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Research article

Periodic event-triggered asynchronous control for almost sure stabilization of hybrid stochastic systems with sampled measurements

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Abstract: In this paper, we were concerned with the periodic event-triggered asynchronous stabilization of a class of hybrid stochastic systems driven by continuous-time Markov chain and Brownian motion, where the measurements of state and mode were available only at sampling instants, and the control was diffusion-dependent. Static and dynamic periodic event-triggered control (PETC) strategies were proposed with a guaranteed minimum interevent time for every sample path solution. Different from the well-known input-to-state stability framework for stability and synthesis of event-triggered control systems, a comparison system approach was developed to show that if the hybrid stochastic system under continuous-time feedback control was pth-moment exponentially stable, then there existed a small sampling period and event-triggering parameters such that the resulting event-triggered control hybrid stochastic system was almost surely exponentially stable. Particularly, the proposed PETC strategies could integrate the beneficial impacts of stochastic noises, which distinguished them from previous results. Two numerical examples were provided to illustrate the efficiency of the theoretical results.

Keywords: stochastic systems; event-triggered control; almost sure stability; stabilization by noise
Mathematics Subject Classification: 93C10, 93D15

1. Introduction

Hybrid systems skillfully incorporate continuous-time evolution and instantaneous changes and have been used to represent a wide variety of real-world processes, such as actuator failures, packet dropouts, cyber-physical systems, power systems, and robotics [1]. As an important class in stochastic modeling, hybrid systems driven by continuous-time Markov chains and Brownian motion have attracted significant attention, particularly in the fields of stability and control. Continuous-time Markov chains capture random switching phenomena arising in networked control, communication, and manufacturing systems, where abrupt mode changes may occur due to failures, environmental variations, or network uncertainties, while Brownian motion models continuous fluctuations caused by environmental noise, measurement errors, or external perturbations. The combination of these two processes provides a powerful framework for describing hybrid stochastic dynamics with discrete random events and continuous uncertainties. Many significant results in this respect can be found in a survey [2].

Sampled-data control is becoming increasingly important and widely employed in modern control structures such as digital control and networked control. The traditional sampled-data control algorithm relies on the time-triggered sampling of the control task executions, which may be inefficient from a computational perspective as the period for sampling and control execution is determined by worst-case estimates and all scenarios that the systems can attain. Event-triggered control, instead, updates control actions after the occurrence of an event generated by some state-dependent triggering rules and has shown the enhanced capability of saving system resources and improving control performance. One of the difficulties in the synthesis of event-triggered control is to guarantee stability/performance properties while preventing the Zeno phenomenon (the infinite number of triggers in finite time). Many works have investigated event-triggered control problems for stochastic systems, e.g., [3–5]. Some scholars have extended the event-triggered control strategies for non-switching stochastic systems to linear Markov jump systems [6–8]. The aforementioned study entailed pth-moment stability ($p \geq 2$), which considers stochastic noises unfavorable system stability factors. Interestingly, stochastic noises can enhance stability, namely, introducing stochastic noise can stabilize unstable systems or make systems more stable [9, 10]. Notably, Li and Liu [11] studied the event-triggered stabilization of a class of stochastic systems with control-depend diffusion terms, and their event-triggered strategy is applicable to stabilization by noise. However, the study in question has a limited scope, concentrating solely on continuous event-triggered control and non-switching stochastic systems. In addition, periodic event-triggered control (PETC) evaluates the event-triggering mechanism (ETM) only at given sampling instants to enable its digital implementation.

In the context of digital control, a mismatch between the system mode and the controller mode obtained in discrete time may arise, resulting in the phenomenon of asynchronous control. It is worth noting that asynchronous control may lead to performance degradation and instability. Therefore, asynchronous control problems of Markov jump systems have attracted extensive attention. Many important results have been obtained, such as asynchronous control described by hidden Markov chains [12, 13] and asynchronous control induced by input delay [14]. For the case of asynchronous control caused by mode sampling, the researchers in [15, 16] solved the optimal sampled-data control problem of linear Markov jump systems via the impulsive system approach. However, these studies are applicable only to the mean square stability of linear hybrid systems with periodic sampling. Moreover, the researchers in [17–19] studied the asynchronous control of stochastic hybrid systems with discrete-time observations of state and mode by the comparison system approach proposed in [10]. However, all these researchers are concerned with stabilization problems via time-triggered control, and only the researchers in [17] focus on the almost sure stabilization by noise. Nevertheless, there is a lack of stochastic event-triggered asynchronous stabilization for hybrid stochastic systems. The major challenges arise from: (i) The asynchronous nature of control updates induced by discrete-time observations of both state and mode, and (ii) the coexistence of mode switching governed by a continuous-time Markov chain and stochastic perturbations modeled by Brownian motion, which make the stability analysis under event-triggered control, especially stabilization by noise, particularly challenging.

Here, we focus on the event-triggered stabilization of a class of hybrid stochastic systems with control-dependent diffusion terms and discrete-time observations of both state and mode. The major contributions are summarized as follows:

- We propose PETC strategies of both static and dynamic types for hybrid stochastic systems, formulated within a time-regularization framework. Since the triggering conditions are monitored only at discrete sampling instants, Zeno behavior is naturally avoided.
- To address the challenges of asynchronous switching control induced by mode sampling, we establish several technical lemmas on the moment properties of the state, sampling error, and control error between synchronous and asynchronous inputs, leveraging properties of Markov chains and stochastic differential equations.
- A comparison system approach inspired by the researchers in [10] is developed to show that almost sure exponential stability of the PETC-based hybrid stochastic system can be guaranteed under suitable sampling periods and event-triggering parameters. Unlike most works [7, 8, 20] that treat stochastic noise solely as detrimental to stability, our approach reveals that stochastic noises may have beneficial effects on stability, thereby reducing conservatism in event-triggered design.

The remainder of the paper is organized as follows. In Section 2, the problem is formulated, and some necessary lemmas are given. In Section 3, we present event-triggering design and stability analysis from static PETC to dynamic ones. Stability verification and numerical examples with comparisons are provided in Section 5, and conclusions are drawn in Section 6.

2. System description and problem formulation

Notation. Let $|\cdot|$ denote the Euclidean norm of a vector. $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ denotes the family of all nonnegative functions $V(t, x)$ on $\mathbb{R}_+ \times \mathbb{R}^n$ that are twice continuously differentiable in x and once in t. For any real numbers a and b, $a \vee b = \max\{a, b\}$. I_A denotes the indicator function of a set A, i.e., $I_A(x) = 1$ if $x \in A$ otherwise 0.

Consider the following hybrid stochastic system

$$\begin{cases} \mathrm{d}x(t) = f_{\sigma(t)}(t, x(t), u(t))\mathrm{d}t + g_{\sigma(t)}(t, x(t), u(t))\mathrm{d}w(t), \ t > t_0, \\ x(t_0) = x_0, \sigma(t_0) = \sigma_0, \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable and $t_0 \geq 0$. $\{\sigma(t), t \geq t_0\}$ is a right-continuous Markov process defined on the probability space which takes values in the finite set $\mathcal{M} = \{1, 2, \ldots, N\}$ with generator $\Pi = (\pi_{ij})$, $i, j \in \mathcal{M}$, given by

$$\mathbb{P}\{\sigma(t + \Delta t) = j \,|\sigma(t) = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & i \neq j, \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & i = j, \end{cases}$$

where $\Delta t > 0$, $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$, and $\pi_{ij} \geq 0$ for $i \neq j$, $\pi_{ii} \leq 0$ with $\sum_{j=1,\, j\neq i}^{N} \pi_{ij} = -\pi_{ii}$. Moreover, transition probabilities for each pair of states $i, j \in \mathcal{M}$ are given by $\mathbb{P}(\sigma(t+\Delta) = j|\sigma(t) = i) = p_{ij}(\Delta)$ for all $t, \Delta \geq 0$, where $p_{ij}(\Delta)$ denotes the (i, j)th element of the matrix $e^{\Pi\Delta}$. $u(t) \in \mathbb{R}^m$ is the control input. $w(t)$ is an one-dimensional Wiener process defined on complete probability space $(\Omega, \mathscr{F}, \mathcal{P})$ with a natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions. For each $i \in \mathcal{M}$, f_i, $g_i \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ satisfy the following assumption.

Assumption 2.1. *For each $i \in \mathcal{M}$, $f_i(t, x, u)$ and $g_i(t, x, u)$ are globally Lipschitz continuous with respect to (x, u), i.e., there exits positive scalar L_1 such that for all $x_1, x_2 \in \mathbb{R}^n$, $u_1, u_2 \in \mathbb{R}^m$, $t \geq 0$*

$$|f_i(t, x_1, u_1) - f_i(t, x_2, u_2)| \vee |g_i(t, x_1, u_1) - g_i(t, x_2, u_2)| \leq L_1(|x_1 - x_2| + |u_1 - u_2|).$$

Remark 2.1. Assumption 2.1 is a standard regularity condition in the theory of stochastic differential equations. It guarantees the existence and uniqueness of strong solutions to system (2.1) under each mode of the Markov chain. Moreover, the global Lipschitz property with respect to the state and the control input ensures that the system trajectories depend continuously on the initial data and inputs, which is fundamental for subsequent stability analysis.

