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Research article

The Sannia ruled surfaces generated by timelike curves under the alternative frame perspective

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Abstract: In three-dimensional Minkowski space, a family of Sannia ruled surfaces is generated from the principal normal ruled surfaces, C-direction ruled surfaces, and Darboux ruled surfaces of timelike curves by employing the Sannia frame. This study thoroughly examines the developability and minimality of these surfaces and establishes the corresponding conditions. Finally, illustrative examples are presented through figures, which vividly demonstrate the geometric characteristics of Sannia ruled surfaces.

Keywords: ruled surfaces; Sannia frame; developable surface; minimal surface; alternative frame
Mathematics Subject Classification: 53A04, 53A05

1. Introduction

Ruled surfaces have been widely studied due to their easily parametrizable nature. A ruled surface is generated by a one-parameter family of rulings sliding along a directrix curve. The striction curve is the intrinsic directrix uniquely embedded within the surface, whereas the directrix curve is an artificially selected reference curve, with infinitely many possible choices. The structure of the striction curve depends solely on the family of rulings of the ruled surface and is independent of parametrization or the choice of the directrix curve, which itself lacks geometric uniqueness. In differential geometry, establishing a moving frame to study the properties of curves and surfaces is a common and essential technique [1–3]. In early studies of ruled surfaces, the conventional Frenet and Darboux frames could only be constructed along directrix curves, and thus failed to simultaneously capture the geometrical information of rulings, imposing restrictions on the study of ruled surfaces. Therefore, in 1925 Sannia chose the striction curve as the directrix by transforming the parameters of the ruled surface and defined a new moving frame – the Sannia frame – of the rulings along the striction curve. This frame incorporated both the striction curve and the rulings into a unified system of differential equations, achieving a synchronous description of their geometric information [4]. In the late 19th century, Wunderlich employed the Sannia frame and suggested its potential applicability to pseudo-Euclidean spaces [5–7]. In 2001 Pottmann and Wallner connected the three vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ of the Sannia frame to the origin, and spherical motion generated three distinct curves: the images of the spherical generator, the central tangent, and the central normal. In this framework, curvature and torsion correspond to the components of angular velocity [8]. This reformulated the twisting of rulings as a problem of spherical geodesic curvature and rotation minimization, yielding two minimal intrinsic variables for computational geometry. In 2006, Hamdoon Omran used the spherical Sannia frame to construct a complete set of Euclidean invariants from the geodesic curvature and distribution parameter, enabling full characterization of developability, minimality, and Weingarten properties for skew ruled surfaces [9]. This established a two-invariant framework for analyzing surface properties.

Recent studies have extended the Sannia frame theory and its applications. Eren Kemal employed the Sannia frame along striction curves of various classical ruled surfaces to define new types of $\boldsymbol{e}_i$–Sannia ruled surfaces in Euclidean space, deriving their developability and minimality conditions, thereby providing a method for constructing novel ruled surfaces [10,11]. Subsequently, Marco Castrillon-López generalized the Sannia frame to three-dimensional Riemannian manifolds and established its equations of motion, invariants and a local existence theorem for ruled surfaces, thereby laying a foundation for studies in non-Euclidean spaces such as the Heisenberg group [12]. Most recently, Bayram constructed a theoretical bridge between the intrinsic geometry of ruled surfaces and line-symmetric rigid body motions via the Sannia frame, revealing its key role in kinematic description and expanding classical motion geometry theory [13].

Existing studies on the Sannia frame have been confined to Euclidean space. Minkowski space, introduced to formulate special relativity, captures the interdependence of time, space, and motion, offering a more accurate description of physical phenomena. The alternative frame further aids in analyzing particle and observer kinematics in curved spacetime, elucidating relativistic and gravitational effects [14–16]. Therefore, based on the alternative frame for timelike curves in Minkowski space, we construct non-lightlike Sannia ruled surfaces from the principal normal, C-direction, and Darboux ruled surfaces. In this paper, we derive their developability and minimality conditions, establish general links between these surface properties and the underlying timelike curve, and illustrate the geometry of these surfaces.

2. Preliminaries

Let E_1^3 be the Minkowski 3-space, E_1^3 is the real vector space $\mathbb{R}^3$ endowed with the standard flat metric

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 - a_3 b_3, \tag{2.1}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ are vectors in E_1^3.

The vector product of $\boldsymbol{a} \times \boldsymbol{b}$ is given by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & -\boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \tag{2.2}$$

where $\boldsymbol{i} = (1, 0, 0)$, $\boldsymbol{j} = (0, 1, 0)$, $\boldsymbol{k} = (0, 0, 1)$ are the natural basis vectors in E_1^3.

Let $\boldsymbol{v}$ be any arbitrary vector in E_1^3. Then, $\boldsymbol{v}$ is said to be spacelike if $\langle \boldsymbol{v}, \boldsymbol{v}\rangle > 0$; lightlike if $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0$; timelike if $\langle \boldsymbol{v}, \boldsymbol{v}\rangle < 0$.

A regular curve is said to be a timelike curve, spacelike curve, or lightlike curve if, the tangent vector is timelike, spacelike, or lightlike, respectively.

Let $\boldsymbol{\alpha} : I \subset E \to E_1^3$ be a timelike curve with arc length parameter s in E_1^3, then the Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ of $\boldsymbol{\alpha}$ satisfies

$$\begin{pmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 \\ \kappa_1 & 0 & \tau_1 \\ 0 & -\tau_1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}, \tag{2.3}$$

where $\langle \boldsymbol{T}, \boldsymbol{T}\rangle = -1$, $\langle \boldsymbol{N}, \boldsymbol{N}\rangle = 1$, $\langle \boldsymbol{B}, \boldsymbol{B}\rangle = 1$.

The alternative frame $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ of $\boldsymbol{\alpha}$ satisfies

$$\begin{pmatrix} \boldsymbol{N}' \\ \boldsymbol{C}' \\ \boldsymbol{W}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_2 & 0 \\ \delta_1\kappa_2 & 0 & \delta_2\tau_2 \\ 0 & \delta_2\tau_2 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{C} \\ \boldsymbol{W} \end{pmatrix}, \tag{2.4}$$

where $\boldsymbol{C} = \dfrac{\kappa_1\boldsymbol{T} + \tau_1\boldsymbol{B}}{\sqrt{|\kappa_1^2 - \tau_1^2|}}$, $\boldsymbol{W} = -\dfrac{\tau_1\boldsymbol{T} + \kappa_1\boldsymbol{B}}{\sqrt{|\kappa_1^2 - \tau_1^2|}}$, $\kappa_2 = \sqrt{|\kappa_1^2 - \tau_1^2|}$, $\tau_2 = \dfrac{\tau_1'\kappa_1 - \kappa_1'\tau_1}{\kappa_1^2 - \tau_1^2}$, $\delta_1 = \operatorname{sgn}(\langle \boldsymbol{W}, \boldsymbol{W}\rangle)$, $\delta_2 = \operatorname{sgn}(\langle \boldsymbol{C}, \boldsymbol{C}\rangle)$.

Without loss of generality, we only investigate the case where $\kappa_1^2 - \tau_1^2 > 0$, that is $\boldsymbol{C} = \frac{\kappa_1\boldsymbol{T}+\tau_1\boldsymbol{B}}{\sqrt{\kappa_1^2-\tau_1^2}}$, $\boldsymbol{W} = -\frac{\tau_1\boldsymbol{T}+\kappa_1\boldsymbol{B}}{\sqrt{\kappa_1^2-\tau_1^2}}$, $\langle \boldsymbol{N}, \boldsymbol{N}\rangle = 1$, $\langle \boldsymbol{C}, \boldsymbol{C}\rangle = -1$, $\langle \boldsymbol{W}, \boldsymbol{W}\rangle = 1$ and $\kappa_2 = \sqrt{\kappa_1^2 - \tau_1^2}$. The case where $\kappa_1^2 - \tau_1^2 < 0$ is analogous and will not be detailed in this paper.

We can define the principal normal ruled surfaces $\boldsymbol{X}_N$, the C-direction ruled surfaces $\boldsymbol{X}_C$ and the Darboux ruled surfaces $\boldsymbol{X}_W$ of $\boldsymbol{\alpha}$ as follows:

$$\boldsymbol{X}_N(s, v) = \boldsymbol{\alpha}(s) + v\boldsymbol{N}, \tag{2.5}$$

$$\boldsymbol{X}_C(s, v) = \boldsymbol{\alpha}(s) + v\boldsymbol{C}, \tag{2.6}$$

$$\boldsymbol{X}_W(s, v) = \boldsymbol{\alpha}(s) + v\boldsymbol{W}, \tag{2.7}$$

where $\{\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{W}\}$ is the alternative moving frame along the curve $\boldsymbol{\alpha}$.

Definition 2.1. *Let $\boldsymbol{r} : I \to S^2(1)$ be a curve on unit sphere (or $\boldsymbol{r} : I \to S^2(-1)$ be a curve on unit pseudosphere), then the Sannia frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a moving frame along $\boldsymbol{r}$, defined by*

$$\boldsymbol{e}_1 = \boldsymbol{r}, \boldsymbol{e}_2 = \frac{\boldsymbol{e}_1'}{\sqrt{\epsilon_1\langle \boldsymbol{e}_1'\boldsymbol{e}_1'\rangle}}, \boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2. \tag{2.8}$$

The Sannia formulas are

$$\begin{pmatrix} \boldsymbol{e}_1' \\ \boldsymbol{e}_2' \\ \boldsymbol{e}_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_3 & 0 \\ -\kappa_3 & 0 & \tau_3 \\ 0 & -\tau_3 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{pmatrix}, \tag{2.9}$$

where $\kappa_3 = \epsilon_2\langle \boldsymbol{e}_1', \boldsymbol{e}_2\rangle$, $\tau_3 = \epsilon_3\langle \boldsymbol{e}_2', \boldsymbol{e}_3\rangle$ are the curvature functions of $\boldsymbol{r}$, and $\epsilon_1 = \operatorname{sgn}\left(\langle \boldsymbol{e}_1', \boldsymbol{e}_1'\rangle\right)$, $\epsilon_2 = \operatorname{sgn}(\langle \boldsymbol{e}_2, \boldsymbol{e}_2\rangle)$, $\epsilon_3 = \operatorname{sgn}(\langle \boldsymbol{e}_3, \boldsymbol{e}_3\rangle)$.

