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Research article

Fixed point theorems without continuity condition in non-solid extended abstract metric spaces with applications

Yan Han[1], **Jin Chen**[1,*], **Shaoyuan Xu**[1] **and Xianhua Xie**[2]

[1] School of Mathematics and Statistics, Zhaotong University, Zhaotong, Yunnan, China
[2] School of Mathematics and Computer Science, Gannan Normal University, Ganzhou, Jiangxi, China

* **Correspondence:** Email: chenjin702@126.com.

Abstract: In this paper, several fixed point results for different contractions in extended abstract metric spaces with non-solid cones are established. The definitions of Cauchy sequences, convergent sequences, and completeness with respect to normality in the non-solid extended abstract metric spaces are reintroduced. We prove that the fixed point exists and is unique in these spaces by removing the orbital continuity assumption. Our results generalize and extend several significant theorems in classical abstract metric spaces and standard metric spaces. A consequence, we present several examples demonstrating that our main theorems serve as powerful tools for solving equations in different space frameworks, for both solid and non-solid cones.

Keywords: non-solid space; orbital continuity; fixed point; Reich type and Kannan type (almost) contractions
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1. Introduction

Fixed point theory plays a crucial role in nonlinear analysis, differential equations, optimization theory, game theory, and related fields. In recent years, as nonlinear problems have become increasing complex, research on fixed point theory has been continuously deepened. New fixed point theorems and iterative algorithms have been developed. The introduction of partial order structures has provided a new perspective for fixed point theory, enabling the study of the existence and uniqueness of fixed points under more general conditions. In the famous Banach contraction principle [1], the existence and uniqueness of fixed points in complete metric spaces (MSs) were established. Specifically, in a complete metric space, any mapping that satisfies the Banach contraction condition possesses a unique fixed point, and the corresponding iterative sequence converges to this fixed point. Subsequently, Kannan [2], Reich [3], Hardy [4] and Ćirić [5] modified the Banach contraction into the Kannan contraction, Reich contraction, Hardy-Rogers contraction and Ćirić quasi-contraction, respectively. In [6], Berinde introduced an important generalization, initially termed "weak contraction", and later renamed "almost contraction", see [7,8]. These mathematicians established fixed point theorems for various types of contractions in complete MSs. Scholars have widely applied these results to investigate the existence and uniqueness of solutions to nonlinear integral equations, nonlinear integral-differential equations and nonlinear functional differential equations.

The concept of MS was initially proposed by Maurice Frechet [9] in 1906, quantifying the distance between any two points in a non-empty set. Subsequently, this concept was continuously expanded and refined by many mathematicians, leading to more advanced and important structures. Firstly, Bakhtin [10] and Czerwik [11] first proposed the concept of b-metric space (b-MS), an extension of the traditional MS. In a standard MS, the distance between points satisfies the triangle inequality, whereas in the b-MSs, by introducing a constant $s \geq 1$, this rule is slightly relaxed, making the metric more flexible. Despite this modification, all b-MSs still maintain symmetry and non-negativity. Extended b-metric spaces (Eb-MSs) were later proposed by Kamran et al. [12], further generalizing these concepts. In Eb-MS, a variable multiplier is allowed to vary according to the involved points rather than remain a fixed constant $s \geq 1$. This multiplier is represented as a function $\eta(f, g) \geq 1$, depending on the configuration of the two points. By replacing the fixed constant with a flexible function, Eb-MSs can be applied to more complex situations. In 2007, Huang et al. [13] proposed cone metric spaces (CMSs) as a generalization of traditional MSs, replacing real number sets with partially ordered Banach spaces. Based on this concept, they defined Cauchy sequences and their convergence in CMSs. Subsequently, Liu et al. [14,15] extended CMSs by replacing the underlying Banach space with a Banach algebra (BA), introducing CMSs over BA. They provided a generalized definition of CMSs and explored the partial order of points within these cones. They also demonstrated, through examples, that the fixed point theorems in a CMS over a BA are generally not analogous to those in a traditional MS. Similarly, Huang et al. [16,17] re-proposed the concept of cone b-metric space (Cb-MS) over a BA, combining CMSs over BA with b-MSs. Meanwhile, scholars successively expanded these concepts to cone b-metric-like spaces over Banach algebras (Cb-MLSs over BAs) [18], and further to extended cone b-metric-like spaces over Banach algebras (ECb-MLSs over BAs) [19]. In these spaces, the abstract metric no longer requires that the distance between coincident points be zero; rather, points with zero distance must coincide. Since then, applications have been developed concerning the existence and uniqueness of solutions to various differential and integral equations, see [20–23]. Through extensive review of references, we find that these conclusions also depend on the condition of the solid cones.

Until 2023, Xu and Cheng et al. [24] established fixed point theorems for certain contractions in non-solid Cb-MSs over BAs, demonstrating applications to both solid cones ($\mathbb{R}^2$) and non-solid cones (the generalized real-valued Lebesgue integrable functions on [0, 1]). These applications highlight the significant value of results in non-solid cones, which are not equivalent to theorems in existing MSs, CMSs, or Cb-MSs over BAs under solid cones. More recently, Shi et al. [25] presented several fixed point theorems for Reich-type and Kannan-type contractions in solid ECb-MLSs over BAs. However, similar to the main results in [19,25,26], these theorems crucially depend on three restrictive assumptions: (i) completeness of the underlying spaces, (ii) solidness of cones, and (iii) orbital continuity of contractions.

In parallel, several studies have further advanced the theory of fixed points in generalized metric spaces. For instance, Belhenniche et al. [27,28] established fixed point theorems for generalized contractive mappings in Eb-MSs and partial metric spaces, with applications to dynamic programming and integral equations. Meanwhile, Alsulami et al. [29] provided a critical analysis of C^*-algebra-valued contraction mappings, demonstrating that many such results can be derived from classical metric space theory.

In this paper, we make substantial and explicit improvements over [19,25,26] by establishing fixed point theorems that weaken or eliminate these limitations. Specifically, our main contributions are as follows:

(1) Weakening completeness to orbital completeness: While [19,25,26] require complete spaces, we establish existence under the significantly weaker condition of orbital completeness.
(2) Extending from solid to non-solid cones: The frameworks in [19,25,26] are restricted to solid cones, whereas our results apply to both solid and non-solid cones, substantially broadening their applicability.
(3) Relaxing or removing continuity requirements: We significantly weaken the continuity assumptions in [19,25,26]. In some of our results, the orbital continuity condition is relaxed to the weaker f_0-orbital continuity; in others, it is removed entirely, thereby establishing fixed point theorems for a broad class of non-continuous operators.

These improvements are not merely incremental; they represent fundamental advances in the theory. Furthermore, we demonstrate the practical significance of our generalizations through examples that apply in both solid and non-solid cone settings, thereby highlighting the universality and enhanced applicability of our approach compared to existing results.

2. Preliminaries

From now on, unless otherwise specified, we assume that $\mathbb{N}$ denotes the collection of all natural numbers, and $\mathcal{A}$ denotes a real Banach algebra endowed with a unit element e (Here "real" indicates that the algebra is defined over the field of $\mathbb{R}$), making it a unital Banach algebra. Let U be a normal cone in $\mathcal{A}$ and $\preceq$ denote the partial order induced by U, defined as follows

$$f \preceq g \Leftrightarrow g - f \in U, f \prec g \Leftrightarrow f \preceq g \text{ and } f \neq g, \forall f, g \in \mathcal{A}.$$

U is called normal if there is a real positive number $K > 0$ such that for all $f,\ g \in \mathcal{A}$,

$$\theta \preceq f \preceq g \Rightarrow \|f\| \leq K\|g\|.$$

The smallest constant K satisfying the above condition is called the normal constant of U. Since these are basic notions already presented in [30], we do not repeat them here.

We now proceed by presenting essential definitions and supporting lemmas.

Definition 2.1. (see [16]) Let the set $M \neq \emptyset, f, g \in M$ and the constant $s \geq 1$. If a mapping $\delta : M \times M \to \mathcal{A}$ satisfies:

(i) $\delta(f, g) = \theta \Leftrightarrow f = g$;

(ii) $\delta(f, g) = \delta(g, f)$;

(iii) $\delta(f,g) \preceq s[\delta(f,h) + \delta(h,g)]$,
then $(M, \mathcal{A}, \delta)$ is called a Cb-MS over a BA.

