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Research article

A characterization theorem for alternating groups

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Abstract: In this paper, we weakened the relevant conditions of order components, and only utilized the even-order components and the orders of centralizers to investigate their impact on the group structure. We demonstrated that alternating groups with disconnected prime graphs can be uniquely determined by the even-order component $m_1(G)$ and the set $\pi_{p_m}(G)$. Here, G represented a finite group, $\pi(G)$ was the set of prime factors of the order of G, p_m was the largest element in $\pi(G)$, and $\pi_{p_m}(G) = \{|C_G(x)| \| x \in G \text{ and } |x| = p_m\}$ denoted the set of orders of centralizers of p_m-order elements in G.

Keywords: finite groups; simple groups; order components; centralizers
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1. Introduction

All groups considered in this paper are finite groups.

The classification theorem of finite simple groups was one of the most significant mathematical achievements in the field of algebra. The lengthy proof process, extensive involvement of participants, multitude of articles, and sheer volume are unprecedented in mathematical history, representing a monumental milestone. Subsequently, the quantitative characterization of finite simple groups became a hot research topic. In recent years, the McKay conjecture and Alperin conjecture in group representation theory have been focused on simple groups and quasi-simple groups (see [1]). Equally notable is the application of certain characterization theorems in the emerging field of generative artificial intelligence. Further understanding and identifying properties of simple groups is imperative.

The origin of the quantitative characterization of finite simple groups traces back to a correspondence between the distinguished group theorist Professor W. J. Shi and Fields Medalist Professor J. G. Thompson around 40 years ago. They proposed the following two conjectures, respectively:

Two-order Conjecture: If G is a finite group, and M is a finite simple group such that $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, where $\pi_e(G)$ denotes the set of element orders in G, then $G \cong M$ (see [2]).

Thompson Conjecture: If G is a finite group with a trivial centralizer, and M is a finite simple group such that $N(G) = N(M)$, where $N(G) = \{n \in N \,|G$ has a conjugacy class of length $n\}$, then $G \cong M$ (see [3]).

Over the past 40 years, group theorists have continuously pushed the boundaries of research in this field, proposing an increasing number of quantitative characterization methods. Examples include: the spectrum characterization of finite simple groups [4]; the order equations of finite simple groups [5]; the characterization of finite simple groups through the width of orders [6–8]; the characterization based on conjugacy class sizes [9–12]; and the OD-characterization of finite simple groups [13, 14]. It is worth mentioning that in 2009, Russian mathematician Fields Prize winner V. D. Mazurov and others finally proved "Two-order conjecture" based on the work of Professor W. J. Shi and his predecessors. Subsequently, the conjecture was transformed into a theorem (see [15]).

In a series of papers addressing the aforementioned Thompson Conjecture and Two-order Conjecture, the concept of the prime graph of a group, denoted as $\Gamma(G)$, played a crucial role. This concept was first introduced by K. W. Gruenberg and O. Kegel in [16]: Let G be a finite group, the vertex set of G is the set of all prime factors of $|G|$, and two vertices p and q are adjacent if and only if G contains an element of order pq . The number of connected components in $\Gamma(G)$ is denoted by $t(G)$, and the set of connected components in the prime graph of G is denoted by $T(G) = \{\pi_i | i = 1, 2, \dots, t(G)\}$. When G is an even-order group, it is stipulated that $2 \in \pi_1$. The prime graph components of all finite simple groups were provided by A. S. Kondrat'ev in [17].

In the 1990s, Professor G. Y. Chen introduced the concept of order components based on prime graph components (see [18]). Let G be a finite group, $\pi_1, \pi_2, \dots, \pi_{t(G)}$ are all the connected components of $\Gamma(G)$, then $|G| = m_1 m_2 \cdots m_{t(G)}$, where the prime factor set of m_i is π_i for $i = 1, 2, \dots, t(G)$. The numbers $m_1, m_2, \dots, m_{t(G)}$ are called the order components of G, and $OC(G) = \{m_1, m_2, \dots, m_{t(G)}\}$ is the set of order components of G (see [18]). For convenience, we denote the even-order component of G as $m_1(G)$. Professor G. Y. Chen gave the order components of all simple groups whose prime graph is disconnected (see [18] Tables 1–4). Naturally, another question arises: Can order components characterize all simple groups?

Regarding this question, group theorists have conducted extensive research. For example, G. Y. Chen proved it holds for Suzuki-Ree groups [19], $PSL_2(q)$ [20], and ${}^3D_4(q)$ [21]. A. Khosravi and B. Khosravi proved it holds for $PSL(p, q)$ [22] and $PSU(p+1, q)$ [23]. B. Khosravi and Bahman Khosravi demonstrated it holds for $E_6(q)$ [24] and ${}^2E_6(q)$ [25]. H. G. Shi proved it holds for $D_p(3) (p \geq 5)$ [26]. A. Iranmanesh and S. Alavi. demonstrated it holds for $PSL(5, q)$ [27], $PSL(3, q)$ [28, 29].

From this, we can see that the order components are very effective tools in characterizing finite simple groups. However, it's important to note that the complexity of order component information varies among different groups, and in complex cases, they can become difficult to handle. Fortunately, among the order components, even-order components are somewhat special. According to the Feit-Thompson theorem [30], all simple groups are of even order. This implies that all simple groups have an even-order component $m_1(G)$. When using order components to characterize finite simple groups, focusing exclusively on the even-order components and disregarding other components can significantly simplify the problem, which is a key motivation for this paper.