Definition 2.2. *Let $X_A(s,v) = \boldsymbol{\alpha}(s) + v\boldsymbol{r}_A(s)$ be a ruled surface with striction curve $\boldsymbol{\beta}_A$, and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be Sannia frame on $\boldsymbol{r}_A$. The ruled surfaces generated in direction $\boldsymbol{e}_i$ are said to be $\boldsymbol{e}_i$-Sannia ruled surfaces as follows:*

$$\boldsymbol{\Phi}_{A_i}(s,v) = \boldsymbol{\beta}_A(s) + v\boldsymbol{e}_i(s)\ (i = 1,2,3). \tag{2.10}$$

For an $\boldsymbol{e}_i$-Sannia ruled surface $\boldsymbol{\Phi}_{A_i}(s,v) = \boldsymbol{\beta}_A(s) + v\boldsymbol{e}_i(s)$, the first and second fundamental quantities are given by

$$E_{\boldsymbol{\Phi}_{A_i}} = \langle(\boldsymbol{\Phi}_{A_i})_s, (\boldsymbol{\Phi}_{A_i})_s\rangle, F_{\boldsymbol{\Phi}_{A_i}} = \langle(\boldsymbol{\Phi}_{A_i})_s, (\boldsymbol{\Phi}_{A_i})_v\rangle, G_{\boldsymbol{\Phi}_{A_i}} = \langle(\boldsymbol{\Phi}_{A_i})_v, (\boldsymbol{\Phi}_{A_i})_v\rangle,$$
$$l_{\boldsymbol{\Phi}_{A_i}} = \langle(\boldsymbol{\Phi}_{A_i})_{ss}, \boldsymbol{u}_{\boldsymbol{\Phi}_{A_i}}\rangle, m_{\boldsymbol{\Phi}_{A_i}} = \langle(\boldsymbol{\Phi}_{A_i})_{sv}, \boldsymbol{u}_{\boldsymbol{\Phi}_{A_i}}\rangle, n_{\boldsymbol{\Phi}_{A_i}} = \langle(\boldsymbol{\Phi}_{A_i})_{vv}, \boldsymbol{u}_{\boldsymbol{\Phi}_{A_i}}\rangle,$$

where $\boldsymbol{u}_{\boldsymbol{\Phi}_{A_i}} = \frac{(\boldsymbol{\Phi}_{A_i})_s \times (\boldsymbol{\Phi}_{A_i})_v}{\|(\boldsymbol{\Phi}_{A_i})_s \times (\boldsymbol{\Phi}_{A_i})_v\|}$ is the unit normal of $\boldsymbol{\Phi}_{A_i}$.

The Gaussian curvature $K_{\boldsymbol{\Phi}_{A_i}}$ and mean curvature $H_{\boldsymbol{\Phi}_{A_i}}$ of $\boldsymbol{\Phi}_{Ai}$ are computed as follows:

$$K_{\boldsymbol{\Phi}_{A_i}} = \frac{l_{\boldsymbol{\Phi}_{A_i}} n_{\boldsymbol{\Phi}_{A_i}} - m^2_{\boldsymbol{\Phi}_{A_i}}}{E_{\boldsymbol{\Phi}_{A_i}} G_{\boldsymbol{\Phi}_{A_i}} - F^2_{\boldsymbol{\Phi}_{A_i}}}, H_{\boldsymbol{\Phi}_{A_i}} = \frac{1}{2}\frac{E_{\boldsymbol{\Phi}_{A_i}} n_{\boldsymbol{\Phi}_{A_i}} - 2F_{\boldsymbol{\Phi}_{A_i}} m_{\boldsymbol{\Phi}_{A_i}} + G_{\boldsymbol{\Phi}_{A_i}} l_{\boldsymbol{\Phi}_{A_i}}}{E_{\boldsymbol{\Phi}_{A_i}} G_{\boldsymbol{\Phi}_{A_i}} - F^2_{\boldsymbol{\Phi}_{A_i}}}.$$

Next, we generate a family of non-lightlike $\boldsymbol{e}_i$-Sannia ruled surfaces by employing the principal normal ruled surface, C-direction ruled surface, and Darboux ruled surface based on the alternative frame, and investigate their developability and minimality.

3. The Sannia ruled surfaces generated by principal normal ruled surface

For a principal normal ruled surface $\boldsymbol{X}_N(s,v) = \boldsymbol{\alpha}(s) + v\boldsymbol{N}(s)$, the striction curve is determined to be $\boldsymbol{\beta}_N = \boldsymbol{\alpha} - \frac{\kappa_1}{\kappa_2^2}\boldsymbol{N}$. Subsequently, the Sannia frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of $\boldsymbol{N}$ can be easily established by $\boldsymbol{e}_1 = \boldsymbol{N}$, $\boldsymbol{e}_2 = \boldsymbol{C}$ and $\boldsymbol{e}_3 = \boldsymbol{W}$. Then the generated $\boldsymbol{e}_i$-Sannia ruled surface are as follows:

$$\boldsymbol{\Phi}_{N_1} = \boldsymbol{\alpha} - \frac{\kappa_1}{\kappa_2^2}\boldsymbol{N} + v\boldsymbol{N}, \tag{3.1}$$

$$\boldsymbol{\Phi}_{N_2} = \boldsymbol{\alpha} - \frac{\kappa_1}{\kappa_2^2}\boldsymbol{N} + v\boldsymbol{C}, \tag{3.2}$$

$$\boldsymbol{\Phi}_{N_3} = \boldsymbol{\alpha} - \frac{\kappa_1}{\kappa_2^2}\boldsymbol{N} + v\boldsymbol{W}. \tag{3.3}$$

Theorem 3.1. *Let $\boldsymbol{\Phi}_{N_1}$ be an $\boldsymbol{e}_1$-Sannia ruled surface of a principal normal ruled surfaces, then the following conclusions hold:*

i) $\boldsymbol{\Phi}_{N_1}$ *is developable if and only if $\boldsymbol{\alpha}$ is a plane curve,*

ii) $\boldsymbol{\Phi}_{N_1}$ *is minimal if and only if $\boldsymbol{\alpha}$ is either a plane curve or a cylindrical helix.*

Proof. Taking partial derivatives of $\boldsymbol{\Phi}_{N_1}$, we obtain

$$(\boldsymbol{\Phi}_{N_1})_s = -\left(\frac{\kappa_1}{\kappa_2^2}\right)'\boldsymbol{N} + v\kappa_2\boldsymbol{C} + \frac{\tau_1}{\kappa_2}\boldsymbol{W}, (\boldsymbol{\Phi}_{N_1})_v = \boldsymbol{N}.$$

Thus, the first fundamental quantities of $\mathbf{\Phi}_{N_1}$ are as follows:

$$E_{\mathbf{\Phi}_{N_1}}=\left(\left(\frac{\kappa_1}{\kappa_2^2}\right)'\right)^2-v^2\kappa_2^2+\frac{\tau_1^2}{\kappa_2^2},F_{\mathbf{\Phi}_{N_1}}=-\left(\frac{\kappa_1}{\kappa_2^2}\right)',G_{\mathbf{\Phi}_{N_1}}=1.$$

The unit normal vector of the ruled surface is

$$\boldsymbol{u}_{\mathbf{\Phi}_{N_1}}=\frac{-\tau_1\boldsymbol{C}-v\kappa_2^2\boldsymbol{W}}{\sqrt{\left|v^2\kappa_2^4-\tau_1^2\right|}}.$$

Taking the second-order partial derivatives of $\mathbf{\Phi}_{N_1}$, we obtain

$$(\mathbf{\Phi}_{N_1})_{ss}=\left(-\left(\frac{\kappa_1}{\kappa_2^2}\right)''+v\kappa_2^2\right)\boldsymbol{N}+\left(-\left(\frac{\kappa_1}{\kappa_2^2}\right)'\kappa_2-\frac{\tau_1\tau_2}{\kappa_2}+v\kappa_2'\right)\boldsymbol{C}+\left(\left(\frac{\tau_1}{\kappa_2}\right)'-v\kappa_2\tau_2\right)\boldsymbol{W},$$

$$(\mathbf{\Phi}_{N_1})_{sv}=\kappa_2\boldsymbol{C},(\mathbf{\Phi}_{N_1})_{vv}=0.$$

Thus, the second fundamental quantities of $\mathbf{\Phi}_{N_1}$ are as follows:

$$l_{\mathbf{\Phi}_{N_1}}=\frac{-\tau_1\left(\left(\frac{\kappa_1}{\kappa_2^2}\right)'\kappa_2+\frac{\tau_1\tau_2}{\kappa_2}-v\kappa_2'\right)+v\kappa_2^2\left(-\left(\frac{\tau_1}{\kappa_2}\right)'+v\kappa_2\tau_2\right)}{\sqrt{\left|v^2\kappa_2^4-\tau_1^2\right|}},$$

$$m_{\mathbf{\Phi}_{N_1}}=\frac{\kappa_2\tau_1}{\sqrt{\left|v^2\kappa_2^4-\tau_1^2\right|}},n_{\mathbf{\Phi}_{N_1}}=0.$$

The Gaussian curvature of the ruled surface $\mathbf{\Phi}_{N_1}$ is given by

$$K_{\mathbf{\Phi}_{N_1}}=\frac{\kappa_2^2\tau_1^2}{(v^2\kappa_2^4-\tau_1^2)|v^2\kappa_2^4-\tau_1^2|},\tag{3.4}$$

it follows that $\mathbf{\Phi}_{N_1}$ is developable if and only if $\boldsymbol{\alpha}$ is planar since $\kappa_2\neq 0$.

The mean curvature of the ruled surface $\mathbf{\Phi}_{N_1}$ is

$$H_{\mathbf{\Phi}_{N_1}}=\frac{B_1}{2(v^2\kappa_2^4-\tau_1^2)\sqrt{|v^2\kappa_2^4-\tau_1^2|}},\tag{3.5}$$

where $B_1=-2\tau_1\kappa_2\left(\frac{\kappa_1}{\kappa_2^2}\right)'+\tau_1\left(\left(\frac{\kappa_1}{\kappa_2^2}\right)'\kappa_2+\frac{\tau_1\tau_2}{\kappa_2}-v\kappa_2'\right)+v\kappa_2^2\left(\left(\frac{\tau_1}{\kappa_2}\right)'-v\kappa_2\tau_2\right)$.

Thus, $\mathbf{\Phi}_{N_1}$ is a minimal surface if and only if $B_1=0$, which is equivalent to

$$v^2\kappa_2^3\tau_2+v\left(\kappa_2'\tau_1-\kappa_2^2\left(\frac{\tau_1}{\kappa_2}\right)'\right)+\left(\left(\frac{\kappa_1}{\kappa_2^2}\right)'\kappa_2\tau_1+\frac{\tau_1^2\tau_2}{\kappa_2}\right)=0,$$

further analysis leads to the system of equations

$$\begin{cases}\kappa_2^3\tau_2=0,\\ \kappa_2'\tau_1-\kappa_2^2\left(\dfrac{\tau_1}{\kappa_2}\right)'=0,\\ \left(\dfrac{\kappa_1}{\kappa_2^2}\right)'\tau_1\kappa_2-\dfrac{\tau_1^2\tau_2}{\kappa_2}=0.\end{cases}\tag{3.6}$$

Since $\tau_2 = \frac{\tau_1'\kappa_1 - \kappa_1'\tau_1}{\kappa_1^2 - \tau_1^2}$, it follows that $\tau_2 = 0$ if and only if $\boldsymbol{\alpha}$ is a plane curve or a general helix.

We observe that for a plane curve or a cylindrical helix, all the equations of (3.6) are satisfied.

However, for a non-cylindrical general helix, the second equation of (3.6) holds if and only if $\kappa_1^2 - \tau_1^2 = 0$, which contradicts our assumption. ■

In the subsequent discussion, we adopt the following conventions: the symbols c and c_i denote nonzero constants, and the symbol g' denotes the derivative of g with respect to s. These conventions will not be restated unless otherwise specified.