Definition 2.2. (see [18]) Let the set $M \neq \emptyset, f, g \in M$ and $\eta : M \times M \to [1, +\infty)$ be a function. If a mapping $\delta : M \times M \to \mathcal{A}$ satisfies:

(i) $\delta(f,g) = \theta \Leftrightarrow f = g$;

(ii) $\delta(f,g) = \delta(g,f)$;

(iii) $\delta(f,g) \preceq \eta(f,g)[\delta(f,h) + \delta(h,g)]$,

then $(M, \mathcal{A}, \delta)$ is called an ECb-MS over a BA.

Definition 2.3. (see [19]) Let the set $M \neq \emptyset, f, g \in M$ and $\eta : M \times M \to [1, +\infty)$ be a function. If a mapping $\delta : M \times M \to \mathcal{A}$ satisfies:

(i) $\delta(f,g) = \theta \Rightarrow f = g$;

(ii) $\delta(f,g) = \delta(g,f)$;

(iii) $\delta(f,g) \preceq \eta(f,g)[\delta(f,h) + \delta(h,g)]$,

then $(M, \mathcal{A}, \delta)$ is called an ECb-MLS over a BA.

Example 2.1. Suppose the Banach algebra $\mathcal{A} = Ł\,[0,1]$ (the set of all generalized real-valued Lebesgue integral functions on $[0,1]$). The cone is defined by

$$U = \{\varphi \in Ł\,[0,1] : \varphi = \varphi(t) \geq 0, a.e.t \in [0,1]\}$$

with the norm $\|\varphi\|_1 = \int_0^1 |\varphi(t)|dt$. Let $M = [0, +\infty)$. Define the cone metric $\delta : M \times M \to \mathcal{A}$ as $\delta(f,g)(t) = |f+g|\varphi(t), f, g \in M, \varphi \in U\backslash\{\theta\}$. Then $(M, \mathcal{A}, \delta)$ is a normal and non-solid ECb-MLS over a BA with $\eta(f,g) = e^{|f+g|} \geq 1$.

Inspired by the related concepts in [24] (defined in Cb-MSs over BAs), we introduced the definitions of Cauchy sequences, convergent sequences, and completeness in non-solid ECb-MLSs over BAs as follows.

Definition 2.4. Suppose $(M, \mathcal{A}, \delta)$ is an ECb-MLS over a BA. Let $f \in M$ and $\{f_n\} \subseteq M$. We say

(i) $\{f_n\}$ converges to f with respect to normality if and only if $\|\delta(f_n, f)\| \to 0$ as $n \to +\infty$, that is $f_n \stackrel{\|\cdot\|}{\to} f (n \to +\infty)$;

(ii) $\{f_n\}$ is called a Cauchy sequence with respect to normality if $\lim_{n,m\to+\infty} \|\delta(f_n, f_m)\| \to 0$;

(iii) $(M, \mathcal{A}, \delta)$ is said to be complete with respect to normality if each Cauchy sequence $\{f_n\}$ in M converges to a point $f \in M$.

The proof of the following lemma is similar to the proof of Lemma 2.1 in [24], so we omit it.

Lemma 2.1. The limit of a convergent sequence with respect to normality in ECb-MLSs over BAs is unique. That is, if for any given sequence $\{f_n\} \subset M$ $(n \in \mathbb{N})$, there exist $f, g \in M$ such that $f_n \stackrel{\|\cdot\|}{\to} f$ and $f_n \stackrel{\|\cdot\|}{\to} g$ as $n \to +\infty$, then $f = g$.

Lemma 2.2. (see [30]) Suppose $\mathcal{A}$ is a unital Banach algebra, $f \in \mathcal{A}$. Then the spectral radius $\rho_{\mathcal{A}}(f)$ of f satisfies

$$\rho_{\mathcal{A}}(f) = \lim_{n\to+\infty} \|f^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|f^n\|^{\frac{1}{n}}.$$

In particular, $\rho_{\mathcal{A}}(f) = \rho_{\mathcal{A}}(-f)$ and $\rho_{\mathcal{A}}(f) \leq \|f\|$.

Lemma 2.3. (see [30]) Suppose $\mathcal{A}$ is a unital Banach algebra with a unit e, $f \in \mathcal{A}$. If $\rho_{\mathcal{A}}(f) < 1$, then $e - f$ is invertible. Moreover,

$$(e-f)^{-1} = \sum_{i=0}^{+\infty} f^i \text{ and } \rho_{\mathcal{A}}\left((e-f)^{-1}\right) \leq \frac{1}{1 - \rho_{\mathcal{A}}(f)}.$$

Lemma 2.4. (see [31]) Suppose $\mathcal{A}$ is a Banach algebra and $f \in \mathcal{A}$. If $\rho_{\mathcal{A}}(f) < 1$, then $\lim_{n\to+\infty} \|f^n\| = 0$.
Lemma 2.5. (see [30]) For $f, g \in \mathcal{A}$, if f commutes with g, then $\rho_{\mathcal{A}}(f+g) \leq \rho_{\mathcal{A}}(f) + \rho_{\mathcal{A}}(g)$ and $\rho_{\mathcal{A}}(fg) \leq \rho_{\mathcal{A}}(f)\rho_{\mathcal{A}}(g)$.

3. Fixed points of different contractions in ECb-MLSs over BAs

In this section, we consider $(M, \mathcal{A}, \delta)$ to be an ECb-MLS over a BA and U a normal cone with normal constant $K \geq 1$. In light of the previous definitions and lemmas, some fixed point results for Reich-type, Kannan-type, and Banach-type contractions in non-solid $(M, \mathcal{A}, \delta)$ are presented. First, define a subset U^* of $\mathcal{A}$ as follows:

$$U^* = \left\{ a \in U, a \neq \theta : \lim_{n\to+\infty} \frac{\|a^{n+1}\|}{\|a^n\|} \text{exists} \right\}.$$

Based on the Reich-type contraction in [3], we give the following Reich-type contraction in ECb-MLSs over BAs.
Definition 3.1. Let the set $M \neq \emptyset$ and U be a non-solid cone. If the mapping $\tau : M \to M$ satisfies

$$\delta(\tau f, \tau g) \preceq a_1\delta(f, g) + a_2\delta(f, \tau f) + a_3\delta(g, \tau g) \tag{3.1}$$

for all $f, g \in M$, where $a_i \in U^*(i = 1, 2, 3)$ such that a_1, a_2 commutes with a_3 and

$$\rho_{\mathcal{A}}(a_1 + a_2) + \rho_{\mathcal{A}}(a_3) < 1, \tag{3.2}$$

then τ is named a Reich-type contraction in $(M, \mathcal{A}, \delta)$.
Definition 3.2. The space $(M, \mathcal{A}, \delta)$ is named τ-orbitally complete if every Cauchy sequence with respect to normality included in $O_\tau(f)$ for some $f \in M$ converges in M, where $O_\tau(f) = \{f, \tau f, \tau^2 f, \tau^3 f, \cdots\}$.
Remark 3.1. Every complete space $(M, \mathcal{A}, \delta)$ is τ-orbitally complete for any mapping τ, but not the converse.
Theorem 3.1. Let $(M, \mathcal{A}, \delta)$ be a non-solid τ-orbitally complete ECb-MLS over a BA, where τ is a Reich-type contraction and $\eta(f, g) \leq \frac{1}{1-\rho_{\mathcal{A}}(a_1+a_2)}$ for any $f, g \in M$. If for any Picard iterating sequence $\{f_n\} = \{\tau^n f_0\}$ generated by $f_0 \in M$,

$$\lim_{n,m\to+\infty} \eta(f_n, f_m) < \frac{1}{\|(e - a_3)^{-1}(a_1 + a_2)\|}, \tag{3.3}$$

then τ admits a fixed point $f \in M$ and the sequence $f_n \overset{\|\cdot\|}{\to} f\ (n \to +\infty)$.
Proof. Let $f_0 \in M$ be given. There is a sequence $\{f_n\} \subset M$ by utilizing $f_n = \tau f_{n-1} = \tau^n f_0, n \geq 1$. If for some natural number n, $\tau f_n = f_{n+1} = f_n$, then f_n is a fixed point of τ in M. So, we assume that $f_{n+1} \neq f_n$ for all $n \in \mathbb{N}$. According to Definition 3.1, we see

$$\begin{aligned} \delta(f_n, f_{n+1}) &= \delta(\tau f_{n-1}, \tau f_n) \\ &\preceq a_1\delta(f_{n-1}, f_n) + a_2\delta(f_{n-1}, \tau f_{n-1}) + a_3\delta(f_n, \tau f_n) \\ &= (a_1 + a_2)\delta(f_{n-1}, f_n) + a_3\delta(f_n, f_{n+1}). \end{aligned} \tag{3.4}$$