Another motivation is the centralizer of a group. The centralizer is often used to describe the symmetrical structure within the group. It consists of the conjugacy class of the group's commutative elements and provides valuable information about the group's structure. For example, the Brauer-Fowler theorem [31], proved in 1955, shows that studying finite groups can be done through their centralizers. Subsequently, Professor Brauer formally proposed using the centralizers of involutions to characterize finite groups at the International Congress of Mathematicians in 1962. Fifty years later, in 2012, Professor L. G. He proposed ONC-characterization (see [32]) and utilized centralizers of the highest order elements to characterize finite simple groups in his doctoral thesis. In 2022, Ms. Z. B. Wang studied simple groups with disconnected prime graphs (see [33, 34]) in her doctoral research.

Based on these two motivations, this paper aims to introduce a novel method for characterizing simple groups by integrating centralizers, order components, and disconnected prime graphs. We weakened the relevant conditions of order components, and only utilized a single order component, along with the orders of centralizers, to investigate their impact on the group structure. We prove that alternating groups with disconnected prime graphs can be precisely characterized by even-order component $m_1(G)$ and $\pi_{p_m}(G)$, where $p_m = max\{\pi(G)\}$ and $\pi_{p_m}(G) =\{|C_G(x)| \,|\, x \in G$ and $|x| = p_m\}$.

Main theorem. *Let G be a finite group, and A be an alternating group with disconnected prime graphs. Then, $G \cong A$ if and only if:*

(a) $m_1(G) = m_1(A)$;

(b) $\pi_{p_m}(G) = \pi_{p_m}(A)$.

2. Preliminaries

The symbols used in this paper are standard and can be referenced in [35]. The main symbols are listed in Table 1.

Table 1. Table of main symbols.

Notation	Meaning
$\|G\|$	The order of the group G
$Aut(G)$	The automorphism group of the group G
$Out(G)$	The outer automorphism group of the group G
$\pi(G)$	The set of prime factors of the order of G
$\|\pi(G)\|$	The number of prime factors of $\|G\|$
p_m	The largest element of $\pi(G)$
$\pi_{p_m}(G)$	The set of orders of centralizers of p_m-order elements in G
G_{p_i}	The sylow p_i-subgroup of G
$t(G)$	The number of connected components of G
$T(G)$	The set of connected components of G
$m_1(G)$	The even-order component of G

Lemma 2.1. *[16, Corollary] Let G be a finite group with $t(G) \geq 2$. Then, the structure of G is as follows:*

(a) G is a Frobenius group;

(b) G is a 2-Frobenius group;

(c) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where H is a nilpotent π_1-group, G/K is a solvable

π_1*-group, K/H is a non-Abelian simple group, and* $|G/K|$ *divides* $|\mathrm{Out}(K/H)|$.

Lemma 2.2. *[36, Theorem 1] Let G be an even-order Frobenius group with Frobenius kernel H and Frobenius complement K. Then,* $t(G) = 2$ *and* $T(G) = \{\pi(H), \pi(K)\}$ *Moreover, the structure of G is one of the following:*

1) If $2 \in \pi(H)$*, then the Sylow subgroups of K are cyclic;*

2) If $2 \in \pi(K)$*, then H is an Abelian group. When K is soluble, the odd-order Sylow subgroups of K are cyclic and the Sylow 2-subgroup is either a cyclic group or a generalized quaternion group. When K is insoluble, there exists* $K_0 \leq K$ *such that* $|K : K_0| \leq 2$ *and* $K_0 \simeq Z \times SL(2, 5)$*, where* $(|Z|, 30) = 1$ *and the Sylow subgroups of Z are cyclic.*

Lemma 2.3. *[37, Lemma 8.3, §IV] Let* $G = HK$ *be a Frobenius group, where H is the Frobenius kernel and K is the Frobenius complement. Then, we have* $|K| \big| |H| - 1$.

Lemma 2.4. *[36, Theorem 2] Let G be an even-order 2-Frobenius group. Then,* $t(G) = 2$*, and G has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$*, such that* $\pi(K/H) = \pi_2$*,* $\pi(H) \cup \pi(G/K) = \pi_1$*,* $|G/K|$ *divides* $|\mathrm{Aut}(K/H)|$*, and both* $|G/K|$ *and* $|K/H|$ *are cyclic groups. In particular,* $|G/K| \leq |K/H|$ *and G is soluble.*

Lemma 2.5. *[38, Lemma 10] Let p be a prime number,* $p > 13$*. Then, there exist two prime numbers,* r_1 *and* r_2*, such that* $\frac{(p-1)}{2} < r_1 < r_2 < p - 1$.

Lemma 2.6. *[39, Lemma 8] Let G be a finite group with* $t(G) \geq 2$*, and let N be a normal subgroup of G. If N is the* π_i*-group of some prime graph component, and* $m_1, m_2, \cdots, m_r$ *are some order components of G that are not* π_i*-numbers, then* $m_1 m_2 \cdots m_r \big| |N| - 1$.

The following lemmas have been proven in [40, Lemma 6], but there are some differences here. We will use the θ function to classify *Lie*-type groups with prime order components.

Lemma 2.7. *If L is a simple group of Lie-type and has prime odd order component* $p \geq 17$ *and* $\pi(L)$ *has at most* $\theta(L)$ *prime numbers t, where* $\frac{(p+1)}{2} < t < p$*. Then,* $\theta(L) \leq 3$.

Throughout the proof of [40, Lemma 6], we can divide simple groups of *Lie*-type L, with prime odd order component $p \geq 17$, into the following cases:

(1)
- $\theta(L) = 0$ if L is isomorphic to $A_{p'-1}(q), A_{p'}(q)$, where $q - 1 \big| p' + 1$;
- $A_2(2)$, ${}^2A_{p'-1}(q)$, ${}^2A_{p'}(q)$, where $q + 1 \big| p' + 1$;
- ${}^2A_3(2), B_n(q)$, where $n = 2^{m'}$ and q is odd;
- $B_{p'}(3), C_n(q)$, where $n = 2^{m'}$ or $(n, q) = (p', 3)$;
- $D_{p'+1}(3), D_{p'}(q)$ for $q = 3, 5$;
- ${}^2D_n(q)$ for $(n, q) = (2^{m'}, q), (p', 3)$, where $5 \leq p' \neq 2^{m'} + 1$ or $(2^{m'} + 1, 3)$;
- $G_2(q)$, where $q \equiv \varepsilon(mod\ 3)$ for $\varepsilon = \pm 1$;
- ${}^3D_4(q), E_6(q)$ or ${}^2E_6(q)$.