Theorem 3.2. *Let $\boldsymbol{\Phi}_{N_2}$ be an $\boldsymbol{e}_2$-Sannia ruled surface of a principal normal ruled surface, then the following conclusions hold:*

i) $\boldsymbol{\Phi}_{N_2}$ *is developable if and only if*

$$\tau_1 - \tau_2\left(\frac{\kappa_1}{\kappa_2^2}\right)' = 0,$$

ii) $\boldsymbol{\Phi}_{N_2}$ *is minimal if and only if $\boldsymbol{\alpha}$ is a plane curve and a cylindrical helix or*

$$\begin{cases} \tau_2 = c\kappa_2, \\ \tau_1 - \tau_2\left(\dfrac{\kappa_1}{\kappa_2^2}\right)' = 0. \end{cases}$$

Proof. Taking partial derivatives of $\boldsymbol{\Phi}_{N_2}$, we obtain

$$(\boldsymbol{\Phi}_{N_2})_s = \left(-\left(\frac{\kappa_1}{\kappa_2^2}\right)' + v\kappa_2\right)\boldsymbol{N} + \left(\frac{\tau_1}{\kappa_2} - v\tau_2\right)\boldsymbol{W}, (\boldsymbol{\Phi}_{N_2})_v = \boldsymbol{C}.$$

Thus, the first fundamental quantities of $\boldsymbol{\Phi}_{N_2}$ are as follows:

$$E_{\boldsymbol{\Phi}_{N_2}} = \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)' - v\kappa_2\right)^2 + \left(\frac{\tau_1}{\kappa_2} - v\tau_2\right)^2, F_{\boldsymbol{\Phi}_{N_2}} = 0, G_{\boldsymbol{\Phi}_{N_2}} = -1.$$

The unit normal vector of the ruled surface is

$$\boldsymbol{u}_{\boldsymbol{\Phi}_{N_2}} = \frac{\left(v\tau_2 - \frac{\tau_1}{\kappa_2}\right)\boldsymbol{N} - \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)' - v\kappa_2\right)\boldsymbol{W}}{\sqrt{\left(\frac{\tau_1}{\kappa_2} - v\tau_2\right)^2 + \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)' - v\kappa_2\right)^2}}.$$

Taking the second-order partial derivatives of $\boldsymbol{\Phi}_{N_2}$, we obtain

$$(\boldsymbol{\Phi}_{N_2})_{ss} = \left(v\kappa_2' - \left(\frac{\kappa_1}{\kappa_2^2}\right)''\right)\boldsymbol{N} + \left(-\kappa_2\left(\frac{\kappa_1}{\kappa_2^2}\right)' + v\kappa_2^2 - \frac{\tau_1\tau_2}{\kappa_2} + v\tau_2^2\right)\boldsymbol{C} + \left(\left(\frac{\tau_1}{\kappa_2}\right)' - v\tau_2'\right)\boldsymbol{W},$$

$$(\boldsymbol{\Phi}_{N_2})_{sv} = \kappa_2\boldsymbol{N} - \tau_2\boldsymbol{W}, (\boldsymbol{\Phi}_{N_2})_{vv} = 0.$$

Thus, the second fundamental quantities of $\boldsymbol{\Phi}_{N_2}$ are as follows:

$$l_{\boldsymbol{\Phi}_{N_2}} = \frac{\left(-\left(\frac{\kappa_1}{\kappa_2^2}\right)'' + v\kappa_2'\right)\left(-\frac{\tau_1}{\kappa_2} + v\tau_2\right) - \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)' - v\kappa_2\right)\left(\left(\frac{\tau_1}{\kappa_2}\right)' - v\tau_2'\right)}{\sqrt{\left(\frac{\tau_1}{\kappa_2} - v\tau_2\right)^2 + \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)' - v\kappa_2\right)^2}},$$

$$m_{\mathbf{\Phi}_{N_2}} = \frac{-\tau_1 + \tau_2\left(\frac{\kappa_1}{\kappa_2^2}\right)'}{\sqrt{\left(\frac{\tau_1}{\kappa_2} - \nu\tau_2\right)^2 + \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)' - \nu\kappa_2\right)^2}}, n_{\mathbf{\Phi}_{N_2}} = 0.$$

The Gaussian curvature of the ruled surface $\mathbf{\Phi}_{N_2}$ is given by

$$K_{\mathbf{\Phi}_{N_2}} = -\frac{\left(\tau_1 - \tau_2\left(\frac{\kappa_1}{\kappa_2^2}\right)'\right)^2}{\left(\left(\frac{\tau_1}{\kappa_2} - \nu\tau_2\right)^2 + \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)' - \nu\kappa_2\right)^2\right)^2}, \tag{3.7}$$

it follows that $\mathbf{\Phi}_{N_2}$ is developable if and only if $\tau_1 - \tau_2\left(\frac{\kappa_1}{\kappa_2^2}\right)' = 0$.

The mean curvature of the ruled surface $\mathbf{\Phi}_{N_2}$ is

$$H_{\mathbf{\Phi}_{N_2}} = \frac{B_2}{2\left(\left(\frac{\tau_1}{\kappa_2} - \nu\tau_2\right)^2 + \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)' - \nu\kappa_2\right)^2\right)^{\frac{3}{2}}}, \tag{3.8}$$

where $B_2 = \left(-\left(\frac{\kappa_1}{\kappa_2^2}\right)'' + \nu\kappa_2'\right)\left(-\frac{\tau_1}{\kappa_2} + \nu\tau_2\right) - \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)' - \nu\kappa_2\right)\left(\left(\frac{\tau_1}{\kappa_2}\right)' - \nu\tau_2'\right)$.

Thus, $\mathbf{\Phi}_{N_2}$ is a minimal surface if and only if $B_2 = 0$, which is equivalent to

$$\nu^2\left(\kappa_2'\tau_2 - \tau_2'\kappa_2\right) + \nu\left(-\tau_2\left(\frac{\kappa_1}{\kappa_2^2}\right)'' - \kappa_2'\frac{\tau_1}{\kappa_2} + \tau_2'\left(\frac{\kappa_1}{\kappa_2^2}\right)' + \kappa_2\left(\frac{\tau_1}{\kappa_2}\right)'\right) + \left(\frac{\tau_1}{\kappa_2}\right)\left(\frac{\kappa_1}{\kappa_2^2}\right)'' - \left(\frac{\tau_1}{\kappa_2}\right)'\left(\frac{\kappa_1}{\kappa_2^2}\right)' = 0.$$

Further, we have

$$\begin{cases} \kappa_2'\tau_2 - \tau_2'\kappa_2 = 0, \\ \tau_2\left(\dfrac{\kappa_1}{\kappa_2^2}\right)'' + \kappa_2'\dfrac{\tau_1}{\kappa_2} - \tau_2'\left(\dfrac{\kappa_1}{\kappa_2^2}\right)' - \kappa_2\left(\dfrac{\tau_1}{\kappa_2}\right)' = 0, \\ \left(\dfrac{\tau_1}{\kappa_2}\right)\left(\dfrac{\kappa_1}{\kappa_2^2}\right)'' - \left(\dfrac{\tau_1}{\kappa_2}\right)'\left(\dfrac{\kappa_1}{\kappa_2^2}\right)' = 0. \end{cases} \tag{3.9}$$

It can be shown that when $\tau_2 \neq 0$, Eq (3.9) is solved as

$$\begin{cases} \tau_2 = c\kappa_2, \\ \tau_1 - \tau_2\left(\frac{\kappa_1}{\kappa_2^2}\right)' = 0. \end{cases}$$

When $\tau_2 = 0$, it is easy to see from (3.9) that $\boldsymbol{\alpha}$ is a plane curve or a cylindrical helix. ■

Corollary 3.3. *The* $\mathbf{e}_2$*-Sannia ruled surface of a principal normal ruled surface of a plane curve is developable and minimal.*

Theorem 3.4. *Let* $\mathbf{\Phi}_{N_3}$ *be an* $\mathbf{e}_3$*-Sannia ruled surface of a principal normal ruled surface, then the following conclusions hold:*

i) $\mathbf{\Phi}_{N_3}$ *is developable if and only if* $\tau_2 = 0$ *or* $\left(\dfrac{\kappa_1}{\kappa_2^2}\right)' = 0$*,*

ii) $\mathbf{\Phi}_{N_3}$ *is non-minimal.*

Proof. Taking partial derivatives of $\mathbf{\Phi}_{N_3}$, we obtain

$$(\mathbf{\Phi}_{N_3})_s = -\left(\frac{\kappa_1}{\kappa_2^2}\right)' \boldsymbol{N} - v\tau_2 \boldsymbol{C} + \frac{\tau_1}{\kappa_2}\boldsymbol{W}, (\mathbf{\Phi}_{N_3})_v = \boldsymbol{W}.$$

Thus, the first fundamental quantities of $\mathbf{\Phi}_{N_3}$ are as follows:

$$E_{\mathbf{\Phi}_{N_3}} = \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)'\right)^2 - v^2\tau_2^2 + \frac{\tau_1^2}{\kappa_2^2}, F_{\mathbf{\Phi}_{N_3}} = \frac{\tau_1}{\kappa_2}, G_{\mathbf{\Phi}_{N_3}} = 1.$$

The unit normal vector of the ruled surface is

$$\boldsymbol{u}_{\mathbf{\Phi}_{N_3}} = \frac{-v\tau_2\boldsymbol{N} - \left(\frac{\kappa_1}{\kappa_2^2}\right)'\boldsymbol{C}}{\sqrt{\left|v^2\tau_2^2 - \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)'\right)^2\right|}}.$$

Taking the second-order partial derivatives of $\mathbf{\Phi}_{N_3}$, we obtain the following results:

$$(\mathbf{\Phi}_{N_3})_{ss} = -\left(\left(\frac{\kappa_1}{\kappa_2^2}\right)'' + v\kappa_2\tau_2\right)\boldsymbol{N} - \left(\kappa_2\left(\frac{\kappa_1}{\kappa_2^2}\right)' + \frac{\tau_1\tau_2}{\kappa_2} - v\tau_2'\right)\boldsymbol{C} - \left(v\tau_2^2 + \left(\frac{\tau_1}{\kappa_2}\right)'\right)\boldsymbol{W},$$

$$(\mathbf{\Phi}_{N_3})_{sv} = -\tau_2\boldsymbol{C}, (\mathbf{\Phi}_{N_3})_{vv} = 0.$$

Thus, the second fundamental quantities of $\mathbf{\Phi}_{N_3}$ are as follows:

$$l_{\mathbf{\Phi}_{N_3}} = \frac{v^2\tau_2^2\kappa_2 + v\tau_2\left(\frac{\kappa_1}{\kappa_2^2}\right)'' - \left(\frac{\kappa_1}{\kappa_2^2}\right)'\left(\kappa_2\left(\frac{\kappa_1}{\kappa_2^2}\right)' + \frac{\tau_1\tau_2}{\kappa_2} + v\tau_2'\right)}{\sqrt{\left|v^2\tau_2^2 - \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)'\right)^2\right|}},$$

$$m_{\mathbf{\Phi}_{N_3}} = \frac{-\tau_2\left(\frac{\kappa_1}{\kappa_2^2}\right)'}{\sqrt{\left|v^2\tau_2^2 - \left(\left(\frac{\kappa_1}{\kappa_2^2}\right)'\right)^2\right|}}, n_{\mathbf{\Phi}_{N_3}} = 0.$$

The Gaussian curvature of the ruled surface $\mathbf{\Phi}_{N_3}$ is given by

$$K_{\mathbf{\Phi}_{N_3}} = \frac{\tau_2^2\left(\frac{\kappa_1}{\kappa_2^2}\right)'^2}{\left(v^2\tau_2^2 - \left(\frac{\kappa_1}{\kappa_2^2}\right)'^2\right)\cdot\left|v^2\tau_2^2 - \left(\frac{\kappa_1}{\kappa_2^2}\right)'^2\right|}, \tag{3.10}$$

it follows that $\mathbf{\Phi}_{N_3}$ is developable if and only if either $\tau_2 = 0$ or $\left(\frac{\kappa_1}{\kappa_2^2}\right)' = 0$, but not both. If both are satisfied, the normal vector $\boldsymbol{U}_{\mathbf{\Phi}_{N_3}} = (\mathbf{\Phi}_{N_3})_s \times (\mathbf{\Phi}_{N_3})_v$ is a lightlike vector, which contradicts our assumption.