By $\rho_{\mathcal{A}}(a_1 + a_2) + \rho_{\mathcal{A}}(a_3) < 1$, we know $e - a_3$ is invertible since $\rho_{\mathcal{A}}(a_3) < 1$. Then, (3.4) implies that

$$\delta(f_n, f_{n+1}) \preceq (e - a_3)^{-1}(a_1 + a_2)\delta(f_{n-1}, f_n). \tag{3.5}$$

Set $a = (e - a_3)^{-1}(a_1 + a_2)$. From (3.5), we deduce

$$\delta(f_n, f_{n+1}) \preceq a\delta(f_{n-1}, f_n) \preceq a^2\delta(f_{n-2}, f_{n-1}) \preceq \cdots \preceq a^n\delta(f_0, f_1). \tag{3.6}$$

For any $n \in \mathbb{N}$ and $p \geq 1$, by (3.6) and Definition 2.3, we have

$$\begin{aligned}
\delta(f_n, f_{n+p}) &\preceq \eta(f_n, f_{n+p})[\delta(f_n, f_{n+1}) + \delta(f_{n+1}, f_{n+p})] \\
&\preceq \eta(f_n, f_{n+p})\delta(f_n, f_{n+1}) + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})[\delta(f_{n+1}, f_{n+2}) + \delta(f_{n+2}, f_{n+p})] \\
&\preceq \eta(f_n, f_{n+p})\delta(f_n, f_{n+1}) + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})\delta(f_{n+1}, f_{n+2}) \\
&\quad + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})\eta(f_{n+2}, f_{n+p})[\delta(f_{n+2}, f_{n+3}) + \delta(f_{n+3}, f_{n+p})] \\
&\preceq \eta(f_n, f_{n+p})\delta(f_n, f_{n+1}) + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})\delta(f_{n+1}, f_{n+2}) \\
&\quad + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})\eta(f_{n+2}, f_{n+p})\delta(f_{n+2}, f_{n+3}) + \cdots \\
&\quad + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})\cdots\eta(f_{n+p-2}, f_{n+p})[\delta(f_{n+p-2}, f_{n+p-1}) + \delta(f_{n+p-1}, f_{n+p})] \\
&\preceq \eta(f_n, f_{n+p})\delta(f_n, f_{n+1}) + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})\delta(f_{n+1}, f_{n+2}) \\
&\quad + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})\eta(f_{n+2}, f_{n+p})\delta(f_{n+2}, f_{n+3}) + \cdots \\
&\quad + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})\cdots\eta(f_{n+p-2}, f_{n+p})\delta(f_{n+p-2}, f_{n+p-1}) \\
&\quad + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})\cdots\eta(f_{n+p-2}, f_{n+p})\eta(f_{n+p-1}, f_n + p)\delta(f_{n+p-1}, f_{n+p}) \\
&\preceq \eta(f_n, f_{n+p})a^n\delta(f_0, f_1) + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})a^{n+1}\delta(f_0, f_1) \\
&\quad + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})\eta(f_{n+2}, f_{n+p})a^{n+2}\delta(f_0, f_1) + \cdots \\
&\quad + \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})\cdots\eta(f_{n+p-2}, f_{n+p})\eta(f_{n+p-1}, f_{n+p})a^{n+p-1}\delta(f_0, f_1) \\
&= \left(\sum_{i=0}^{p-1}\left(a^i\prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\right)\right)a^n\delta(f_0, f_1) \\
&\preceq \left(\sum_{i=0}^{+\infty}\left(a^i\prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\right)\right)a^n\delta(f_0, f_1),
\end{aligned} \tag{3.7}$$

where $\prod$ is a product symbol, that is

$$\prod_{j=n}^{n+i}\eta(f_j, f_{n+p}) = \eta(f_n, f_{n+p})\eta(f_{n+1}, f_{n+p})\cdots\eta(f_{n+i}, f_{n+p}).$$

Then

$$\left\|\sum_{i=0}^{+\infty}\left(a^i\prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\right)\right\| \leq \sum_{i=0}^{+\infty}\left(\prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\|a^i\|\right) \leq \sum_{i=0}^{+\infty}\left(\prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\|a\|^i\right). \tag{3.8}$$

For $i \in \mathbb{N}$, we define

$$V_n^{n+p}(i) = \prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\|a\|^i, 0 \leq i \leq p-1,$$

which implies

$$\frac{V_n^{n+p}(i+1)}{V_n^{n+p}(i)} = \eta(f_{n+i+1}, f_{n+p})\frac{\|a\|^{i+1}}{\|a\|^i} = \eta(f_{n+i+1}, f_{n+p})\|a\|.$$

By (3.3), we see

$$\lim_{i\to+\infty} \eta(f_{n+i+1}, f_{n+p})\|a\| < 1.$$

So, the series $\sum_{i=0}^{+\infty}\left(\prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\|a\|^i\right)$ converges. Due to (3.8), we know $\left\|\sum_{i=0}^{p-1}\left(a^i\prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\right)\right\|$ is bounded.

Since a_1, a_2 commute with a_3, that is, $a_1a_3 = a_3a_1, a_2a_3 = a_3a_2$, by Lemma 2.3, we gain

$$\begin{aligned}(e-a_3)^{-1}(a_1+a_2) &= \left(\sum_{i=0}^{+\infty}(a_3)^i\right)(a_1+a_2)\\ &= \left(\sum_{i=0}^{+\infty}(a_3)^i\right)a_1 + \left(\sum_{i=0}^{+\infty}(a_3)^i\right)a_2\\ &= a_1\left(\sum_{i=0}^{+\infty}(a_3)^i\right) + a_2\left(\sum_{i=0}^{+\infty}(a_3)^i\right)\\ &= (a_1+a_2)\left(\sum_{i=0}^{+\infty}(a_3)^i\right)\\ &= (a_1+a_2)(e-a_3)^{-1},\end{aligned}$$

which implies $(e-a_3)^{-1}$ commutes with a_1+a_2. By Lemma 2.5 and $\rho_{\mathcal{A}}(a_1+a_2)+\rho_{\mathcal{A}}(a_3) < 1$, we get

$$\begin{aligned}\rho_{\mathcal{A}}(a) &= \rho_{\mathcal{A}}\left((e-a_3)^{-1}(a_1+a_2)\right)\\ &\le \rho_{\mathcal{A}}((e-a_3)^{-1})\rho_{\mathcal{A}}(a_1+a_2)\\ &\le \frac{\rho_{\mathcal{A}}(a_1+a_2)}{1-\rho_{\mathcal{A}}(a_3)} < 1.\end{aligned}$$

It means $\|a^n\| \to 0$ $(n\to+\infty)$ by Lemma 2.4. By (3.7) and (3.8), we have

$$\begin{aligned}\left\|\delta(f_n, f_{n+p})\right\| &\le \left\|\left(\sum_{i=0}^{+\infty}\left(a^i\prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\right)\right)a^n\delta(f_0, f_1)\right\|\\ &\le \|a^n\|\cdot\|\delta(f_0, f_1)\|\cdot\left\|\sum_{i=0}^{+\infty}\left(a^i\prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\right)\right\|\\ &\le \|a^n\|\cdot\|\delta(f_0, f_1)\|\cdot\sum_{i=0}^{+\infty}\left(\prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\|a\|^i\right).\end{aligned}$$

Thus, by $\|a^n\| \to 0$ $(n\to+\infty)$ and the series $\sum_{i=0}^{+\infty}\left(\prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\|a\|^i\right)$ converges, we know for any $p \ge 1$, $\|\delta(f_n, f_{n+p})\| \to 0$ $(n\to+\infty)$, and $\{f_n\}$ is a Cauchy sequence in M.