Moreover, in the setup of emulation design framework, it is assumed that system (2.1) is exponentially stabilizable in the pth moment ($p > 0$) sense.

Assumption 2.2. *There exists a mode-dependent controller $u(t) = k_{\sigma(t)}(x(t))$ such that the solution $x(t)$ of the closed-loop system*

$$dx(t) = f_{\sigma(t)}(t, x(t), k_{\sigma(t)}(x(t)))dt + g_{\sigma(t)}(t, x(t), k_{\sigma(t)}(x(t)))dw(t), \tag{2.2}$$

satisfies

$$\mathbb{E}|x(t)|^p \leq M|x_0|^p e^{-\gamma(t-t_0)}, \ t \geq t_0,$$

where M, γ, p are some positive constants, and it is further assumed that for each $i \in \mathcal{M}$, $k_i \in C(\mathbb{R}^n, \mathbb{R}^m)$ satisfies the Lipschitz condition, i.e., there exists positive scalar L_2 such that for all $x_1, x_2 \in \mathbb{R}^n$, $i \in \mathcal{M}$,

$$|k_i(x_1) - k_i(x_2)| \leq L_2|x_1 - x_2|.$$

Remark 2.2. Assumption 2.2 postulates the existence of a mode-dependent state-feedback controller that renders the closed-loop system (2.2) exponentially stable in the pth moment. This ensures that the state trajectories decay in expectation at an exponential rate, which is a standard notion of stochastic stability. The additional Lipschitz continuity of each feedback map $k_i(\cdot)$ guarantees the well-posedness of the closed-loop system and provides a technical foundation for the subsequent stability analysis.

In this paper, we assume that the configuration of the hybrid stochastic systems synthesized by discrete-time feedback controller over networked communication is illustrated in Figure 1. The state $x(t)$ and the mode $\sigma(t)$ are observed only at discrete times or say that the outputs of the state $x(t)$ and the mode $\sigma(t)$ are measured periodically, i.e., the system output is $\{(x(s_\ell), \sigma(s_\ell))\}_{\ell\in\mathbb{N}_0}$, where $s_\ell = t_0 + \ell\tau$, and $\tau > 0$ is the sampling period. Based on the discrete-time measurements, the control input with time-triggered mechanism has the following form

$$u(t) = k_{\sigma(s_\ell)}(x(s_\ell)), \ t \in [s_\ell, s_{\ell+1}), \ \ell \in \mathbb{N}_0.$$

To reduce the consumption of communication and energy resources, we consider the event-triggered diagram for generating the times for controller updates from a triggering condition involving the current state or output measurement of the plant and the last transmitted data. For this case, the control input with sample-and-hold measurements, which has the following form

$$u(t) = k_{\tilde{\sigma}(t)}(\tilde{x}(t)), \tag{2.3}$$

where $\tilde{\sigma}(t) = \sigma(\hat{t}_l)$, $t \in [\hat{t}_l, \hat{t}_{l+1})$, and $\tilde{x}(t) = x(t_i)$, $t \in [t_i, t_{i+1})$ in which $\{\hat{t}_l\}_{l\in\mathbb{N}_0}$ and $\{t_i\}_{i\in\mathbb{N}_0}$ are subsequences (stopping time sequences) of $\{s_\ell\}_{\ell\in\mathbb{N}_0}$, will be computed later by the designed ETMs.

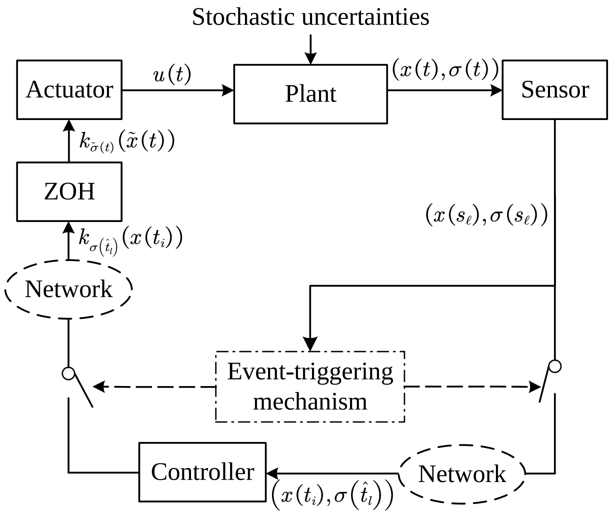

Figure 1. Diagram of periodic event-triggered control for hybrid stochastic systems.

Remark 2.3. Unlike the existing event-triggered strategies that use sampled state and continuous-time mode information for feedback, i.e., $u(t) = k_{\sigma(t)}(\tilde{x}(t))$ [7, 8, 20], the periodic event-triggered controller (2.3) may not align with the system mode due to mode sampling.

The following lemmas will be used in the proofs of the main theorems.

Lemma 2.1. *[21] For all $t \geq 0$ and $i \in \mathcal{M}$, we have $p_{ii}(t) \geq e^{\pi_{ii}t}$.*

Lemma 2.2. *Let $z(t) \in \mathbb{R}^n$ be a continuous and $\mathscr{F}_t$-adapted stochastic process. If there exist positive constants c_0, c_1, h, and $\kappa \in (0,1)$ such that for any $t \geq t_0$, $p > 0$,*

$$\mathbb{E}\left[\sup_{t_0\leq s\leq t}|z(s)|^p\right] \leq c_0\mathbb{E}|z(t_0)|^p e^{c_1(t-t_0)},$$

and

$$\mathbb{E}|z(t_0 + (\ell+1)h)|^p \leq \kappa\mathbb{E}|z(t_0 + \ell h)|^p, \ \ell \in \mathbb{N}_0,$$

then the stochastic process $z(t)$ has the following properties:

$$\mathbb{E}|z(t)|^p \leq c_0 e^{c_1 h + \ln(\frac{1}{\kappa})}\mathbb{E}|z(t_0)|^p e^{-\frac{\ln(\frac{1}{\kappa})}{h}t},$$

$$\limsup_{t\to+\infty}\frac{1}{t}\ln(|z(t)|) \leq -\frac{\ln(\frac{1}{\kappa})}{2ph}, \ a.s..$$

Proof. For any $t > t_0$, there must exist $\ell \in \mathbb{N}_0$ such that $t - t_0 \in [\ell h, (\ell+1)h)$. Then, $\mathbb{E}\left[\sup_{t_0+\ell h \le s \le t_0+(\ell+1)h} |z(s)|^p\right] \le c_0 e^{c_1 h} \mathbb{E}|z(t_0+\ell h)|^p \le c_0 e^{c_1 h} \kappa^\ell \mathbb{E}|z(t_0)|^p \le c_0 e^{c_1 h} e^{-\ell \ln \frac{1}{\kappa}} \mathbb{E}|z(t_0)|^p$. Using the relation that $\ell \le \frac{t-t_0}{h} \le \ell+1$, one has $\mathbb{E}\left[\sup_{t_0+\ell h \le s \le t_0+(\ell+1)h} |z(s)|^p\right] \le c_0 e^{c_1 h + \ln \frac{1}{\kappa}} \mathbb{E}|z(t_0)|^p e^{-\frac{\ln \frac{1}{\kappa}}{h}(t-t_0)}$. On the other hand, using Chebyshev's inequality,

$$\mathbb{P}\left(\sup_{t_0+\ell h \le s \le t_0+(\ell+1)h} |z(s)|^p > e^{-\frac{\ell}{2}\ln\frac{1}{\kappa}}\right) \le c_0 e^{c_1 h} \mathbb{E}|z(t_0)|^p e^{-\frac{\ell}{2}\ln\frac{1}{\kappa}}.$$

Since $\sum_{\ell=0}^{+\infty} e^{-\frac{\ell}{2}\ln\frac{1}{\kappa}} < +\infty$, the Borel-Cantelli lemma yields that for almost all $\omega \in \Omega$, there exists a positive integer $\ell_0(\omega)$ such that

$$\sup_{t_0+\ell h \le s \le t_0+(\ell+1)h} |z(s,\omega)|^p \le e^{-\frac{\ell}{2}\ln\frac{1}{\kappa}}, \quad \ell \ge \ell_0(\omega),$$

which implies that for $t \in [t_0+\ell h, t_0+(\ell+1)h]$ and $\ell \ge \ell_0(\omega)$,

$$\frac{\ln(|z(t,\omega)|)}{t-t_0} \le -\frac{\ell \ln \frac{1}{\kappa}}{2p(\ell+1)h}.$$

Letting $\ell \to +\infty$, we get

$$\limsup_{t\to+\infty} \frac{1}{t}\ln(|z(t,\omega)|) = \limsup_{t\to+\infty} \frac{1}{t-t_0}\ln(|z(t,\omega)|) \le -\lim_{\ell\to+\infty} \frac{\ell \ln\frac{1}{\kappa}}{2p(\ell+1)h} = -\frac{\ln\frac{1}{\kappa}}{2ph}, \text{a.s..}$$

□

3. Event-triggering design and stability analysis

In this section, we propose systematic design procedures for static and dynamic periodic event-triggered control schemes for stochastic systems (2.1) with the sampled-data controller (2.3).

