\hat{\beta}_{\mathrm{BJRRE}}) - \mathrm{MMSE}(\hat{\beta}_{\mathrm{BAUMRTE}}) = \varphi\xi\Big[&\Lambda^{-1}(I_r - (\Lambda + kI_r)^{-2}k^2)^{-2} - \Lambda^{-1}(I_r - (\Lambda \\ &+ k(1+d)I_r)^{-2} \times (k(1+d))^2)^2\Big]\xi^T + R_{JR}R_{JR}^T - RR^T.\end{aligned} \tag{3.16}$$

Equation (3.16) can be described using the MSE as:

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BJRRE}}) - \mathrm{MSE}(\hat{\beta}_{\mathrm{BAUMRTE}}) = \varphi\xi\mathrm{diag}\left(\frac{\left(\lambda_j^2 + 2\lambda_j k\right)^2}{(\lambda_j + k)^4\lambda_j} - \frac{(\lambda_j^2 + 2\lambda_j k(1+d))^2}{(\lambda_j + k(1+d))^4\lambda_j}\right)\xi^T + R_{JR}R_{JR}^T - RR^T.$$

The matrix $\left[\Lambda^{-1}(I_r - (\Lambda + kI_r)^{-2}k^2)^2 - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd iff

$$(\lambda_j^2 + 2k\lambda_j)^2(\lambda_j + k(1+d))^4 - (\lambda_j + k)^4\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2 > 0,$$

which is equivalent to

$$(\lambda_j^2 + 2k\lambda_j)^2(\lambda_j + k(1+d))^4 > (\lambda_j + k)^4\left(\lambda_j^2 + 2\lambda_j k(1+d)\right)^2$$

being non-negative. Therefore the matrix $\left[\Lambda^{-1}(I_r - (\Lambda + kI_r)^{-2}k^2)^{-2} - \Lambda^{-1}(I_r - (\Lambda + k(1+d)I_r)^{-2}(k(1+d))^2)^2\right]$ is pd, and the proof is completed by Lemma 3.1. □

Theorem 3.11. *Let $k > 0$, $0 < d < 1$, and*

$$\left(\left(\lambda_j + k\right)^2 - 4k^2\right)^2 (\lambda_j - k)^2(\lambda_j + k(1 + d))^4 > (\lambda_j + k)^6 \left(\lambda_j^2 + 2\lambda_j k(1 + d)\right)^2,$$

then $\hat{\beta}_{BAUMRTE}$ is superior to $\hat{\beta}_{BJKLE}$ iff

$$MMSE(\hat{\beta}_{BJKLE}) - MMSE(\hat{\beta}_{BAUMRTE}) > 0, \quad R_{JKL} = Bias(\hat{\beta}_{BJKLE})$$

and

$$R = Bias(\hat{\beta}_{BAUMRTE}).$$

Proof. The difference between $\mathrm{MMSE}(\hat{\beta}_{\mathrm{BJKLE}})$ and $\mathrm{MMSE}(\hat{\beta}_{\mathrm{BAUMRTE}})$, as computed using Eqs (2.38) and (3.4), is

$$\begin{aligned}\mathrm{MMSE}(\hat{\beta}_{\mathrm{BJKLE}}) - \mathrm{MMSE}(\hat{\beta}_{\mathrm{BAUMRTE}}) = \varphi\xi[(I_r - (2k(\Lambda + kI_r)^{-1})^2)^2\Lambda(I_r - (2k(\Lambda + kI_r)^{-1})^2) - \Lambda^{-1}(I_r \\ - (\Lambda + k(1 + d)I_r)^{-2}(k(1 + d))^2)^2\big]\xi^T + R_{JKL}R_{JKL}^T - RR^T.\end{aligned} \tag{3.17}$$

Equation (3.17) can be described using the MSE as:

$$\begin{aligned}\mathrm{MSE}(\hat{\beta}_{\mathrm{BJKLE}}) - \mathrm{MSE}(\hat{\beta}_{\mathrm{BAUMRTE}}) = \varphi\xi\mathrm{diag}\left(\frac{\left(\left(\lambda_j + k\right)^2 - 4k^2\right)^2 (\lambda_j - k)^2}{(\lambda_j + k)^6\lambda_j} - \frac{(\lambda_j^2 + 2\lambda_j k(1 + d))^2}{(\lambda_j + k(1 + d))^4\lambda_j}\right)\xi^T \\ + R_{JKL}R_{JKL}^T - RR^T.\end{aligned}$$

The matrix $\left[(I_r - (2k(\Lambda + kI_r)^{-1})^2)^2\Lambda(I_r - (2k(\Lambda + kI_r)^{-1})^2) - \Lambda^{-1}(I_r - (\Lambda + k(1 + d)I_r)^{-2}(k(1 + d))^2)^2\right]$ is pd iff

$$\left(\left(\lambda_j + k\right)^2 - 4k^2\right)^2 (\lambda_j - k)^2(\lambda_j + k(1 + d))^4 - (\lambda_j + k)^6 \left(\lambda_j^2 + 2\lambda_j k(1 + d)\right)^2 > 0,$$

which is equivalent to

$$\left(\left(\lambda_j + k\right)^2 - 4k^2\right)^2 (\lambda_j - k)^2(\lambda_j + k(1 + d))^4 > (\lambda_j + k)^6 \left(\lambda_j^2 + 2\lambda_j k(1 + d)\right)^2$$

being non-negative. Therefore the matrix $\left[(I_r - (2k(\Lambda + kI_r)^{-1})^2)^2\Lambda(I_r - (2k(\Lambda + kI_r)^{-1})^2) - \Lambda^{-1}(I_r - (\Lambda + k(1 + d)I_r)^{-2}(k(1 + d))^2)^2\right]$ is pd, and the proof is completed by Lemma 3.1. □

3.2. Selection biasing parameter estimator

Table 1 illustrates three parameter values for the proposed estimator $(\hat{k}_1, \hat{d}_1)$, $(\hat{k}_2, \hat{d}_2)$, and $(\hat{k}_3, \hat{d}_3)$ based on the work of Lukman et al. [24], along with the corresponding parameter estimates for each estimator introduced in the BRM. The effectiveness and strength of the proposed estimator are inherently influenced by the choice of these shrinkage parameters. Specifically, the values of $\hat{k}$ and $\hat{d}$ govern the trade-off between bias and variance: larger values may reduce variance but increase bias, while smaller values may do the opposite. Hence, careful tuning is essential to achieve an optimal balance that leads to superior estimation accuracy, particularly under high multicollinearity. These parameters are essential for both simulation and application purposes. The optimal values for each estimator were selected based on previous studies referenced in Table 1.

Table 1. Summary of biasing parameters for different estimators in BRM.

Estimator	k values	d values	Author
BRRE	$\hat{k} = \frac{\min(\lambda_j)}{\varphi \min(\hat{\alpha}_j^2)}$	-	Abonazel and Taha [19]
BLE	-	$\hat{d} = \max\left(0, \min\left(\frac{\sum_{j=1}^{r}\left(\frac{\varphi^{-1}-\hat{\alpha}_j^2}{(1+\lambda_j)^2}\right)}{\sum_{j=1}^{r}\left(\frac{(\varphi\lambda_j)^{-1}+\hat{\alpha}_j^2}{(1+\lambda_j)^2}\right)}, 1\right)\right)$	Karlsson et al. [21]
BJSE	-	$\hat{c} = \min\left(\frac{\hat{\alpha}_j^2\lambda_j}{\varphi+\hat{\alpha}_j^2\lambda_j}\right)$	Amin et al. [22]
BLTE	$\hat{k} = \sum_{j=1}^{r}\left(\frac{1}{\varphi\hat{\alpha}_j^2}\right)$	$\hat{d} = \max\left(\frac{\sum_{j=1}^{r} \frac{\varphi^{-1}-\hat{k}\hat{\alpha}_j^2}{(\hat{k}+\lambda_j)^2}}{\sum_{j=1}^{r} \frac{\varphi^{-1}+\lambda_j\hat{\alpha}_j^2}{\lambda_j(\hat{k}+\lambda_j)^2}}, 0\right)$	Algamal and Abonazel [29]
BOKE	$\hat{k} = \frac{1}{r}\sum_{j=1}^{r}\left(\frac{\lambda_j}{\varphi(\lambda_j\hat{\alpha}_j^2(1-d))-(d/\varphi)}\right)$	$\hat{d} = \frac{1}{2}\min\left(\frac{\lambda_j\hat{\alpha}_j^2}{(\lambda_j\hat{\alpha}_j^2+(1/\varphi))}\right)$	Abonazel et al. [30]
BDKE	$\hat{k} = \left(\frac{1}{r}\sum_{j=1}^{r}\frac{1}{\varphi(1+\hat{d})\left(\frac{1}{\varphi\lambda_j}+2\hat{\alpha}_j^2\right)}\right)^{1/r}$	$\min\left(\frac{\hat{\alpha}_j^2}{(1/(\lambda_j\varphi)+\hat{\alpha}_j^2)}\right)$	Abonazel et al. [31]
BMRTE	$\hat{k} = \min\left(\frac{\varphi}{(1+\hat{d})\hat{\alpha}_j^2}\right)$	$\hat{d} = \min\left(\frac{\varphi}{\hat{k}\hat{\alpha}_j^2}\right) - 1$	Lukman et al. [24]
BKLE	$\hat{k} = \min\left(\frac{\varphi\lambda_j}{2\hat{\alpha}_j^2+(\varphi\lambda_j)}\right)$	-	Koç and Dünder [25]
BJRRE	$\hat{k} = \sqrt{\max\left(k_j = \frac{\varphi\left(1+\sqrt{1+\frac{\hat{\alpha}_j^2\lambda_j}{\varphi}}\right)}{\hat{\alpha}_j^2}\right)/r}$	-	Farghali [26]
BJKLE	$\hat{k} = \min\left(\frac{\varphi}{2\hat{\alpha}_j^2+(\varphi/\lambda_j)}\right)$	-	Koç and Dünder [25]
		Proposed parameters for BAUMRTE	
	$\hat{k}_1 = \frac{r\varphi}{\max((1+d)\hat{\alpha}_j^2)}$	$\hat{d}_1 = \frac{1}{\max\left(\frac{1}{\frac{\varphi}{\hat{k}_1\hat{\alpha}_j^2}-1}\right)}$	
	$\hat{k}_2 = \mathrm{median}\left(\frac{\varphi}{(1+d)\hat{\alpha}_j^2}\right)$	$\hat{d}_2 = \frac{1}{\max\left(\frac{1}{\frac{\varphi}{\hat{k}_2\hat{\alpha}_j^2}-1}\right)}$	
BAUMRTE	$\hat{k}_3 = \mathrm{median}\left(\frac{r}{(1+d)\hat{\alpha}_j^2}\right)$	$\hat{d}_3 = \frac{1}{\max\left(\frac{1}{\frac{\varphi}{\hat{k}_3\hat{\alpha}_j^2}-1}\right)}$	

4. Simulation study

This section evaluates the performance of the proposed estimator through a comprehensive simulation experiment. The study compares the proposed BAUMRTE with other established estimators, including the BMLE, BRRE, BLE, BJSE, BLTE, BOKE, BDKE, BMRTE, and BKLE.

4.1. Simulation design

The response variable was generated from a beta distribution using the logit link function, defined as [38]:

$$\mu_i = \frac{\exp\left(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ir}\right)}{1 + \exp\left(\beta_0 \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ir}\right)}, \quad i = 1, \ldots, n; \quad j = 1, \ldots, r,$$

where x_{ij} represents the correlated independent variables and β_j denotes the true parameter vector of the BRM. The parameters were selected such that

$$\sum_{j=1}^{r} \beta_j^2 = 1.$$

The correlated explanatory variables were generated using the formula [8, 39]

$$x_{ij} = \left(1 - \rho^2\right)^{\frac{1}{2}} z_{ij} + \rho z_{i(j+1)}, \quad i = 1, \ldots, n; \quad j = 1, \ldots, r,$$

where ρ represents the degree of correlation among the regressors and z_{ij} are independent standard normal pseudo-random variables.

The study investigated the performance of the proposed estimator under various conditions as described in Table 2.

Table 2. Simulation setup parameters used in the study.