(2)
- $\theta(L) = 1$ if L is isomorphic to one of the simple groups $A_1(q)$, where $2 \big| q$;
- $A_2(4)$, ${}^2A_5(2), C_{p'}(2), D_n(2)$, where $n = p'$ or $p' + 1$;
- ${}^2D_n(2)$, where $(n, q) = (2^{m'} + 1, 2)$ or $(p' = 2^{m'} + 1, 3), m' \geq 2$;

- $E_7(3)$, $F_4(q)$, ${}^2F_4(q)$, where $q = 2^{2n+1} > 2$, or $C_2(q)$, where$3|q$.

(3) • $\theta(L) = 2$ if L is isomorphic to one of the simple groups $A_1(q)$, where $q \equiv \varepsilon(mod\ 4)$, for $\varepsilon = \pm 1$;

- ${}^2B_2(q)$, where $q = 2^{2m'} + 1 > 2$;
- ${}^2G_2(q)$, where $q = 3^{2m'+1} > 3$.

(4) • $\theta(L) = 3$ if L is isomorphic to one of the simple groups $E_8(q)$ or ${}^2E_6(2)$.

Lemma 2.8. *[40, Lemma 1] If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i, such that $\frac{(n+1)}{2} < p_i < n$. Here,*

$s(n) = 6$ *for* $n \geq 48$*;*
$s(n) = 5$ *for* $42 \leq n \leq 47$*;*
$s(n) = 4$ *for* $38 \leq n \leq 41$*;*
$s(n) = 3$ *for* $18 \leq n \leq 37$*;*
$s(n) = 2$ *for* $14 \leq n \leq 17$*;*
$s(n) = 1$ *for* $6 \leq n \leq 13$.

Lemma 2.9. *When p is a prime number greater than or equal to 5 and n is a positive integer, it holds that $\sum_{i=1}^{n} p^{n-i} \neq 2n$.*

Proof. Let $S_1 = \sum_{i=1}^{n} p^{n-i} = p^{n-1} + p^{n-2} + p^{n-3} + \cdots + p + 1$. From the formula for the sum of a geometric series, we know that $S_1 = \frac{p^n-1}{p-1}$. We will prove this conclusion in different cases.

Case 1: When $n = 1$, $S_1 = \frac{p^n-1}{p-1} = 1 < 2n$.

Case 2: When $n = 2$, $p = 5$, $S_1 = \frac{p^n-1}{p-1} = 3n$.

Case 3: When $n \neq 1$ or $n = 2, p > 5$, we prove that $\frac{p^n-1}{p-1} > 3n$. We will use mathematical induction for n. When $n = 2$, $\frac{p^2-1}{p-1} = p + 1$, and since $p > 5$, it follows that $\frac{p^2-1}{p-1} = p + 1 > 6 > 3n$. Assume $\frac{p^n-1}{p-1} = p^{n-1} + p^{n-2} + p^{n-3} + \cdots + p + 1 > 3n$, then $\frac{p^{n+1}-1}{p-1} = p^n + p^{n-1} + \cdots + p^2 + p + 1 > p^n + 3n = p^n + 3(n+1) - 3 = 3(n+1) + p^n - 3$. Since p is a prime greater than 5, it is clear that $p^n - 3 > 0$, thus, $\frac{p^{n+1}-1}{p-1} > 3(n+1)$, which proves the result. □

Lemma 2.10. *When $p = 3$ and $n = 2m + 1$, it holds that $\sum_{i=1}^{n} p^{n-i} \pm \sqrt{3} \sum_{j=1}^{\frac{n}{2}} p^{\frac{n}{2}-j}$ is larger than $4n$.*

Proof. Let $S_1 = \sum_{i=1}^{n} p^{n-i}$, $S_2 = \sum_{j=1}^{\frac{n}{2}} p^{\frac{n}{2}-j}$, then the original expression equals $S_1 \pm \sqrt{3}S_2$. By the formula for the sum of a geometric series, we can obtain the following:

$$S_1 = \sum_{i=1}^{n} p^{n-i} = \frac{p^n - 1}{p - 1}, S_2 = \sum_{j=1}^{\frac{n}{2}} p^{\frac{n}{2}-j} = \frac{p^{\frac{n}{2}} - 1}{p - 1}.$$

At this time, let

$$\begin{aligned} S(n) &= S_1 - \sqrt{3}S_2 \\ &= \frac{p^n - 1}{p - 1} - \sqrt{3}\frac{p^{\frac{n}{2}} - 1}{p - 1} \\ &= \frac{3^n - 1}{2} - \sqrt{3}\frac{3^{\frac{n}{2}} - 1}{2}. \end{aligned}$$