3.1. Static event-triggered control strategy

The static ETM is designed by the following rule

$$t_{i+1} = \min_{\ell\in\mathbb{N}} \{s_\ell > t_i \mid |x(s_\ell) - x(t_i)| > \delta |x(s_\ell)|\}, \tag{3.1}$$

and

$$\tilde{\sigma}(t) = \begin{cases} \sigma(s_\ell), & \text{if } \sigma(s_\ell) \ne \tilde{\sigma}(s_\ell^-) \\ \tilde{\sigma}(s_\ell^-), & \text{if } \sigma(s_\ell) = \tilde{\sigma}(s_\ell^-) \end{cases}, t \in [s_\ell, s_{\ell+1}), \tag{3.2}$$

where $\delta > 0$ denotes the event triggering parameter.

Remark 3.1. Note that the ETM (3.1) is monitored only at the sampling instants $\{s_\ell\}_{\ell\in\mathbb{N}}$. Consequently, the event-generator inherently excludes the occurrence of Zeno behavior.

For ease of modeling, we define an auxiliary variable

$$\tilde{x}(t) = \begin{cases} x(s_\ell), & \text{if } \mathcal{J}(s_\ell^-) > 0 \\ \tilde{x}(s_\ell^-), & \text{if } \mathcal{J}(s_\ell^-) \le 0 \end{cases}, t \in [s_\ell, s_{\ell+1}), \tag{3.3}$$

with the initial values $\tilde{x}(s_0^-) = x(s_0)$, where $\mathcal{J}(t) := |x(t) - \tilde{x}(t)| - \delta|x(t)|$. Under the auxiliary variable, the periodic ETM (3.1) has become to $t_{i+1} = \min_{\ell\in\mathbb{N}} \left\{ s_\ell > t_i \,|\mathcal{J}(s_\ell^-) > 0 \right\}$. Then, the control input (2.3) is modified by

$$u(t) = k_{\tilde{\sigma}(t)}(\tilde{x}(t))$$

which leads to the following closed-loop system

$$\mathrm{d}x(t) = f_{\sigma(t)}(t, x(t), k_{\tilde{\sigma}(t)}(\tilde{x}(t)))\mathrm{d}t + g_{\sigma(t)}(t, x(t), k_{\tilde{\sigma}(t)}(\tilde{x}(t)))\mathrm{d}w(t). \tag{3.4}$$

In what follows, the moment properties of the state $x(t)$, sampling error $x(t) - \tilde{x}(t)$, and the control error between synchronous and asynchronous control inputs, i.e., $k_{\tilde{\sigma}(t)}(\tilde{x}(t)) - k_{\sigma(t)}(\tilde{x}(t))$, will be established.

Lemma 3.1. *Let $x(t)$ be the solution of closed-loop system* (3.4)*, then there holds that for any $t \geq s_\ell$, $\ell \in \mathbb{N}_0$,*

$$\sup_{s_\ell\leq v\leq t} \mathbb{E}\left[|x(v)|^2|\mathscr{F}_{s_\ell}\right] \leq |x(s_\ell)|^2 \mathrm{e}^{\underline{\alpha}_L(t-s_\ell)}, \tag{3.5}$$

$$\mathbb{E}\left[\sup_{s_\ell\leq v\leq t} |x(v)|^2|\mathscr{F}_{s_\ell}\right] \leq 2|x(s_\ell)|^2 \mathrm{e}^{2\overline{\alpha}_L(t-s_\ell)}, \tag{3.6}$$

$$\mathbb{E}\left[|x(t) - \tilde{x}(t)|^2|\mathscr{F}_{s_\ell}\right] \leq \varrho(\tau, \delta)|x(s_\ell)|^2 \mathrm{e}^{\underline{\alpha}_L(t-s_\ell)}, \tag{3.7}$$

where $\underline{\alpha}_L = 2L_1 + 3L_1^2 + 2L_2^2(1 + 2L_1^2)(\delta^2 + 1)$, $\overline{\alpha}_L = 2L_1 + 131L_1^2 + 2L_2^2(1 + 130L_1^2)(\delta^2 + 1)$, *and* $\varrho(\tau, \delta) = 3\delta^2 + 6\tau(\tau + 1)L_1^2(1 + 2L_2^2(\delta^2 + 1))$.

Proof. By Itô's formula [9, Page 36], Assumptions 2.1 and 2.2, one can show that for any $t \geq s_\ell$,

$$\begin{aligned}
|x(t)|^2 &= |x(s_\ell)|^2 + \int_{s_\ell}^t [2x^{\mathrm{T}}(s) f_{\sigma(s)}(s, x(s), k_{\tilde{\sigma}(s)}(\tilde{x}(s))) + |g_{\sigma(s)}(s, x(s), k_{\tilde{\sigma}(s)}(\tilde{x}(s)))|^2]\mathrm{d}s \\
&\quad + \int_{s_\ell}^t 2x^{\mathrm{T}}(s) g_{\sigma(s)}(s, x(s), k_{\tilde{\sigma}(s)}(\tilde{x}(s)))\mathrm{d}w(s) \\
&\leq |x(s_\ell)|^2 + \int_{s_\ell}^t [(2L_1 + 3L_1^2)|x(s)|^2 + L_2^2(1 + 2L_1^2)|\tilde{x}(s)|^2]\mathrm{d}s \\
&\quad + \int_{s_\ell}^t 2x^{\mathrm{T}}(s) g_{\sigma(s)}(s, x(s), k_{\tilde{\sigma}(s)}(\tilde{x}(s)))\mathrm{d}w(s).
\end{aligned} \tag{3.8}$$

Note that the event-triggering rule (3.1) implies that for any $t \in [s_\ell, s_{\ell+1})$, $|\tilde{x}(t)|^2 \leq 2|\tilde{x}(s_\ell) - x(s_\ell)|^2 + 2|x(s_\ell)|^2 \leq 2(\delta^2 + 1)|x(s_\ell)|^2$ which can be rewritten as

$$|\tilde{x}(t)|^2 \leq 2(\delta^2 + 1)|x(\varsigma_t)|^2, \tag{3.9}$$

where $\varsigma_t = \lfloor \frac{t-t_0}{\tau} \rfloor \tau$, and $\lfloor \frac{t}{\tau} \rfloor$ rounds $\frac{t}{\tau}$ to the nearest integer less than or equal to $\frac{t}{\tau}$.

For any $t \geq s_\ell$, Eq (3.8) has the following estimate

$$\begin{aligned}
\mathbb{E}\left[|x(t)|^2|\mathcal{F}_{s_\ell}\right] &\leq |x(s_\ell)|^2 + \int_{s_\ell}^t [(2L_1 + 3L_1^2)|x(s)|^2 + 2L_2^2(1 + 2L_1^2)(\delta^2 + 1)|x(\varsigma_s)|^2]\mathrm{d}s \\
&\leq |x(s_\ell)|^2 + \underline{\alpha}_L \int_{s_\ell}^t \sup_{s_\ell\leq v\leq s} \mathbb{E}\left[|x(v)|^2|\mathcal{F}_{s_\ell}\right] \mathrm{d}s,
\end{aligned}$$

which can be further deduced that $\sup_{s_\ell\le v\le t}\mathbb{E}\left[|x(v)|^2|\mathcal{F}_{s_\ell}\right]\le|x(s_\ell)|^2+\underline{\alpha}_L\int_{s_\ell}^t\sup_{s_\ell\le v\le s}\mathbb{E}\left[|x(v)|^2|\mathcal{F}_{s_\ell}\right]\mathrm{d}s$. Hence, the Gronwall inequality yields that (3.5).