Because $(M, \mathcal{A}, \delta)$ is τ-orbitally complete, the sequence $\{f_n\}$ converges to some point $f \in M$. We claim that f is the fixed point in M. In fact, by (3.1), we have

$$\delta(\tau f_n, \tau f) \le a_1\delta(f_n, f) + a_2\delta(f_n, \tau f_n) + a_3\delta(f, \tau f)$$

$$\leq a_1\delta(f_n, f) + a_2\delta(f_n, f_{n+1}) + \eta(f, \tau f)a_3\left[\delta(f, f_{n+1}) + \delta(f_{n+1}, \tau f)\right].$$

We gain

$$(e - \eta(f, \tau f)a_3)\,\delta(f_{n+1}, \tau f) \leq a_1\delta(f_n, f) + a_2\delta(f_n, f_{n+1}) + \eta(f, \tau f)a_3\delta(f, f_{n+1}).$$

Owing to $\eta(f, \tau f) \leq \frac{1}{1-\rho_{\mathcal{A}}(a_1+a_2)}$, we conclude that

$$\rho_{\mathcal{A}}(\eta(f, \tau f)a_3) \leq \eta(f, \tau f)\rho_{\mathcal{A}}(a_3) \leq \frac{\rho_{\mathcal{A}}(a_3)}{1 - \rho_{\mathcal{A}}(a_1 + a_2)} < 1.$$

Thus, $e - \eta(f, \tau f)a_3$ is invertible. Therefore, we have

$$\delta(f_{n+1}, \tau f) \leq (e - \eta(f, \tau f)a_3)^{-1}\left[a_1\delta(f_n, f) + a_2\delta(f_n, f_{n+1}) + \eta(f, \tau f)a_3\delta(f, f_{n+1})\right].$$

As U is normal, we get

$$\|\delta(f_{n+1}, \tau f)\| \leq K\|(e - \eta(f, \tau f)a_3)^{-1}\|(\|a_1\| \cdot \|\delta(f_n, f)\| + \|a_2\| \cdot \|\delta(f_n, f_{n+1})\| + \eta(f, \tau f)\|a_3\| \cdot \|\delta(f, f_{n+1})\|).$$

Because $f_n \overset{\|\cdot\|}{\to} f, f_{n+1} \overset{\|\cdot\|}{\to} f$ $(n \to +\infty)$ and $\{f_n\}$ is a Cauchy sequence, we know $\|\delta(f_n, f)\| \to 0$, $\|\delta(f_n, f_{n+1})\| \to 0$ and $\|\delta(f, f_{n+1})\| \to 0$ as $n \to +\infty$. As a result, $\|\delta(f_{n+1}, \tau f)\| \to 0$, that is, $f_{n+1} \overset{\|\cdot\|}{\to} \tau f$ $(n \to +\infty)$. So, $\tau f = f$ by Lemma 2.1, and $f \in M$ is the fixed point.

Now we give the definition of Kannan-type contraction and the corresponding fixed point theorems as follows.

Definition 3.3. Let the set $M \neq \emptyset$ and U be a non-solid cone. If the mapping $\tau : M \to M$ satisfies

$$\delta(\tau f, \tau g) \leq a\left[\delta(f, \tau f) + \delta(g, \tau g)\right] \tag{3.9}$$

for all $f, g \in M$, where $a \in U^*$ such that

$$\rho_{\mathcal{A}}(a) < \frac{1}{2},$$

then τ is named a Kannan-type contraction in $(M, \mathcal{A}, \delta)$.

Theorem 3.2. Let $(M, \mathcal{A}, \delta)$ be a non-solid τ-orbitally complete ECb-MLS over a BA, where τ is a Kannan-type contraction and $\eta(f, g) \leq \frac{1}{1-\rho_{\mathcal{A}}(a)}$ for any $f, g \in M$. If for any Picard iterating sequence $\{f_n\} = \{\tau^n f_0\}$ generated by $f_0 \in M$,

$$\lim_{n,m\to+\infty} \eta(f_n, f_m) < \frac{1}{\|(e - a)^{-1}a\|},$$

then τ admits a fixed point $f \in M$ and the sequence $f_n \overset{\|\cdot\|}{\to} f$ $(n \to +\infty)$.

Proof. Let $f_0 \in M$ be given. There is a sequence $\{f_n\} \subset M$ by utilizing $f_n = \tau f_{n-1} = \tau^n f_0, n \geq 1$. If for some natural number n, $\tau f_n = f_{n+1} = f_n$, then f_n is a fixed point of τ in M. So, we assume that $f_{n+1} \neq f_n$ for all $n \in \mathbb{N}$. According to Definition 3.3, we see

$$\begin{aligned}\delta(f_n, f_{n+1}) &= \delta(\tau f_{n-1}, \tau f_n)\\ &\leq a(\delta(f_{n-1}, \tau f_{n-1}) + \delta(f_n, \tau f_n))\end{aligned}$$

$$= a\delta(f_{n-1}, f_n) + a\delta(f_n, f_{n+1}).$$

It follows that

$$\delta(f_n, f_{n+1}) \preceq k\delta(f_{n-1}, f_n) \preceq k^2\delta(f_{n-2}, f_{n-1}) \preceq \cdots \preceq k^n\delta(f_0, f_1),$$

where $k = (e - a)^{-1}a$ and $\rho_{\mathcal{A}}(k) \leq \frac{\rho_{\mathcal{A}}(a)}{1-\rho_{\mathcal{A}}(a)} < 1$.

Similar to the proof of Theorem 3.1, we know the series $\sum_{i=0}^{+\infty}\left(\prod_{j=n}^{n+i}\eta(f_j, f_{n+p})\|k\|^i\right)$ converges. The condition $\rho_{\mathcal{A}}(k) < 1$ means $\|k^n\| \to 0$ $(n \to +\infty)$ by Lemma 2.4. It deduces that $\{f_n\}$ is a Cauchy sequence in M.

Because $(M, \mathcal{A}, \delta)$ is τ-orbitally complete, the sequence $\{f_n\}$ converges to some point $f \in M$. We claim that f is the fixed point in M. In fact, by (3.9), we have

$$\begin{aligned}\delta(\tau f_n, \tau f) &\preceq a[\delta(f_n, \tau f_n) + \delta(f, \tau f)]\\ &\preceq a\delta(f_n, f_{n+1}) + \eta(f, \tau f)a\left[\delta(f, f_{n+1}) + \delta(f_{n+1}, \tau f)\right].\end{aligned}$$

We gain

$$(e - \eta(f, \tau f)a)\,\delta(f_{n+1}, \tau f) \preceq a\delta(f_n, f_{n+1}) + \eta(f, \tau f)a\delta(f, f_{n+1}).$$

Owing to $\eta(f, \tau f) \leq \frac{1}{1-\rho_{\mathcal{A}}(a)}$, we conclude that

$$\rho_{\mathcal{A}}(\eta(f, \tau f)a) \leq \eta(f, \tau f)\rho_{\mathcal{A}}(a) \leq \frac{\rho_{\mathcal{A}}(a)}{1 - \rho_{\mathcal{A}}(a)} < 1.$$

Thus, $e - \eta(f, \tau f)a$ is invertible. Therefore, we have

$$\delta(f_{n+1}, \tau f) \preceq (e - \eta(f, \tau f)a)^{-1}\left[a\delta(f_n, f_{n+1}) + \eta(f, \tau f)a\delta(f, f_{n+1})\right].$$

As U is normal, we get

$$\|\delta(f_{n+1}, \tau f)\| \leq K\|\,(e - \eta(f, \tau f)a)^{-1}\,\|\,(\|a\| \cdot \|\delta(f_n, f_{n+1})\| + \eta(f, \tau f)\|a\| \cdot \|\delta(f, f_{n+1})\|).$$

Because $f_n \xrightarrow{\|\cdot\|} f, f_{n+1} \xrightarrow{\|\cdot\|} f (n \to +\infty)$ and $\{f_n\}$ is a Cauchy sequence, it follows that $\|\delta(f_n, f)\| \to 0$, $\|\delta(f_n, f_{n+1})\| \to 0$ and $\|\delta(f, f_{n+1})\| \to 0$ as $n \to +\infty$. As a result, $\|\delta(f_{n+1}, \tau f)\| \to 0$, that is, $f_{n+1} \xrightarrow{\|\cdot\|} \tau f$ $(n \to +\infty)$. So, $\tau f = f$ by Lemma 2.1 and $f \in M$ is the fixed point.

Remark 3.2. In Theorem 3.1 and Theorem 3.2, we establish the existence of fixed points without requiring these contractions to satisfy continuity (Theorem 3.13 in [26]), orbital continuity (Theorem 2 in [19]) or δ-lower orbital continuity (Theorems 3.2, 3.6, 4.1 in [25]). These results improve upon the main theorems in [19,25,26], which depend crucially on continuity assumptions of τ.

If we further assume that τ is f_0-orbital continuity, then the condition $\eta(f, g) \leq \frac{1}{1-\rho_{\mathcal{A}}(a_1+a_2)}$ for any $f, g \in M$ in the above theorems can be deleted. We now give these definitions.

Definition 3.4. Let $(M, \mathcal{A}, \delta)$ be an ECb-MLS over a BA. Let a point $f_0 \in M$ and $O_\tau(f_0) = \{f_0, \tau f_0, \tau^2 f_0, \tau^3 f_0, \cdots\}$. If for any sequence $\{f_n\} \subseteq O_\tau(f_0), f_n \xrightarrow{\|\cdot\|} f$ as $n \to +\infty$ implies $\tau f_n \xrightarrow{\|\cdot\|} \tau f$ as $n \to +\infty$, then we say τ is f_0-orbitally continuous.