If $S(n) > 4n$ holds for all values of n, then the minimum value of $S(n)$ is greater than $4n$, that is, $S(n) - 4n > 0$. Let $3^{\frac{n}{2}} = x$, then $n = 2\log_3(x)$. Therefore, $S(n) - 4n = \frac{x^2 - \sqrt{3}x + \sqrt{3} - 1}{2} - 8\log_3(x)$. Let $g(x) = \frac{x^2 - \sqrt{3}x + \sqrt{3} - 1}{2} - 8\log_3(x)$. If $g(x) > 0$, then $S(n) - 4n > 0$ holds for all values of n, which implies that $S(n) > 4n$ holds for all values of n. Taking the derivative of the function $g(x)$, we get $g'(x) = x - \sqrt{3} - \frac{8}{x\ln 3} = \frac{\ln 3\cdot x^2 - \sqrt{3}\cdot\ln 3\cdot x - 8}{x\ln 3}$. Let $g'(x) = x - \sqrt{3} - \frac{8}{x\ln 3} = \frac{\ln 3\cdot x^2 - \sqrt{3}\cdot\ln 3\cdot x - 8}{x\ln 3} = 0$, where $x > 1$. Here, $\ln 3\cdot x^2 - \sqrt{3}\cdot\ln 3\cdot x - 8$ is a quadratic function with $\triangle < 0$ and $a > 0$, so $\ln 3\cdot x^2 - \sqrt{3}\cdot\ln 3\cdot x - 8 > 0$, which implies that $g'(x) > 0$ holds for all values of x. At this point, $g(x)$ is a strictly increasing function. Since $g(x) > g(1) = 0$, we can conclude that $g(x) > 0 \Leftrightarrow S(n) > 4n$. Since $S_1 - \sqrt{3}S_2 > 4n$, it follows that $S_1 + \sqrt{3}S_2 > 4n$, thus proving that $\sum_{i=1}^{n} p^{n-i} \pm \sqrt{3}\sum_{j=1}^{\frac{n}{2}} p^{\frac{n}{2}-j} > 4n$. □

Lemma 2.11. *When p is a prime number and n is a positive integer, it holds that* $\sum_{i=1}^{8n} p^{8n-i} + \sum_{i=1}^{7n} p^{7n-i} - \sum_{i=1}^{5n} p^{5n-i} - \sum_{i=1}^{4n} p^{4n-i} - \sum_{i=1}^{3n} p^{3n-i} + \sum_{i=1}^{n} p^{n-i}$ *is larger than* $121n$.

Proof. Let $\alpha = \sum_{i=1}^{8n} p^{8n-i} + \sum_{i=1}^{7n} p^{7n-i} - \sum_{i=1}^{5n} p^{5n-i} - \sum_{i=1}^{4n} p^{4n-i} - \sum_{i=1}^{3n} p^{3n-i} + \sum_{i=1}^{n} p^{n-i}$. By using the formula for the sum of a geometric series, we can obtain $\alpha = \frac{1}{p-1}(p^{8n} + p^{7n} - p^{5n} - p^{4n} - p^{3n} + p^n)$. To prove that $\alpha > 121n$, it suffices to prove that $\alpha - 121n > 0$. Let us define $f(n) = \frac{1}{p-1}(p^{8n} + p^{7n} - p^{5n} - p^{4n} - p^{3n} + p^n) - 121n$. Taking the derivative of the function $f(n)$, we get $f'(n) = \frac{\ln p}{p-1}(8p^{8n} + 7p^{7n} - 5p^{5n} - 4p^{4n} - 3p^{3n} + p^n) - 121 > \frac{\ln p}{p-1}(8p^{8n} - 5p^{7n}) - 121 = \frac{p^n}{p-1}\cdot(p^n \ln p)(8p^{6n} - 5p^{5n}) - 121 > 8p^{6n} - 5p^{5n} - 121$. Using mathematical induction, when $n = 1$, $8p^6 - 5p^5 - 121 > 0$ holds; assume that the conclusion holds for $n = k$, i.e., $8p^{6k} - 5p^{5k} - 121 > 0$; when $n = k+1$,with $8p^{6(k+1)} - 5p^{5(k+1)} - 121 = 8\cdot p^6\cdot p^{6k} - 5\cdot p^5\cdot p^{5k} - 121 > p^6(8p^{6k} - 5p^{5k}) - 121 > 8p^{6k} - 5p^{5k} - 121 > 0$, the conclusion holds, i.e., $f'(n) > 0$, and the function $f(n)$ is monotonically increasing. Since p is at least 2, and it is easy to verify that when $p \geq 2$, the function is monotonically increasing with respect to p. When $n = 1$ and $p = 2$, $f(1) = 210$, that is, $f(n) > f(1) > 0$. Hence, it is proven that $\alpha - 121n > 0$. □

Definition 2.12. *If G is a finite group with a subgroup H such that* $H \cap H^x$ *for all x in* $G \setminus H$*, then G is called a Frobenius group. Let* $N = G \setminus \cup_{x\in G}(H \setminus 1)^x$*. Then,* $G = HN$*, and* $H \cap N = 1$*. H is called a Frobenius complememt and N the Frobenius kernel.*

Definition 2.13. *A group G is called a 2-Frobenius group if G has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ *such that K is a Frobenius group with Frobenius kernel H and G/H is also a Frobenius group with kernel K/H (see [36]).*

3. Proof of main theorem

According to the literature [18], it is known that the cases of disconnected prime graphs in alternating simple groups can be divided into two types:

(a) Alternating group A_p, where p or $p-2$ are prime numbers;

(b) Alternating group A_n, where $n = p, p+1, p+2$, with p being a prime number, and either n nor $n-2$ are prime numbers.

The even-order component $m_1(A_p)$ of the former is given by $\frac{(p-1)(p-3)!}{2}$, while the even-order component $m_1(A_n)$ of the latter is given by $\frac{n!}{2p}$. The orders of the centralizers of their highest prime factors are both equal to p, that is, $\pi_{p_m}(A_p) = \pi_{p_m}(A_n) = \{p\}$, so p is isolated in $\Gamma(G)$.

In this paper, we only discuss the cases of alternating groups A_n for $n \geq 17$. For the case where $n < 17$, refer to [41].

Theorem 3.1. *Let G be a finite group and A be an alternating group with disconnected prime graphs. Then, $G \cong A$ if and only if:*

(a) $m_1(G) = m_1(A)$;

(b) $\pi_{p_m}(G) = \pi_{p_m}(A)$.

Proof. The necessity of the theorem is obvious, so we only need to prove the sufficiency.