On the other hand, for any $t\ge s_\ell$,

$$\mathbb{E}\left[\sup_{s_\ell\le v\le t}|x(v)|^2\Big|\mathscr{F}_{s_\ell}\right]\le|x(s_\ell)|^2+\int_{s_\ell}^t\mathbb{E}\left[(2L_1+3L_1^2)|x(s)|^2+L_2^2(1+2L_1^2)|\tilde{x}(s)|^2|\mathscr{F}_{s_\ell}\right]\mathrm{d}s$$
$$+\mathbb{E}\left[\sup_{s_\ell\le v\le t}\left|\int_{s_\ell}^v 2x^{\mathrm{T}}(s)g_{\sigma(s)}(s,x(s),k_{\tilde{\sigma}(s)}(\tilde{x}(s)))\mathrm{d}w(s)\right|\mathscr{F}_{s_\ell}\right].\tag{3.10}$$

By the Burkholder-Davis-Gundy inequality , we have that

$$\begin{aligned}&\mathbb{E}\left[\sup_{s_\ell\le v\le t}\left|\int_{s_\ell}^v 2x^{\mathrm{T}}(s)g_{\sigma(s)}(s,x(s),k_{\tilde{\sigma}(s)}(\tilde{x}(s)))\mathrm{d}w(s)\right|\mathscr{F}_{s_\ell}\right]\\
&\le 4\sqrt{2}\mathbb{E}\left[\int_{s_\ell}^t|2x^{\mathrm{T}}(s)g_{\sigma(s)}(s,x(s),k_{\tilde{\sigma}(s)}(\tilde{x}(s)))|^2\mathrm{d}s\Big|\mathscr{F}_{s_\ell}\right]^{\frac{1}{2}}\\
&\le 8\sqrt{2}\mathbb{E}\left[\sup_{s_\ell\le v\le t}|x(v)|^2\int_{s_\ell}^t|g_{\sigma(s)}(s,x(s),k_{\tilde{\sigma}(s)}(\tilde{x}(s)))|^2\mathrm{d}s\Big|\mathscr{F}_{s_\ell}\right]^{\frac{1}{2}}\\
&\le\frac{1}{2}\mathbb{E}\left[\sup_{s_\ell\le v\le t}|x(v)|^2|\mathscr{F}_{s_\ell}\right]+64\int_{s_\ell}^t\mathbb{E}\left[|g_{\sigma(s)}(s,x(s),k_{\tilde{\sigma}(s)}(\tilde{x}(s)))|^2\Big|\mathscr{F}_{s_\ell}\right]\mathrm{d}s\\
&\le\frac{1}{2}\mathbb{E}\left[\sup_{s_\ell\le v\le t}|x(v)|^2|\mathscr{F}_{s_\ell}\right]+128L_1^2\int_{s_\ell}^t\mathbb{E}\left[|x(s)|^2+L_2^2|\tilde{x}(s)|^2\Big|\mathscr{F}_{s_\ell}\right]\mathrm{d}s.\end{aligned}$$

Substituting this into (3.10) yields

$$\mathbb{E}\left[\sup_{s_\ell\le v\le t}|x(v)|^2\Big|\mathscr{F}_{s_\ell}\right]\le 2|x(s_\ell)|^2+2\int_{s_\ell}^t\mathbb{E}\left[(2L_1+131L_1^2)|x(v)|^2\right.$$
$$\left.+L_2^2(1+130L_1^2)|\tilde{x}(v)|^2|\mathscr{F}_{s_\ell}\right]\mathrm{d}v.\tag{3.11}$$

Using (3.9) and the Gronwall inequality gives the required assertion (3.6).

Moreover, $\varsigma_t\ge s_\ell$, $\forall t\ge s_\ell$. Thus, using Hölder's inequality and Theorem 7.1 of [9, Page 39] again, we get that

$$\begin{aligned}\mathbb{E}\left[|x(t)-\tilde{x}(t)|^2|\mathscr{F}_{s_\ell}\right]&=\mathbb{E}\left[\left|x(\varsigma_t)-\tilde{x}(t)+\int_{\varsigma_t}^t f_{\sigma(s)}(s,x(s),k_{\tilde{\sigma}(s)}(\tilde{x}(s)))\mathrm{d}s\right.\right.\\
&\quad\left.\left.+\int_{\varsigma_t}^t g_{\sigma(s)}(s,x(s),k_{\tilde{\sigma}(s)}(\tilde{x}(s)))\mathrm{d}w(s)\right|^2\Big|\mathscr{F}_{s_\ell}\right]\\
&\le 3|x(\varsigma_t)-\tilde{x}(t)|^2+3\tau\int_{\varsigma_t}^t\mathbb{E}\left[\left|f_{\sigma(s)}(s,x(s),k_{\tilde{\sigma}(s)}(\tilde{x}(s)))\right|^2\Big|\mathscr{F}_{s_\ell}\right]\mathrm{d}s\\
&\quad+3\int_{\varsigma_t}^t\mathbb{E}\left[\left|g_{\sigma(s)}(s,x(s),k_{\tilde{\sigma}(s)}(\tilde{x}(s)))\right|^2\Big|\mathscr{F}_{s_\ell}\right]\mathrm{d}s\\
&\le 3|x(\varsigma_t)-\tilde{x}(t)|^2+3L_1^2(\tau+1)\int_{\varsigma_t}^t\mathbb{E}\left[(|x(s)|+L_2|\tilde{x}(s)|)^2|\mathscr{F}_{s_\ell}\right]\mathrm{d}s\\
&\le 3|x(\varsigma_t)-\tilde{x}(t)|^2+6L_1^2(\tau+1)\int_{\varsigma_t}^t\mathbb{E}\left[(|x(s)|^2+L_2^2|\tilde{x}(s)|^2)|\mathscr{F}_{s_\ell}\right]\mathrm{d}s.\end{aligned}$$

By event-triggering mechanisms (3.1) and (3.5), the above equation can deduce that

$$\begin{aligned}\mathbb{E}\left[|x(t)-\tilde{x}(t)|^2|\mathscr{F}_{s_\ell}\right] &\le 3\delta^2|x(\varsigma_t)|^2+6L_1^2(\tau+1)\int_{\varsigma_t}^{t}\mathbb{E}\left[(|x(s)|^2+2L_2^2(\delta^2+1)|x(\varsigma_s)|^2)|\mathscr{F}_{s_\ell}\right]\mathrm{d}s\\ &\le 3\delta^2|x(\varsigma_t)|^2+6\tau(\tau+1)L_1^2(1+2L_2^2(\delta^2+1))\sup_{s_\ell\le s\le t}\mathbb{E}\left[|x(s)|^2|\mathscr{F}_{s_\ell}\right]\\ &\le \varrho(\tau,\delta)|x(s_\ell)|^2\mathrm{e}^{\underline{\alpha}_L(t-s_\ell)}.\end{aligned}$$

The proof is complete. □

Lemma 3.2. *Let $x(t)$ be the solution of closed-loop system* (3.4)*, then for any $t\ge s_\ell$, $\ell\in\mathbb{N}_0$, the following estimates hold*

$$\mathbb{E}\left[\int_{s_\ell}^{t}|k_{\sigma(s)}(\tilde{x}(s))-k_{\tilde{\sigma}(s)}(\tilde{x}(s))|^2\mathrm{d}s\middle|\mathscr{F}_{s_\ell}\right]\le\varphi_0(\tau,\delta,t-s_\ell)|x(s_\ell)|^2, \tag{3.12}$$

$$\mathbb{E}\left[\int_{s_\ell}^{t}|x(s)-\tilde{x}(s)|^2\mathrm{d}s\middle|\mathscr{F}_{s_\ell}\right]\le\varphi_1(\tau,\delta,t-s_\ell)|x(s_\ell)|^2, \tag{3.13}$$

where $\varphi_0(\tau,\delta,\Delta)=8L_2^2(1-\mathrm{e}^{\underline{\pi}\tau})(\delta^2+1)\frac{1}{\underline{\alpha}_L}\left(\mathrm{e}^{\underline{\alpha}_L\Delta}-1\right)$, $\underline{\pi}=\min_{i\in\mathcal{M}}\{\pi_{ii}\}$*, and* $\varphi_1(\tau,\delta,\Delta)=\varrho(\tau,\delta)\frac{1}{\underline{\alpha}_L}\times\left(\mathrm{e}^{\underline{\alpha}_L\Delta}-1\right)$.

Proof. Note that for any $t\in[s_i,s_{i+1})$, $i\in\mathbb{N}_0$, $\tilde{x}(t)=\tilde{x}(s_i)$, $\tilde{\sigma}(t)=\sigma(s_i)$. Hence, $(\tilde{x}(t),\tilde{\sigma}(t))$ is $\mathscr{F}_{s_i}$-measurable.

$$\begin{aligned}\mathbb{E}\left[|k_{\sigma(t)}(\tilde{x}(t))-k_{\tilde{\sigma}(t)}(\tilde{x}(t))|^2\mathscr{F}_{s_i}\right] &=\mathbb{E}\left[|k_{\sigma(t)}(\tilde{x}(t))-k_{\sigma(s_i)}(\tilde{x}(t))|^2I_{\{\sigma(t)\neq\sigma(s_i)\}}|\mathscr{F}_{s_i}\right]\\ &\le 4L_2^2|\tilde{x}(t)|^2\mathbb{E}\left[I_{\{\sigma(t)\neq\sigma(s_i)\}}|\mathscr{F}_{s_i}\right]\\ &=4L_2^2|\tilde{x}(t)|^2\mathbb{E}\left[\sum_{j=1}^{N}\sum_{l=1,l\neq j}^{N}I_{\{\sigma(s_i)=j\}}I_{\{\sigma(t)=l\}}\middle|\mathscr{F}_{s_i}\right]\\ &=4L_2^2|\tilde{x}(t)|^2\sum_{j=1}^{N}\sum_{l=1,l\neq j}^{N}I_{\{\sigma(s_i)=j\}}\mathbb{P}\{\sigma(t)=l|\sigma(s_i)=j\}\\ &=4L_2^2|\tilde{x}(t)|^2\sum_{j=1}^{N}I_{\{\sigma(s_i)=j\}}(1-p_{jj}(t-s_i))\\ &\le 4L_2^2|\tilde{x}(t)|^2\sum_{j=1}^{N}I_{\{\sigma(s_i)=j\}}(1-\mathrm{e}^{\pi_{jj}(t-s_i)})\\ &\le 4L_2^2|\tilde{x}(t)|^2\sum_{j=1}^{N}I_{\{\sigma(s_i)=j\}}(1-\mathrm{e}^{\pi_{jj}\tau})\\ &\le 4L_2^2|\tilde{x}(t)|^2(1-\mathrm{e}^{\underline{\pi}\tau}),\end{aligned}$$

where Lemma 2.1 is used.