The mean curvature of the ruled surface $\mathbf{\Phi}_{N_3}$ is

$$H_{\mathbf{\Phi}_{N_3}} = \frac{B_3}{-2\left(v^2\tau_2^2 - \left(\frac{\kappa_1}{\kappa_2^2}\right)'^2\right)\sqrt{\left|v^2\tau_2^2 - \left(\frac{\kappa_1}{\kappa_2^2}\right)'^2\right|}}, \tag{3.11}$$

where $B_3 = \frac{\tau_1\tau_2}{\kappa_2}\left(\frac{\kappa_1}{\kappa_2^2}\right)' + v\tau_2\left(\left(\frac{\kappa_1}{\kappa_2^2}\right)'' + v\kappa_2\tau_2\right) - \left(\frac{\kappa_1}{\kappa_2^2}\right)'\left(\kappa_2\left(\frac{\kappa_1}{\kappa_2^2}\right)' + v\tau_2'\right)$.

Thus, $\mathbf{\Phi}_{N_3}$ is a minimal surface if and only if $B_3 = 0$, which is equivalent to

$$v^2\tau_2^2\kappa_2 + v\left(\tau_2\left(\frac{\kappa_1}{\kappa_2^2}\right)'' - \tau_2'\left(\frac{\kappa_1}{\kappa_2^2}\right)'\right) + \left(\frac{\tau_1\tau_2}{\kappa_2}\left(\frac{\kappa_1}{\kappa_2^2}\right)' - \kappa_2\left(\left(\frac{\kappa_1}{\kappa_2^2}\right)'\right)^2\right) = 0.$$

Further, we have

$$\begin{cases} \tau_2 = 0, \\ \left(\frac{\kappa_1}{\kappa_2^2}\right)' = 0. \end{cases} \tag{3.12}$$

However, it implies that $\mathbf{\Phi}_{N_3}$ is lightlike, which contradicts our assumption. Therefore, $\mathbf{\Phi}_{N_3}$ is non-minimal. ∎

Based on the equivalent condition for developability established in Theorem 3.4, we now examine the following cases to derive Corollary 3.5 and Corollary 3.6:

Corollary 3.5. $\tau_2 = 0$ *if and only if* $\boldsymbol{\alpha}$ *is a non-circular curve or a non-cylindrical general helix.*

Proof. Since $\tau_2 = 0$ holds if and only if $\boldsymbol{\alpha}$ is a plane curve or a generalized helix. According to the analysis of Theorem 3.4, thus $\boldsymbol{\alpha}$ must be a non-circular curve or a non-cylindrical generalized helix. ∎

By performing a simple transformation on $\left(\frac{\kappa_1}{\kappa_2^2}\right)' = 0$, we obtain another equivalent condition for $\mathbf{\Phi}_{N_3}$ to be developable:

Corollary 3.6. $\tau_2 \neq 0$ *if and only if* $\boldsymbol{\alpha}$ *is a curve with* $\kappa_1 = c_1 e^{\int 2f(s)\,ds}$, $\tau_1 = e^{\int f(s)\,ds}\sqrt{c_1^2 e^{\int 2f(s)\,ds} - c_2^2}$.

Proof. The equation $\left(\frac{\kappa_1}{\kappa_2^2}\right)' = 0$ implies $\kappa_1'\kappa_2 - 2\kappa_1\kappa_2' = 0$. Further analysis yields

$$\begin{cases} \frac{\kappa_1'}{\kappa_1} = 2f(s), \\ \frac{\kappa_2'}{\kappa_2} = f(s), \end{cases}$$

where $f(s)$ is nonzero.

Since $\kappa_2 = \sqrt{\kappa_1^2 - \tau_1^2}$, solving the above system of differential equations yields

$$\begin{cases} \kappa_1 = c_1 e^{\int 2f(s)\,ds}, \\ \tau_1 = e^{\int f(s)\,ds}\sqrt{c_1^2 e^{\int 2f(s)\,ds} - c_2^2}. \end{cases} \tag{3.13}$$

Substituting (3.13) into τ_2 shows that $\tau_2 \neq 0$. ∎

The equivalent conditions for developability, derived from Corollary 3.5 and Corollary 3.6, can be summarized as follows:

Corollary 3.7. $\boldsymbol{\Phi}_{N_3}$ *is developable if and only if* $\boldsymbol{\alpha}$ *is one of the followings:*

i) a non-circular curve,
ii) a non-cylindrical general helix,
iii) a curve with $\kappa_1 = c_1 e^{\int 2f(s)ds}$, $\tau_1 = e^{\int f(s)ds}\sqrt{c_1^2 e^{\int 2f(s)ds} - c_2^2}$.

4. The Sannia ruled surfaces generated by C-direction ruled surface

For a C-direction ruled surface $\boldsymbol{X}_C = \boldsymbol{\alpha}(s) + v\boldsymbol{C}(s)$, the striction curve is determined to be $\boldsymbol{\beta}_C = \boldsymbol{\alpha} + \frac{\tau_1\tau_2}{\kappa_2(\kappa_2^2+\tau_2^2)}\boldsymbol{C}$. Subsequently, the Sannia frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of $\boldsymbol{C}$ can be easily established by $\boldsymbol{e}_1 = \boldsymbol{C}$, $\boldsymbol{e}_2 = \frac{\kappa_2\boldsymbol{N}-\tau_2\boldsymbol{W}}{\sqrt{\kappa_2^2+\tau_2^2}}$ and $\boldsymbol{e}_3 = \frac{-\tau_2\boldsymbol{N}-\kappa_2\boldsymbol{W}}{\sqrt{\kappa_2^2+\tau_2^2}}$. Then the generated $\boldsymbol{e}_i$-Sannia ruled surfaces are as follows:

$$\boldsymbol{\Phi}_{C_1} = \boldsymbol{\alpha} + Q\boldsymbol{C} + v\boldsymbol{C}, \tag{4.1}$$

$$\boldsymbol{\Phi}_{C_2} = \boldsymbol{\alpha} + Q\boldsymbol{C} + v\frac{\kappa_2\boldsymbol{N} - \tau_2\boldsymbol{W}}{\sqrt{\kappa_2^2 + \tau_2^2}}, \tag{4.2}$$

$$\boldsymbol{\Phi}_{C_3} = \boldsymbol{\alpha} + Q\boldsymbol{C} + v\frac{-\tau_2\boldsymbol{N} - \kappa_2\boldsymbol{W}}{\sqrt{\kappa_2^2 + \tau_2^2}}, \tag{4.3}$$

where $Q = \frac{\tau_1\tau_2}{\kappa_2(\kappa_2^2+\tau_2^2)}$.

Theorem 4.1. *Let* $\boldsymbol{\Phi}_{C_1}$ *be an* $\boldsymbol{e}_1$*-Sannia ruled surface of a* C*-direction ruled surface, then the following conclusions hold:*
i) $\boldsymbol{\Phi}_{C_1}$ *is developable if and only if* $\boldsymbol{\alpha}$ *is planar,*
ii) $\boldsymbol{\Phi}_{C_1}$ *is minimal if and only if* $\boldsymbol{\alpha}$ *is planar.*

Proof. Taking partial derivatives of $\boldsymbol{\Phi}_{C_1}$, we obtain

$$(\boldsymbol{\Phi}_{C_1})_s = \lambda_1\boldsymbol{N} + \lambda_2\boldsymbol{C} + \lambda_3\boldsymbol{W}, (\boldsymbol{\Phi}_{C_1})_v = \boldsymbol{C},$$

where $\lambda_1 = \kappa_2(Q + v)$, $\lambda_2 = Q' + \frac{\kappa_1}{\kappa_2}$, $\lambda_3 = -v\tau_2 + \frac{\tau_1}{\kappa_2} - Q\tau_2$.

Thus, the first fundamental quantities of $\boldsymbol{\Phi}_{C_1}$ are as follows:

$$E_{\boldsymbol{\Phi}_{C_1}} = \lambda_1^2 - \lambda_2^2 + \lambda_3^2, F_{\boldsymbol{\Phi}_{C_1}} = -\lambda_2, G_{\boldsymbol{\Phi}_{C_1}} = -1.$$

The unit normal vector of the ruled surface is

$$\boldsymbol{u}_{\boldsymbol{\Phi}_{C_1}} = -\frac{\lambda_3\boldsymbol{N} - \lambda_1\boldsymbol{W}}{\sqrt{\lambda_1^2 + \lambda_3^2}}.$$

Taking the second-order partial derivatives of

$$(\boldsymbol{\Phi}_{C_1})_{ss} = (\lambda_1' + \lambda_2\kappa_2)\boldsymbol{N} + (\lambda_1\kappa_2 + \lambda_2' - \lambda_3\tau_2)\boldsymbol{C} + (-\lambda_2\tau_2 + \lambda_3')\boldsymbol{W},$$

$$(\boldsymbol{\Phi}_{C_1})_{sv} = \kappa_2\boldsymbol{N} - \tau_2\boldsymbol{W}, (\boldsymbol{\Phi}_{C_1})_{vv} = 0.$$

Thus, the second fundamental quantities of $\boldsymbol{\Phi}_{C_1}$ are as follows:

$$l_{\boldsymbol{\Phi}_{C_1}} = \frac{-\lambda_3(\lambda_1' + \lambda_2\kappa_2) - \lambda_1(\lambda_2\tau_2 - \lambda_3')}{\sqrt{\lambda_1^2 + \lambda_3^2}},$$

$$m_{\boldsymbol{\Phi}_{C_1}} = \frac{-\tau_1}{\sqrt{\lambda_1^2 + \lambda_3^2}}, n_{\boldsymbol{\Phi}_{C_1}} = 0.$$

The Gaussian curvature of the ruled surface $\boldsymbol{\Phi}_{C_1}$ is given by

$$K_{\boldsymbol{\Phi}_{C_1}} = \frac{\tau_1^2}{\left(\lambda_1^2 + \lambda_3^2\right)^2}, \tag{4.4}$$

it follows that $\boldsymbol{\Phi}_{C_1}$ is developable if and only if $\boldsymbol{\alpha}$ is planar.