Definition 3.5. If τ is f_0-orbitally continuous at every point $f_0 \in M$, then we say τ is orbitally continuous in M.

Remark 3.3. From the above definitions, it is not difficult to observe that f_0-orbital continuity is weaker than orbital continuity and continuity.

Example 3.1. In order to show our theorems are applicable in examples with weaker conditions, we construct a f_0-orbitally continuous mapping τ in this example. It does not satisfy the conditions of orbital continuity or continuity. Let $\mathcal{A} = \mathbb{R}^2$ with the norm $\|f\| = \|(f_1, f_2)\| = |f_1| + |f_2|$ and the multiplication defined by

$$fg = (f_1, f_2)(g_1, g_2) = (f_1g_1, f_1g_2 + f_2g_1),$$

where $f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{A}$. Direct verification shows that $\mathcal{A}$ is a unital Banach algebra with its unit $e = (1, 0)$. Define a cone $P = \{(f_1, f_2) \in \mathbb{R}^2 : f_1, f_2 \geq 0\}$. Put $M = [0, 1] \times [0, 1]$ and construct a mapping $\delta : M \times M \to \mathcal{A}$ by

$$\delta((f_1, f_2), (g_1, g_2)) = (|f_1 + g_1|^2, |f_2 + g_2|^2).$$

We can prove that $(M, \mathcal{A}, \delta)$ is an ECb-MLS over a BA with $\eta(f, g) = 1 + \max\{\|f\|, \|g\|, 1\}$ by a detail checking. Define a self-mapping τ on M as

$$\tau f = \tau(f_1, f_2) = \begin{cases} (1, 1), & (f_1, f_2) = (0, 0); \\ (1, 1), & (f_1, f_2) = (1, 1); \\ (\frac{1}{3}f_1, \frac{1}{3}f_2), & otherwise. \end{cases}$$

It is clear that τ is f_0-orbitally continuous. Actually, let $f_0 = (0, 0)$, then

$$\tau^n f_0 \to (1, 1) \text{ and } \tau(\tau^n f_0) \to (1, 1) = \tau(1, 1)\ (n \to +\infty).$$

However, τ is not orbitally continuous, this is because

$$\tau^n(f_1, f_2) = \left(\left(\frac{1}{3}\right)^n f_1, \left(\frac{1}{3}\right)^n f_2\right) \to (0, 0)\ (n \to +\infty),$$

which follows that, for all $f_1, f_2 \in (0, 1)$, $\tau(\tau^n(f_1, f_2)) \to (0, 0) \neq \tau(0, 0)$ as $n \to +\infty$. Accordingly, we claim that the condition of orbital continuity is stronger than f_0-orbital continuity for the mapping τ.

Theorem 3.3. Let $(M, \mathcal{A}, \delta)$ be a non-solid τ-orbitally complete ECb-MLS over a BA, where τ is a Reich-type contraction and f_0-orbitally continuous for some $f_0 \in M$. If for any Picard iterating sequence $\{f_n\} = \{\tau^n f_0\}$ generated by f_0,

$$\lim_{n,m\to+\infty} \eta(f_n, f_m) < \frac{1}{\|(e - a_3)^{-1}(a_1 + a_2)\|},$$

then τ admits a fixed point $f \in M$ and the sequence $f_n \overset{\|\cdot\|}{\to} f\ (n \to +\infty)$.

Proof. Let $f_0 \in M$ be given. There is a sequence $\{f_n\} \subset M$ by utilizing $f_n = \tau f_{n-1} = \tau^n f_0, n \geq 1$. If for some natural number n, $\tau f_n = f_{n+1} = f_n$, then f_n is a fixed point of τ in M. So, we assume that $f_{n+1} \neq f_n$ for all $n \in \mathbb{N}$. By Theorem 3.1, we see $f_n \overset{\|\cdot\|}{\to} f$ for some $f \in M$ as $n \to +\infty$. Since τ is f_0-orbitally continuous, $f_{n+1} = \tau f_n \overset{\|\cdot\|}{\to} \tau f\ (n \to +\infty)$. Thus, f is the fixed point of τ by Lemma 2.1 and the sequence $\{f_n\}$ converges to f.

Naturally, the following theorem holds.

Theorem 3.4. Let $(M, \mathcal{A}, \delta)$ be a non-solid τ-orbitally complete ECb-MLS over a BA, where τ is a Kannan-type contraction and f_0-orbitally continuous for some $f_0 \in M$. If for any Picard iterating sequence $\{f_n\} = \{\tau^n f_0\}$ generated by f_0,

$$\lim_{n,m\to+\infty} \eta(f_n, f_m) < \frac{1}{\|(e-a)^{-1}a\|},$$

then τ admits a fixed point $f \in M$ and the sequence $f_n \overset{\|\cdot\|}{\to} f$ $(n \to +\infty)$.

At the end of this section, we give the following theorem of Banach-type contraction in ECb-MLSs over BAs, without any continuity condition.

Theorem 3.5. Let $(M, \mathcal{A}, \delta)$ be a non-solid τ-orbitally complete ECb-MLS over a BA. Suppose that the mapping τ is a Banach-type contraction, which satisfies

$$\delta(\tau f, \tau g) \preceq a\delta(f, g) \tag{3.10}$$

for all $f, g \in M$, where $a \in U^*$ such that $\rho_{\mathcal{A}}(a) < 1$. If for any Picard iterating sequence $\{f_n\} = \{\tau^n f_0\}$ generated by f_0,

$$\lim_{n,m\to+\infty} \eta(f_n, f_m) < \frac{1}{\|a\|}, \tag{3.11}$$

then τ admits a fixed point $f \in M$ and the sequence $f_n \overset{\|\cdot\|}{\to} f$ $(n \to +\infty)$.

Proof. Let $f_0 \in M$ be given. There is a sequence $\{f_n\} \subset M$ by utilizing $f_n = \tau f_{n-1} = \tau^n f_0, n \geq 1$. If for some natural number n, $\tau f_n = f_{n+1} = f_n$, then f_n is a fixed point of τ in M. So, we assume that $f_{n+1} \neq f_n$ for all $n \in \mathbb{N}$. According to (3.10), we see

$$\delta(f_n, f_{n+1}) = \delta(\tau f_{n-1}, \tau f_n) \preceq a\delta(f_{n-1}, f_n),$$

which implies that

$$\delta(f_n, f_{n+1}) \preceq a\delta(f_{n-1}, f_n) \preceq a^2\delta(f_{n-2}, f_{n-1}) \preceq \cdots \preceq a^n\delta(f_0, f_1).$$

Similar to the proof of Theorem 3.1, we know the series $\sum_{i=0}^{+\infty} \left(\prod_{j=n}^{n+i} \eta(f_j, f_{n+p})\|a\|^i\right)$ converges. The condition $\rho_{\mathcal{A}}(a) < 1$ means $\|a^n\| \to 0$ $(n \to +\infty)$ by Lemma 2.4. It deduces that $\{f_n\}$ is a Cauchy sequence in M.

Because $(M, \mathcal{A}, \delta)$ is τ-orbitally complete, the sequence $\{f_n\}$ converges to some point $f \in M$. We claim that f is the fixed point in M. In fact, by (3.10), we have

$$\delta(\tau f_n, \tau f) \preceq a\delta(f_n, f).$$

As U is normal, we get

$$\|\delta(f_{n+1}, \tau f)\| \leq K\|a\| \cdot \|\delta(f_n, f)\|.$$

Because $f_n \overset{\|\cdot\|}{\to} f$, we see $\|\delta(f_n, f)\| \to 0$ as $n \to +\infty$. As a result, $\|\delta(f_{n+1}, \tau f)\| \to 0$, that is, $f_{n+1} \overset{\|\cdot\|}{\to} \tau f$ $(n \to +\infty)$. So, $\tau f = f$ by Lemma 2.1 and $f \in M$ is the fixed point.

4. Fixed points of Reich-type and Kannan-type almost contractions in ECb-MSs over BAs

In this section, we consider $(M, \mathcal{A}, \delta)$ to be a complete ECb-MS over a BA and U a normal cone with normal constant $K \geq 1$. In light of the previous definitions and lemmas, we present fixed point results for Reich-type and Kannan-type almost contractions in non-solid $(M, \mathcal{A}, \delta)$ establishing, not only the existence of fixed points but also their uniqueness. These results generalize the main theorems in [13,32].