Denote $\bar{n} = \lfloor \frac{t-s_\ell}{\tau} \rfloor$. By the Fubini theorem, we have

$$\begin{aligned}\mathbb{E}\left[\int_{s_\ell}^t |k_{\sigma(s)}(\tilde{x}(s)) - k_{\tilde{\sigma}(s)}(\tilde{x}(s))|^2 \mathrm{d}s \middle| \mathscr{F}_{s_\ell}\right] &= \sum_{i=0}^{\bar{n}-1} \int_{s_{\ell+i}}^{s_{\ell+i+1}} \mathbb{E}\left[|k_{\sigma(s)}(\tilde{x}(s)) - k_{\tilde{\sigma}(s)}(\tilde{x}(s))|^2 |\mathscr{F}_{s_\ell}\right] \mathrm{d}s \\ &\quad + \int_{s_{\ell+\bar{n}}}^t \mathbb{E}\left[|k_{\sigma(s)}(\tilde{x}(s)) - k_{\tilde{\sigma}(s)}(\tilde{x}(s))|^2 |\mathscr{F}_{s_\ell}\right] \mathrm{d}s \\ &\leq 4L_2^2(1 - \mathrm{e}^{\pi\tau}) \int_{s_\ell}^t \mathbb{E}\left[|\tilde{x}(s)|^2 |\mathscr{F}_{s_\ell}\right] \mathrm{d}s. \end{aligned} \tag{3.14}$$

By Lemma 3.1 and using the fact $|\tilde{x}(t)|^2 \leq 2(\delta^2+1)|x(\varsigma_t)|^2$, we have

$$\begin{aligned}\int_{s_\ell}^t \mathbb{E}\left[|\tilde{x}(s)|^2 |\mathscr{F}_{s_\ell}\right] \mathrm{d}s &\leq 2(\delta^2+1) \int_{s_\ell}^t \mathbb{E}\left[|x(\varsigma_s)|^2 \mathscr{F}_{s_\ell}\right] \mathrm{d}s \\ &\leq 2(\delta^2+1) \int_{s_\ell}^t |x(s_\ell)|^2 \mathrm{e}^{\underline{\alpha}_L(s-s_\ell)} \mathrm{d}s \\ &= 2(\delta^2+1) \frac{1}{\underline{\alpha}_L} \left(\mathrm{e}^{\underline{\alpha}_L(t-s_\ell)} - 1\right) |x(s_\ell)|^2. \end{aligned}$$

Substituting the above inequality into (3.14), we have (3.12).

On the other hand, using Lemma 3.1,

$$\begin{aligned}\mathbb{E}\left[\int_{s_\ell}^t |x(s)) - \tilde{x}(s)|^2 \mathrm{d}s \middle| \mathscr{F}_{s_\ell}\right] \mathrm{d}s &\leq \varrho(\tau,\delta)|x(s_\ell)|^2 \int_{s_\ell}^t \mathrm{e}^{\underline{\alpha}_L(v-s_\ell)} \mathrm{d}v \\ &= \frac{\varrho(\tau,\delta)}{\underline{\alpha}_L} (\mathrm{e}^{\underline{\alpha}_L(t-s_\ell)} - 1)|x(s_\ell)|^2, \end{aligned}$$

which confirms to (3.13). □

Lemma 3.3. *Let $x(t)$ be the solution of system* (3.4)*, and $y(t)$ be the solution of system* (2.2) *with initial value $(x(s_\ell), \sigma(s_\ell))$. Then, there holds that for any $t \geq s_\ell$,*

$$\mathbb{E}\left[|y(t) - x(t)|^2 |\mathscr{F}_{s_\ell}\right] \leq \phi(\tau, \delta, t - s_\ell)|x(s_\ell)|^2, \tag{3.15}$$

where

$$\phi(\tau,\delta,\Delta) = 4(\Delta+1)L_1^2(\varphi_0(\tau,\delta,\Delta) + 2L_2^2\varphi_1(\tau,\delta,\Delta))\mathrm{e}^{8\Delta(\Delta+1)L_1^2(1+L_2)^2}. \tag{3.16}$$

Proof. By Hölder's inequality and Theorem 7.1 of [9, Page 39], one can obtain that

$$\begin{aligned}\mathbb{E}\left[|y(t) - x(t)|^2 |\mathscr{F}_{s_\ell}\right] &= \mathbb{E}\Bigg[\Bigg|\int_{s_\ell}^t [f_{\sigma(s)}(s, y(s), k_{\sigma(s)}(y(s))) - f_{\sigma(s)}(s, x(s), k_{\tilde{\sigma}(s)}(\tilde{x}(s)))] \mathrm{d}s \\ &\quad + \int_{s_\ell}^t [g_{\sigma(s)}(s, y(s), k_{\sigma(s)}(y(s))) - g_{\sigma(s)}(s, x(s), k_{\tilde{\sigma}(s)}(\tilde{x}(s)))] \mathrm{d}w(s)\Bigg|^2 \Bigg| \mathscr{F}_{s_\ell}\Bigg] \\ &\leq 4(t - s_\ell + 1)\mathbb{E}\left[\int_{s_\ell}^t [|f_{\sigma(s)}(s, y(s), k_{\sigma(s)}(y(s))) - f_{\sigma(s)}(s, x(s), k_{\sigma(s)}(\tilde{x}(s)))|^2\right. \end{aligned}$$

$$
\begin{aligned}
&\vee|[g_{\sigma(s)}(s,y(s),k_{\sigma(s)}(y(s)))-g_{\sigma(s)}(s,x(s),k_{\sigma(s)}(\tilde{x}(s)))|^2]\mathrm{d}s\Big|\mathscr{F}_{s_\ell}\Big]\\
&+4(t-s_\ell+1)\mathbb{E}\left[\int_{s_\ell}^{t}[|f_{\sigma(s)}(s,x(s),k_{\sigma(s)}(\tilde{x}(s)))-f_{\sigma(s)}(s,x(s),k_{\tilde{\sigma}(s)}(\tilde{x}(s)))|^2\right.\\
&\left.\vee|g_{\sigma(s)}(s,x(s),k_{\sigma(s)}(\tilde{x}(s)))-g_{\sigma(s)}(s,x(s),k_{\tilde{\sigma}(s)}(\tilde{x}(s)))|^2]\mathrm{d}s\Big|\mathscr{F}_{s_\ell}\right]\\
\leq\ &4(t-s_\ell+1)L_1^2\mathbb{E}\left[\int_{s_\ell}^{t}(|y(s)-x(s)|+L_2|y(s)-\tilde{x}(s)|)^2\mathrm{d}s\Big|\mathscr{F}_{s_\ell}\right]\\
&+4(t-s_\ell+1)L_1^2\mathbb{E}\left[\int_{s_\ell}^{t}|k_{\sigma(s)}(\tilde{x}(s))-k_{\tilde{\sigma}(s)}(\tilde{x}(s))|^2\mathrm{d}s\Big|\mathscr{F}_{s_\ell}\right]\\
\leq\ &8(t-s_\ell+1)L_1^2\mathbb{E}\left[\int_{s_\ell}^{t}[(1+L_2)^2|y(s)-x(s)|^2+L_2^2|x(s)-\tilde{x}(s)|^2]\mathrm{d}s\Big|\mathscr{F}_{s_\ell}\right]\\
&+4(t-s_\ell+1)L_1^2\mathbb{E}\left[\int_{s_\ell}^{t}|k_{\sigma(s)}(\tilde{x}(s))-k_{\tilde{\sigma}(s)}(\tilde{x}(s))|^2\mathrm{d}s\Big|\mathscr{F}_{s_\ell}\right].
\end{aligned}\tag{3.17}
$$

Substituting inequalities (3.12) and (3.13) into (3.17), and using the Gronwall inequality, we obtain (3.15). □

The main result for the case of static PETC is given as follows.