Parameter	Values
Correlation levels (ρ)	0.80, 0.85, 0.90, 0.95, 0.99
Sample sizes (n)	30, 75, 150, 200, 300, 400
Number of covariates (p)	3, 6, 9
Precision parameter (φ)	0.5, 1.0, 1.5
Number of replications	1000 for each combination of (n, ρ, p, φ)

To assess the performance of the proposed estimator, the estimated MSE was used as the evaluation criterion. The estimated MSE is defined as

$$\mathrm{MSE}(\hat{\beta}) = \frac{\sum_{i=1}^{R} \left(\hat{\beta}_i - \beta\right)^t \left(\hat{\beta}_i - \beta\right)}{R},$$

where $\left(\hat{\beta}_i - \beta\right)$ represents the difference between the estimated and true parameter vectors at the i-th replication, and R denotes the total number of replications. A detailed description of the simulation procedure and parameter estimation is provided in Algorithm 1. All computational tasks were carried out using the R programming language version 4.4.1, with the assistance of the "betareg" package for BRM.

Algorithm 1 Simulation steps for the proposed estimators in the R programming language.

1: Start
2: Load the required libraries and set the random seed to ensure reproducibility.
3: Input the simulation factors: sample size (n), number of explanatory variables (p), precision parameter (φ), correlation coefficient (ρ), and true parameter vector (β).
4: Generate multicollinear explanatory variables using

$$x_{ij} = \left(1-\rho^2\right)^{\frac{1}{2}} z_{ij} + \rho z_{i(j+1)}, \quad i = 1,\ldots,n; \quad j = 1,\ldots,r.$$

5: Compute the mean of the beta distribution

$$\mu_i = \frac{\exp\left(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ir}\right)}{1+\exp\left(\beta_0 \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ir}\right)}, \quad i = 1,\ldots,n; \quad j = 1,\ldots,r, \quad \text{with } \sum_{j=1}^{p} \beta_j^2 = 1.$$

6: Generate the response variable from the generalized Poisson distribution

$$y_i \sim \text{Beta}(\mu_i, \varphi).$$

7: Fit the BRM using the 'betareg' package to obtain

$$\texttt{betareg}(y \sim X).$$

8: Compute the intermediate matrices $\hat{W}$ and $\hat{z}$.
9: Compute the biasing parameters for the proposed estimators along with the existing estimator as described in Table 1.
10: Determine $\hat{\beta}$, representing the selected estimator (BMLE, BRRE, BLE, BJSE, BLTE, BOKE, BDKE, BMRTE, BKLE, and BAUMRTE).
11: Compute the squared distance

$$D = \left(\hat{\beta}_i - \beta\right)^t \left(\hat{\beta}_i - \beta\right).$$

12: Repeat Steps 6–10 for 1000 replications.
13: Compute the estimated MSE

$$\text{MSE}(\hat{\beta}) = \frac{\sum_{i=1}^{R} \left(\hat{\beta}_i - \beta\right)^t \left(\hat{\beta}_i - \beta\right)}{R}.$$

14: Stop

4.2. Simulation results discussion

Tables 3–11 present the estimated MSE values for various biased estimators in a BRM, including the proposed BAUMRTE with three parameters, along with existing methods.

The tables are organized based on different values of the correlation coefficient (ρ), sample size (n), number of explanatory variables p, and dispersion parameter (φ). The estimators compared include BMLE, BRRE, BLE, BJSE, BLTE, BOKE, BDKE, BMRTE, BKLE, BJRRE, BJKLE, and the proposed BAUMRTE with three parameters ($\hat{k}_1,\ \hat{d}_1$), ($\hat{k}_2,\ \hat{d}_2$), and ($\hat{k}_3,\ \hat{d}_3$).

- As the sample size (n) increases, the MSE values for all estimators tend to decrease. This is expected, as larger sample sizes typically yield more accurate estimates of the parameters.
- The dispersion parameter φ also affects the MSE values. Lower values of φ (e.g., 0.5) tend to result in higher MSEs, while higher values (e.g., 1.5) lead to lower MSEs.
- Higher values of ρ (e.g., 0.95, 0.99) result in higher MSE values for most estimators, indicating that high correlation among predictors can degrade estimation performance.
- Increasing the number of covariates p generally increases the MSE for all estimators, especially in small sample sizes and high multicollinearity scenarios.
- The traditional BMLE has higher MSE values in each scenario, indicating the presence of multicollinearity.

  Using existing biased estimators introduces an MSE lower than the BMLE. Furthermore, the BAUMRTE outperforms these existing methods, particularly in challenging conditions (high ρ, low φ, and small n).
- The proposed BAUMRTE consistently shows lower MSE values compared to other estimators across different scenarios. This indicates that BAUMRTE is more efficient in estimating the parameters of the BRM.

  Among the three variants of BAUMRTE, the BAUMRTE($\hat{k}_2,\ \hat{d}_2$) generally performs the best in most scenarios, often yielding the lowest MSE values. This suggests that the second variant of the proposed estimator is the most effective in reducing bias and variance.

Simulation results show that the proposed BAUMRTE, particularly the variable BAUMRTE($\hat{k}_2,\ \hat{d}_2$), is highly effective at reducing the MSE across a wide range of scenarios.

This estimator outperforms existing ones, especially under conditions of high multicollinearity and low dispersion, making it the best choice for coefficient estimation in BRM under multicollinearity. BAUMRTE's performance indicates that it is a valuable addition to the range of feature estimation tools, as it offers the lowest MSE, meaning it achieves the greatest balance between bias and variance, especially in cases where traditional methods might fail to provide accurate estimates.

Table 3. Estimated MSE values when p = 3 and $\varphi = 0.5$.

ρ	n	BMLE	BRRE	BLE	BJSE	BLTE	BOKE	BDKE	BMRTE	BKLE	BJRRE	BJKLE	BAUMRTE		
		-	$\hat{k}$	$\hat{d}$	$\hat{c}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}$	$\hat{k}$	$\hat{k}$	$\hat{k}_1,\ \hat{d}_1$	$\hat{k}_2,\ \hat{d}_2$	$\hat{k}_3,\ \hat{d}_3$
0.80	30	1.0788	0.8022	0.5213	0.7982	0.4311	0.8358	0.6460	0.9295	0.7931	0.3848	0.3790	0.3013	0.2830	0.3129
	75	0.4907	0.4270	0.3626	0.4135	0.2651	0.4351	0.3791	0.4635	0.4255	0.2195	0.2187	0.1800	0.1695	0.1866
	150	0.1847	0.1740	0.1590	0.1775	0.1332	0.1754	0.1639	0.1812	0.1739	0.1018	0.1017	0.0879	0.0844	0.0902
	200	0.1281	0.1229	0.1173	0.1258	0.1018	0.1234	0.1174	0.1266	0.1229	0.0755	0.0755	0.0667	0.0642	0.0683
	300	0.1387	0.1340	0.1271	0.1360	0.1138	0.1346	0.1292	0.1372	0.1339	0.0864	0.0864	0.0744	0.0708	0.0766
	400	0.0826	0.0807	0.0778	0.0831	0.0718	0.0809	0.0786	0.0821	0.0806	0.0564	0.0564	0.0499	0.0478	0.0511
0.85	30	1.2482	0.9698	0.6299	0.9843	0.6106	1.0036	0.8336	1.0820	0.9611	0.5119	0.5050	0.4086	0.3835	0.4266
	75	0.5978	0.5046	0.4074	0.5093	0.2807	0.5169	0.4299	0.5609	0.5012	0.2383	0.2372	0.1936	0.1829	0.2006
	150	0.2386	0.2190	0.1955	0.2260	0.1508	0.2214	0.1998	0.2324	0.2186	0.1170	0.1169	0.0987	0.0940	0.1016
	200	0.1771	0.1669	0.1539	0.1720	0.1248	0.1682	0.1558	0.1743	0.1668	0.0865	0.0864	0.0724	0.0686	0.0748
	300	0.1384	0.1323	0.1243	0.1324	0.1072	0.1329	0.1265	0.1365	0.1323	0.0754	0.0753	0.0644	0.0605	0.0666
	400	0.0931	0.0900	0.0860	0.0907	0.0760	0.0903	0.0866	0.0922	0.0899	0.0537	0.0537	0.0457	0.0429	0.0473
0.90	30	2.2226	1.5509	0.6456	1.5264	0.9684	1.6169	1.2713	1.8181	1.5315	0.7455	0.7167	0.5623	0.5301	0.5993
	75	0.9208	0.7357	0.4809	0.7333	0.3522	0.7539	0.5949	0.8495	0.7278	0.3537	0.3510	0.2813	0.2607	0.2941
	150	0.3948	0.3389	0.2738	0.3543	0.1622	0.3436	0.2866	0.3774	0.3370	0.1369	0.1367	0.1092	0.1004	0.1145
	200	0.2819	0.2622	0.2323	0.2648	0.1849	0.2649	0.2407	0.2762	0.2618	0.1371	0.1370	0.1128	0.1059	0.1168
	300	0.1909	0.1782	0.1643	0.1779	0.1283	0.1796	0.1648	0.1873	0.1780	0.0812	0.0812	0.0641	0.0589	0.0672
	400	0.1631	0.1528	0.1418	0.1567	0.1108	0.1538	0.1414	0.1603	0.1526	0.0717	0.0717	0.0580	0.0536	0.0606
0.95	30	2.3864	1.6519	0.5491	1.5407	0.6983	1.7195	1.2203	2.0587	1.6296	0.7719	0.7491	0.5856	0.5409	0.6190
	75	2.4742	1.7168	0.6157	1.6361	0.8177	1.7861	1.3040	2.1352	1.6970	0.8533	0.8271	0.6406	0.5917	0.6653
	150	0.6825	0.5449	0.4040	0.5625	0.2342	0.5579	0.4327	0.6354	0.5398	0.2371	0.2363	0.1862	0.1700	0.1957
	200	0.6107	0.5312	0.3924	0.5355	0.2798	0.5385	0.4545	0.5869	0.5286	0.2655	0.2649	0.2226	0.2071	0.2315
	300	0.3857	0.3456	0.2825	0.3628	0.2104	0.3494	0.3028	0.3746	0.3443	0.1814	0.1812	0.1546	0.1455	0.1597
	400	0.2627	0.2380	0.2074	0.2444	0.1437	0.2403	0.2125	0.2558	0.2374	0.0979	0.0979	0.0780	0.0713	0.0819
0.99	30	12.0145	7.9712	0.4312	6.0239	3.6898	8.3107	6.1247	10.0219	7.9767	3.9536	2.6399	2.2860	2.6979	2.6679
	75	8.7413	5.7251	0.3805	4.4730	2.6077	5.9589	4.3266	7.3354	5.7138	2.7507	2.5115	1.9789	1.8839	2.1044
	150	3.6023	2.3597	0.4729	2.0484	0.8154	2.4614	1.6773	3.1126	2.3436	1.0703	1.0468	0.7947	0.7163	0.8414
	200	2.7489	1.9133	0.5570	1.7554	0.6981	1.9898	1.3680	2.4510	1.8955	0.9019	0.8943	0.6685	0.5987	0.7091
	300	1.4617	0.9922	0.4523	1.0198	0.3230	1.0340	0.6675	1.2953	0.9711	0.3948	0.3933	0.2884	0.2558	0.3075
	400	1.4752	1.0642	0.5194	1.0664	0.3624	1.1062	0.7450	1.3432	1.0490	0.4780	0.4766	0.3635	0.3269	0.3846

Table 4. Estimated MSE values when p = 3 and φ = 1.