When A is an alternating simple group with disconnected prime graph, according to the condition, we know that $m_1(G) = m_1(A)$, so $m_1(G) = \frac{(p-1)(p-3)!}{2}$ or $m_1(G) = \frac{n!}{2p}$, and $\pi_{p_m}(G) = \pi_{p_m}(A) = \{p\}$. Since $m_1(G) = \frac{(p-1)(p-3)!}{2}$ or $m_1(G) = \frac{n!}{2p}$, we conclude that G has more than one component, i.e., $t(G) \geq 2$. Therefore, by Lemma 2.1, we can determine the structure of G. Next, we prove that the structure of G does not satisfy Lemma 2.1 (a) and (b).

If G is a Frobenius group, by Lemma 2.2, we know that $G = HK$, with the prime graph $T(G) = \{\pi(H), \pi(K)\}$, where H is the Frobenius kernel and K is the Frobenius complement. By Lemma 2.3, we know that $|K|\big||H| - 1$, so $|K| < |H|$; considering the given conditions, we have 2 divides $|H|$, i.e., $2 \in \pi(H)$. We immediately obtain $\pi(H) = \pi_1$ and $|K| = p$. Let r be a prime number such that $\frac{(p-1)}{2} < r < p - 1$. By Lemma 2.5, such an r always exists and $r \in \pi(H)$. Let G_r be the Sylow r-subgroup of G. Since H is nilpotent, we have $G_r \trianglelefteq G$. According to Lemma 2.6, $p\big|r - 1$, implying $p < r$, which leads to a contradiction.

If G is a 2-Frobenius group, by Lemma 2.4, we have $t(G) = 2$, and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$ and $\pi(H) \cup \pi(G/K) = \pi_1$. Since $m_1(G) = \frac{(p-1)(p-3)!}{2}$ or $m_1(G) = \frac{n!}{2p}$, it follows that $p \in \pi_2$. Therefore, $|K/H| = p$ and $|G/K|\big||\mathrm{Aut}(K/H)| = p - 1$. Let r be a prime number such that $\frac{(p-1)}{2} < r < p - 1$, and let G_r be a Sylow r-subgroup of G. Since $r \nmid p - 1$, we have $r \in \pi(K)$, implying that $G_r \leq H$ and $G_r \trianglelefteq G$. If we consider the action of p-order elements of K on the Sylow r-subgroup of H, we obtain $p\big|r - 1$, which is a contradiction.

Thus, the structure of G can only be determined by Lemma 2.1(c). Since $m_1(G) = \frac{(p-1)(p-3)!}{2}$ or $m_1(G) = \frac{n!}{2p}$, it follows that $t(G) \geq 2$, hence, $p \in \pi(K/H)$. If H is nontrivial, then from the proof in the previous paragraph, we know that there exists a prime number r such that $\frac{(p-1)}{2} < r < p - 1$. Let G_r be a Sylow r-subgroup of G such that $G_r \leq H$ and $G_r \trianglelefteq G$. If we consider the action of p-order elements of K on the Sylow r-subgroup of H, we obtain $p\big|r - 1$, which leads to a contradiction. Therefore, we have $H = 1$.

Since $H = 1$, K is a normal non-Abelian simple subgroup of G, such that $\pi(G/K) \subseteq \pi_1$, where $\pi_1 = \pi(\frac{(p-1)(p-3)!}{2})$ or $\pi_1 = \pi(\frac{n!}{2p})$, and $p \in \pi(K)$. By utilizing $\pi_{p_m}(G) = \{p\}$, we can conclude that $t(K) \geq 2$.

All simple groups with $t(G) \geq 2$ can be found in the articles by J. S. Williams [16] and A. S. Kondrat'ev [17], and their order components can be found in Tables 1–4 of [18]. Their outer automorphism groups can be referenced in the literature [42]. For a finite group G, if $H \leq G$, it is evident that $m_1(H)\big|m_1(G)$, which will be used subsequently without further notice. □

We will prove in two steps that, except for the cases of alternating groups with disconnected prime graphs, K is not isomorphic to any other simple group.

Step 1. K cannot be isomorphic to any of the 26 sporadic simple groups.

In this case, we only consider the situation where $m_1(G) = \frac{(p-1)(p-3)!}{2}$, and the situation for $m_1(G) = \frac{n!}{2p}$ is similar. Their even-order components $m_1(G)$ and $\pi_{p_m}(G)$ can be found in Table 2.

Table 2. The $m_1(G)$ and $\pi_{p_m}(G)$ for 26 sporadic simple groups.

G	$m_1(G)$	$\pi_{p_m}(G)$
M_{11}	$2^4 \cdot 3^2$	11
M_{12}	$2^6 \cdot 3^3 \cdot 5$	11
M_{22}	$2^7 \cdot 3^2$	11
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	23
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7$	23
J_1	$2^3 \cdot 3 \cdot 5$	19
J_2	$2^7 \cdot 3^3 \cdot 5^2$	7
J_3	$2^7 \cdot 3^5 \cdot 5$	19
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3$	43
HS	$2^9 \cdot 3^2 \cdot 5^3$	11
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$	29
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7$	13
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3$	17
ON	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3$	31
McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7$	11
Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11$	67
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13$	23
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7$	23
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11$	23
F_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11$	13
F_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	23
F'_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13$	29
M	$2^{46} \cdot 3^{30} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 47$	71
B	$2^{41} \cdot 3^{31} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	47
Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	31
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11$	19

If $K \cong J_4$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $p = 43$. In this case, we have $m_1(G) = 2^{38} \cdot 3^{20} \cdot 5^9 \cdot 7^5 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37$ and $m_1(K) = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3$. Therefore, $2^{17} \cdot 3^{17} \cdot 5^8 \cdot 7^4 \,\big|\, |G/K| |Out(J_4)| = 1$, leading to a contradiction. Hence, $K \ncong J_4$.