Theorem 3.1. *Consider closed-loop system* (3.4) *satisfying Assumptions 2.1 and 2.2. For given* $\epsilon \in (0,1)$*, if the sampling period* τ *and the event-triggering parameter* δ *of ETM* (3.1) *satisfy that*

$$
\epsilon+\phi^{\frac{p}{2}}(\tau,\delta,\ln(\mathrm{e}^{\gamma\tau}M/\epsilon)/\gamma)<1, \tag{3.18}
$$

where $\phi(\tau,\delta,\Delta)$ *is defined in* (3.16)*, then the trivial solution of system* (3.4) *is almost surely exponentially stable. In other words, there exists a periodic event-triggering controller* (2.3) *with static PETM* (3.1) *such that event-triggered control stochastic system* (2.1) *is almost surely exponentially stable.*

Proof. It can be seen that for any given $\epsilon \in (0,1)$, $\phi(\tau,\delta,\ln(\mathrm{e}^{\gamma\tau}M/\epsilon)/\gamma)$ is a continuous strictly increasing function respect to (τ,δ), and $\lim_{(\tau,\delta)\to(+\infty,+\infty)}\phi(\tau,\delta,\ln(\mathrm{e}^{\gamma\tau}M/\epsilon)/\gamma)=+\infty$, $\lim_{(\tau,\delta)\to(0,0)}\phi(\tau,\delta,$ $\ln(\mathrm{e}^{\gamma\tau}M/\epsilon)/\gamma)=0$. Therefore, there exist small enough positive constants τ^* and δ^* such that

$$
\phi(\tau^*,\delta^*,\ln(\mathrm{e}^{\gamma\tau^*}M/\epsilon)/\gamma)=(1-\epsilon)^{\frac{2}{p}}.
$$

For notation briefly, set $T=\frac{\ln(\mathrm{e}^{\gamma\tau}M/\epsilon)}{\gamma}$, $\iota=\lfloor\frac{T}{\tau}\rfloor$, and $\tilde{T}=\iota\tau$. Then, for any $\tau\in(0,\tau^*)$ and $\delta\in(0,\delta^*)$, it can be verified that

$$
\phi(\tau,\delta,T)<(1-\epsilon)^{\frac{2}{p}},\ \text{and}\ M\mathrm{e}^{-\gamma T+\gamma\tau}=\epsilon. \tag{3.19}
$$

By the hölder inequality, inequality (3.15) implies that for any $p\in(0,1)$,

$$
\mathbb{E}\left[|y(t)-x(t)|^p|\mathscr{F}_{s_\ell}\right]\leq\phi^{\frac{p}{2}}(\tau,\delta,t-s_\ell)|x(s_\ell)|^p.
$$

Using the well-known inequality $(a+b)^p\leq a^p+b^p$, $p\in(0,1)$,

$$\begin{aligned}\mathbb{E}\left[|x(s_\ell+\tilde{T})|^p|\mathscr{F}_{s_\ell}\right] &\le \left[|y(s_\ell+\tilde{T})-x(s_\ell+\tilde{T})|^p|\mathscr{F}_{s_\ell}\right]+\mathbb{E}\left[|y(s_\ell+\tilde{T})|^p|\mathscr{F}_{s_\ell}\right]\\ &\le \left(\phi^{\frac{p}{2}}(\tau,\delta,\tilde{T})+M\mathrm{e}^{-\gamma\tilde{T}}\right)|x(s_\ell)|^p\\ &\le \left(\phi^{\frac{p}{2}}(\tau,\delta,T)+M\mathrm{e}^{-\gamma T+\gamma\tau}\right)|x(s_\ell)|^p.\end{aligned}$$

It follows from (3.18) and (3.19) that $\mathbb{E}|x(s_{\iota+\ell})|^p \le (\phi^{\frac{p}{2}}(\tau,\delta,T)+\epsilon)\mathbb{E}|x(s_\ell)|^p$. By Lemma 2.2, we conclude that the zero solution of the closed-loop systems (2.1)-(2.3)-(3.1) is almost surely exponentially stable. □

Remark 3.2. The stability conditions established via the comparison system approach are mainly dependent on the Lipschitz constants of the system dynamics. This inevitably introduces a certain degree of conservatism, since (i) global Lipschitz constants are used in the analysis, which may overestimate the actual nonlinear growth, (ii) relaxation steps in constructing the comparison system further enlarge the bounds, and (iii) the event-triggering mechanism is characterized in terms of worst-case inter-event intervals. Nevertheless, the comparison system approach offers the advantage of yielding tractable and verifiable stability criteria without requiring explicit solutions of the original stochastic system. Therefore, a trade-off exists between conservatism and simplicity: While the conditions may be conservative, they provide a general framework that facilitates rigorous stability guarantees under event-triggered control.

3.2. Dynamic event-triggered control strategy

In this subsection, we design a dynamic ETM aimed at further reducing the number of control updates while preserving system stability. Inspired by the researchers in [5, 22], the ETM is designed by the following rule:

$$t_{i+1} = \min_{\ell\in\mathbb{N}_0}\{s_\ell > t_i \mid \eta(s_\ell)+\theta(\delta|x(s_\ell)|-|x(s_\ell)-x(t_i)|)\le 0\}, \tag{3.20}$$

and

$$\begin{cases}\dot{\eta}(t) = -\lambda_1\eta(t)+\lambda_2\left(\delta|x(s_\ell)|-|x(s_\ell)-x(t_i)|\right),\ t\in[s_\ell,s_{\ell+1}),\\ \eta(t_0)=\eta_0,\end{cases} \tag{3.21}$$

where $\theta>0$, $\lambda_1>0$, $\lambda_2>0$, and $\eta_0>0$.

Lemma 3.4. *Under ETM* (3.20) *with* $\theta \ge \frac{\lambda_2}{\lambda_1}(\mathrm{e}^{\lambda_1\tau}-1)$*, the solution* $\eta(t)$ *of dynamic equation* (3.21) *has the properties that* $\eta(t)>0$, $\forall t>0$, *and*

$$\eta^2(t) \le \mathrm{e}^{-2\lambda_0(t-s_\ell)}\eta^2(s_\ell)+\frac{\lambda_2^2\delta^2}{2(\lambda_1-\lambda_0)}\int_{s_\ell}^t \mathrm{e}^{2\lambda_0(s-t)}|x(\varsigma_s)|^2\mathrm{d}s, \tag{3.22}$$

holds for any $t\ge s_\ell$, $\ell\in\mathbb{N}_0$, *and* $\lambda_0\in[0,\lambda_1)$.

Proof. The positivity of $\eta(t)$ for all $t\ge t_0$ was proved in [5]. Moreover, choosing a Lyapunov function candidate $V_\eta(t)=\mathrm{e}^{2\lambda_0(t-s_\ell)}\eta^2$ for system (3.21) by Itô's formula, we have

$$V_\eta(t) = V_\eta(s_\ell)+\int_{s_\ell}^t \mathrm{e}^{2\lambda_0(s-s_\ell)}\{2\lambda_0\eta^2(t)+2\eta(t)[-\lambda_1\eta(t)+\lambda_2\left(\delta|x(\varsigma_s)|-|x(\varsigma_s)-\tilde{x}(s)|\right)]\}\mathrm{d}s$$

$$
\begin{aligned}
&\le V_\eta(s_\ell) + \int_{s_\ell}^{t} e^{2\lambda_0(s-s_\ell)}[-2(\lambda_1-\lambda_0)\eta^2(t) + 2\lambda_2\delta\eta(t)|x(\varsigma_s)|]ds \\
&\le V_\eta(s_\ell) + \frac{\lambda_2^2\delta^2}{2(\lambda_1-\lambda_0)}\int_{s_\ell}^{t} e^{2\lambda_0(s-s_\ell)}|x(\varsigma_s)|^2 ds,
\end{aligned}
$$

which gives the required assertion. □

Remark 3.3. Compared with the static ETM (3.1), the dynamic ETM (3.20) introduces the internal dynamic variable (3.21), which enables adaptive adjustment of the triggering threshold. Specifically, the event-triggering condition of (3.20) can be expressed as $\frac{\eta(s_\ell)}{\theta}+\delta|x(s_\ell)|-|x(s_\ell)-x(t_i)|\le 0$. It is worth noting that the dynamic ETM (3.20) reduces to the static ETM (3.1) when $\theta\to\infty$. Since the solution of (3.21) remains positive, the adaptive mechanism can enlarge the average inter-event intervals, thereby reducing unnecessary transmissions or control updates and enhancing resource efficiency while preserving the desired stability properties. Hence, dynamic ETMs generally provide a more flexible and less conservative alternative to static ETMs.

For ease of modeling, we define an auxiliary variable as (3.3), where $\mathcal{J}(t)$ is replaced by $\bar{\mathcal{J}}(t) := \eta(t)+\theta(\delta|x(t)|-|x(t)-\tilde{x}(t)|)$. Under the auxiliary variable, the closed-loop system can be modeled as (3.4). Furthermore, set $\varsigma_t = \lfloor\frac{t-t_0}{\tau}\tau\rfloor$, the dynamic system (3.21) can be rewritten as $\dot{\eta}(t) = -\lambda_1\eta(t) + \lambda_2\left(\delta|\varsigma_t)| - |x(\varsigma_t)-\tilde{x}(t)|\right)$.

The main result for the dynamic ETC is given as follows.