ρ	n	BMLE	BRRE	BLE	BJSE	BLTE	BOKE	BDKE	BMRTE	BKLE	BJRRE	BJKLE	BAUMRTE		
		-	$\hat{k}$	$\hat{d}$	$\hat{c}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}$	$\hat{k}$	$\hat{k}$	$\hat{k}_1,\ \hat{d}_1$	$\hat{k}_2,\ \hat{d}_2$	$\hat{k}_3,\ \hat{d}_3$
0.80	30	0.8812	0.4796	0.4283	0.5796	0.4087	0.5211	0.4784	0.3930	0.4638	0.2109	0.2100	0.1804	0.1802	0.1824
	75	0.4458	0.3539	0.3407	0.3869	0.2993	0.3685	0.3493	0.3272	0.3493	0.1618	0.1615	0.1424	0.1415	0.1442
	150	0.1832	0.1653	0.1601	0.1724	0.1466	0.1681	0.1645	0.1602	0.1648	0.0855	0.0854	0.0754	0.0748	0.0763
	200	0.0971	0.0914	0.0904	0.0967	0.0852	0.0921	0.0908	0.0910	0.0913	0.0520	0.0520	0.0501	0.0512	0.0499
	300	0.1032	0.0966	0.0960	0.1004	0.0901	0.0976	0.0959	0.0958	0.0964	0.0516	0.0516	0.0452	0.0444	0.0460
	400	0.0833	0.0794	0.0790	0.0831	0.0754	0.0802	0.0789	0.0790	0.0793	0.0469	0.0469	0.0412	0.0405	0.0417
0.85	30	0.9914	0.6201	0.5581	0.7926	0.6160	0.6812	0.6582	0.4165	0.6003	0.2740	0.2725	0.2243	0.2204	0.2295
	75	0.4917	0.3697	0.3573	0.4238	0.3015	0.3878	0.3639	0.3366	0.3627	0.1704	0.1701	0.1562	0.1566	0.1573
	150	0.2430	0.2070	0.2045	0.2236	0.1758	0.2138	0.2021	0.2024	0.2056	0.0924	0.0924	0.0806	0.0804	0.0814
	200	0.1518	0.1365	0.1342	0.1481	0.1199	0.1384	0.1342	0.1367	0.1361	0.0663	0.0663	0.0595	0.0588	0.0602
	300	0.1385	0.1266	0.1258	0.1331	0.1153	0.1288	0.1252	0.1251	0.1264	0.0576	0.0576	0.0480	0.0466	0.0490
	400	0.0822	0.0774	0.0771	0.0798	0.0729	0.0782	0.0770	0.0767	0.0773	0.0408	0.0407	0.0349	0.0340	0.0357
0.90	30	1.7463	0.9608	0.6226	1.2599	0.9724	1.0770	0.9749	0.7501	0.9387	0.4107	0.4044	0.3060	0.2932	0.3210
	75	0.7629	0.4901	0.4337	0.6156	0.3992	0.5305	0.4771	0.4239	0.4728	0.1809	0.1804	0.1407	0.1360	0.1448
	150	0.3780	0.2853	0.2720	0.3310	0.2138	0.2980	0.2762	0.2727	0.2801	0.0919	0.0918	0.0735	0.0703	0.0759
	200	0.2107	0.1816	0.1790	0.1992	0.1538	0.1865	0.1777	0.1794	0.1806	0.0681	0.0681	0.0567	0.0555	0.0579
	300	0.1769	0.1573	0.1561	0.1696	0.1391	0.1610	0.1546	0.1558	0.1567	0.0631	0.0631	0.0518	0.0502	0.0530
	400	0.1405	0.1257	0.1249	0.1349	0.1116	0.1282	0.1238	0.1244	0.1253	0.0449	0.0449	0.0365	0.0350	0.0376
0.95	30	4.7316	2.6877	1.8482	2.9236	1.0910	1.8300	1.6459	1.4533	1.6492	0.9294	0.9688	0.5072	0.4294	0.4326
	75	1.9818	1.0232	0.6025	1.3430	0.9523	1.1584	0.9979	0.8491	0.9975	0.3890	0.3837	0.2753	0.2543	0.2921
	150	0.5639	0.3762	0.3554	0.4603	0.2752	0.4034	0.3556	0.3507	0.3626	0.1202	0.1200	0.0910	0.0847	0.0951
	200	0.4283	0.3161	0.2990	0.3741	0.2372	0.3337	0.3011	0.3060	0.3096	0.0984	0.0983	0.0721	0.0667	0.0755
	300	0.2886	0.2301	0.2259	0.2605	0.1809	0.2390	0.2222	0.2266	0.2273	0.0689	0.0688	0.0515	0.0476	0.0540
	400	0.2055	0.1711	0.1680	0.1957	0.1386	0.1765	0.1661	0.1692	0.1695	0.0451	0.0451	0.0352	0.0335	0.0364
0.99	30	10.9418	5.5100	0.4468	5.6873	5.4857	6.3302	5.4645	4.5860	5.5208	2.1280	1.7697	1.0788	1.1961	1.3162
	75	6.0788	2.5616	0.3554	2.8455	2.2180	2.9175	2.4596	2.3008	2.5695	0.8578	0.8215	0.5535	0.5081	0.6070
	150	3.0440	1.4396	0.4806	1.8456	1.2332	1.6586	1.3387	1.3052	1.4235	0.5225	0.5162	0.3282	0.2915	0.3539
	200	2.3885	1.2308	0.5470	1.5728	1.0328	1.4056	1.1107	1.1678	1.1995	0.4556	0.4531	0.3006	0.2660	0.3223
	300	1.3782	0.7374	0.4857	0.9979	0.5588	0.8214	0.6588	0.7147	0.7065	0.2561	0.2554	0.1745	0.1539	0.1871
	400	1.1838	0.6734	0.4668	0.9123	0.4985	0.7506	0.5948	0.6603	0.6430	0.2359	0.2355	0.1547	0.1365	0.1658

Table 5. Estimated MSE values when p = 3 and $\varphi = 1.5$.

ρ	n	BMLE	BRRE	BLE	BJSE	BLTE	BOKE	BDKE	BMRTE	BKLE	BJRRE	BJKLE	BAUMRTE		
		-	$\hat{k}$	$\hat{d}$	$\hat{c}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}$	$\hat{k}$	$\hat{k}$	$\hat{k}_1,\ \hat{d}_1$	$\hat{k}_2,\ \hat{d}_2$	$\hat{k}_3,\ \hat{d}_3$
0.80	30	0.8216	0.4542	0.4987	0.6620	0.5400	0.5206	0.5389	0.2358	0.4317	0.1995	0.1990	0.1905	0.1973	0.1888
	75	0.3627	0.2706	0.2902	0.3216	0.2668	0.2914	0.2923	0.1717	0.2656	0.1265	0.1264	0.1243	0.1309	0.1218
	150	0.1745	0.1500	0.1546	0.1641	0.1456	0.1548	0.1561	0.1200	0.1491	0.0754	0.0754	0.0744	0.0778	0.0732
	200	0.0992	0.0908	0.0926	0.0977	0.0891	0.0923	0.0928	0.0807	0.0906	0.0494	0.0494	0.0505	0.0536	0.0494
	300	0.0840	0.0768	0.0786	0.0812	0.0755	0.0783	0.0784	0.0682	0.0767	0.0399	0.0399	0.0368	0.0377	0.0366
	400	0.0716	0.0670	0.0681	0.0701	0.0661	0.0680	0.0680	0.0614	0.0669	0.0375	0.0375	0.0342	0.0347	0.0343
0.85	30	0.9856	0.5573	0.5947	0.7973	0.6980	0.6596	0.6707	0.2614	0.5306	0.2553	0.2546	0.2268	0.2319	0.2266
	75	0.4349	0.2945	0.3221	0.3709	0.2901	0.3178	0.3243	0.1791	0.2852	0.1256	0.1255	0.1286	0.1354	0.1261
	150	0.2020	0.1635	0.1731	0.1881	0.1596	0.1708	0.1726	0.1213	0.1615	0.0730	0.0730	0.0738	0.0783	0.0721
	200	0.1053	0.0914	0.0946	0.1032	0.0889	0.0942	0.0946	0.0756	0.0909	0.0432	0.0432	0.0440	0.0468	0.0429
	300	0.1001	0.0894	0.0920	0.0960	0.0873	0.0915	0.0916	0.0776	0.0890	0.0388	0.0388	0.0364	0.0377	0.0361
	400	0.0812	0.0744	0.0764	0.0802	0.0735	0.0756	0.0761	0.0656	0.0742	0.0348	0.0348	0.0314	0.0319	0.0314
0.90	30	1.6420	0.7338	0.6442	1.1996	1.0466	0.8936	0.9141	0.3170	0.7009	0.2907	0.2883	0.2315	0.2300	0.2375
	75	0.6665	0.3903	0.4029	0.5631	0.4025	0.4388	0.4319	0.2333	0.3676	0.1585	0.1582	0.1507	0.1557	0.1494
	150	0.3727	0.2560	0.2772	0.3285	0.2418	0.2794	0.2720	0.1732	0.2469	0.0789	0.0788	0.0697	0.0715	0.0696
	200	0.1876	0.1538	0.1617	0.1780	0.1466	0.1600	0.1605	0.1204	0.1522	0.0508	0.0508	0.0484	0.0505	0.0478
	300	0.1637	0.1406	0.1463	0.1547	0.1381	0.1458	0.1454	0.1151	0.1398	0.0504	0.0504	0.0418	0.0417	0.0423
	400	0.1271	0.1091	0.1140	0.1225	0.1062	0.1124	0.1130	0.0899	0.1085	0.0330	0.0330	0.0285	0.0289	0.0287
0.95	30	2.5522	1.1826	0.7847	1.6984	1.5582	1.4499	1.4045	0.5513	1.1508	0.4361	0.4314	0.2914	0.2745	0.3088
	75	2.2285	0.9727	0.7130	1.5008	1.3612	1.2302	1.1472	0.3936	0.9351	0.3259	0.3226	0.1985	0.1822	0.2121
	150	0.4727	0.2782	0.3110	0.4029	0.2660	0.3098	0.3024	0.1630	0.2605	0.0803	0.0803	0.0701	0.0703	0.0708
	200	0.3488	0.2354	0.2536	0.3130	0.2155	0.2509	0.2530	0.1565	0.2263	0.0638	0.0638	0.0526	0.0515	0.0538
	300	0.2179	0.1601	0.1738	0.2048	0.1508	0.1699	0.1690	0.1148	0.1558	0.0409	0.0408	0.0355	0.0357	0.0357
	400	0.2256	0.1750	0.1866	0.2083	0.1655	0.1837	0.1839	0.1309	0.1719	0.0422	0.0422	0.0337	0.0328	0.0345
0.99	30	8.8874	3.4092	0.5233	4.4427	5.0327	4.1831	4.3221	1.3775	3.4327	1.0918	0.9824	0.5782	0.6128	0.7057
	75	6.2378	2.3026	0.4255	3.3065	3.4000	2.9007	2.8596	0.9572	2.3030	0.7218	0.6991	0.4250	0.3936	0.4739
	150	2.8047	1.0404	0.4849	1.7019	1.3797	1.3183	1.1773	0.5237	1.0274	0.3291	0.3262	0.1972	0.1749	0.2135
	200	1.7176	0.6457	0.4330	1.1470	0.7787	0.7802	0.7279	0.3285	0.6190	0.1904	0.1898	0.1232	0.1089	0.1326
	300	1.3369	0.5818	0.5130	0.9662	0.6873	0.7153	0.6338	0.3004	0.5439	0.1747	0.1744	0.1097	0.0966	0.1180
	400	0.9814	0.4178	0.4093	0.7325	0.4554	0.4978	0.4567	0.2170	0.3808	0.1112	0.1111	0.0743	0.0653	0.0799

Table 6. Estimated MSE values when p = 6 and φ = 0.5.