If $K \cong M_{23}$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $p = 23$. In this case, we have $m_1(G) = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19$ and $m_1(K) = 2^7 \cdot 3^2 \cdot 5 \cdot 7$. Therefore, $2^{11} \cdot 3^6 \cdot 5^3 \cdot 7 \,\big|\, |G/K| |Out(M_{23})| = 1$, leading to a contradiction. Hence, $K \ncong M_{23}$. Similarly, we can conclude that $K \ncong M_{24}, F_{23}, Co_1, Co_2, Co_3$.

If $K \cong J_1$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $p = 19$. In this case, we have $m_1(G) = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ and $m_1(K) = 2^3 \cdot 3 \cdot 5$. Therefore, $2^{15} \cdot 3^5 \cdot 5^2 \,\big|\, |G/K| |Out(J_1)| = 1$, leading to a contradiction. Hence, $K \ncong J_1$. Similarly, we can conclude that $K \ncong J_3, HN$.

If $K \cong Ru$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $p = 29$. In this case, we have $m_1(G) = 2^{25} \cdot 3^{10} \cdot 5^6 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$ and $m_1(K) = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$. Therefore, $2^{11} \cdot 3^7 \cdot 5^3 \cdot 7^3 \cdot 13 \,\big|\, |G/K| |Out(Ru)| = 1$, leading to a contradiction. Hence, $K \ncong Ru$. Similarly, we can conclude that $K \ncong F'_{24}$.

If $K \cong He$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $p = 17$. In this case, we have $m_1(G) = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$ and $m_1(K) = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7$. Therefore, $2^4 \cdot 3^2 \cdot 7 \,\big|\, |G/K| \big| |Out(He)| = 1$, leading to a contradiction. Hence, $K \ncong He$.

If $K \cong ON$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $p = 31$. In this case, we have $m_1(G) = 2^{25} \cdot 3^{14} \cdot 5^6 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$ and $m_1(K) = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3$. Therefore, $2^{16} \cdot 3^{10} \cdot 5^5 \cdot 7 \,\big|\, |G/K| \big| |Out(ON)| = 1$, leading to a contradiction. Hence, $K \ncong ON$.

If $K \cong Ly$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $p = 67$. In this case, we have $m_1(G) = 2^{63} \cdot 3^{32} \cdot 5^{14} \cdot 7^{10} \cdot 11^6 \cdot 13^4 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61$ and $m_1(K) = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3$. Therefore, $2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \,\big|\, |G/K| \big| |Out(Ly)| = 1$, leading to a contradiction. Hence, $K \ncong Ly$.

If $K \cong M$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $p = 71$. In this case, we have $m_1(G) = 2^{66} \cdot 3^{34} \cdot 5^{16} \cdot 7^{10} \cdot 11^6 \cdot 13^4 \cdot 17^4 \cdot 19^3 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67$ and $m_1(K) = 2^{46} \cdot 3^{30} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 47$. Therefore, $2^{20} \cdot 3^4 \cdot 5^7 \cdot 7^4 \cdot 11^4 \cdot 13^2 \cdot 17^3 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \,\big|\, |G/K| \big| |Out(M)| = 1$, leading to a contradiction. Hence, $K \ncong M$.

If $K \cong B$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $p = 47$. In this case, we have $m_1(G) = 2^{41} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ and $m_1(K) = 2^{41} \cdot 3^{31} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$. This leads to a contradiction, Hence, $K \ncong B$.

Step 2. K is not isomorphic to any *Lie*-type simple groups.

Since $m_1(G) = \frac{(p-1)(p-3)!}{2}$ or $m_1(G) = \frac{n!}{2p}$, and $\pi_{p_m}(G) = \{p\}$ with $p \in \pi(K)$, we obtain $\pi_{p_m}(K) = \{p\}$, $p \geq 17$. Therefore, K has prime odd order components. According to Lemma 2.8, we know that $\theta(K) \geq 2$. Hence, by Lemma 2.7, K is isomorphic to one of the following *Lie*-type simple groups:

(a) $A_1(q)$, where $q \equiv \varepsilon(mod\ 4)$, for $\varepsilon = \pm 1$;

(b) ${}^2B_2(q)$, where $q = 2^{2m'} + 1 > 2$;

(c) ${}^2G_2(q)$, where $q = 3^{2m'+1} > 3$;

(d) $E_8(q)$ or ${}^2E_6(2)$.

If $K \cong A_1(q)$, $q \equiv \varepsilon(mod\ 4), \varepsilon = \pm 1$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $\pi_{p_m}(K) = \{p\}$, thus, $p = q$. In this case, $m_1(K) = p \pm 1, m_1(G) = \frac{(p-1)(p-3)!}{2}$, or $m_1(G) = \frac{n!}{2p}$, so we can conclude that $\frac{m_1(G)}{m_1(K)} = \frac{(p-3)!}{2}$ or $\frac{(p-1)(p-3)!}{2(p+1)}$ or $\frac{n!}{2p(p-1)}$ or $\frac{n!}{2p(p+1)}$. Since $n = p, p+1, p+2$, and $p \geq 17$, we have $\frac{m_1(G)}{m_1(K)} \neq 2$. However, $\frac{m_1(G)}{m_1(K)} \,\big|\, |G/K| \big| |Out(A_1(q))| = (2, p-1) = 2$, which is a contradiction. Therefore, $K \ncong A_1(q), q \equiv \varepsilon(mod\ 4), \varepsilon = \pm 1$.

If $K \cong {}^2E_6(2)$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $p = 19$. Taking $m_1(G) = \frac{(p-1)(p-3)!}{2}$, in this case, we have $m_1(G) = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ and $m_1(K) = 2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11$. This leads to a contradiction, and the situation for $m_1(G) = \frac{n!}{2p}$ is similar. Hence, $K \ncong {}^2E_6(2)$.