Theorem 3.2. *Consider closed-loop system* (3.4) *satisfying Assumptions 2.1 and 2.2. For given $\epsilon\in(0,1)$, if the sampling period τ and the event-triggering parameter $(\delta, \theta, \lambda_1, \lambda_2)$ of ETM* (3.20)–(3.21) *satisfy that $\lambda_1 > \frac{\gamma}{p}$, $\lambda_1 > \lambda_0 \ge \frac{\gamma}{p}\frac{\ln(e^{\gamma\tau}/\epsilon)}{\ln(e^{\gamma\tau}M/\epsilon)}$, $\theta \ge \frac{\lambda_2}{\lambda_1}(e^{\lambda_1\tau}-1)$, and*

$$
\epsilon + \overline{\phi}^{\frac{p}{2}}\left(\tau,\delta,\theta,\frac{\ln(e^{\gamma\tau}M/\epsilon)}{\gamma}\right) + \overline{\psi}^{\frac{p}{2}}\left(\delta,\lambda_2,\lambda_0,\frac{\ln(e^{\gamma\tau}M/\epsilon)}{\gamma}\right) < 1, \tag{3.23}
$$

where $\overline{\phi}(\tau,\delta,\theta,\Delta)$ and $\overline{\psi}(\delta,\lambda_2,\lambda_0,\Delta)$ are defined in (3.32) *and* (3.36)*, respectively. Then the trivial solution of system* (3.4) *is almost surely exponentially stable. In other words, there exists a periodic event-triggering controller* (2.3) *with dynamic ETM* (3.20)–(3.21) *such that event-triggered control stochastic system* (2.1) *is almost surely exponentially stable.*

In order to give a strict and explicit proof of Theorem 3.2, several important lemmas are first introduced.

Lemma 3.5. *Let $x(t)$ be the solution of closed-loop system* (3.4)*, then there holds that for any $t\ge s_\ell$, $\ell\in\mathbb{N}_0$,*

$$
\sup_{s_\ell\le v\le t}\mathbb{E}\left[|x(v)|^2|\mathscr{F}_{s_\ell}\right] + \sup_{s_\ell\le v\le t}\mathbb{E}\left[\eta^2(v)|\mathscr{F}_{s_\ell}\right] \le (|x(s_\ell)|^2+\eta^2(s_\ell))e^{\underline{\beta}_L(t-s_\ell)}, \tag{3.24}
$$

$$
\mathbb{E}\left[\sup_{s_\ell\le s\le t}|x(s)|^2 + \sup_{s_\ell\le s\le t}\eta^2(s)\middle|\mathscr{F}_{s_\ell}\right] \le 2(|x(s_\ell)|^2+\eta^2(s_\ell)e^{2\overline{\beta}_L(t-s_\ell)}, \tag{3.25}
$$

$$
\mathbb{E}\left[|x(t)-\tilde{x}(t)|^2|\mathscr{F}_{s_\ell}\right] \le \overline{\varrho}(\tau,\delta,\theta)(|x(s_\ell)|^2+\eta^2(s_\ell))e^{\underline{\beta}_L(t-s_\ell)}, \tag{3.26}
$$

where $\underline{\beta}_L = (2L_1 + 3L_1^2 + 2L_2^2(1 + 2L_1^2)(\delta + 1)^2 + \frac{\lambda_2^2\delta^2}{2\lambda_1}) \vee \frac{2L_2^2(1+2L_1^2)}{\theta^2}$, $\overline{\beta}_L = (2L_1 + 131L_1^2 + L_2^2(1 + 130L_1^2)(\delta + 1)^2 + \frac{\lambda_2^2\delta^2}{4\lambda_1}) \vee \frac{L_2^2(1+130L_1^2)}{\theta^2}$, *and* $\overline{\varrho}(\tau, \delta, \theta) = (12\tau(\tau + 1)L_1^2L_2^2e^{6\tau(\tau+1)L_1^2} + 4)/\theta^2 \vee (12\tau(\tau + 1)L_1^2(1 + L_2(\delta + 1))^2e^{6\tau(\tau+1)L_1^2} + 4\delta^2)$.

Proof. By dynamic event-triggering mechanism (3.20), and the notations ς_t and $\tilde{x}(t)$, one can obtain that for any $t \geq t_0$,

$$|\tilde{x}(t)| \leq |\tilde{x}(t) - x(\varsigma_t)| + |x(\varsigma_t)| \leq \frac{\eta(\varsigma_t)}{\theta} + (\delta + 1)|x(\varsigma_t)|. \tag{3.27}$$

Then, it is easy to show from (3.8) that

$$\begin{aligned}\mathbb{E}[|x(t)|^2|\mathscr{F}_{s_\ell}] \leq |x(s_\ell)|^2 + \mathbb{E}\left[\int_{s_\ell}^t [(2L_1 + 3L_1^2)|x(s)|^2 + 2L_2^2(1 + 2L_1^2)(\delta + 1)^2|x(\varsigma_s)|^2]\mathrm{d}s \middle| \mathscr{F}_{s_\ell}\right] \\ + \frac{2L_2^2(1 + 2L_1^2)}{\theta^2}\int_{s_\ell}^t \mathbb{E}[\eta^2(\varsigma_s)|\mathscr{F}_{s_\ell}]\mathrm{d}s.\end{aligned}$$

Using the fact that $\sup_{s_\ell\leq s\leq t}\mathbb{E}[|x(s)|^2|\mathscr{F}_{s_\ell}] \geq \mathbb{E}[|x(\varsigma_t)|^2|\mathscr{F}_{s_\ell}] \vee \mathbb{E}[|x(t)|^2|\mathscr{F}_{s_\ell}]$, the above inequality yields

$$\begin{aligned}\sup_{s_\ell\leq s\leq t}\mathbb{E}[|x(s)|^2|\mathscr{F}_{s_\ell}] \leq |x(s_\ell)|^2 + (2L_1 + 3L_1^2 + 2L_2^2(1 + 2L_1^2)(\delta + 1)^2)\int_{s_\ell}^t \sup_{s_\ell\leq s\leq v}\mathbb{E}[|x(s)|^2|\mathscr{F}_{s_\ell}]\mathrm{d}v \\ + \frac{2L_2^2(1 + 2L_1^2)}{\theta^2}\int_{s_\ell}^t \mathbb{E}[\eta^2(\varsigma_s)|\mathscr{F}_{s_\ell}]\mathrm{d}s.\end{aligned} \tag{3.28}$$

Based on the fact that $\mathbb{E}[|x(\zeta_s)|^2|\mathscr{F}_{s_\ell}] \leq \sup_{s_\ell\leq v\leq s}\mathbb{E}[|x(v)|^2|\mathscr{F}_{s_\ell}]$ holds for any $s \in [s_\ell, t]$. Letting $\lambda_0 = 0$ in (3.22), and taking the conditional mathematical expectation on the two sides of (3.22), one can obtain

$$\sup_{s_\ell\leq s\leq t}\mathbb{E}[\eta^2(s)|\mathscr{F}_{s_\ell}] \leq \eta^2(s_\ell) + \frac{\lambda_2^2\delta^2}{2\lambda_1}\int_{s_\ell}^t \sup_{s_\ell\leq s\leq v}\mathbb{E}[|x(s)|^2|\mathscr{F}_{s_\ell}]\mathrm{d}v. \tag{3.29}$$

Combining (3.28) and (3.29), we have

$$\begin{aligned}\sup_{s_\ell\leq s\leq t}\mathbb{E}[|x(s)|^2|\mathscr{F}_{s_\ell}] + \sup_{s_\ell\leq s\leq t}\mathbb{E}[\eta^2(s)|\mathscr{F}_{s_\ell}] \leq |x(s_\ell)|^2 + \eta^2(s_\ell) + \underline{\beta}_L\int_{s_\ell}^t\{\sup_{s_\ell\leq s\leq v}\mathbb{E}[|x(s)|^2|\mathscr{F}_{s_\ell}] \\ + \sup_{s_\ell\leq s\leq v}\mathbb{E}[\eta^2(s)|\mathscr{F}_{s_\ell}]\}\mathrm{d}v.\end{aligned}$$

An application of the Gronwall inequality gives the required assertion (3.24).