ρ	n	BMLE	BRRE	BLE	BJSE	BLTE	BOKE	BDKE	BMRTE	BKLE	BJRRE	BJKLE	BAUMRTE		
		-	$\hat{k}$	$\hat{d}$	$\hat{c}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}$	$\hat{k}$	$\hat{k}$	$\hat{k}_1,\ \hat{d}_1$	$\hat{k}_2,\ \hat{d}_2$	$\hat{k}_3,\ \hat{d}_3$
0.80	30	2.9670	2.1730	1.3108	1.8209	1.3791	2.2349	1.9782	2.2459	2.1562	0.5814	0.5733	0.4886	0.4693	0.5054
	75	1.1348	0.9683	0.7679	0.8061	0.4901	0.9732	0.8550	1.0612	0.9635	0.2680	0.2666	0.2426	0.2333	0.2485
	150	0.4743	0.4408	0.4029	0.4082	0.3146	0.4419	0.4114	0.4636	0.4402	0.1552	0.1551	0.1470	0.1447	0.1486
	200	0.3705	0.3512	0.3269	0.3372	0.2690	0.3520	0.3328	0.3649	0.3510	0.1385	0.1385	0.1275	0.1236	0.1298
	300	0.2607	0.2497	0.2374	0.2344	0.2035	0.2500	0.2394	0.2575	0.2496	0.0927	0.0927	0.0853	0.0825	0.0870
	400	0.2154	0.2096	0.1998	0.2036	0.1813	0.2097	0.2042	0.2137	0.2095	0.0978	0.0978	0.0908	0.0879	0.0925
0.85	30	4.2530	3.0135	1.3790	2.3261	1.7211	3.0709	2.6616	3.3601	2.9955	0.7896	0.7708	0.6744	0.6476	0.7028
	75	2.3497	1.8554	1.0922	1.5247	0.9595	1.8834	1.5785	2.1133	1.8416	0.5668	0.5421	0.4763	0.4619	0.4857
	150	0.6814	0.6176	0.5474	0.5544	0.3908	0.6197	0.5621	0.6609	0.6163	0.1714	0.1712	0.1559	0.1506	0.1591
	200	0.3965	0.3715	0.3412	0.3491	0.2705	0.3721	0.3486	0.3892	0.3711	0.1144	0.1143	0.1067	0.1036	0.1086
	300	0.3320	0.3170	0.2967	0.3109	0.2530	0.3182	0.3033	0.3270	0.3169	0.1172	0.1172	0.1061	0.1030	0.1079
	400	0.4966	0.4831	0.3815	0.4663	0.4244	0.4836	0.4711	0.4922	0.4830	0.2760	0.2752	0.2553	0.2477	0.2597
0.90	30	4.9709	3.7110	1.4768	2.7929	2.2004	3.7468	3.3038	4.1461	3.6884	1.0957	1.0595	0.9437	0.9125	0.9842
	75	2.5026	1.9482	1.1622	1.5370	0.9378	1.9748	1.6861	2.1589	1.9328	0.4903	0.4849	0.4179	0.4006	0.4332
	150	1.1218	0.9694	0.7592	0.8232	0.4922	0.9758	0.8483	1.0689	0.9649	0.2663	0.2654	0.2323	0.2216	0.2385
	200	2.2286	1.7443	0.6623	1.1577	0.6171	1.7955	1.1280	2.1644	1.7425	0.4780	0.3485	0.3514	0.3879	0.3538
	300	0.5628	0.5260	0.4477	0.4983	0.3898	0.5268	0.4963	0.5490	0.5255	0.1828	0.1825	0.1671	0.1602	0.1713
	400	0.4376	0.4138	0.3799	0.3843	0.3053	0.4143	0.3906	0.4311	0.4135	0.1218	0.1218	0.1099	0.1046	0.1129
0.95	30	11.2098	7.6255	1.7086	5.2815	4.8911	7.8688	6.8563	8.5009	7.6506	2.1556	1.9709	1.6719	1.6867	1.7928
	75	4.5551	3.4717	1.5166	2.4152	1.3457	3.4918	2.9256	4.0698	3.4602	0.9397	0.9251	0.8197	0.7754	0.8503
	150	2.3293	1.8995	1.1573	1.4488	0.6775	1.9089	1.5847	2.1825	1.8880	0.5398	0.5338	0.4729	0.4462	0.4902
	200	1.2322	1.0315	0.7911	0.9087	0.4234	1.0407	0.8568	1.1688	1.0249	0.2514	0.2510	0.2183	0.2066	0.2248
	300	1.1592	1.0233	0.8130	0.8967	0.5473	1.0293	0.9024	1.1069	1.0193	0.3158	0.3154	0.2766	0.2627	0.2843
	400	0.7794	0.6938	0.5854	0.6095	0.3745	0.6953	0.6160	0.7536	0.6916	0.1595	0.1594	0.1414	0.1328	0.1462
0.99	30	47.3470	32.9630	1.1357	20.7060	19.4817	33.5061	29.7588	36.2039	33.0441	8.8560	6.4381	5.7973	7.1081	5.6944
	75	27.3408	24.0516	0.8918	17.5186	19.9658	24.0987	22.9284	25.9883	24.0676	4.1547	3.4578	3.0145	3.5478	2.9874
	150	11.0345	7.9775	1.2261	4.5935	2.4974	8.0554	6.4902	9.7487	7.9670	1.9082	1.8585	1.6041	1.5102	1.6595
	200	5.9728	4.1633	1.2786	2.8238	1.3494	4.2237	3.2335	5.2614	4.1442	0.9252	0.9182	0.7672	0.7137	0.7979
	300	6.0841	4.8339	1.4143	3.5761	2.1256	4.8604	4.0712	5.6534	4.8130	1.9470	1.9169	1.7627	1.7003	1.8017
	400	3.5227	2.5810	1.1915	1.8195	0.6369	2.6019	1.9773	3.2035	2.5557	0.5970	0.5952	0.5128	0.4765	0.5328

Table 7. Estimated MSE values when p = 6 and $\varphi = 1$.

ρ	n	BMLE	BRRE	BLE	BJSE	BLTE	BOKE	BDKE	BMRTE	BKLE	BJRRE	BJKLE	BAUMRTE		
		-	$\hat{k}$	$\hat{d}$	$\hat{c}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}$	$\hat{k}$	$\hat{k}$	$\hat{k}_1,\ \hat{d}_1$	$\hat{k}_2,\ \hat{d}_2$	$\hat{k}_3,\ \hat{d}_3$
0.80	30	2.6417	1.6188	1.2868	1.6734	1.6067	1.6891	1.8096	1.0279	1.5999	0.3930	0.3907	0.3502	0.3441	0.3581
	75	0.9045	0.6727	0.6577	0.6657	0.5307	0.6871	0.6851	0.5843	0.6600	0.1548	0.1547	0.1518	0.1540	0.1511
	150	0.4567	0.4026	0.3955	0.3826	0.3532	0.4053	0.4044	0.3873	0.4012	0.1251	0.1251	0.1245	0.1265	0.1239
	200	0.2652	0.2414	0.2394	0.2421	0.2192	0.2424	0.2414	0.2375	0.2409	0.0804	0.0804	0.0818	0.0836	0.0811
	300	0.1964	0.1827	0.1820	0.1849	0.1703	0.1834	0.1827	0.1807	0.1825	0.0695	0.0695	0.0715	0.0734	0.0707
	400	0.1749	0.1661	0.1642	0.1634	0.1561	0.1667	0.1658	0.1653	0.1660	0.0627	0.0627	0.0601	0.0599	0.0604
0.85	30	4.3190	2.6353	1.6421	2.6307	2.7420	2.7906	2.9784	1.6094	2.6099	0.5963	0.5893	0.4901	0.4765	0.5102
	75	1.9850	1.2363	0.9165	1.3118	1.1477	1.2948	1.3136	0.9307	1.2124	0.2647	0.2605	0.2346	0.2333	0.2393
	150	0.5648	0.4712	0.4637	0.4647	0.3935	0.4768	0.4705	0.4519	0.4677	0.1103	0.1103	0.1096	0.1112	0.1092
	200	0.3413	0.3021	0.2987	0.2987	0.2653	0.3041	0.3015	0.2957	0.3009	0.0791	0.0791	0.0786	0.0800	0.0782
	300	0.2944	0.2703	0.2657	0.2731	0.2433	0.2720	0.2689	0.2687	0.2698	0.0860	0.0860	0.0842	0.0849	0.0841
	400	0.2150	0.2026	0.1985	0.2053	0.1876	0.2034	0.2016	0.2025	0.2024	0.0747	0.0747	0.0710	0.0709	0.0713
0.90	30	4.4002	2.5546	1.5056	2.5816	2.7104	2.6666	2.8830	1.5700	2.5411	0.5611	0.5520	0.4566	0.4455	0.4797
	75	2.0928	1.3688	1.1164	1.2879	1.0061	1.4075	1.4229	1.0993	1.3364	0.2811	0.2803	0.2583	0.2544	0.2619
	150	0.8688	0.6515	0.6239	0.6594	0.4819	0.6660	0.6426	0.6118	0.6388	0.1242	0.1242	0.1177	0.1183	0.1178
	200	0.6608	0.5415	0.5275	0.5312	0.4270	0.5467	0.5360	0.5271	0.5362	0.1016	0.1016	0.0949	0.0933	0.0960
	300	0.5146	0.4532	0.4284	0.4407	0.3888	0.4569	0.4504	0.4474	0.4513	0.1094	0.1093	0.0972	0.0946	0.0989
	400	0.3645	0.3285	0.3195	0.3270	0.2896	0.3298	0.3269	0.3266	0.3277	0.0762	0.0761	0.0712	0.0700	0.0721
0.95	30	9.5361	4.8702	1.6527	4.6055	5.6494	5.1299	5.7809	2.9194	4.9875	0.9694	0.9194	0.7359	0.7435	0.8057
	75	3.2469	1.8671	1.2360	1.7471	1.3737	1.9196	1.9534	1.5263	1.8326	0.3331	0.3304	0.2850	0.2702	0.2962
	150	1.6243	1.0275	0.9299	1.0421	0.6692	1.0597	1.0198	0.9292	0.9978	0.1549	0.1546	0.1321	0.1253	0.1363
	200	1.7468	1.4190	1.0578	1.3898	1.1260	1.4410	1.3956	1.3777	1.4016	0.4570	0.4548	0.3794	0.3637	0.3898
	300	1.0316	0.8251	0.7501	0.8133	0.6166	0.8347	0.8040	0.8259	0.8160	0.1845	0.1843	0.1620	0.1541	0.1666
	400	0.7546	0.6067	0.5865	0.5945	0.4739	0.6146	0.6002	0.5913	0.6007	0.0991	0.0991	0.0842	0.0796	0.0869
0.99	30	38.1912	19.9095	1.1040	16.4236	21.5827	21.2035	23.7256	12.0603	20.1210	3.4220	2.9865	2.1619	2.4558	2.3520
	75	17.5440	10.0358	1.1270	7.1146	7.6831	10.3084	10.5371	8.4968	10.0732	1.8161	1.7547	1.4401	1.3902	1.5371
	150	8.6985	4.6973	1.1318	3.5960	3.1040	4.8252	4.8131	4.2352	4.7094	0.7406	0.7302	0.6055	0.5646	0.6353
	200	4.7580	2.6511	1.1335	2.1995	1.5442	2.7045	2.6511	2.5197	2.6357	0.4124	0.4105	0.3444	0.3179	0.3603
	300	4.1901	2.4543	1.2632	2.1289	1.4840	2.5231	2.4026	2.3874	2.4282	0.4304	0.4290	0.3560	0.3296	0.3716
	400	3.2814	1.9780	1.2306	1.8382	1.2104	2.0353	1.9129	1.9397	1.9447	0.3284	0.3279	0.2648	0.2440	0.2767

Table 8. Estimated MSE values when p = 6 and $\varphi = 1.5$.