We know that when $p' \neq 2$, the Sylow p'-subgroups of the symmetric group S_p and the alternating group A_p have the same order. Using the formula in [43, Chapter VII, Section §4], we can determine the order of the Sylow p'-subgroup $G_{p'}$ of S_p. The order is given by

$$|G_{p'}| = |S_p|_{p'} = |p!|_{p'} = {p'}^{s(p)},$$

where

$$s(p) = \left[\frac{p}{p'}\right] + \left[\frac{p}{{p'}^2}\right] + \cdots,$$

and $[a]$ represents the integer part of a real number a. Let

$$p = a_r(p')^r + a_{r-1}(p')^{r-1} + \cdots + a_1 p' + a_0,$$

where $0 \leq a_i \leq p' - 1$ for $i = 0, 1, \cdots, r$, and $a_r \neq 0$. It can be easily verified that

$$s(p) = a_r s((p')^r) + a_{r-1} s((p')^{r-1}) + \cdots + a_1 s(p') + a_0 s(1).$$

In this way, we can determine the order of the Sylow p'-subgroup of A_p, which is also the order of the Sylow p'-subgroup of the group G, where p' is a prime number and the subsequent q represents a power of p', i.e., $q = (p')^n$.

If $K \cong {}^2B_2(q)$, where $q = 2^{2m'} + 1 > 2$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $\pi_{p_m}(K) = \{p\}$. Thus, $p = q - 1$ or $q - \sqrt{2q} + 1$ or $q + \sqrt{2q} + 1$. When $p = q - 1$, we have $s(p) = s((p')^n) - s(1)$, where

$$\begin{aligned} s((p')^n) &= \left[\frac{(p')^n}{p'}\right] + \left[\frac{(p')^n}{(p')^2}\right] + \left[\frac{(p')^n}{(p')^3}\right] + \cdots + \left[\frac{(p')^n}{(p')^{n-1}}\right] + \left[\frac{(p')^n}{(p')^n}\right] \\ &= (p')^{n-1} + (p')^{n-2} + (p')^{n-3} + \cdots + p' + 1, \end{aligned}$$

and

$$s(1) = \left[\frac{1}{p'}\right] = 0.$$

Thus, the order of the Sylow p'-subgroup of G is $(p')^{\sum_{i=1}^{n} (p')^{(n-i)}}$. Since $m_1(K) = q^2$, the order of the Sylow p'-subgroup of K is $(p')^{2n}$. Since $q = 2^{2m'} + 1 > 2$, the smallest possible value of p' is 5, then we have $\frac{|G_{p'}|}{|K_{p'}|} = (p')^{\alpha} \| |G/K| |Out({}^2B_2(q))| = n$, where $\alpha = (p')^{n-1} + (p')^{n-2} + (p')^{n-3} + \cdots + p' + 1 - 2n$. By Lemma 2.9, we know that $(p')^{n-1} + (p')^{n-2} + (p')^{n-3} + \cdots + p' + 1 \neq 2n$. In Lemma 2.9 Case 1, we have $(p')^{n-1} + (p')^{n-2} + (p')^{n-3} + \cdots + p' + 1 < 2n$, i.e., $\alpha < 0$, which is obviously contradictory. In Lemma 2.9 Case 2, we have $\alpha = (p')^{n-1} + (p')^{n-2} + (p')^{n-3} + \cdots + p' + 1 - 2n = n$ and $p' \geq 5$, so $(p')^{\alpha} \nmid n$, a contradiction. In Lemma 2.9 Case 3, we have $\alpha = (p')^{n-1} + (p')^{n-2} + (p')^{n-3} + \cdots + p' + 1 - 2n > n$. So we can conclude that $(p')^{\alpha} \nmid n$, which leads to a contradiction. Hence, $K \ncong {}^2B_2(q)$. Similarly, when $p = q \pm \sqrt{2q} + 1$, we have $s(p) = s((p')^n) \pm \sqrt{2} s((p')^{\frac{n}{2}}) + s(1)$, where

$$\begin{aligned} s((p')^{\frac{n}{2}}) &= \left[\frac{(p')^{\frac{n}{2}}}{p'}\right] + \left[\frac{(p')^{\frac{n}{2}}}{(p')^2}\right] + \left[\frac{(p')^{\frac{n}{2}}}{(p')^3}\right] + \cdots + \left[\frac{(p')^{\frac{n}{2}}}{(p')^{\frac{n}{2}-1}}\right] + \left[\frac{(p')^{\frac{n}{2}}}{(p')^{\frac{n}{2}}}\right] \\ &= (p')^{\frac{n-2}{2}} + (p')^{\frac{n-4}{2}} + (p')^{\frac{n-6}{2}} + \cdots + p' + 1. \end{aligned}$$

Thus, the order of the Sylow p'-subgroup of G is $(p')^{\sum_{i=1}^{n} (p')^{(n-i)} \pm \sqrt{2} \sum_{j=1}^{\frac{n}{2}} (p')^{(\frac{n}{2}-j)}}$, which is a contradiction. Therefore, $K \ncong {}^2B_2(q)$.