Definition 4.1. Let the set $M \neq \emptyset$ and U be a non-solid cone. If the mapping $\tau : M \to M$ satisfies

$$\delta(\tau f, \tau g) \preceq a_1\delta(f, g) + a_2\delta(f, \tau f) + a_3\delta(g, \tau g) + a_4\delta(g, \tau f) \tag{4.1}$$

for all $f, g \in M$, where $a_i \in U^*(i = 1, 2, 3, 4)$ such that a_1, a_2 commutes with a_3 and

$$\rho_{\mathcal{A}}(a_1 + a_2) + \rho_{\mathcal{A}}(a_3) < 1, \tag{4.2}$$

then τ is named a Reich-type almost contraction in $(M, \mathcal{A}, \delta)$.

Definition 4.2. Let the set $M \neq \emptyset$ and U be a non-solid cone. If the mapping $\tau : M \to M$ satisfies

$$\delta(\tau f, \tau g) \preceq a\left[\delta(f, \tau f) + \delta(g, \tau g)\right] + k\delta(g, \tau f)$$

for all $f, g \in M$, where $a, k \in U^*$ such that

$$\rho_{\mathcal{A}}(a) < \frac{1}{2},$$

then τ is named a Kannan-type almost contraction in $(M, \mathcal{A}, \delta)$.

Theorem 4.1. Let $(M, \mathcal{A}, \delta)$ be a non-solid τ-orbitally complete ECb-MS over a BA, where τ is a Reich-type almost contraction and $\eta(f, g) \leq \frac{1}{1-\rho_{\mathcal{A}}(a_1+a_2)}$ for any $f, g \in M$. If for any Picard iterating sequence $\{f_n\} = \{\tau^n f_0\}$ generated by $f_0 \in M$,

$$\lim_{n,m\to+\infty} \eta(f_n, f_m) < \frac{1}{\|(e - a_3)^{-1}(a_1 + a_2)\|}, \tag{4.3}$$

then τ admits a fixed point $f \in M$ and the sequence $f_n \overset{\|\cdot\|}{\to} f$ $(n \to +\infty)$. Furthermore, if $\rho_{\mathcal{A}}(a_1 + a_4) < 1$, then the fixed point is unique.

Proof. Let $f_0 \in M$ be given. There is a sequence $\{f_n\} \subset M$ by utilizing $f_n = \tau f_{n-1} = \tau^n f_0, n \geq 1$. If for some natural number n, $\tau f_n = f_{n+1} = f_n$, then f_n is a fixed point of τ in M. So, we assume that $f_{n+1} \neq f_n$ for all $n \in \mathbb{N}$. According to Definition 4.1, we see

$$\begin{aligned}\delta(f_n, f_{n+1}) &= \delta(\tau f_{n-1}, \tau f_n)\\ &\preceq a_1\delta(f_{n-1}, f_n) + a_2\delta(f_{n-1}, \tau f_{n-1}) + a_3\delta(f_n, \tau f_n) + a_4\delta(f_n, \tau f_{n-1})\\ &= (a_1 + a_2)\delta(f_{n-1}, f_n) + a_3\delta(f_n, f_{n+1}).\end{aligned} \tag{4.4}$$

Similar to the analysis of Theorem 3.1, we can obtain $\{f_n\}$ is a Cauchy sequence in M. Because $(M, \mathcal{A}, \delta)$ is τ-orbitally complete, the sequence $\{f_n\}$ converges to some point $f \in M$. We claim that f is the fixed point in M. In fact, by (4.1), we have

$$\delta(\tau f_n, \tau f) \preceq a_1\delta(f_n, f) + a_2\delta(f_n, \tau f_n) + a_3\delta(f, \tau f) + a_4\delta(f, \tau f_n)$$

$$\leq a_1\delta(f_n, f) + a_2\delta(f_n, f_{n+1}) + \eta(f, \tau f)a_3\left[\delta(f, f_{n+1}) + \delta(f_{n+1}, \tau f)\right] + a_4\delta(f, f_{n+1}).$$

It follows that

$$\left[e - \eta(f, \tau f)a_3\right]\delta(f_{n+1}, \tau f) \leq a_1\delta(f_n, f) + a_2\delta(f_n, f_{n+1}) + (\eta(f, \tau f)a_3 + a_4)\delta(f, f_{n+1}).$$

Owing to $\eta(f, \tau f) \leq \frac{1}{1-\rho_{\mathcal{A}}(a_1+a_2)}$, we conclude that

$$\rho_{\mathcal{A}}(\eta(f, \tau f)a_3) \leq \eta(f, \tau f)\rho_{\mathcal{A}}(a_3) \leq \frac{\rho_{\mathcal{A}}(a_3)}{1 - \rho_{\mathcal{A}}(a_1 + a_2)} < 1.$$

Thus, $e - \eta(f, \tau f)a_3$ is invertible. Therefore, we have

$$\delta(f_{n+1}, \tau f) \leq \left[e - \eta(f, \tau f)a_3\right]^{-1}\left[a_1\delta(f_n, f) + a_2\delta(f_n, f_{n+1}) + (\eta(f, \tau f)a_3 + a_4)\,\delta(f, f_{n+1})\right].$$

As U is normal, we get

$$\begin{aligned}\|\delta(f_{n+1}, \tau f)\| \leq{}& K\|(e - \eta(f, \tau f)a_3)^{-1}\|(\|a_1\| \cdot \|\delta(f_n, f)\| \\ &+ \|a_2\| \cdot \|\delta(f_n, f_{n+1})\| + \|\eta(f, \tau f)a_3 + a_4\| \cdot \|\delta(f, f_{n+1})\|).\end{aligned}$$

Because $f_n \overset{\|\cdot\|}{\to} f, f_{n+1} \overset{\|\cdot\|}{\to} f$ $(n \to +\infty)$ and $\{f_n\}$ is a Cauchy sequence, $\|\delta(f_n, f)\| \to 0, \|\delta(f_n, f_{n+1})\| \to 0$ and $\|\delta(f, f_{n+1})\| \to 0$ as $n \to +\infty$. As a result, $\|\delta(f_{n+1}, \tau f)\| \to 0$, that is, $f_{n+1} \overset{\|\cdot\|}{\to} \tau f$ $(n \to +\infty)$. So, $\tau f = f$ by Lemma 2.1. Hence, $f \in M$ is the fixed point and the sequence $f_n \overset{\|\cdot\|}{\to} f$ $(n \to +\infty)$.

We now turn to the proof of uniqueness for the fixed point. Otherwise, if there is another point $g \in M, g \neq f$ such that $\tau g = g$, then

$$\begin{aligned}\delta(\tau f, \tau g) &\leq a_1\delta(f, g) + a_2\delta(f, \tau f) + a_3\delta(g, \tau g) + a_4\delta(g, \tau f) \\ &= (a_1 + a_4)\delta(f, g).\end{aligned}$$

Repeating this progress, we have

$$\delta(f, g) \leq (a_1 + a_4)\delta(f, g) = (a_1 + a_4)\delta(\tau f, \tau g) \leq (a_1 + a_4)^2\delta(f, g) \leq \cdots \leq (a_1 + a_4)^n\delta(f, g).$$

Thus, $\delta(f, g) \leq (a_1 + a_4)^n\delta(f, g)$ for any $n \geq 1$. It implies that

$$\|\delta(f, g)\| \leq K\|(a_1 + a_4)^n\| \cdot \|\delta(f, g)\|.$$

Using the fact that $\rho_{\mathcal{A}}(a_1+a_4) < 1$, we know $\|(a_1+a_4)^n\| \to 0$ $(n \to +\infty)$ by Lemma 2.4. So, $\delta(f, g) = \theta$ and $f = g$.

Next, we consider the fixed point theorem for Reich-type almost contraction in non-solid $(M, \mathcal{A}, \delta)$ under the condition of f_0-orbital continuity.

Theorem 4.2. Let $(M, \mathcal{A}, \delta)$ be a non-solid τ-orbitally complete ECb-MS over a BA, where τ is a Reich-type almost contraction and f_0-orbitally continuous for some $f_0 \in M$. If for any Picard iterating sequence $\{f_n\} = \{\tau^n f_0\}$ generated by $f_0 \in M$,

$$\lim_{n,m\to+\infty} \eta(f_n, f_m) < \frac{1}{\|(e - a_3)^{-1}(a_1 + a_2)\|},$$

then τ admits a fixed point $f \in M$ and the sequence $f_n \overset{\|\cdot\|}{\to} f$ $(n \to +\infty)$. Furthermore, if $\rho_{\mathcal{A}}(a_1+a_4) < 1$, then the fixed point is unique.