ρ	n	BMLE	BRRE	BLE	BJSE	BLTE	BOKE	BDKE	BMRTE	BKLE	BJRRE	BJKLE	BAUMRTE		
		-	$\hat{k}$	$\hat{d}$	$\hat{c}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}$	$\hat{k}$	$\hat{k}$	$\hat{k}_1,\ \hat{d}_1$	$\hat{k}_2,\ \hat{d}_2$	$\hat{k}_3,\ \hat{d}_3$
0.80	30	2.3450	1.2272	1.2395	1.5200	1.6630	1.2917	1.6502	0.4235	1.2017	0.3102	0.3195	0.3044	0.3067	0.3058
	75	0.7324	0.5077	0.5649	0.5737	0.4939	0.5165	0.5776	0.2853	0.4909	0.2249	0.2349	0.2001	0.2090	0.1957
	150	0.3355	0.2815	0.2946	0.2885	0.2734	0.2854	0.2984	0.2151	0.2795	0.1051	0.1052	0.1008	0.1045	0.1046
	200	0.2993	0.2626	0.2724	0.2654	0.2569	0.2651	0.2732	0.2187	0.2616	0.0811	0.0811	0.0768	0.0706	0.0751
	300	0.2047	0.1852	0.1910	0.1879	0.1832	0.1864	0.1913	0.1595	0.1848	0.0685	0.0685	0.0649	0.0686	0.0632
	400	0.1351	0.1263	0.1283	0.1305	0.1246	0.1268	0.1290	0.1148	0.1261	0.0555	0.0555	0.0594	0.0619	0.0582
0.85	30	3.8135	1.7783	1.6127	2.3202	2.6853	1.9662	2.5204	0.5119	1.7611	0.3388	0.3374	0.2952	0.2923	0.3013
	75	1.5625	0.9104	0.9384	1.0772	0.9942	0.9597	1.0701	0.4747	0.8790	0.2355	0.2353	0.2288	0.2305	0.2288
	150	0.5621	0.4440	0.4735	0.4688	0.4223	0.4523	0.4746	0.3267	0.4380	0.1157	0.1157	0.1137	0.1188	0.1113
	200	0.2973	0.2521	0.2632	0.2654	0.2445	0.2552	0.2655	0.1987	0.2504	0.0786	0.0786	0.0663	0.0609	0.0642
	300	0.2796	0.2492	0.2549	0.2549	0.2414	0.2516	0.2574	0.2151	0.2484	0.0723	0.0723	0.0714	0.0707	0.0719
	400	0.1972	0.1813	0.1840	0.1878	0.1780	0.1827	0.1856	0.1633	0.1811	0.0659	0.0659	0.0654	0.0654	0.0655
0.90	30	3.8261	1.9567	1.4500	2.3316	2.6978	2.0710	2.5924	0.6519	1.9412	0.4308	0.4264	0.3788	0.3760	0.3937
	75	2.0685	1.1309	1.1514	1.3381	1.2645	1.2049	1.4085	0.4816	1.0839	0.2509	0.2507	0.2488	0.2529	0.2476
	150	0.7041	0.4808	0.5317	0.5565	0.4461	0.4922	0.5361	0.2981	0.4628	0.1099	0.1099	0.1006	0.1064	0.1079
	200	0.5100	0.3897	0.4164	0.4201	0.3710	0.3974	0.4215	0.2733	0.3831	0.0722	0.0721	0.0700	0.0717	0.0713
	300	0.4373	0.3586	0.3779	0.3768	0.3450	0.3644	0.3785	0.2813	0.3552	0.0709	0.0709	0.0696	0.0706	0.0694
	400	0.3463	0.2985	0.3096	0.3052	0.2887	0.3014	0.3119	0.2455	0.2969	0.0533	0.0533	0.0533	0.0544	0.0529
0.95	30	8.2790	3.7263	1.5840	4.1862	5.5546	4.0239	5.1002	1.4072	3.9149	0.6908	0.6653	0.5355	0.5395	0.5819
	75	3.0689	1.4222	1.2674	1.7767	1.7304	1.5329	1.8331	0.5503	1.3862	0.2397	0.2387	0.2006	0.1953	0.2056
	150	1.4298	0.8196	0.8571	0.9585	0.7318	0.8599	0.9350	0.4601	0.7761	0.1221	0.1220	0.1080	0.1062	0.1094
	200	0.9120	0.5860	0.6377	0.6952	0.5086	0.6025	0.6535	0.3618	0.5600	0.0864	0.0864	0.0855	0.0865	0.0853
	300	0.8062	0.5816	0.6154	0.6344	0.5250	0.5943	0.6302	0.4020	0.5651	0.0915	0.0915	0.0824	0.0806	0.0836
	400	0.6483	0.4773	0.5184	0.5130	0.4397	0.4872	0.5150	0.3401	0.4659	0.0533	0.0533	0.0481	0.0472	0.0488
0.99	30	35.8401	15.5501	1.3770	16.2436	24.2616	17.3908	23.0194	3.9106	15.9000	2.1494	1.9794	1.3155	1.4759	1.5334
	75	16.1869	7.3850	1.1626	6.8037	9.0311	7.9701	9.5649	3.0471	7.4950	1.0361	1.0137	0.7715	0.7367	0.8291
	150	8.3322	3.8358	1.2716	3.7389	4.1483	4.1154	4.7473	1.7985	3.8511	0.4929	0.4884	0.3782	0.3514	0.3990
	200	4.5578	2.1581	1.2157	2.2927	2.1546	2.2872	2.6087	1.0878	2.1467	0.3135	0.3124	0.2483	0.2296	0.2600
	300	3.1966	1.4510	1.0478	1.6457	1.3335	1.4986	1.7543	0.7661	1.4138	0.1870	0.1866	0.1569	0.1447	0.1642
	400	2.9480	1.4599	1.2207	1.6785	1.3928	1.5426	1.7376	0.7494	1.4259	0.2007	0.2005	0.1562	0.1440	0.1634

Table 9. Estimated MSE values when p = 9 and $\varphi = 0.5$.

ρ	n	BMLE	BRRE	BLE	BJSE	BLTE	BOKE	BDKE	BMRTE	BKLE	BJRRE	BJKLE	BAUMRTE		
		-	$\hat{k}$	$\hat{d}$	$\hat{c}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}$	$\hat{k}$	$\hat{k}$	$\hat{k}_1,\ \hat{d}_1$	$\hat{k}_2,\ \hat{d}_2$	$\hat{k}_3,\ \hat{d}_3$
0.80	30	6.6238	4.8580	2.4912	3.6714	3.6422	4.9250	4.7363	4.6370	4.8454	1.5477	1.5438	0.9785	0.9592	1.0052
	75	2.0240	1.6713	1.3174	1.2605	0.9625	1.6786	1.5202	1.8249	1.6650	0.6584	0.6582	0.5465	0.5431	0.5490
	150	1.0120	0.9184	0.8077	0.7806	0.6453	0.9203	0.8526	0.9770	0.9165	0.4469	0.4468	0.4205	0.4222	0.4201
	200	0.8332	0.7892	0.6266	0.7077	0.6369	0.7909	0.7532	0.8168	0.7887	0.4479	0.4477	0.4083	0.4084	0.4091
	300	0.4811	0.4638	0.4186	0.4340	0.4012	0.4641	0.4497	0.4750	0.4636	0.3082	0.3082	0.3003	0.3025	0.3014
	400	0.3118	0.3014	0.2894	0.2791	0.2595	0.3017	0.2921	0.3087	0.3014	0.1938	0.1938	0.1791	0.1722	0.1876
0.85	30	9.2185	6.8710	3.0176	4.9183	4.7850	6.9619	6.6360	6.7984	6.8647	2.2704	2.2648	1.4444	1.4128	1.4815
	75	2.5429	2.0614	1.5513	1.5317	1.1486	2.0714	1.8777	2.2464	2.0518	0.7505	0.7501	0.5857	0.5789	0.5907
	150	1.0260	0.9261	0.8036	0.7279	0.5935	0.9278	0.8523	0.9905	0.9241	0.3447	0.3447	0.2916	0.2907	0.2924
	200	0.6741	0.6258	0.5712	0.5300	0.4543	0.6268	0.5860	0.6585	0.6250	0.2854	0.2853	0.2739	0.2755	0.2732
	300	0.5245	0.4960	0.4670	0.4524	0.3911	0.4967	0.4709	0.5159	0.4956	0.2838	0.2838	0.2607	0.2634	0.2694
	400	0.4219	0.4066	0.3864	0.3759	0.3412	0.4069	0.3927	0.4176	0.4065	0.2502	0.2502	0.2362	0.2387	0.2351
0.90	30	10.1124	7.4642	2.5798	5.2013	5.3101	7.5391	7.0623	7.8001	7.4791	2.5477	2.5351	1.5904	1.5697	1.6440
	75	4.7131	3.6866	2.0644	2.5040	1.8216	3.7050	3.2527	4.1639	3.6746	1.2349	1.2334	0.8276	0.8066	0.8431
	150	1.7716	1.5514	1.2263	1.1698	0.8127	1.5540	1.3779	1.6965	1.5450	0.5721	0.5720	0.4346	0.4269	0.4390
	200	0.9631	0.8681	0.7673	0.7023	0.5623	0.8694	0.7926	0.9327	0.8663	0.3827	0.3827	0.3425	0.3418	0.3430
	300	0.9274	0.8563	0.7128	0.7323	0.6131	0.8576	0.7955	0.9057	0.8549	0.4400	0.4399	0.3864	0.3844	0.3879
	400	0.5993	0.5597	0.5160	0.4974	0.4253	0.5605	0.5266	0.5857	0.5591	0.2903	0.2903	0.2873	0.2890	0.2866
0.95	30	17.4071	12.3405	2.5557	7.6540	7.9465	12.4777	11.7150	12.8132	12.3655	3.6306	3.6078	2.0901	2.0599	2.1787
	75	10.6396	8.1689	2.2085	5.0289	4.2766	8.1963	7.1534	9.4596	8.1784	2.8729	2.8595	1.8154	1.7848	1.8817
	150	3.2837	2.6344	1.7328	1.7843	1.0357	2.6436	2.2404	3.0300	2.6189	0.8686	0.8683	0.6067	0.5917	0.6154
	200	2.2699	1.8463	1.3589	1.3060	0.7378	1.8542	1.5542	2.1224	1.8360	0.6011	0.6009	0.4347	0.4263	0.4395
	300	1.4676	1.2841	1.0447	0.9800	0.6952	1.2862	1.1348	1.4108	1.2791	0.5197	0.5196	0.4211	0.4157	0.4241
	400	1.3093	1.1712	0.9560	0.9425	0.6975	1.1736	1.0518	1.2685	1.1677	0.5072	0.5072	0.4152	0.4102	0.4181
0.99	30	113.6662	76.9805	1.7681	48.4301	57.2785	77.8915	76.9529	72.3421	77.4805	21.8890	21.0987	10.2108	11.9418	11.7974
	75	48.5493	36.3516	1.6967	19.4982	17.7876	36.5367	32.4142	41.1616	36.3615	10.6700	10.4963	5.6897	5.7901	6.0230
	150	16.4735	12.3199	2.0607	6.0582	4.0359	12.3730	10.3378	14.6946	12.3302	3.4201	3.4109	1.9444	1.8729	2.0003
	200	12.6094	10.0699	2.2765	6.1722	4.4170	10.0955	8.6965	11.6645	10.0713	3.8412	3.8281	2.4902	2.4388	2.5594
	300	9.2413	7.4179	2.2461	4.6712	3.0516	7.4383	6.3244	8.6228	7.4060	3.0576	3.0509	2.1755	2.1289	2.2149
	400	7.0114	5.6136	2.1055	3.5682	2.2961	5.6340	4.7660	6.5101	5.5976	1.9838	1.9811	1.2432	1.2068	1.2745

Table 10. Estimated MSE values when p = 9 and $\varphi = 1$.