The mean curvature of the ruled surface $\boldsymbol{\Phi}_{C_1}$ is

$$H_{\boldsymbol{\Phi}_{C_1}} = -\frac{B_4}{2(\lambda_1^2 + \lambda_3^2)^{\frac{3}{2}}}, \tag{4.5}$$

where $B_4 = -\lambda_2\tau_1 + \lambda_3\lambda_1' - \lambda_1\lambda_3'$.

Thus, $\boldsymbol{\Phi}_{C_1}$ is a minimal surface if and only if $B_4 = 0$, which is equivalent to

$$\begin{cases} \tau_2'\kappa_2 - \kappa_2'\tau_2 = 0, \\ \frac{\tau_1\kappa_1'}{\kappa_2} - \kappa_2\left(\frac{\tau_1}{\kappa_2}\right)' = 0, \\ \frac{\tau_1\kappa_1}{\kappa_2} = 0. \end{cases} \tag{4.6}$$

Thus, Eq (4.6) holds if and only if $\boldsymbol{\alpha}$ is planar. ∎

Theorem 4.2. *Let $\boldsymbol{\Phi}_{C_2}$ be an $\boldsymbol{e}_2$-Sannia ruled surface of a C-direction ruled surface, then the following conclusions hold:*

i) $\boldsymbol{\Phi}_{C_2}$ *is developable if and only if*

$$\left(\frac{\kappa_1}{\kappa_2} + Q'\right)(\kappa_2\tau_2' - \tau_2\kappa_2') + \tau_1(\kappa_2^2 + \tau_2^2) = 0,$$

ii) $\boldsymbol{\Phi}_{C_2}$ *is minimal, then*

$$\frac{\left(\frac{\kappa_1}{\kappa_2} + Q'\right)'}{\frac{\kappa_1}{\kappa_2} + Q'} = \frac{\kappa_2\tau_2' + \kappa_2'\tau_2}{\left(\kappa_2^2 + \tau_2^2\right)^{\frac{3}{2}}}.$$

Proof. Taking partial derivatives of $\boldsymbol{\Phi}_{C_2}$, we obtain

$$(\boldsymbol{\Phi}_{C_2})_s = \eta_1\boldsymbol{N} + \eta_2\boldsymbol{C} + \eta_3\boldsymbol{W}, (\boldsymbol{\Phi}_{C_2})_v = \frac{\kappa_2}{\sqrt{\kappa_2^2 + \tau_2^2}}\boldsymbol{N} - \frac{\tau_2}{\sqrt{\kappa_2^2 + \tau_2^2}}\boldsymbol{W},$$

where $\eta_1 = Q\kappa_2 + v\left(\frac{\kappa_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'$, $\eta_2 = \frac{\kappa_1}{\kappa_2} + Q' + v\sqrt{\kappa_2^2+\tau_2^2}$, $\eta_3 = \frac{\tau_1}{\kappa_2} - Q\tau_2 - v\left(\frac{\tau_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'$.

Thus, the first fundamental quantities of $\mathbf{\Phi}_{C_2}$ are as follows:

$$E_{\mathbf{\Phi}_{C_2}} = \eta_1^2 - \eta_2^2 + \eta_3^2, F_{\mathbf{\Phi}_{C_2}} = 0, G_{\mathbf{\Phi}_{C_2}} = 1.$$

The unit normal vector of the ruled surface is

$$\boldsymbol{u}_{\mathbf{\Phi}_{C_2}} = \frac{-\eta_2\tau_2\boldsymbol{N} - (\eta_1\tau_2 + \eta_3\kappa_2)\boldsymbol{C} - \eta_2\kappa_2\boldsymbol{W}}{\sqrt{\left|\eta_2^2(\kappa_2^2+\tau_2^2) - (\eta_3\kappa_2+\eta_1\tau_2)^2\right|}}.$$

Taking the second-order partial derivatives of $\mathbf{\Phi}_{C_2}$, we obtain

$$(\mathbf{\Phi}_{C_2})_{ss} = (\eta_1' + \eta_2\kappa_2)\boldsymbol{N} + (\eta_1\kappa_2 + \eta_2' - \eta_3\tau_2)\boldsymbol{C} + (\eta_3' - \eta_2\tau_2)\boldsymbol{W},$$

$$(\mathbf{\Phi}_{C_2})_{sv} = \left(\frac{\kappa_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'\boldsymbol{N} + \sqrt{\kappa_2^2+\tau_2^2}\,\boldsymbol{C} - \left(\frac{\tau_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'\boldsymbol{W}, (\mathbf{\Phi}_{C_2})_{vv} = 0.$$

Thus, the second fundamental quantities of $\mathbf{\Phi}_{C_2}$ are as follows:

$$l_{\mathbf{\Phi}_{C_2}} = \frac{-\eta_2\eta_1'\tau_2 + (\eta_3\kappa_2+\eta_1\tau_2)\eta_2' - \eta_2\eta_3'\kappa_2}{\sqrt{\left|\eta_2^2\left(\kappa_2^2+\tau_2^2\right) - (\eta_3\kappa_2+\eta_1\tau_2)^2\right|}},$$

$$m_{\mathbf{\Phi}_{C_2}} = \frac{\left(\frac{\kappa_1}{\kappa_2}+Q'\right)\frac{\kappa_2\tau_2'-\tau_2\kappa_2'}{\sqrt{\kappa_2^2+\tau_2^2}} + \tau_1\sqrt{\kappa_2^2+\tau_2^2}}{\sqrt{\left|\eta_2^2(\kappa_2^2+\tau_2^2) - (\eta_3\kappa_2+\eta_1\tau_2)^2\right|}}, n_{\mathbf{\Phi}_{C_2}} = 0.$$

The Gaussian curvature of the ruled surface $\mathbf{\Phi}_{C_2}$ is given by

$$K_{\mathbf{\Phi}_{C_2}} = \frac{\left(\kappa_2^2+\tau_2^2\right)\left(\left(\frac{\kappa_1}{\kappa_2}+Q'\right)\frac{\kappa_2\tau_2'-\tau_2\kappa_2'}{\sqrt{\kappa_2^2+\tau_2^2}} + \sqrt{\kappa_2^2+\tau_2^2}\,\tau_1\right)^2}{\left(\eta_2^2(\kappa_2^2+\tau_2^2) - (\eta_3\kappa_2+\eta_1\tau_2)^2\right)\cdot\left|\eta_2^2(\kappa_2^2+\tau_2^2) - (\eta_3\kappa_2+\eta_1\tau_2)^2\right|}, \tag{4.7}$$

it follows that $\mathbf{\Phi}_{C_2}$ is developable if and only if $\left(\frac{\kappa_1}{\kappa_2}+Q'\right)(\kappa_2\tau_2' - \tau_2\kappa_2') + \tau_1(\kappa_2^2+\tau_2^2) = 0$.

The mean curvature of the ruled surface $\mathbf{\Phi}_{C_2}$ is

$$H_{\mathbf{\Phi}_{C_2}} = \frac{\left(\kappa_2^2+\tau_2^2\right)B_5}{-2\left(\eta_2^2\left(\kappa_2^2+\tau_2^2\right) - (\eta_3\kappa_2+\eta_1\tau_2)^2\right)\sqrt{\left|\eta_2^2\left(\kappa_2^2+\tau_2^2\right) - (\eta_3\kappa_2+\eta_1\tau_2)^2\right|}}, \tag{4.8}$$

where $B_5 = -\eta_2\eta_1'\tau_2 + (\eta_3\kappa_2+\eta_1\tau_2)\eta_2' - \eta_2\eta_3'\kappa_2$.

Thus, $\mathbf{\Phi}_{C_2}$ is a minimal surface if and only if $B_5 = 0$, which is equivalent to

$$v^2 b_1 + v b_2 + b_3 = 0,$$

where

$$b_1 = \sqrt{\kappa_2^2+\tau_2^2}\left(\kappa_2\left(\frac{\tau_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'' - \tau_2\left(\frac{\kappa_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)''\right) - \frac{(\kappa_2\tau_2'-\kappa_2'\tau_2)(\kappa_2\kappa_2'+\tau_2\tau_2')}{(\kappa_2^2+\tau_2^2)^{\frac{3}{2}}},$$

$$b_2 = \sqrt{\kappa_2^2+\tau_2^2}\left(-\kappa_2\left(\frac{\tau_2}{\kappa_2}\right)' + Q(\kappa_2\tau_2'+\kappa_2'\tau_2)\right) - \frac{\kappa_2\tau_2'-\kappa_2'\tau_2}{\sqrt{\kappa_2^2+\tau_2^2}}\left(\left(\frac{\kappa_1}{\kappa_2}\right)' + Q''\right)$$
$$+\tau_1\frac{\kappa_2\kappa_2'+\tau_2\tau_2'}{\kappa_2^2+\tau_2^2} + \left(\frac{\kappa_1}{\kappa_2}+Q'\right)\left(\kappa_2\left(\frac{\tau_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'' - \tau_2\left(\frac{\kappa_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)''\right),$$

$$b_3 = \left(\frac{\kappa_1}{\kappa_2}+Q'\right)\left(-\kappa_2\left(\frac{\kappa_1}{\kappa_2}\right)' + Q(\kappa_2\tau_2'-\kappa_2'\tau_2)\right) + \tau_1\left(\frac{\kappa_1}{\kappa_2}+Q'\right)'.$$

Thus, $\mathbf{\Phi}_{C_2}$ is minimal if and only if $b_1 = b_2 = b_3 = 0$. From $b_1 = b_3 = 0$, we obtain

$$\begin{cases} \kappa_2\left(\frac{\tau_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'' - \tau_2\left(\frac{\kappa_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'' = \frac{(\kappa_2\tau_2'-\kappa_2'\tau_2)(\kappa_2\kappa_2'+\tau_2\tau_2')}{(\kappa_2^2+\tau_2^2)^2}, \\ -\kappa_2\left(\frac{\kappa_1}{\kappa_2}\right)' + Q(\kappa_2\tau_2'-\kappa_2'\tau_2) = \frac{-\tau_1\left(\frac{\kappa_1}{\kappa_2}+Q'\right)'}{\frac{\kappa_1}{\kappa_2}+Q'}. \end{cases} \tag{4.9}$$

Substituting (4.9) into $b_2 = 0$, and thus we obtain

$$\frac{\left(\frac{\kappa_1}{\kappa_2}+Q'\right)'}{\frac{\kappa_1}{\kappa_2}+Q'} = \frac{\kappa_2\tau_2'+\kappa_2'\tau_2}{\left(\kappa_2^2+\tau_2^2\right)^{\frac{3}{2}}}.\blacksquare$$

Particularly, if $\boldsymbol{\alpha}$ is planar, it is easy to verify that $K_{\mathbf{\Phi}_{C_2}} = H_{\mathbf{\Phi}_{C_2}} = 0$, so $\mathbf{\Phi}_{C_2}$ is developable minimal. If $\boldsymbol{\alpha}$ is a cylindrical helix, $K_{\mathbf{\Phi}_{C_2}} \neq 0, H_{\mathbf{\Phi}_{C_2}} = 0$, so $\mathbf{\Phi}_{C_2}$ is a non-developable minimal surface.