Proof. Let $f_0 \in M$ be given. There is a sequence $\{f_n\} \subset M$ by utilizing $f_n = \tau f_{n-1} = \tau^n f_0, n \geq 1$. If for some natural number n, $\tau f_n = f_{n+1} = f_n$, then f_n is a fixed point of τ in M. So, we assume that $f_{n+1} \neq f_n$ for all $n \in \mathbb{N}$. Similar to Theorem 4.1, we see the sequence $\{f_n\}$ converges to some point $f \in M$. Since τ is f_0-orbitally continuous, $f_{n+1} = \tau f_n \overset{\|\cdot\|}{\to} \tau f$ $(n \to +\infty)$. Thus, f is the fixed point of τ by Lemma 2.1. The proof of uniqueness closely follows that of Theorem 4.1.

The fixed point theorems for Kannan-type almost contraction with vector-valued coefficients in non-solid ECb-MS over BA can be obtained similar to Theorem 4.1 and Theorem 4.2. Accordingly, the proof is omitted.

Theorem 4.3. Let $(M, \mathcal{A}, \delta)$ be a non-solid τ-orbitally complete ECb-MS over a BA, where τ is a Kannan-type almost contraction and $\eta(f, g) \leq \frac{1}{1-\rho_{\mathcal{A}}(a)}$ for any $f, g \in M$. If for any Picard iterating sequence $\{f_n\} = \{\tau^n f_0\}$ generated by $f_0 \in M$,

$$\lim_{n,m\to+\infty} \eta(f_n, f_m) < \frac{1}{\|(e-a)^{-1}a\|},$$

then τ admits a fixed point $f \in M$ and the sequence $f_n \overset{\|\cdot\|}{\to} f$ $(n \to +\infty)$. Furthermore, if $\rho_{\mathcal{A}}(k) < 1$, then the fixed point is unique.

Theorem 4.4. Let $(M, \mathcal{A}, \delta)$ be a non-solid τ-orbitally complete ECb-MS over a BA, where τ is a Kannan-type almost contraction and f_0-orbitally continuous for some $f_0 \in M$. If for any Picard iterating sequence $\{f_n\} = \{\tau^n f_0\}$ generated by $f_0 \in M$,

$$\lim_{n,m\to+\infty} \eta(f_n, f_m) < \frac{1}{\|(e-a)^{-1}a\|},$$

then τ admits a fixed point $f \in M$ and the sequence $f_n \overset{\|\cdot\|}{\to} f$ $(n \to +\infty)$. Furthermore, if $\rho_{\mathcal{A}}(k) < 1$, then the fixed point is unique.

5. Applications

This section provides illustrative examples demonstrating that our main results serve as effective tools for establishing solutions in different equations, regardless of whether the underlying cones are solid or non-solid.

Example 5.1. Let the set $M = [0, 1)$ and the Banach algebra $\mathcal{A} = \mathbb{R}$. The multiplication of $\mathcal{A}$ is defined by its usual pointwise multiplication. Then $\mathcal{A}$ is a Banach algebra with $e = 1$. The cone is denoted by

$$U = \{f \in \mathcal{A} \mid f \geq 0\} \subset \mathbb{R}.$$

For any non-negative real function $\varphi(t) \in Ł[0, 1]$ (the set of all generalized real-valued Lebesgue integral functions on $[0, 1]$). Define $\delta : M \times M \to \mathcal{A}$ by

$$\delta(f, g) = \alpha \int_0^1 |f + g|\varphi(t)dt,$$

where $\alpha \geq 0$ is a constant and the function $\eta(f,g) = 1 + \frac{1}{4}|f+g|$ for any $f,g \in M$. Then $(M,\mathcal{A},\delta)$ is a normal and solid ECb-MLS over a BA. Define the mapping $\tau : M \to M$ by

$$\tau f = \begin{cases} \frac{2}{3}f, f \in [0, \frac{9}{17}); \\ \frac{1}{10}f + \frac{3}{10}, f \in [\frac{9}{17}, 1). \end{cases}$$

Then τ is f_0-orbitally continuous for any $f_0 \in [0, \frac{9}{17})$, because $\tau^n f_0 \to 0, \tau(\tau^n f_0) \to 0 = \tau 0$. However, τ is not orbitally continuous. In fact, for any $f \in [\frac{9}{17}, 1)$, letting n tend to $+\infty$, we have

$$\tau^n f \to \frac{1}{3}, \tau(\tau^n f) \to \frac{1}{3} \neq \tau\frac{1}{3} = \frac{2}{9},$$

which yields that τ is not orbitally continuous. Moreover, the space $(M,\mathcal{A},\delta)$ is τ-orbitally complete but not complete. Next, we prove that τ is a Kannan-type contraction in three different cases.
Case 1: for all $f,g \in [\frac{9}{17}, 1)$, we have $\tau f = \frac{1}{10}f + \frac{3}{10}, \tau g = \frac{1}{10}g + \frac{3}{10}$. Then,

$$\begin{aligned}
\delta(\tau f, \tau g) &= \alpha \int_0^1 \left|\frac{1}{10}f + \frac{3}{10} + \frac{1}{10}g + \frac{3}{10}\right| \varphi(t)dt \\
&= \alpha \int_0^1 \left|\frac{1}{10}(f+g) + \frac{3}{5}\right| \varphi(t)dt \\
&\leq \alpha \int_0^1 \left|\frac{11}{25}(f+g) + \frac{6}{25}\right| \varphi(t)dt \\
&= \frac{2}{5}\alpha\left(\int_0^1 \left|f + \frac{1}{10}f + \frac{3}{10}\right| \varphi(t)dt + \int_0^1 \left|g + \frac{1}{10}g + \frac{3}{10}\right| \varphi(t)dt\right) \\
&= \frac{2}{5}\left(\delta(f,\tau f) + \delta(g,\tau g)\right),
\end{aligned}$$

since

$$\frac{11}{25}(f+g) + \frac{6}{25} - \frac{1}{10}(f+g) - \frac{3}{5} = \frac{17}{50}(f+g) - \frac{9}{25} \geq \frac{17}{50}\cdot\frac{18}{17} - \frac{9}{25} = 0.$$

Case 2: for all $f,g \in [0, \frac{9}{17})$, we have $\tau f = \frac{2}{3}f, \tau g = \frac{2}{3}g$. Then,

$$\begin{aligned}
\delta(\tau f, \tau g) &= \alpha \int_0^1 \left|\frac{2}{3}f + \frac{2}{3}g\right| \varphi(t)dt \\
&= \frac{2}{3}\alpha \int_0^1 |f+g|\, \varphi(t)dt \\
&\leq \frac{2}{5}\alpha\left(\int_0^1 \left|f + \frac{2}{3}f\right| \varphi(t)dt + \int_0^1 \left|g + \frac{2}{3}g\right| \varphi(t)dt\right) \\
&= \frac{2}{5}\left(\delta(f,\tau f) + \delta(g,\tau g)\right).
\end{aligned}$$

Case 3: for all $f \in [0, \frac{9}{17}), g \in [\frac{9}{17}, 1)$, we have $\tau f = \frac{2}{3}f, \tau g = \frac{1}{10}g + \frac{3}{10}$. Then,

$$\delta(\tau f, \tau g) = \alpha \int_0^1 \left|\frac{2}{3}f + \frac{1}{10}g + \frac{3}{10}\right| \varphi(t)dt$$

$$
\begin{aligned}
&\leq \alpha \int_0^1 \left|\frac{2}{3}f\right| \varphi(t)dt + \alpha \int_0^1 \left|\frac{1}{10}g + \frac{3}{10}\right| \varphi(t)dt \\
&\leq \frac{2}{5}\alpha \left(\int_0^1 \left|f + \frac{2}{3}f\right| \varphi(t)dt + \int_0^1 \left|g + \frac{1}{10}g + \frac{3}{10}\right| \varphi(t)dt\right) \\
&= \frac{2}{5}\left(\delta(f, \tau f) + \delta(g, \tau g)\right).
\end{aligned}
$$

The third inequality holds because

$$\frac{2}{5}\left(\frac{11}{10}g + \frac{3}{10}\right) - \frac{1}{10}g - \frac{3}{10} = \frac{17}{50}g - \frac{9}{50} \geq \frac{17}{50} \cdot \frac{9}{17} - \frac{9}{50} = 0.$$

Let $a = \frac{2}{5} \in U, \rho_{\mathcal{A}}(a) = \frac{2}{5} < \frac{1}{2}$. Moreover, we have

$$\eta(f, g) = 1 + \frac{1}{4}|f + g| < 1 + \frac{1}{4} \times 2 = \frac{3}{2} < \frac{5}{3} = \frac{1}{1 - \rho_{\mathcal{A}}(a)}.$$

To verify that $\lim_{n,m\to+\infty} \eta(f_n, f_m) < \frac{1}{\|(e-a)^{-1}a\|}$, we analyze two cases.