ρ	n	BMLE	BRRE	BLE	BJSE	BLTE	BOKE	BDKE	BMRTE	BKLE	BJRRE	BJKLE	BAUMRTE		
		-	$\hat{k}$	$\hat{d}$	$\hat{c}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}$	$\hat{k}$	$\hat{k}$	$\hat{k}_1,\ \hat{d}_1$	$\hat{k}_2,\ \hat{d}_2$	$\hat{k}_3,\ \hat{d}_3$
0.80	30	5.5125	3.2282	2.4053	3.1323	3.7941	3.3603	3.9359	1.6454	3.2366	0.8635	0.8626	0.6370	0.6328	0.6454
	75	1.6401	1.1809	1.1435	1.0506	0.9892	1.1942	1.2467	0.9619	1.1658	0.4576	0.4576	0.4269	0.4329	0.4139
	150	0.7877	0.6565	0.6552	0.6061	0.5619	0.6616	0.6607	0.6250	0.6519	0.3261	0.3261	0.3052	0.3042	0.3107
	200	0.4764	0.4220	0.4239	0.4025	0.3796	0.4241	0.4216	0.4136	0.4204	0.2378	0.2378	0.2307	0.2315	0.2302
	300	0.3439	0.3205	0.3202	0.3086	0.3009	0.3215	0.3204	0.3175	0.3201	0.2335	0.2335	0.2207	0.2276	0.2273
	400	0.2736	0.2570	0.2561	0.2449	0.2422	0.2577	0.2570	0.2549	0.2568	0.1773	0.1773	0.1694	0.1659	0.1663
0.85	30	7.3404	4.4194	2.7573	3.8961	4.7699	4.5640	5.1949	2.6192	4.4264	1.2238	1.2226	0.8596	0.8475	0.8733
	75	2.2020	1.5254	1.4223	1.3438	1.2814	1.5482	1.6197	1.2166	1.5064	0.5402	0.5401	0.5122	0.5150	0.5112
	150	0.8607	0.7170	0.6939	0.6357	0.5939	0.7209	0.7255	0.6734	0.7119	0.2616	0.2616	0.2598	0.2556	0.2570
	200	0.6043	0.5299	0.5239	0.4898	0.4669	0.5329	0.5312	0.5141	0.5275	0.2572	0.2572	0.2479	0.2504	0.2549
	300	0.4217	0.3810	0.3788	0.3686	0.3455	0.3824	0.3799	0.3775	0.3800	0.2499	0.2499	0.2305	0.2311	0.2301
	400	0.3314	0.3086	0.3059	0.2915	0.2863	0.3093	0.3083	0.3064	0.3082	0.1988	0.1988	0.1807	0.1842	0.1850
0.90	30	8.9253	5.2196	2.7308	4.5318	5.9565	5.3999	6.2588	3.0732	5.3293	1.4493	1.4460	0.9354	0.9236	0.9620
	75	4.0449	2.6352	1.9801	2.2199	2.2374	2.6885	2.8420	2.0190	2.6163	0.7237	0.7233	0.5502	0.5451	0.5546
	150	1.5068	1.1775	1.0898	1.0161	0.9034	1.1869	1.1887	1.0925	1.1616	0.3687	0.3687	0.3328	0.3334	0.3328
	200	0.8356	0.6962	0.6860	0.6357	0.5797	0.6999	0.6951	0.6786	0.6911	0.3352	0.3352	0.3278	0.3222	0.3257
	300	0.6468	0.5516	0.5461	0.5080	0.4712	0.5547	0.5500	0.5411	0.5483	0.2853	0.2853	0.2699	0.2647	0.2675
	400	0.5707	0.5035	0.4949	0.4705	0.4484	0.5061	0.5026	0.4954	0.5016	0.2685	0.2685	0.2440	0.2481	0.2420
0.95	30	17.1095	9.8603	2.7524	8.2089	10.9831	10.1339	11.9021	5.4907	10.0002	2.2576	2.2496	1.2923	1.3006	1.3493
	75	10.6635	6.9302	2.4024	5.2919	6.1695	7.0294	7.3138	5.8350	6.9880	2.0748	2.0557	1.2648	1.2523	1.3274
	150	2.6314	1.7564	1.5337	1.4972	1.2208	1.7760	1.7866	1.5814	1.7239	0.5208	0.5208	0.4333	0.4302	0.4353
	200	1.8202	1.2812	1.1899	1.1193	0.8793	1.2921	1.2783	1.2206	1.2611	0.4171	0.4171	0.3709	0.3702	0.3715
	300	1.2897	1.0134	0.9742	0.8932	0.7751	1.0191	1.0082	0.9840	1.0009	0.3875	0.3875	0.3620	0.3622	0.3621
	400	1.0553	0.8370	0.8271	0.7439	0.6526	0.8414	0.8299	0.8206	0.8274	0.3101	0.3101	0.3022	0.3034	0.3017
0.99	30	105.3956	57.5323	1.8903	47.9468	70.0062	59.7886	73.4346	27.4129	59.0222	12.4044	12.1492	5.2065	6.1984	6.3389
	75	39.1924	23.3492	1.8098	15.2414	19.9328	23.7928	25.8277	17.4031	23.5598	5.0983	5.0609	2.6181	2.6264	2.7884
	150	14.4624	8.9305	2.1867	5.5013	5.7434	9.0402	9.2077	8.1210	8.9547	1.9136	1.9110	1.0896	1.0516	1.1193
	200	9.1147	5.5530	2.1759	3.6316	3.3698	5.6255	5.6486	5.2216	5.5625	1.1788	1.1783	0.7293	0.7058	0.7437
	300	8.2397	5.6807	2.2553	4.3130	3.9355	5.7185	5.7259	5.4714	5.6729	2.1071	2.1033	1.5098	1.4870	1.5436
	400	5.5460	3.6353	2.0652	2.5667	2.1292	3.6774	3.6326	3.5156	3.6151	0.8877	0.8876	0.5845	0.5683	0.5938

Table 11. Estimated MSE values when p = 9 and $\varphi = 1.5$.

ρ	n	BMLE	BRRE	BLE	BJSE	BLTE	BOKE	BDKE	BMRTE	BKLE	BJRRE	BJKLE	BAUMRTE		
		-	$\hat{k}$	$\hat{d}$	$\hat{c}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}$	$\hat{k}$	$\hat{k}$	$\hat{k}_1,\ \hat{d}_1$	$\hat{k}_2,\ \hat{d}_2$	$\hat{k}_3,\ \hat{d}_3$
0.80	30	4.7003	2.2477	2.2401	2.7079	3.4892	2.4578	3.3329	0.8047	2.2947	0.5982	0.5980	0.5335	0.5362	0.5333
	75	1.5385	0.9804	1.1045	1.0128	1.0754	1.0074	1.1790	0.5373	0.9558	0.4622	0.4622	0.5540	0.5647	0.5483
	150	0.7462	0.5872	0.6289	0.5852	0.5773	0.5975	0.6352	0.4265	0.5796	0.3391	0.3391	0.4421	0.4535	0.4364
	200	0.4390	0.3730	0.3936	0.3667	0.3678	0.3768	0.3926	0.3012	0.3707	0.2384	0.2384	0.3368	0.3473	0.3315
	300	0.3087	0.2791	0.2891	0.2786	0.2774	0.2806	0.2881	0.2459	0.2784	0.2411	0.2411	0.3274	0.3366	0.3228
	400	0.2413	0.2230	0.2283	0.2205	0.2209	0.2240	0.2285	0.2020	0.2227	0.1916	0.1916	0.2672	0.2755	0.2631
0.85	30	6.2976	3.2725	2.5593	3.4880	4.6073	3.4587	4.5306	1.1506	3.3138	0.9034	0.9030	0.7226	0.7196	0.7285
	75	1.8156	1.1327	1.2500	1.1756	1.2273	1.1595	1.3615	0.6517	1.1090	0.5336	0.5336	0.5100	0.5079	0.5059
	150	0.8598	0.6711	0.7062	0.6395	0.6416	0.6787	0.7305	0.4672	0.6619	0.2673	0.2673	0.2339	0.2427	0.2329
	200	0.5792	0.4827	0.5051	0.4668	0.4686	0.4861	0.5127	0.3774	0.4790	0.2538	0.2538	0.3194	0.3275	0.3154
	300	0.4068	0.3531	0.3700	0.3521	0.3477	0.3559	0.3683	0.2978	0.3514	0.2500	0.2500	0.3284	0.3368	0.3242
	400	0.3157	0.2857	0.2931	0.2786	0.2809	0.2874	0.2946	0.2503	0.2850	0.2052	0.2052	0.2772	0.2851	0.2732
0.90	30	6.5951	4.6222	4.2157	4.8315	5.2147	4.6857	5.5147	2.5784	4.2547	1.1457	1.2457	0.9875	0.8547	0.8350
	75	3.4552	1.8943	1.8085	1.9105	2.2114	1.9682	2.4064	0.8216	1.8663	0.5338	0.5337	0.4901	0.4925	0.4894
	150	1.2718	0.8856	0.9534	0.8656	0.8292	0.9006	0.9943	0.5544	0.8597	0.3901	0.3901	0.3193	0.3248	0.3166
	200	0.7981	0.6275	0.6655	0.6057	0.6003	0.6352	0.6736	0.4789	0.6199	0.3892	0.3892	0.3781	0.3844	0.3749
	300	0.6324	0.5104	0.5399	0.5007	0.4915	0.5159	0.5446	0.3950	0.5048	0.3689	0.3686	0.3438	0.3502	0.3405
	400	0.4656	0.3903	0.4147	0.3901	0.3836	0.3935	0.4115	0.3169	0.3873	0.2533	0.2533	0.2145	0.2212	0.2112
0.95	30	15.3716	7.1506	2.9260	7.7953	11.3986	7.6376	10.7395	1.4492	7.4229	1.4233	1.4207	0.8169	0.8108	0.8499
	75	7.8699	3.9559	2.2923	3.5410	4.8685	4.0776	5.1038	1.9225	4.1114	1.0315	1.0307	0.7435	0.7325	0.7547
	150	2.4303	1.4224	1.4583	1.4032	1.3383	1.4491	1.6661	0.8314	1.3856	0.4391	0.4391	0.4055	0.4059	0.4055
	200	1.8223	1.1175	1.2344	1.1222	1.0524	1.1389	1.2959	0.6648	1.0910	0.3680	0.3680	0.3605	0.3623	0.3598
	300	1.1736	0.8278	0.8889	0.8109	0.7655	0.8398	0.9123	0.5746	0.8093	0.3255	0.3255	0.3159	0.3187	0.3135
	400	0.9528	0.7050	0.7646	0.6968	0.6625	0.7143	0.7682	0.5067	0.6915	0.3035	0.3035	0.2923	0.2955	0.2907
0.99	30	89.0642	41.2734	2.1422	38.9094	62.9950	44.4374	61.1816	10.7459	43.0926	7.1514	7.0387	2.8948	3.3283	3.4721
	75	36.1971	17.7977	1.9807	14.0208	21.6102	18.4760	23.4643	7.1655	18.2070	3.2455	3.2319	1.6581	1.6436	1.7534
	150	13.2611	6.6965	2.2358	5.2543	6.9363	6.8810	8.3759	3.2490	6.7594	1.2378	1.2368	0.7540	0.7331	0.7709
	200	8.5521	4.5055	2.2344	3.5782	4.1863	4.6173	5.4290	2.5045	4.5163	0.8693	0.8690	0.5727	0.5585	0.5816
	300	6.4692	3.5318	2.1398	2.8727	3.1459	3.6210	4.1728	2.0863	3.5356	0.7587	0.7586	0.5195	0.5081	0.5263
	400	5.0095	2.7926	1.9486	2.4160	2.4613	2.8571	3.2582	1.7119	2.7758	0.7003	0.7001	0.4876	0.4785	0.4934

5. Application to real-life data

In this section, we assess the performance of the proposed estimator using two real-world datasets. The MSE serves as the primary metric, allowing for a direct comparison with existing methods and a practical assessment of the proposed estimator's accuracy under realistic conditions.

5.1. Boston housing data

To further examine the practicality of our new proposed estimator, we apply the proposed estimator to the Boston housing data, which contains information on median housing values from 506 census tracts in the Boston suburbs during the 1970 census. This data was originally compiled by Sosa [40] and includes 506 observations across 12 variables, whose summary statistics are described in Table 12.

Table 12. Summary statistics for variables in Boston housing data.

Variable	Mean	Median	SD	Variance	Min	Max	Range	Q1.25%	Q3.75%	Skewness	Kurtosis
y	0.1265	0.1136	0.0714	0.0051	0.0173	0.3797	0.3624	0.0695	0.1696	0.9038	3.4765
x_1	3.6135	0.2565	8.6015	73.9866	0.0063	88.9762	88.9699	0.0820	3.6771	5.2077	39.7528
x_2	11.3636	0.0000	23.3225	543.9368	0.0000	100.0000	100.0000	0.0000	12.5000	2.2191	6.9799
x_3	11.1368	9.6900	6.8604	47.0644	0.4600	27.7400	27.2800	5.1900	18.1000	0.2941	1.7668
x_4	0.5547	0.5380	0.1159	0.0134	0.3850	0.8710	0.4860	0.4490	0.6240	0.7271	2.9241
x_5	6.2846	6.2085	0.7026	0.4937	3.5610	8.7800	5.2190	5.8855	6.6235	0.4024	4.8610
x_6	68.5749	77.5000	28.1489	792.3584	2.9000	100.0000	97.1000	45.0250	94.0750	-0.5972	2.0300
x_7	3.7950	3.2074	2.1057	4.4340	1.1296	12.1265	10.9969	2.1002	5.1884	1.0088	3.4713
x_8	408.2372	330.0000	168.5371	28404.7595	187.0000	711.0000	524.0000	279.0000	666.0000	0.6680	1.8570
x_9	18.4555	19.0500	2.1649	4.6870	12.6000	22.0000	9.4000	17.4000	20.2000	-0.7999	2.7059
x_{10}	356.6740	391.4400	91.2949	8334.7523	0.3200	396.9000	396.5800	375.3775	396.2250	-2.8818	10.1438
x_{11}	22.5289	21.2000	9.1822	84.3124	5.0000	50.0000	45.0000	17.0250	25.0000	1.1076	4.4900

The response variable is the percentage of the population with lower socioeconomic status (y). The explanatory variables include: per capita crime rate by town (x_1), the proportion of residential land zoned for lots over 25,000 square feet (x_2), the proportion of non-retail business acres per town (x_3), nitric oxide concentration (parts per 10 million) (x_4), average number of rooms per dwelling (x_5), proportion of owner-occupied units built before 1940 (x_6), weighted distances to five Boston employment centers (x_7), full-value property-tax rate per USD 10,000 (x_8), pupil-teacher ratio by town (x_9), $1000(B - 0.63)^2$, where B is the proportion of Black residents by town (x_{10}), corrected median value of owner-occupied homes in USD 1000's (x_{11}).

This data provides a comprehensive set of variables to evaluate the performance of the proposed estimator in a real-world context.