Corollary 4.3. *The $\mathbf{e}_2$-Sannia ruled surface of a C-direction ruled surface of a plane curve is developable and minimal. However, the $\mathbf{e}_2$-Sannia ruled surface of the C-direction ruled surface of a cylindrical helix is a non-developable minimal.*

Theorem 4.4. *Let $\mathbf{\Phi}_{C_3}$ be an $\mathbf{e}_3$-Sannia ruled surface of a C-direction ruled surface, then the following conclusions hold:*

i) $\mathbf{\Phi}_{C_3}$ *is developable if and only if*

$$\left(\frac{\kappa_1}{\kappa_2}+Q'\right)(\tau_2\kappa_2'-\kappa_2\tau_2') = 0,$$

ii) $\mathbf{\Phi}_{C_3}$ *is minimal if and only if*

$$\begin{cases} c_2\tau_2 - c_1\kappa_2 = 0, \\ \left(\frac{\tau_1}{\kappa_2}\right)' = c_1\kappa_1 + c_2(2Q'\kappa_2+\kappa_2'Q). \end{cases}$$

Proof. Taking partial derivatives of $\mathbf{\Phi}_{C_3}$, we obtain

$$(\mathbf{\Phi}_{C_3})_s = \pi_1 \boldsymbol{N} + \pi_2 \boldsymbol{C} + \pi_3 \boldsymbol{W}, (\mathbf{\Phi}_{C_3})_v = \frac{-\tau_2 \boldsymbol{N} - \kappa_2 \boldsymbol{W}}{\sqrt{\kappa_2^2 + \tau_2^2}},$$

where $\pi_1 = Q\kappa_2 - v\left(\frac{\tau_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'$, $\pi_2 = \frac{\kappa_1}{\kappa_2} + Q'$, $\pi_3 = \frac{\tau_1}{\kappa_2} - Q\tau_2 - v\left(\frac{\kappa_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'$.

Thus, the first fundamental quantities of $\mathbf{\Phi}_{C_3}$ are as follows:

$$E_{\mathbf{\Phi}_{C_3}} = \pi_1^2 - \pi_2^2 + \pi_3^2, F_{\mathbf{\Phi}_{C_3}} = \frac{-\tau_1}{\sqrt{\kappa_2^2+\tau_2^2}}, G_{\mathbf{\Phi}_{C_3}} = 1.$$

The unit normal vector of the ruled surface is

$$\boldsymbol{u}_{\mathbf{\Phi}_{C_3}} = \frac{-\pi_2\kappa_2 \boldsymbol{N} - (\pi_1\kappa_2 - \pi_3\tau_2)\boldsymbol{C} + \pi_2\tau_2 \boldsymbol{W}}{\sqrt{\left|\pi_2^2(\kappa_2^2+\tau_2^2) - (\pi_1\kappa_2 - \pi_3\tau_2)^2\right|}}.$$

Taking the second-order partial derivatives of $\mathbf{\Phi}_{C_3}$, we obtain

$$(\mathbf{\Phi}_{C_3})_{ss} = (\pi_1' + \pi_2\kappa_2)\boldsymbol{N} + (\pi_1\kappa_2 + \pi_2' - \pi_3\tau_2)\boldsymbol{C} + (\pi_3' - \pi_2\tau_2)\boldsymbol{W},$$

$$(\mathbf{\Phi}_{C_3})_{sv} = -\left(\frac{\tau_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'\boldsymbol{N} - \left(\frac{\kappa_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'\boldsymbol{W}, (\mathbf{\Phi}_{C_3})_{vv} = 0.$$

Thus, the second fundamental quantities of $\mathbf{\Phi}_{C_3}$ are as follows:

$$l_{\mathbf{\Phi}_{C_3}} = \frac{-\pi_2\kappa_2(\pi_1' + \pi_2\kappa_2) + (\pi_1\kappa_2 - \pi_3\tau_2)(\pi_1\kappa_2 + \pi_2' - \pi_3\tau_2) + \pi_2\tau_2(\pi_3' - \pi_2\tau_2)}{\sqrt{\left|\pi_2^2(\kappa_2^2+\tau_2^2) - (\pi_1\kappa_2 - \pi_3\tau_2)^2\right|}},$$

$$m_{\mathbf{\Phi}_{C_3}} = \frac{-\left(\frac{\kappa_1}{\kappa_2} + Q'\right)\cdot\frac{\tau_2\kappa_2' - \kappa_2\tau_2'}{\sqrt{\kappa_2^2+\tau_2^2}}}{\sqrt{\left|\pi_2^2(\kappa_2^2+\tau_2^2) - (\pi_1\kappa_2 - \pi_3\tau_2)^2\right|}}, n_{\mathbf{\Phi}_{C_3}} = 0.$$

The Gaussian curvature of the ruled surface $\mathbf{\Phi}_{C_3}$ is given by

$$K_{\mathbf{\Phi}_{C_3}} = \frac{(\kappa_2^2+\tau_2^2)\left(\left(\frac{\kappa_1}{\kappa_2} + Q'\right)\cdot\frac{\tau_2\kappa_2' - \kappa_2\tau_2'}{\sqrt{\kappa_2^2+\tau_2^2}}\right)^2}{\left(\pi_2^2(\kappa_2^2+\tau_2^2) - (\pi_1\kappa_2 - \pi_3\tau_2)^2\right)\cdot\left|\pi_2^2(\kappa_2^2+\tau_2^2) - (\pi_1\kappa_2 - \pi_3\tau_2)^2\right|}, \tag{4.10}$$

it follows that $\mathbf{\Phi}_{C_3}$ is developable if and only if

$$\left(\frac{\kappa_1}{\kappa_2} + Q'\right)(\tau_2\kappa_2' - \kappa_2\tau_2') = 0.$$

The mean curvature of the ruled surface $\mathbf{\Phi}_{C_3}$ is

$$H_{\mathbf{\Phi}_{C_3}} = \frac{2\sqrt{\kappa_2^2+\tau_2^2}B_6 + (\kappa_2^2+\tau_2^2)B_7}{-2\left(\pi_2^2(\kappa_2^2+\tau_2^2) - (\pi_1\kappa_2+\pi_3\tau_2)^2\right)\sqrt{\left|\pi_2^2(\kappa_2^2+\tau_2^2) - (\pi_1\kappa_2+\pi_3\tau_2)^2\right|}}, \tag{4.11}$$

where $B_6 = \tau_1\left(\frac{\kappa_1}{\kappa_2} + Q'\right)\cdot\frac{\tau_2\kappa_2'-\kappa_2\tau_2'}{\sqrt{\kappa_2^2+\tau_2^2}}$,
$B_7 = -\pi_2\kappa_2(\pi_1' + \pi_2\kappa_2) + (\pi_1\kappa_2 - \pi_3\tau_2)(\pi_1\kappa_2 + \pi_2' - \pi_3\tau_2) - \pi_2\tau_2(\pi_2\tau_2 - \pi_3')$.

Thus, $\mathbf{\Phi}_{C_3}$ is a minimal surface if and only if $2\sqrt{\kappa_2^2+\tau_2^2}B_6 + (\kappa_2^2+\tau_2^2)B_7 = 0$, which is equivalent to

$$v^2 b_4 + v b_5 + b_6 = 0,$$

where

$$b_4 = \kappa_2\tau_2' - \kappa_2'\tau_2,$$

$$b_5 = -(\kappa_2^2+\tau_2^2)\left(\frac{\kappa_1}{\kappa_2} + Q'\right)'\cdot\frac{\kappa_2\tau_2'-\kappa_2'\tau_2}{\sqrt{\kappa_2^2+\tau_2^2}} + (\kappa_2^2+\tau_2^2)\left(\frac{\kappa_1}{\kappa_2} + Q'\right)\left(\kappa_2\left(\frac{\tau_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)'' - \tau_2\left(\frac{\kappa_2}{\sqrt{\kappa_2^2+\tau_2^2}}\right)''\right),$$

$$b_6 = \left(\frac{\kappa_1}{\kappa_2} + Q'\right)^2(\kappa_2^2+\tau_2^2) + \left(\frac{\kappa_1}{\kappa_2} + Q'\right)\left(Q'(\kappa_2^2+\tau_2^2) + Q(\kappa_2\kappa_2' + \tau_2'\tau_2) - \tau_2\left(\frac{\tau_1}{\kappa_2}\right)'\right).$$

Thus, $\mathbf{\Phi}_{C_3}$ is minimal if and only if $b_4 = b_5 = b_6 = 0$. From $b_4 = 0$, it follows that $b_5 = \kappa_2\tau_2' - \kappa_2'\tau_2 = 0$. Substituting this into $b_6 = 0$ gives

$$\begin{cases} c_2\tau_2 - c_1\kappa_2 = 0, \\ \left(\frac{\tau_1}{\kappa_2}\right)' = c_1\kappa_1 + c_2(2Q'\kappa_2 + \kappa_2' Q). \blacksquare \end{cases}$$

Particularly, if $\boldsymbol{\alpha}$ is a plane curve or a cylindrical helix , it is easy to verify that $K_{\mathbf{\Phi}_{C_3}} = 0, H_{\mathbf{\Phi}_{C_3}} \neq 0$, so $\mathbf{\Phi}_{C_3}$ is non-minimal developable.

Corollary 4.5. *The $\boldsymbol{e}_3$-Sannia ruled surface of a C-direction ruled surface of a plane curve or a cylindrical helix is non-minimal developable.*

5. The Sannia ruled surfaces generated by Darboux ruled surface

For a Darboux ruled surface $\boldsymbol{X}_W = \boldsymbol{\alpha} + v\boldsymbol{W}$, the striction curve is determined to be $\boldsymbol{\beta}_W = \boldsymbol{\alpha} + \frac{\kappa_1}{\kappa_2\tau_2}\boldsymbol{W}$. Subsequently, the Sannia frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of $\boldsymbol{W}$ can be easily established by $\boldsymbol{e}_1 = \boldsymbol{W}$, $\boldsymbol{e}_2 = \boldsymbol{C}$ and $\boldsymbol{e}_3 = -\boldsymbol{N}$. Then the generated $\boldsymbol{e}_i$-Sannia ruled surfaces are as follows:

$$\mathbf{\Phi}_{W_1} = \boldsymbol{\alpha} + \frac{\kappa_1}{\kappa_2\tau_2}\boldsymbol{W} + v\boldsymbol{W}, \tag{5.1}$$

$$\mathbf{\Phi}_{W_2} = \boldsymbol{\alpha} + \frac{\kappa_1}{\kappa_2\tau_2}\boldsymbol{W} + v\boldsymbol{C}, \tag{5.2}$$

$$\boldsymbol{\Phi}_{W_3} = \boldsymbol{\alpha} + \frac{\kappa_1}{\kappa_2\tau_2}\boldsymbol{W} - v\boldsymbol{N}. \tag{5.3}$$

When $\boldsymbol{\alpha}$ is a plane curve or a general helix, it is easy to verify that $\tau_2 = 0$, and the striction curve $\boldsymbol{\beta}_W$ does not exist. Therefore, an $\boldsymbol{e}_i$-Sannia ruled surface cannot be generated. Next, we assume that $\boldsymbol{\alpha}$ is neither a plane curve nor a general helix.