(1) If $f_0 \in [0, \frac{9}{17}), f_n = \left(\frac{2}{3}\right)^n f_0, f_m = \left(\frac{2}{3}\right)^m f_0$, then

$$\lim_{n,m\to+\infty} \eta(f_n, f_m) = \lim_{n,m\to+\infty} \left(1 + \frac{1}{4}\left|\left(\frac{2}{3}\right)^n f_0 + \left(\frac{2}{3}\right)^m f_0\right|\right) = 1 < \frac{3}{2} = \frac{1}{\|(e-a)^{-1}a\|},$$

(2) If $f_0 \in [\frac{9}{17}, 1)$, then

$$f_n = \left(\frac{1}{10}\right)^n f_0 + \frac{3}{10} \times \frac{1 - \left(\frac{1}{10}\right)^n}{1 - \frac{1}{10}} = \left(\frac{1}{10}\right)^n f_0 + \frac{1}{3}\left(1 - \left(\frac{1}{10}\right)^n\right)$$

and

$$f_m = \left(\frac{1}{10}\right)^m f_0 + \frac{1}{3}\left(1 - \left(\frac{1}{10}\right)^m\right).$$

One get

$$
\begin{aligned}
\lim_{n,m\to+\infty} \eta(f_n, f_m) &= \lim_{n,m\to+\infty} \left(1 + \frac{1}{4}\left|\left(\frac{1}{10}\right)^n f_0 + \frac{1}{3}\left(1 - \left(\frac{1}{10}\right)^n\right) + \left(\frac{1}{10}\right)^m f_0 + \frac{1}{3}\left(1 - \left(\frac{1}{10}\right)^m\right)\right|\right) \\
&= \frac{7}{6} < \frac{3}{2} = \frac{1}{\|(e-a)^{-1}a\|}.
\end{aligned}
$$

This establishes all requirements of Theorem 3.4, from which it follows that τ has a fixed point in M.

Remark 5.1. The main results of [19,25,26] are not applicable in Example 5.1 since they require that the spaces are complete and the mappings satisfy certain continuity conditions. Therefore, conclusions are very different from the existing conclusions in the literature.

Example 5.2. Let $F : [0, 1] \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a generalized real-valued Lebesgue integral function where $\overline{\mathbb{R}} = [-\infty, +\infty]$ is the set on all generalized real numbers. Consider the following nonlinear integral equation

$$\int_0^1 F(t, f(s))\,\mathrm{d}s = f(t). \tag{5.1}$$

If for a.e. $t \in [0,1]$, and a.e. $f, g \in \overline{\mathbb{R}}$, one has

$$|F(t,f) - F(t,g)| \leq H(x)|f-g|, \tag{5.2}$$

where the function $H(x)$ satisfying $0 < \int_0^1 H(x)\,\mathrm{d}x < 1$. Then Eq. (5.1) has a unique non-negative solution in Ł $[0,1]$.
Proof. Let $\mathcal{A} =$ Ł $[0,1]$ and the norm be defined by $\|f\|_1 = \int_0^1 \left|f(t)\right|\mathrm{d}t$ for $f \in \mathcal{A}$. The multiplication in $\mathcal{A}$ defined as the pointwise multiplication of functions. Then $\mathcal{A}$ is a real Banach algebra with unit element $e = e(t) = 1$ for a.e. $t \in [0,1]$. The cone is denoted by

$$U = \{f \in \text{Ł}\,[0,1] \mid f = f(t) \geq 0, \text{for a.e.}\, t \in [0,1]\}.$$

Then U is normal but non-solid. Let $M =$ Ł $[0,1]$. Define $\delta : M \times M \to \mathcal{A}$ by

$$\delta(f,g)(t) = \left(\int_0^1 |f(t) + g(t)|\,\mathrm{d}t\right) \cdot \varphi(t),$$

where $\varphi(t) \in U \backslash \{\theta\}$ with $\eta(f,g) = 1 + \max\{|f|, |g|\}$ for all $f, g \in M$. Then $(M, \mathcal{A}, \delta)$ is a normal and nonsolid ECb-MLS over BA. Consider the nonlinear mapping $\tau : M \to M$ defined by

$$\tau f(t) = \int_0^1 F(t, f(s))\,\mathrm{d}s.$$

Similar to Example 5.1, we can verify that condition (3.11) is satisfied. According to (5.2) and the fact that $\|fg\|_1 \leq \|f\|_1\|g\|_1$, one get

$$\begin{aligned}
\delta(\tau f, \tau g)(t) &= \left(\int_0^1 |\tau f(t) + \tau g(t)|\,\mathrm{d}t\right)\varphi(t) \\
&= \left(\int_0^1 \left|\int_0^1 (F(t,f(s)) + F(t,g(s)))\mathrm{d}s\right| \mathrm{d}t\right)\varphi(t) \\
&\leq \left(\int_0^1 \left(\int_0^1 |F(t,f(s)) + F(t,g(s))|\,\mathrm{d}s\right)\mathrm{d}t\right)\varphi(t) \\
&\leq \left(\int_0^1 \left(\int_0^1 H(t)\left|f(s) + g(s)\right|\mathrm{d}s\right)\mathrm{d}t\right)\varphi(t) \\
&\leq \left(\int_0^1 H(t)\mathrm{d}t\right) \cdot \left(\int_0^1 \left|f(s) + g(s)\right|\mathrm{d}s\right)\varphi(t) \\
&= a\left(\int_0^1 \left|f(s) + g(s)\right|\mathrm{d}s\right)\varphi(t) \\
&= a\delta(f,g),
\end{aligned}$$

where $a = \int_0^1 H(t)\,\mathrm{d}t \in (0,1)$ satisfying the spectral radius $\rho_{\mathcal{A}}(a) < 1$. In fact, $0 < a < 1$ implies $\rho_{\mathcal{A}}(a) = \lim_{n\to+\infty} \|a^n\|^{\frac{1}{n}} < 1$. By Theorem 3.5, we see all the condition of Theorem 3.5 are satisfied. Therefore, it follows from Theorem 3.5 that the Eq (5.1) has a unique non-negative solution in Ł $[0,1]$.
Remark 5.2. The results presented in this paper extend and generalize the corresponding results of Reich-type and Kannan-type contractions given in [13,18,19,25,26,32]. Our results are established

within the framework of non-solid ECb-MLSs and ECb-MSs over BAs, which are not equivalent to any metric spaces, even when the cone is normal, this distinction arises because all existing methods for proving such equivalence fundamentally depend on the solidness of cones, thereby rendering both these approaches and their associated techniques inapplicable in our case.

Remark 5.3. The key results of this paper enrich the literature on fixed point theory in metric and abstract metric spaces, complementing previous work such as [31–35].

6. Conclusions

This paper addresses the existence and uniqueness of Reich-type and Kannan-type (almost) contractions in non-solid ECb-MLSs and ECb-MSs over BAs, together with the convergence of the associated iterated sequences. By introducing new notions of Cauchy sequence, convergent sequence, and completeness with respect to normality in these abstract spaces, we establish several fixed point theorems for such contractions with vector-valued coefficients. The main novelty lies in deducing new fixed-point results without any orbital continuity in non-solid abstract spaces by new concepts. Another novelty is the weakening of completeness to orbital completeness. In the illustrative applications, we still obtain existence, uniqueness, and iterative convergence of fixed points for these contractions, even without continuity of mappings or completeness of spaces.

Future research is expected to focus the fixed points of Hardy-Rogers contractions and Ćirić quasi-contractions in these abstract spaces, together with related applications to nonlinear integral functions and neural networks.

Author contributions

Conceptualization, Y. H., J. C., S. X. and X. X.; methodology, Y. H., J. C. and S. X.; validation, Y. H., J. C., S. X. and X. X.; formal analysis, Y. H., J. C., S. X. and X. X.; investigation, Y. H. and S. X.; writing-original draft preparation, Y. H. and S. X.; writing-review and editing, Y. H., J. C. and S. X.; supervision, S. X; project administration, Y. H., S. X. and X. X.; funding acquisition, Y. H., S. X. and X. X. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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