The response variable in this data represents a proportion ranging between 0 and 1 as presented in Figure 1, making it suitable for BRM. Additionally, when fitting this data for different unit regression models, the BRM achieves the best performance by achieving the highest log-likelihood (LL) and the lowest Akaike information criterion (AIC), corrected AIC (AICc), and Bayesian information criterion (BIC), which makes it the best model for fitting this data as detailed in Table 13.

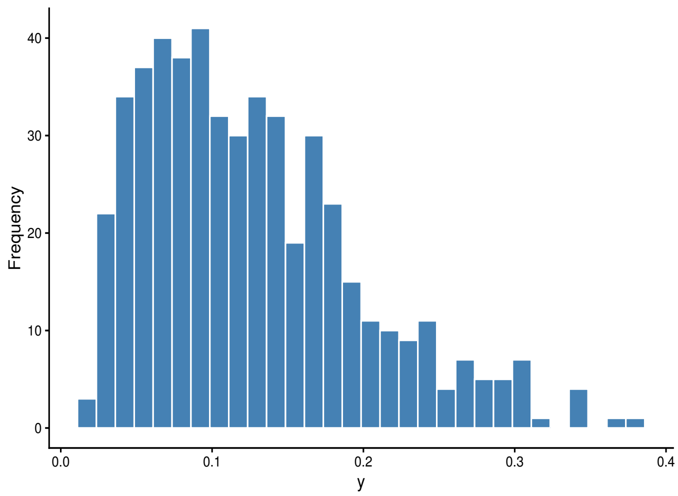

Figure 1. Histogram of response variable y in Boston housing data.

Table 13. LL, AIC, AICc, and BIC values for the fitted models in Boston housing data.

Criterion	BRM	UKRM	UWRM	ULRM
LL	1073.615	1031.931	1070.303	369.6943
AIC	-2121.230	-2037.862	-2114.607	-709.3886
AICc	-2120.490	-2035.571	-2112.316	-709.3886
BIC	-2066.285	-1982.917	-2059.662	-656.4472

*Note: UKRM = Unit Kumaraswamy regression model, UWRM = Unit Weibull regression model, ULRM = Unit Logistic regression model.

Additionally, to check multicollinearity in the Boston housing data, we use the condition number, which is calculated as $\sqrt{\max(\lambda_j)/\min(\lambda_j)}$ equal to 15215.05. This indicates the presence of severe multicollinearity among the covariates. This issue is further supported by the correlation matrix presented in Figure 2, which highlights strong correlation between the covariates. These findings underscore the importance of addressing multicollinearity when applying the BRM to this data.

The response variable (y_i) was modeled using a beta regression as follows:

$$y_i \sim \text{Beta}(\mu_i, \phi),$$

where the mean $\mu_i \in (0, 1)$ is linked to the linear predictor via a logistic link function:

$$\text{logit}(\mu_i) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_{11} x_{11,i}.$$

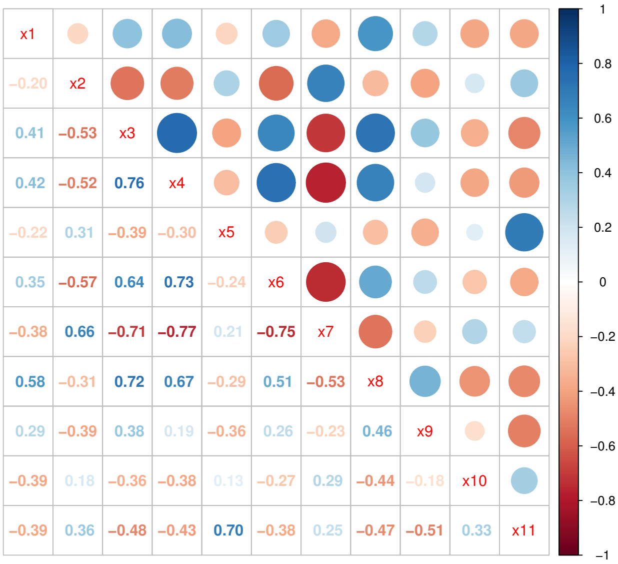

Figure 2. Correlation matrix between explanatory variables in Boston housing data.

Table 14 presents the estimates and MSEs for BMLE along with biased estimators, including BRRE, BLE, BJSE, BLTE, BOKE, BMRTE, BKLE, BJRRE, BJKLE, and BAUMRTE, which are fitted to the Boston housing data via the BRM. Coefficients were obtained by evaluating the estimating Eqs (2.6), (2.10)–(2.17), and (3.1), and the corresponding MSEs were computed with the closed-form expressions provided in Eqs (2.9), (2.21), (2.23), (2.25), (2.27), (2.29), (2.31), (2.33), (2.35), and (3.5).

Table 14. Estimated coefficients and MSEs of the different estimators for the Boston housing data.

Coefficient	BMLE	BRRE	BLE	BJSE	BLTE	BOKE	BDKE	BMRTE	BKLE	BJRRE	BJKLE	BAUMRTE		
	-	$\hat{k}$	$\hat{d}$	$\hat{c}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}$	$\hat{k}$	$\hat{k}$	$\hat{k}_1,\ \hat{d}_1$	$\hat{k}_2,\ \hat{d}_2$	$\hat{k}_3,\ \hat{d}_3$
Intercept	0.243391	-0.000013	0.212730	0.121730	-0.000065	0.001006	0.125747	-0.009859	-0.056273	0.000769	0.246595	0.001746	0.000405	0.001572
x_1	-0.001019	0.000116	-0.000999	-0.000509	0.000667	0.002951	-0.000944	-0.000978	-0.000831	-0.000992	-0.001207	-0.001013	-0.000916	-0.001012
x_2	0.000111	-0.000476	0.000113	0.000056	-0.002739	-0.007625	0.000120	-0.000160	0.000130	0.000174	0.000093	0.000138	0.000232	0.000141
x_3	0.003130	0.000030	0.003101	0.001565	0.000429	0.002234	0.003010	0.001845	0.002707	0.002980	0.003555	0.003161	0.002416	0.003156
x_4	-0.331875	-0.000004	-0.308984	-0.165984	-0.000018	-0.002176	-0.242825	-0.016542	-0.088867	-0.000754	-0.467452	-0.002481	-0.000299	-0.002146
x_5	-0.199082	-0.000100	-0.198365	-0.099569	-0.000552	-0.005350	-0.196352	-0.153000	-0.192336	-0.031729	-0.206287	-0.071997	-0.016340	-0.065484
x_6	0.006953	-0.000191	0.006948	0.003477	0.000537	0.001211	0.006931	0.006152	0.006862	0.006992	0.007043	0.006975	0.007007	0.006977
x_7	-0.025262	-0.000094	-0.024554	-0.012634	-0.000522	-0.003470	-0.022546	-0.024499	-0.018358	-0.008251	-0.032294	-0.015586	-0.004342	-0.014634
x_8	0.000144	-0.000977	0.000139	0.000072	-0.000643	-0.000625	0.000127	0.000064	0.000098	0.000144	0.000190	0.000144	0.000145	0.000144
x_9	-0.015977	-0.000208	-0.015409	-0.007991	-0.000963	-0.007501	-0.013803	-0.018286	-0.010513	-0.010367	-0.021542	-0.012403	-0.007757	-0.012155
x_{10}	-0.000157	-0.003491	-0.000155	-0.000078	-0.004078	-0.002784	-0.000148	-0.000143	-0.000136	-0.000156	-0.000178	-0.000157	-0.000151	-0.000157
x_{11}	-0.044085	-0.000532	-0.044002	-0.022049	-0.003228	-0.022571	-0.043767	-0.047141	-0.043310	-0.042181	-0.044861	-0.043070	-0.039934	-0.042995
MSE	14.642443	0.211831	12.275684	4.655137	0.211403	0.206828	6.803567	0.222002	1.324709	0.202321	1.095835	0.200294	0.204668	0.200104

The estimated coefficients reveal critical insights into the behavior of the estimators: A clear pattern of coefficient shrinkage is evident. The BMLE produces coefficient estimates with the largest magnitudes (e.g., -0.332 for x_4, -0.199 for x_5). In contrast, the best-performing estimators (e.g., BRRE, BAUMRTE) shrink these values dramatically towards zero. For example, the coefficient for x_4 is shrunk from -0.332 (BMLE) to approximately -0.002 by the top performers. This is the hallmark of L_2-type regularization, which mitigates the inflationary effect of multicollinearity on coefficient variance. The top-performing estimators (BRRE, BOKE, BAUMRTE, etc.) show remarkable consistency in their coefficient estimates. For many predictors (e.g., x_1, x_6, x_8, x_{10}), the estimates are nearly identical across these methods. This convergence suggests a stable region in the parameter space that is optimal for prediction under multicollinearity, and these estimators are successfully locating it.

The application results demonstrate a pronounced superiority of shrinkage estimators over the baseline BMLE, which yielded an MSE of 14.642. This substantial error strongly suggests the presence of multicollinearity, a condition that inflates the variance of the BMLE and compromises its predictive accuracy. In contrast, all biased estimators achieved a drastic reduction in MSE, validating their core function of introducing controlled bias to secure a greater reduction in variance, thereby optimizing the bias-variance trade-off. Performance can be divided into groups based on the size of the MSE. The top group includes ridge-type, Liu-type, and hybrid estimators, such as the BRRE, BLE, BOKE, and BJRRE, which have moderate MSE values.

Among all methods, the BAUMRTE ensemble achieved the lowest recorded MSEs, ranging from 0.2001 to 0.2047, establishing it as the preeminent estimator in this analysis. The minimal performance variation across its three parameter configurations further indicates a desirable robustness to specific tuning parameter selections. This superior performance is likely attributable to its adaptive or ensemble structure, which affords a more flexible navigation of the bias-variance trade-off than its constituent estimators. These findings underscore the critical importance of estimator selection in the presence of multicollinearity, with the BAUMRTE showing significant potential for applications in BRM.

5.2. *Body fat data*

To emphasize the practical benefit and flexibility of the proposed estimator, we use other data for the body fat data, which was originally introduced by Penrose et al. [41] and later utilized by Hammad et al. [8]. This dataset consists of n = 252 observations, with one response and 13 covariates, whose summary statistics are described in Table 15.

The response variable y is the body-fat percentage determined by underwater weighing; the thirteen predictors are age in years (x_1), weight in pounds (x_2), height in inches (x_3), neck circumference in cm (x_4), chest circumference in cm (x_5), abdomen circumference in cm (x_6), hip circumference in cm (x_7), thigh circumference in cm (x_8), knee circumference in cm (x_9), ankle circumference in cm (x_{10}), biceps (extended) circumference in cm (x_{11}), forearm circumference in cm (x_{12}), and wrist circumference in cm (x_{13}). This suite of anthropometric measurements yields a rich multivariate profile of body morphology, permitting regression analyses that relate body composition to easily obtained covariates; all variables have been scaled to comparable units so that coefficient estimates remain interpretable and numerical stability is preserved during model fitting.

Table 15. Summary statistics for variables in body fat data.

Variable	Mean	Median	SD	Variance	Min	Max	Range	Q1.25%	Q3.75%	Skewness	Kurtosis
y	0.1915	0.1920	0.0837	0.0070	0.000	0.4750	0.4750	0.1248	0.2530	0.1455	2.6490
x_1	0.4488	0.4300	0.1260	0.0159	0.220	0.8100	0.5900	0.3575	0.5400	0.2818	2.5681
x_2	1.7892	1.7650	0.2939	0.0864	1.185	3.6315	2.4465	1.5900	1.9700	1.1981	8.1418
x_3	0.7015	0.7000	0.0366	0.0013	0.295	0.7775	0.4825	0.6825	0.7225	-5.3529	61.3457
x_4	0.3799	0.3800	0.0243	0.0006	0.311	0.5120	0.2010	0.3640	0.3942	0.5493	5.6422
x_5	1.0082	0.9965	0.0843	0.0071	0.793	1.3620	0.5690	0.9435	1.0537	0.6775	3.9441
x_6	0.9256	0.9095	0.1078	0.0116	0.694	1.4810	0.7870	0.8458	0.9932	0.8334	5.1807
x_7	0.9990	0.9930	0.0716	0.0051	0.850	1.4770	0.6270	0.9550	1.0352	1.4882	10.3002
x_8	0.5941	0.5900	0.0525	0.0028	0.472	0.8730	0.4010	0.5600	0.6235	0.8163	5.5894
x_9	0.3859	0.3850	0.0241	0.0006	0.330	0.4910	0.1610	0.3697	0.3992	0.5137	4.0169
x_{10}	0.2310	0.2280	0.0169	0.0003	0.191	0.3390	0.1480	0.2200	0.2400	2.2417	14.6858
x_{11}	0.3227	0.3205	0.0302	0.0009	0.248	0.4500	0.2020	0.3020	0.3432	0.2838	3.4649
x_{12}	0.2866	0.2870	0.0202	0.0004	0.210	0.3490	0.1390	0.2730	0.3000	-0.2180	3.8255
x_{13}	0.1823	0.1830	0.0093	0.0001	0.158	0.2140	0.0560	0.1760	0.1880	0.2799	3.3642

The response variable is a proportion confined to the open unit interval (0, 1), as displayed in Figure 3, which thus satisfies the distributional requirements of the BRM. The BRM yields the largest LL and simultaneously attains the smallest AIC, AICc, and BIC, thereby providing the best-supported model for these data as detailed in Table 16.