Theorem 5.1. *Let $\boldsymbol{\Phi}_{W_1}$ be an $\boldsymbol{e}_1$-Sannia ruled surface of a Darboux ruled surface, then $\boldsymbol{\Phi}_{W_1}$ is non-minimal developable.*

Proof. Taking partial derivatives of $\boldsymbol{\Phi}_{W_1}$, we obtain

$$(\boldsymbol{\Phi}_{W_1})_s = -v\tau_2\boldsymbol{C} + \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)\boldsymbol{W}, (\boldsymbol{\Phi}_{W_1})_v = \boldsymbol{W}.$$

Thus, the first fundamental quantities of $\boldsymbol{\Phi}_{W_1}$ are as follows:

$$E_{\boldsymbol{\Phi}_{W_1}} = -(v\tau_2)^2 + \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)^2, F_{\boldsymbol{\Phi}_{W_1}} = \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}, G_{\boldsymbol{\Phi}_{W_1}} = 1.$$

The unit normal vector of the ruled surface is

$$\boldsymbol{u}_{\boldsymbol{\Phi}_{W_1}} = \pm\boldsymbol{N}.$$

Taking the second-order partial derivatives of $\boldsymbol{\Phi}_{W_1}$, we obtain the following results

$$(\boldsymbol{\Phi}_{W_1})_{ss} = -v\kappa_2\tau_2\boldsymbol{N} + \left(-v\tau_2' - \tau_2\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' - \frac{\tau_1\tau_2}{\kappa_2}\right)\boldsymbol{C} + \left(v\tau_2^2 + \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'' + \left(\frac{\tau_1}{\kappa_2}\right)'\right)\boldsymbol{W},$$

$$(\boldsymbol{\Phi}_{W_1})_{sv} = -\tau_2\boldsymbol{C}, (\boldsymbol{\Phi}_{W_1})_{vv} = 0.$$

Thus, the second fundamental quantities of $\boldsymbol{\Phi}_{W_1}$ are as follows:

$$l_{\boldsymbol{\Phi}_{W_1}} = \pm v\kappa_2\tau_2, m_{\boldsymbol{\Phi}_{W_1}} = 0, n_{\boldsymbol{\Phi}_{W_1}} = 0.$$

The Gaussian curvature of the ruled surface $\boldsymbol{\Phi}_{W_1}$ is given by

$$K_{\boldsymbol{\Phi}_{W_1}} = 0, \tag{5.4}$$

it follows that $\boldsymbol{\Phi}_{W_1}$ is always developable.

The mean curvature of the ruled surface $\boldsymbol{\Phi}_{W_1}$ is

$$H_{\boldsymbol{\Phi}_{W_1}} = \pm\frac{\kappa_2}{2v\tau_2}, \tag{5.5}$$

since $\kappa_2 \neq 0$, it follows that $\boldsymbol{\Phi}_{W_1}$ cannot be minimal. ■

Theorem 5.2. *Let $\boldsymbol{\Phi}_{W_2}$ be an $\boldsymbol{e}_2$-Sannia ruled surface of a Darboux ruled surface, then the following conclusions hold:*

i) $\boldsymbol{\Phi}_{W_2}$ *is developable if and only if* $\frac{\tau_1}{\kappa_2} = -\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'$,

ii) $\boldsymbol{\Phi}_{W_2}$ *is minimal if and only if*

$$\begin{cases} \tau_2 = c_1\kappa_2, \\ \dfrac{\tau_1}{\kappa_2} + \left(\dfrac{\kappa_1}{\kappa_2\tau_2}\right)' = c_2\kappa_2. \end{cases}$$

Proof. Taking partial derivatives of $\boldsymbol{\Phi}_{W_2}$, we obtain

$$(\boldsymbol{\Phi}_{W_2})_s = v\kappa_2 \boldsymbol{N} + \left(-v\tau_2 + \frac{\tau_1}{\kappa_2} + \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'\right)\boldsymbol{W}, (\boldsymbol{\Phi}_{W_2})_v = \boldsymbol{C}.$$

Thus, the first fundamental quantities of $\boldsymbol{\Phi}_{W_2}$ are as follows:

$$E_{\boldsymbol{\Phi}_{W_2}} = v^2\kappa_2^2 + \left(\frac{\tau_1}{\kappa_2} + \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' - v\tau_2\right)^2, F_{\boldsymbol{\Phi}_{W_2}} = 0, G_{\boldsymbol{\Phi}_{W_2}} = -1.$$

The unit normal vector of the ruled surface is

$$\boldsymbol{u}_{\boldsymbol{\Phi}_{W_2}} = \frac{\left(-\frac{\tau_1}{\kappa_2} - \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + v\tau_2\right)\boldsymbol{N} + v\kappa_2\boldsymbol{W}}{\sqrt{v^2\kappa_2^2 + \left(-\frac{\tau_1}{\kappa_2} - \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + v\tau_2\right)^2}}.$$

Taking the second-order partial derivatives of $\boldsymbol{\Phi}_{W_2}$, we obtain:

$$(\boldsymbol{\Phi}_{W_2})_{ss} = v\kappa_2'\boldsymbol{N} + \left(v\kappa_2^2 - \tau_2\left(\frac{\tau_1}{\kappa_2} - \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' - v\tau_2\right)\right)\boldsymbol{C} + \left(\left(\frac{\tau_1}{\kappa_2}\right)' + \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'' - v\tau_2'\right)\boldsymbol{W},$$

$$(\boldsymbol{\Phi}_{W_2})_{sv} = \kappa_2\boldsymbol{N} - \tau_2\boldsymbol{W}, (\boldsymbol{\Phi}_{W_2})_{vv} = 0.$$

Thus, the second fundamental quantities of $\boldsymbol{\Phi}_{W_2}$ are as follows:

$$l_{\boldsymbol{\Phi}_{W_2}} = \frac{v\kappa_2'\left(-\frac{\tau_1}{\kappa_2} - \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + v\tau_2\right) - v\kappa_2\left(-\left(\frac{\tau_1}{\kappa_2}\right)' - \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'' + v\tau_2'\right)}{\sqrt{v^2\kappa_2^2 + \left(-\frac{\tau_1}{\kappa_2} - \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + v\tau_2\right)^2}},$$

$$m_{\boldsymbol{\Phi}_{W_2}} = \frac{-\tau_1 - \kappa_2\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'}{\sqrt{v^2\kappa_2^2 + \left(-\frac{\tau_1}{\kappa_2} - \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + v\tau_2\right)^2}}, n_{\boldsymbol{\Phi}_{W_2}} = 0.$$

The Gaussian curvature of the ruled surface $\boldsymbol{\Phi}_{W_2}$ is given by

$$K_{\boldsymbol{\Phi}_{W_2}} = \frac{\kappa_2^2\left(\frac{\tau_1}{\kappa_2} + \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'\right)^2}{\left(v^2\kappa_2^2 + \left(-\frac{\tau_1}{\kappa_2} - \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + v\tau_2\right)^2\right)^2}, \tag{5.6}$$

it follows that $\boldsymbol{\Phi}_{W_2}$ is developable if and only if $\frac{\tau_1}{\kappa_2} = -\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'$.

The mean curvature of the ruled surface $\boldsymbol{\Phi}_{W_2}$ is

$$H_{\boldsymbol{\Phi}_{W_2}} = \frac{B_8}{2\left(v^2\kappa_2^2 + \left(-\frac{\tau_1}{\kappa_2} - \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + v\tau_2\right)^2\right)^{\frac{3}{2}}}, \tag{5.7}$$

where $B_8 = -v\kappa_2'\left(-\frac{\tau_1}{\kappa_2} - \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + v\tau_2\right) + v\kappa_2\left(-\left(\frac{\tau_1}{\kappa_2}\right)' - \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'' + v\tau_2'\right)$.

Thus, $\mathbf{\Phi}_{W_2}$ is a minimal surface if and only if $B_8 = 0$, which is equivalent to

$$v(\kappa_2'\tau_2 - \tau_2'\kappa_2) - \kappa_2'\frac{\tau_1}{\kappa_2} - \kappa_2'\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \kappa_2\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'' + \kappa_2\left(\frac{\tau_1}{\kappa_2}\right)' = 0,$$

further, we have

$$\begin{cases} \kappa_2'\tau_2 - \tau_2'\kappa_2 = 0, \\ \kappa_2'\frac{\tau_1}{\kappa_2} + \kappa_2'\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' - \kappa_2\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'' - \kappa_2\left(\frac{\tau_1}{\kappa_2}\right)' = 0, \end{cases} \tag{5.8}$$

further from (5.8), we obtain the equivalent equations

$$\begin{cases} \tau_2 = c_1\kappa_2, \\ \dfrac{\tau_1}{\kappa_2} + \left(\dfrac{\kappa_1}{\kappa_2\tau_2}\right)' = c_2\kappa_2 (c_2 \text{ is an arbitrary constant}). \blacksquare \end{cases}$$

Theorem 5.3. *Let $\mathbf{\Phi}_{W_3}$ be an $\boldsymbol{e}_3$-Sannia ruled surface of a Darboux ruled surface, then the following conclusions hold:*

i) $\mathbf{\Phi}_{W_3}$ *is developable if and only if* $\frac{\tau_1}{\kappa_2} = -\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'$,

ii) $\mathbf{\Phi}_{W_3}$ *is non-minimal.*

Proof. Taking partial derivatives of $\mathbf{\Phi}_{W_3}$, we obtain

$$(\mathbf{\Phi}_{W_3})_s = -v\kappa_2\boldsymbol{C} + \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)\boldsymbol{W}, (\mathbf{\Phi}_{W_3})_v = -\boldsymbol{N}.$$

Thus, the first fundamental quantities of $\mathbf{\Phi}_{W_3}$ are as follows:

$$E_{\mathbf{\Phi}_{W_3}} = -v^2\kappa_2^2 + \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)^2, F_{\mathbf{\Phi}_{W_3}} = 0, G_{\mathbf{\Phi}_{W_3}} = 1.$$

The unit normal vector of the ruled surface is

$$\boldsymbol{u}_{\mathbf{\Phi}_{W_3}} = \frac{\left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)\boldsymbol{C} - v\kappa_2\boldsymbol{W}}{\sqrt{\left|(v\kappa_2)^2 - \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)^2\right|}}.$$

Taking the second-order partial derivatives of $\mathbf{\Phi}_{W_3}$, we obtain:

$$(\mathbf{\Phi}_{W_3})_{ss} = -v\kappa_2^2\boldsymbol{N} + \left(-v\kappa_2' - \tau_2\left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)\right)\boldsymbol{C} + \left(v\kappa_2\tau_2 + \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'' + \left(\frac{\tau_1}{\kappa_2}\right)'\right)\boldsymbol{W},$$

$$(\mathbf{\Phi}_{W_3})_{sv} = -\kappa_2\boldsymbol{C}, (\mathbf{\Phi}_{W_3})_{vv} = 0.$$

Thus, the second fundamental quantities of $\mathbf{\Phi}_{W_3}$ are as follows:

$$l_{\mathbf{\Phi}_{W_3}} = \frac{-v\kappa_2\left(v\kappa_2\tau_2 + \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'' + \left(\frac{\tau_1}{\kappa_2}\right)'\right) + \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)\left(v\kappa_2' + \tau_2\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1\tau_2}{\kappa_2}\right)}{\sqrt{\left|(v\kappa_2)^2 - \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)^2\right|}},$$

$$m_{\mathbf{\Phi}_{W_3}} = \frac{\kappa_2\left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)}{\sqrt{\left|(\nu\kappa_2)^2 - \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)^2\right|}}, n_{\mathbf{\Phi}_{W_3}} = 0.$$

The Gaussian curvature of the ruled surface $\mathbf{\Phi}_{W_3}$ is given by

$$K_{\mathbf{\Phi}_{W_3}} = \frac{\kappa_2^2\left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)^2}{\left((\nu\kappa_2)^2 - \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)^2\right)\cdot\left|(\nu\kappa_2)^2 - \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)^2\right|}, \tag{5.9}$$

it follows that $\mathbf{\Phi}_{W_3}$ is developable if and only if $\frac{\tau_1}{\kappa_2} = -\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'$.