If $K \cong {}^2G_2(q)$, where $q = 3^{2m'+1} > 3$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $\pi_{p_m}(K) = \{p\}$. Thus, $p = q - \sqrt{3q} + 1$ or $q + \sqrt{3q} + 1$. When $p = q \pm \sqrt{3q} + 1$, we have $s(p) = s((p')^n) \pm \sqrt{3} s((p')^{\frac{n}{2}}) + s(1)$, where

$$\begin{aligned} s((p')^{\frac{n}{2}}) &= \left[\frac{(p')^{\frac{n}{2}}}{p'}\right] + \left[\frac{(p')^{\frac{n}{2}}}{(p')^2}\right] + \left[\frac{(p')^{\frac{n}{2}}}{(p')^3}\right] + \cdots + \left[\frac{(p')^{\frac{n}{2}}}{(p')^{\frac{n}{2}-1}}\right] + \left[\frac{(p')^{\frac{n}{2}}}{(p')^{\frac{n}{2}}}\right] \\ &= (p')^{\frac{n-2}{2}} + (p')^{\frac{n-4}{2}} + (p')^{\frac{n-6}{2}} + \cdots + p' + 1. \end{aligned}$$

Thus, the order of the Sylow p'-subgroup of G is $(p')^{\sum_{i=1}^{n} (p')^{(n-i)} \pm \sqrt{3} \sum_{j=1}^{\frac{n}{2}} (p')^{(\frac{n}{2}-j)}}$, while $m_1(K) = q^3$, so the order of the Sylow p'-subgroup of K is $(p')^{3n}$. Since $q = 3^{2m'+1} > 3$, we have $p' = 3$ and

$n = 2m' + 1$. At this time, by Lemma 2.10, we know that $\sum_{i=1}^{n}(p')^{(n-i)} \pm \sqrt{3}\sum_{j=1}^{\frac{n}{2}}(p')^{(\frac{n}{2}-j)}$ is larger than $4n$. We have $(p')^{\alpha}\|\,|G/K|\,|Out(\,^2G_2(q))| = n$, where $\alpha = \sum_{i=1}^{n}(p')^{(n-i)} \pm \sqrt{3}\sum_{j=1}^{\frac{n}{2}}(p')^{(\frac{n}{2}-j)} - 3n > n$. So, we can conclude that $(p')^{\alpha} \nmid n$, which is a contradiction. Therefore, $K \ncong \,^2G_2(q)$.

If $K \cong E_8(q)$, then from $p \in \pi(K)$ and $\pi_{p_m}(G) = \{p\}$, we have $\pi_{p_m}(K) = \{p\}$. Thus, p is equal to one of the following four numbers:

$$
\begin{aligned}
&q^8 + q^7 - q^5 - q^4 - q^3 + q + 1,\\
&q^8 - q^7 + q^5 - q^4 + q^3 - q + 1,\\
&q^8 - q^6 + q^4 - q^2 + 1,\\
&q^8 - q^4 + 1.
\end{aligned}
$$

When $p = q^8 + q^7 - q^5 - q^4 - q^3 + q + 1$, we have $s(p) = s((p')^{8n}) + s((p')^{7n}) - s((p')^{5n}) - s((p')^{4n}) - s((p')^{3n}) + s((p')^{n}) + s(1)$. Thus, the order of the Sylow p'-subgroup of G is

$$(p')^{\sum_{i=1}^{8n}(p')^{(8n-i)}+\sum_{i=1}^{7n}(p')^{(7n-i)}-\sum_{i=1}^{5n}(p')^{(5n-i)}-\sum_{i=1}^{4n}(p')^{(4n-i)}-\sum_{i=1}^{3n}(p')^{(3n-i)}+\sum_{i=1}^{n}(p')^{(n-i)}}.$$

While $m_1(K) = q^{120}(q^{18}-1)(q^{14}-1)(q^{12}-1)^2(q^{10}-1)^2(q^8-1)^2(q^4+q^2+1)$, then the order of the Sylow p'-subgroup of K is $(p')^{120n}$. Since $q \equiv 0, 1, 2, 3, 4(mod\ 5)$, by Lemma 2.11, we know that $\sum_{i=1}^{8n}(p')^{(8n-i)} + \sum_{i=1}^{7n}(p')^{(7n-i)} - \sum_{i=1}^{5n}(p')^{(5n-i)} - \sum_{i=1}^{4n}(p')^{(4n-i)} - \sum_{i=1}^{3n}(p')^{(3n-i)} + \sum_{i=1}^{n}(p')^{(n-i)}$ is larger than $121n$, which implies that $(p')^{\alpha}\nmid|G/K|\,|Out(E_8(q))| = n$, where

$$
\begin{aligned}
\alpha = &\sum_{i=1}^{8n}(p')^{(8n-i)} + \sum_{i=1}^{7n}(p')^{(7n-i)} - \sum_{i=1}^{5n}(p')^{(5n-i)} - \sum_{i=1}^{4n}(p')^{(4n-i)} - \sum_{i=1}^{3n}(p')^{(3n-i)}\\
&+ \sum_{i=1}^{n}(p')^{(n-i)} - 120n > n.
\end{aligned}
$$

Therefore, $K \ncong E_8(q)$. Similarly, we can prove that when $p = q^8 - q^7 + q^5 - q^4 + q^3 - q + 1$ or $q^8 - q^6 + q^4 - q^2 + 1$ or $q^8 - q^4 + 1$, $K \ncong E_8(q)$.

Therefore, we have $K \cong A$, implying $1 \trianglelefteq A \trianglelefteq G$. In this case, it is clear that $C_G(A) = 1$ and $Out(A) = Z_2$. As a result, we have either $G \cong A$ or $G \cong Aut(A)$. However, in this latter case $m_1(G) > m_1(A)$, a contradiction. Hence, we conclude that $G \cong A$.

4. Conclusions

Regarding the characterization problem of finite simple groups, this paper proposes a new characterization method based on the even-order components and centralizers. It proves the effectiveness of this method for alternating groups with disconnected prime graphs, but, we only investigate the case of alternating simple groups with disconnected prime graphs. In reality, a significant number of alternating simple groups have connected prime graphs. Whether the methods proposed in this paper are equally effective for this subset of groups is still under verification. We have already tested a small portion and found that the method is effective. We look forward to proving one day that this method is universally effective for all simple groups, which would be an exciting accomplishment.

Author contributions

Zhangjia Han: Conceptualization, methodology, validation, writing–review and editing, funding acquisition; Dongyang He: Methodology, validation, writing–original draft preparation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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