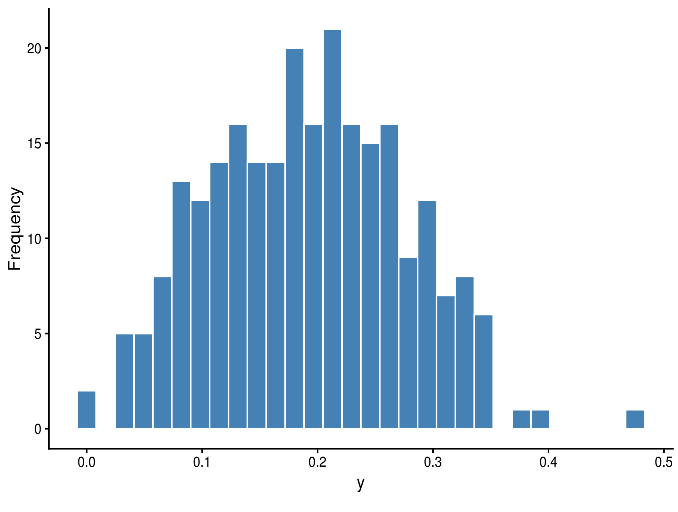

Figure 3. Histogram of response variable y in body fat data.

Table 16. LL, AIC, AICc, and BIC values for the fitted models in body fat data.

Criterion	BRM	UKRM	UWRM	ULRM
LL	395.7120	-761.4240	-759.3901	-708.4826
AIC	380.6258	-731.2517	-726.4096	-678.3102
AICc	277.3610	-524.7219	-519.8798	-471.7805
BIC	369.6943	-709.3886	-709.3886	-656.4472

The condition number is 582.99, signaling severe multicollinearity among the covariates; this conclusion is corroborated by the correlation matrix displayed in Figure 4, where numerous pairwise correlations exceed 0.80. Taken together, these diagnostics underscore the necessity of remedial action against collinearity before any regression technique is applied to this data set.

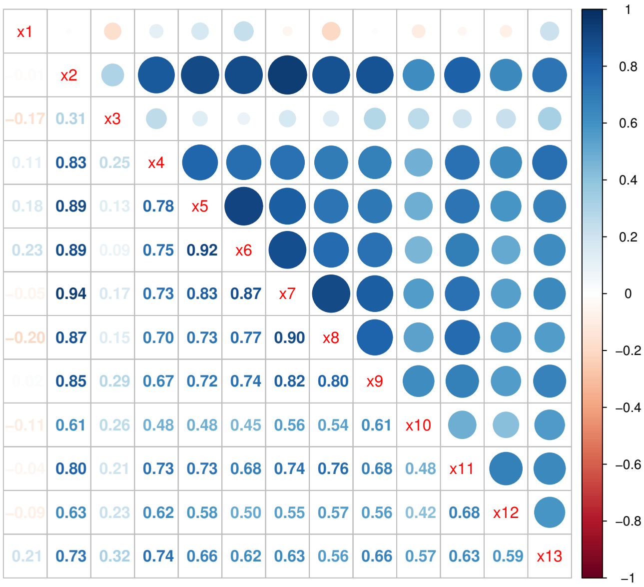

Figure 4. Correlation matrix between explanatory variables in the body fat data.

The response variable (y_i) was modeled using a beta regression as follows:

$$y_i \sim \text{Beta}(\mu_i, \phi),$$

where the mean $\mu_i \in (0, 1)$ is linked to the linear predictor via a logistic link function:

$$\text{logit}(\mu_i) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_{13} x_{13,i}.$$

Analysis of the body fat data in Table 17 starkly illustrates how the choice of the estimator affects the model when covariates are multicollinear.

The baseline BMLE returns an MSE of 2056.14, indicating severe instability driven by inflated coefficient variance resulting from the consequences of multicollinearity. Introducing a controlled amount of bias emerges as an effective remedy: Every regularized alternative slashes the MSE, quantitatively confirming that bias-variance rebalancing is indispensable in this setting.

Table 17. Estimated coefficients and MSEs of the different estimators for the body fat data.

Coefficient	BMLE	BRRE	BLE	BJSE	BLTE	BOKE	BDKE	BMRTE	BKLE	BJRRE	BJKLE	BAUMRTE		
	-	$\hat{k}$	$\hat{d}$	$\hat{c}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}, \hat{d}$	$\hat{k}$	$\hat{k}$	$\hat{k}$	$\hat{k}_1,\ \hat{d}_1$	$\hat{k}_2,\ \hat{d}_2$	$\hat{k}_3,\ \hat{d}_3$
Intercept	-6.04713	-5.86905	-3.59520	-1.94853	-3.49298	-5.68821	-1.13565	-3.84281	-5.69887	-4.14063	-6.18608	-3.87337	-3.96054	-4.48669
x_1	0.59196	0.49288	0.38519	0.19074	0.39050	0.47238	0.17877	0.37621	0.39578	0.52247	0.79357	0.49300	0.50233	0.55620
x_2	-1.01820	-0.95353	-0.09924	-0.32809	-0.05072	-0.89495	0.82329	-0.21128	-0.89149	-1.00052	-1.13032	-1.06455	-1.04326	-0.95275
x_3	0.41960	0.16398	-1.31217	0.13521	-1.37112	0.04308	-3.04839	-1.16384	-0.08452	0.61236	0.94031	0.57340	0.59086	0.60347
x_4	-3.61075	-3.66336	-1.60011	-1.16347	-1.51966	-3.48481	0.41659	-1.80205	-3.72077	-0.56625	-3.06125	-0.36874	-0.42002	-0.99225
x_5	0.49157	0.45852	0.27368	0.15840	0.27193	0.43152	0.05566	0.27813	0.42678	0.28793	0.55724	0.25703	0.26523	0.35155
x_6	6.32854	6.26187	4.52648	2.03920	4.39497	6.17725	2.71485	4.81824	6.19854	4.22414	6.45204	3.36664	3.63131	5.21148
x_7	-2.21088	-2.22225	-1.52966	-0.71240	-1.47648	-2.20214	-0.84452	-1.65262	-2.23395	-1.02959	-2.13386	-0.81357	-0.87507	-1.35889
x_8	3.57569	3.38767	1.31407	1.15217	1.21021	3.24382	-0.95529	1.56409	3.20647	1.51818	3.89324	1.11193	1.22824	2.14527
x_9	1.71547	1.41875	-0.06435	0.55276	-0.10976	1.27899	-1.84764	0.05445	1.13032	0.35537	2.13603	0.25700	0.28382	0.54885
x_{10}	1.39685	0.72461	-0.48343	0.45010	-0.49682	0.56644	-2.36480	-0.44090	0.06647	0.23845	2.67205	0.15388	0.17633	0.41531
x_{11}	0.94867	0.90375	0.22955	0.30568	0.19109	0.86969	-0.49245	0.32248	0.86037	0.22601	0.98140	0.15068	0.17130	0.36389
x_{12}	4.14463	3.35558	0.54565	1.33550	0.46436	3.06305	-3.05949	0.75473	2.58635	0.85056	5.48706	0.55225	0.63202	1.44296
x_{13}	-8.70369	-4.96782	-0.98425	-2.80453	-0.93371	-4.42969	6.73904	-1.11736	-1.29938	-0.36452	-3.00184	-0.24480	-0.27570	-0.64430
MSE	2056.13905	1111.12601	205.37265	1007.25748	197.42971	941.35703	1502.10499	229.31655	678.97995	170.88255	611.04257	168.19936	167.47079	213.05168

The performance of the estimators can be stratified into distinct tiers. The best-performing group, which includes the BLE, BLTE, BJRRE, and the BAUMRTE ensemble, achieves MSEs below 230. Notably, the BLE and BLTE (MSEs 205 and 197, respectively) establish a strong foundational performance, while the more sophisticated BJRRE and BAUMRTE variants push the performance boundary even further, with BAUMRTE $(\hat{k}_2, \hat{d}_2)$ attaining the lowest MSE of 167.47. This positions BAUMRTE as the most effective estimator for this dataset, with its ensemble nature likely providing a more robust optimization of the bias-variance trade-off. A second tier of estimators, including the BRRE, BOKE, and BKLE, delivers significant improvements over the BMLE (MSEs ranging from 679 to 1111) but is substantially outperformed by the top tier. The BJSE and BDKE form a middle group, reducing the BMLE's MSE by over half but failing to match the efficiency of the BAUMRTE.

An examination of the coefficient estimates reveals the mechanism behind this performance disparity. The BMLE coefficients exhibit large magnitudes and high variability (e.g., x_{13}: -8.70, x_6: 6.33), which are hallmarks of an overfit model. In contrast, the top-performing estimators consistently apply a strong shrinkage effect, pulling these extreme coefficients towards zero and, in some cases, even reversing their signs to more plausible values (e.g., x_{13} is shrunk to approximately -0.27 by the best BAUMRTE). The resulting vectors are both more stable and more parsimonious. Consistency of sign and relative magnitude within the leading group corroborates the existence of a well-defined, low-variance region in the parameter space that is conducive to prediction. These findings highlight the effectiveness of the BAUMRTE in reducing MSE, making it a suitable choice for regression analysis in the BRM under multicollinearity.

6. Conclusions

The BRM is an appropriate model to use for predicting the response variable when it is in the form of ratios or proportions and follows the beta distribution. Sometimes, the covariates of the model are highly correlated, and this is known as multicollinearity. The MLE estimating model parameters become unreliable under multicollinearity. To address this, alternative methods, such as BRRE, BLE, and other estimation methods, have been considered. This study addressed the issue of multicollinearity by proposing the use of the BAUMRTE for the BRM. Furthermore, we derived its mathematical properties and compared its performance theoretically with the available methods (MLE, BRRE, BLE, and others) in terms of MSE. A simulation experiment was conducted by varying different factors to evaluate the efficiency of the proposed estimator over other estimators. Two real-life applications were also analyzed to illustrate the findings of the simulation experiment. From the results of the simulation, it was observed that for all the scenarios, the suggested estimator outperformed its competitive estimators in terms of smaller MSE. Moreover, the findings of both applications revealed the efficiency of the proposed estimator over the other considered estimators. This whole study presented evidence that, in the case of severe multicollinearity, almost unbiased estimation methods performed better than biased estimators in the balance between bias and variance, resulting in reduced MSE. Results of both the simulation study and empirical applications provide evidence that the BAUMRTE is superior to other estimators due to its smaller MSE as compared to other considered estimators. Hence, based on the findings of the simulation experiment and real-life applications, we recommend practitioners utilize the BAUMRTE($\hat{k}_2$, $\hat{d}_2$) for estimating BRM parameters due to its better results in the presence of multicollinearity. Despite its superior performance, the proposed BAUMRTE estimator has certain limitations. Its reliance on a shrinkage parameter introduces sensitivity to the calibration of this factor. Furthermore, this study is limited to specific models and conditions, leaving its efficacy in other complex scenarios, such as high-dimensional data and other models, in need. Future work could explore the applicability of this estimator to high-dimensional data exhibiting multicollinearity or investigate its performance in other regression models, such as the Poisson regression model, negative binomial regression model, and gamma regression model [42, 43]. A robust version of this estimator could also be developed to handle multicollinearity and outliers [44].

Author contributions

Randa Alharbi: writing-original draft, validation, methodology, formal analysis, conceptualization, writing-review, investigation; Abdulaziz S. Alghamdi: writing-original draft, validation, methodology, formal analysis, conceptualization, writing-review, investigation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (IPP: 292-662-2025). The authors, therefore, acknowledge with thanks DSR for technical and financial support.

Conflict of interest

The authors declare no conflicts of interest.

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