The mean curvature of the ruled surface $\mathbf{\Phi}_{W_3}$ is

$$H_{\mathbf{\Phi}_{W_3}} = \frac{B_9}{2\left((\nu\kappa_2)^2 - \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)^2\right)\sqrt{\left|(\nu\kappa_2)^2 - \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)^2\right|}}, \tag{5.10}$$

where $B_9 = \nu\kappa_2\left(\nu\kappa_2\tau_2 + \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'' + \left(\frac{\tau_1}{\kappa_2}\right)'\right) - \left(\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1}{\kappa_2}\right)\left(\nu\kappa_2' + \tau_2\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' + \frac{\tau_1\tau_2}{\kappa_2}\right)$.

Thus, $\mathbf{\Phi}_{W_3}$ is a minimal surface if and only if $B_9 = 0$, which is equivalent to

$$\nu^2\kappa_2^2\tau_2 - \nu\left(-\kappa_2\left(\frac{\tau_1}{\kappa_2}\right)' - \kappa_2\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'' + \kappa_2'\frac{\tau_1}{\kappa_2} + \kappa_2'\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'\right) - \tau_2\left(\left(\frac{\tau_1}{\kappa_2}\right) + \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'\right)^2 = 0,$$

further, we have

$$\begin{cases}\tau_2 = 0,\\ \kappa_2\left(\frac{\tau_1}{\kappa_2}\right)' + \kappa_2\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'' - \kappa_2'\frac{\tau_1}{\kappa_2} - \kappa_2'\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)' = 0,\\ \tau_2\left(\left(\frac{\tau_1}{\kappa_2}\right) + \left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'\right) = 0.\end{cases} \tag{5.11}$$

Clearly, according to the system of Eq (5.11), this contradicts the assumption $\tau_2 \neq 0$. Therefore, $\mathbf{\Phi}_{W_3}$ is not a minimal surface. ■

Corollary 5.4. *The $\boldsymbol{e}_i$-Sannia ruled surfaces generated by Darboux ruled surface are developable if and only if $\frac{\tau_1}{\kappa_1} = c_1 s + c_2$.*

Proof. First, convert the differential equation $\frac{\tau_1}{\kappa_2} = -\left(\frac{\kappa_1}{\kappa_2\tau_2}\right)'$ into the following equation

$$\frac{\kappa_1^2 - \tau_1^2}{\tau_1'\kappa_1 - \kappa_1'\tau_1} = -2\int\frac{\tau_1}{\kappa_1}\,ds.$$

Let $u(s) = \frac{\tau_1}{\kappa_1}$, the above indefinite integral equation becomes

$$\frac{u^2 - 1}{u'} = 2\int u\,ds.$$

Differentiating both sides of the above indefinite integral equation yields the general solutions $u_1(s) = \pm 1$ and $u_2(s) = c_1 s + c_2$(c_2 is an arbitrary constant).

If $u_1(s) = \pm 1$, then $\frac{\tau_1}{\kappa_1} = \pm 1$, and $\boldsymbol{\alpha}$ is a general helix; however striction curve of the Darboux ruled surface does not exist, so we discard this case. ■

6. Applications with the Sannia ruled surfaces

Example. *Let* $\alpha(s) = (\sqrt{2}\,\text{ch}\, s, s, \sqrt{2}\,\text{sh}\, s)$ *be a cylindrical helix. Its alternative frame vectors and curvature functions are as follows:*

$$\boldsymbol{N} = (\text{ch}\, s, 0, \text{sh}\, s), \boldsymbol{C} = (\text{sh}\, s, 0, \text{ch}\, s), \boldsymbol{W} = (0, -1, 0), \kappa_1 = \sqrt{2}, \tau_1 = -1, \kappa_2 = 1, \tau_2 = 0.$$

The striction curve of the principal normal ruled surface and the Sannia frame vectors of the principal normal ruled surface are

$$\boldsymbol{\beta}_N = (0, s, 0), \boldsymbol{e}_{1X_N} = (\text{ch}\, s, 0, \text{sh}\, s), \boldsymbol{e}_{2X_N} = (\text{sh}\, s, 0, \text{ch}\, s), \boldsymbol{e}_{3X_N} = (0, -1, 0).$$

The $\boldsymbol{e}_1, \boldsymbol{e}_2$-Sannia ruled surfaces (Figures 1 and 2) are given by

$$\boldsymbol{\Phi}_{N_1} = (0, s, 0) + v(\text{ch}\, s, 0, \text{sh}\, s), \boldsymbol{\Phi}_{N_2} = (0, s, 0) + v(\text{sh}\, s, 0, \text{ch}\, s).$$

The Gaussian curvature and mean curvature are calculated as

$$K_{\boldsymbol{\Phi}_{N_1}} = \frac{1}{(v^2-1)\cdot|v^2-1|}, H_{\boldsymbol{\Phi}_{N_1}} = 0, K_{\boldsymbol{\Phi}_{N_2}} = \left(\frac{1}{v^2+1}\right)^2, H_{\boldsymbol{\Phi}_{N_2}} = 0.$$

It is evident that $\boldsymbol{\Phi}_{N_1}$ and $\boldsymbol{\Phi}_{N_2}$ are both non-developable minimal surfaces.

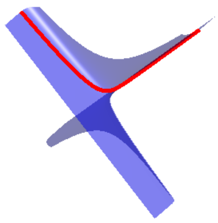

Figure 1. The red curve is $\alpha(s)$, and the blue surface is the $\boldsymbol{e}_1$-Sannia ruled surface $\boldsymbol{\Phi}_{N_1}$ generated by a principal normal ruled surface.

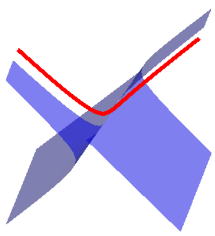

Figure 2. The red curve is $\alpha(s)$, and the blue surface is the $\boldsymbol{e}_2$-Sannia ruled surface $\boldsymbol{\Phi}_{N_2}$ generated by a principal normal ruled surface.

The striction curve of Darboux ruled surfaces $\boldsymbol{\beta}_C(s)$, and the Sannia frame vectors of Darboux ruled surfaces are

$$\boldsymbol{\beta}_C = (\sqrt{2}\,\mathrm{ch}\, s, s, \sqrt{2}\,\mathrm{sh}\, s), \boldsymbol{e}_{1X_C} = (\mathrm{sh}\, s, 0, \mathrm{ch}\, s), \boldsymbol{e}_{2X_C} = (\mathrm{ch}\, s, 0, \mathrm{sh}\, s), \boldsymbol{e}_{3X_C} = (0, 1, 0).$$

The $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$-Sannia ruled surfaces (Figures 3–5) are respectively

$$\boldsymbol{\Phi}_{C_1} = (\sqrt{2}\,\mathrm{ch}\, s, s, \sqrt{2}\,\mathrm{sh}\, s) + v(\mathrm{sh}\, s, 0, \mathrm{ch}\, s),$$
$$\boldsymbol{\Phi}_{C_2} = (\sqrt{2}\,\mathrm{ch}\, s, s, \sqrt{2}\,\mathrm{sh}\, s) + v(\mathrm{ch}\, s, 0, \mathrm{sh}\, s),$$
$$\boldsymbol{\Phi}_{C_3} = (\sqrt{2}\,\mathrm{ch}\, s, s, \sqrt{2}\,\mathrm{sh}\, s) + v(0, 1, 0).$$

The Gaussian and mean curvatures are calculated as

$$K_{\boldsymbol{\Phi}_{C_1}} = \frac{-1}{(v^2+1)^2}, H_{\boldsymbol{\Phi}_{C_1}} = \frac{\sqrt{2}}{2(v^2+1)^{3/2}},$$

$$K_{\boldsymbol{\Phi}_{C_2}} = -\frac{1}{((\sqrt{2}+v)^2-1)\cdot|(\sqrt{2}+v)^2-1|}, H_{\boldsymbol{\Phi}_{C_2}} = 0, K_{\boldsymbol{\Phi}_{C_3}} = 0, H_{\boldsymbol{\Phi}_{C_3}} = \frac{\sqrt{2}}{4}.$$

It is evident that $\boldsymbol{\Phi}_{C_1}$ is a non-developable non-minimal surface, $\boldsymbol{\Phi}_{C_2}$ is a non-developable minimal surface, and $\boldsymbol{\Phi}_{C_3}$ is a non-minimal developable surface.

Since $\alpha(s)$ is a cylindrical helix, the striction curve of the Darboux ruled surface does not exist; clearly, the $\boldsymbol{e}_i$-Sannia ruled surfaces cannot be generated by the Darboux ruled surface. Moreover, according to Theorem 3.4, $\boldsymbol{\Phi}_{N_3}$ is a lightlike ruled surface.

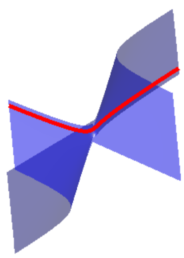

Figure 3. The red curve is $\alpha(s)$, and the blue surface is the $\mathbf{e}_1$-Sannia ruled surface $\boldsymbol{\Phi}_{C_1}$ generated by a C-direction ruled surface.

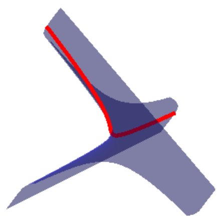

Figure 4. The red curve is $\alpha(s)$, and the blue surface is the $\boldsymbol{e}_2$-Sannia ruled surface $\boldsymbol{\Phi}_{C_2}$ generated by a C-direction ruled surface.

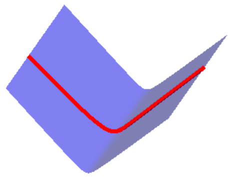

Figure 5. The red curve is $\alpha(s)$, and the blue surface is the $\boldsymbol{e}_3$-Sannia ruled surface $\boldsymbol{\Phi}_{C_3}$ generated by a C-direction ruled surface.

7. Conclusions

This paper investigates the developability and minimality of non-lightlike Sannia ruled surfaces generated from principal normal, C-direction, and Darboux ruled surfaces of timelike curves. It demonstrates that the developability and minimality of the generated Sannia ruled surfaces are fully determined by the curvature functions of the directrix curves α. This implies that the intrinsic properties of the directrix curves α govern the corresponding properties of the generated Sannia ruled surfaces.

Author contributions

Wenke Zhang: Conceptualization, Methodology, Formal analysis, Writing–Original Draft; Na Hu: Validation, Investigation, Writing–Review Editing, Supervision, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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