Similarly, set $\lambda_0 = 0$ in (3.22), we have

$$\sup_{s_\ell\leq s\leq t}\eta^2(s) \leq \eta^2(s_\ell) + \frac{\lambda_2^2\delta^2}{2\lambda_1}\int_{s_\ell}^t \sup_{s_\ell\leq s\leq v}|x(s)|^2\mathrm{d}v.$$

Then, it follows from (3.11) and (3.27) that

$$\begin{aligned}\mathbb{E}[\sup_{s_\ell\leq s\leq t}|x(s)|^2 + \sup_{s_\ell\leq s\leq t}\eta^2(s)|\mathscr{F}_{s_\ell}] \leq 2(|x(s_\ell)|^2 + \eta^2(s_\ell)) \\ + 2\overline{\beta}_L\int_{s_\ell}^t \mathbb{E}[\sup_{s_\ell\leq s\leq v}|x(s)|^2| + \sup_{s_\ell\leq s\leq v}\eta^2(s)|\mathscr{F}_{s_\ell}]\mathrm{d}v\end{aligned}$$

$$\leq 2(|x(s_\ell)|^2 + \eta^2(s_\ell))e^{2\overline{\beta}_L(t-s_\ell)}.$$

Finally, let us proceed to prove the assertion (3.26). By Yong's inequality and (3.27),

$$\begin{aligned}\mathbb{E}[|x(t) - \tilde{x}(t)|^2|\mathscr{F}_{s_\ell}] &\leq 2\mathbb{E}[|x(t) - x(\varsigma_t)|^2|\mathscr{F}_{s_\ell}] + 2\mathbb{E}[|x(\varsigma_t) - \tilde{x}(t)|^2|\mathscr{F}_{s_\ell}] \\ &\leq 2\mathbb{E}[|x(t) - x(\varsigma_t)|^2|\mathscr{F}_{s_\ell}] + \frac{4}{\theta^2}\mathbb{E}[\eta^2(\varsigma_t)|\mathscr{F}_{s_\ell}] + 4\delta^2\mathbb{E}[|x(\varsigma_t)|^2|\mathscr{F}_{s_\ell}]. \qquad (3.30)\end{aligned}$$

Now, we use Hölder's inequality and Burkholder-Davis-Gundy inequality to estimate the first term of (3.30).

$$\begin{aligned}\mathbb{E}\left[|x(t) - x(\varsigma_t)|^2|\mathscr{F}_{s_\ell}\right] &= \mathbb{E}\Bigg[\bigg|\int_{\varsigma_t}^t f_{\sigma(s)}(s, x(s), k_{\tilde{\sigma}(s)}(\tilde{x}(s)))\mathrm{d}s \\ &\quad + \int_{\varsigma_t}^t g_{\sigma(s)}(s, x(s), k_{\tilde{\sigma}(s)}(\tilde{x}(s)))\mathrm{d}w(s)\bigg|^2 \Big|\mathscr{F}_{s_\ell}\Bigg] \\ &\leq 2L_1^2(\tau + 1)\int_{\varsigma_t}^t \mathbb{E}\left[(|x(s)| + L_2|\tilde{x}(s)|)^2|\mathscr{F}_{s_\ell}\right]\mathrm{d}s \\ &\leq 2L_1^2(\tau + 1)\int_{\varsigma_t}^t \mathbb{E}\left[(|x(s) - x(\varsigma_s)| + \frac{L_2}{\theta}\eta(\varsigma_s) + [1 + L_2(\delta + 1)]|x(\varsigma_s)|)^2|\mathscr{F}_{s_\ell}\right]\mathrm{d}s \\ &\leq 6\tau L_1^2(\tau + 1)e^{6\tau(\tau+1)L_1^2}\mathbb{E}\left[\frac{L_2^2}{\theta^2}\eta^2(\varsigma_t) + [1 + L_2(\delta + 1)]^2|x(\varsigma_t)|^2|\mathscr{F}_{s_\ell}\right].\end{aligned}$$

Substituting the aforementioned inequality into (3.30) gives

$$\mathbb{E}[|x(t) - \tilde{x}(t)|^2|\mathscr{F}_{s_\ell}] \leq \overline{\varrho}(\tau, \delta, \theta)\mathbb{E}\left[|x(\varsigma_t)|^2 + \eta^2(\varsigma_t)|\mathscr{F}_{s_\ell}\right].$$

It follows from (3.24) that (3.26) holds for any $t \geq s_\ell$. □

Lemma 3.6. *Let $x(t)$ be the solution of system* (3.4)*, and $y(t)$ be the solution of system* (2.2) *with initial value $(x(s_\ell), \sigma(s_\ell))$. Then, there holds that for any $t \geq s_\ell$,*

$$\mathbb{E}\left[|y(t) - x(t)|^2|\mathscr{F}_{s_\ell}\right] \leq \overline{\phi}(\tau, \delta, \theta, t - s_\ell)(|x(s_\ell)|^2 + \eta^2(s_\ell)), \qquad (3.31)$$

where

$$\overline{\phi}(\tau, \delta, \theta, \Delta) = \frac{8(\Delta + 1)L_1^2L_2^2}{\underline{\beta}_L}\left[\overline{\varrho}(\tau, \delta, \theta) + 4(1 - e^{\underline{\pi}\tau})(\frac{1}{\theta^2} \vee (\delta + 1)^2)\right]\left(e^{\underline{\beta}_L\Delta} - 1\right)e^{8\Delta(\Delta+1)L_1^2(1+L_2)^2}. \qquad (3.32)$$

Proof. With the help of (3.5), (3.14), and (3.17), we see that

$$\begin{aligned}\mathbb{E}\left[|y(t) - x(t)|^2|\mathscr{F}_{s_\ell}\right] &\leq 8(t - s_\ell + 1)L_1^2(1 + L_2)^2\int_{s_\ell}^t \mathbb{E}[|y(s) - x(s)|^2|\mathscr{F}_{s_\ell}]\mathrm{d}s \\ &\quad + \frac{8(t - s_\ell + 1)L_1^2L_2^2\overline{\varrho}(\tau, \delta, \theta)}{\underline{\beta}_L}\left(e^{\underline{\beta}_L(t-s_\ell)} - 1\right)(|x(s_\ell)|^2 + \eta^2(s_\ell)) \\ &\quad + 16(t - s_\ell + 1)L_1^2L_2^2(1 - e^{\underline{\pi}\tau})\int_{s_\ell}^t \mathbb{E}\left[|\tilde{x}(s)|^2|\mathscr{F}_{s_\ell}\right]\mathrm{d}s. \qquad (3.33)\end{aligned}$$

In view of Lemma 3.4 and (3.27), we see that

$$\begin{aligned}\mathbb{E}\left[|\tilde{x}(s)|^2|\mathscr{F}_{s_\ell}\right] &\le 2\int_{s_\ell}^{t}\mathbb{E}\left[\frac{1}{\theta^2}(\delta+1)^2\eta^2(\varsigma_s)+(\delta+1)^2|x(\varsigma_s)|^2\bigg|\mathscr{F}_{s_\ell}\right]\mathrm{d}s\\ &\le 2(\frac{1}{\theta^2}\vee(\delta+1)^2)\int_{s_\ell}^{t}\{\sup_{s_\ell s\le v}\mathbb{E}\left[\eta^2(s)|\mathscr{F}_{s_\ell}\right]+\sup_{s_\ell s\le v}\mathbb{E}\left[|x(s)|^2|\mathscr{F}_{s_\ell}\right]\}\mathrm{d}v\\ &\le \frac{2(\frac{1}{\theta^2}\vee(\delta+1)^2)}{\underline{\beta}_L}\left(\mathrm{e}^{\underline{\beta}_L(t-s_\ell)}-1\right)(|x(s_\ell)|^2+\eta^2(s_\ell)).\end{aligned}$$

Substituting the above estimation to (3.33) and using the Gronwall inequality, one can obtain the assertion (3.15). □

Now, we are ready to prove Theorem 3.2.

Proof. By the hölder inequality and the inequality $(a+b)^p\le a^p+b^p$, $p\in(0,1)$, inequality (3.15) implies that for any $p\in(0,1)$,

$$\mathbb{E}\left[|y(t)-x(t)|^p|\mathscr{F}_{s_\ell}\right]\le\overline{\phi}^{\frac{p}{2}}(\tau,\delta,\theta,t-s_\ell)(|x(s_\ell)|^p+\eta^p(s_\ell)). \tag{3.34}$$

On the other hand, it follows from Lemma 3.5 that for any $t\ge s_\ell$,

$$\mathbb{E}\left[\eta^2(t)|\mathscr{F}_{s_\ell}\right]\le[\mathrm{e}^{-2\lambda_0(t-s_\ell)}+\overline{\psi}(\delta,\lambda_2,\lambda_0,t-s_\ell)]\eta^2(s_\ell)+\overline{\psi}(\delta,\lambda_2,\lambda_0,t-s_\ell)|x(s_\ell)|^2$$

which implies that

$$\mathbb{E}\left[\eta^p(t)|\mathscr{F}_{s_\ell}\right]\le[\mathrm{e}^{-p\lambda_0(t-s_\ell)}+\overline{\psi}^{\frac{p}{2}}(\delta,\lambda_2,\lambda_0,t-s_\ell)]\eta^p(s_\ell)+\overline{\psi}^{\frac{p}{2}}(\delta,\lambda_2,\lambda_0,t-s_\ell)|x(s_\ell)|^p, \tag{3.35}$$

where $\overline{\psi}(\delta,\lambda_2,\lambda_0,\Delta)$ is defined as

$$\overline{\psi}(\delta,\lambda_2,\lambda_0,\Delta)=\frac{\lambda_2^2\delta^2}{2(\lambda_1-\lambda_0)(2\lambda_0+\underline{\beta}_L)}(\mathrm{e}^{\underline{\beta}_L\Delta}-\mathrm{e}^{-2\lambda_0\Delta}). \tag{3.36}$$

Let $\chi(\tau,\delta,\theta,\epsilon)$ denote the left side of (3.23). It can be seen that for any given $\epsilon\in(0,1)$ and $\theta>0$, $\chi(\tau,\delta,\theta,\epsilon)$ is a continuous strictly increasing function with respect to (τ,δ), and satisfies that $\lim_{\theta\to+\infty}\chi(0,0,\theta,\epsilon)=\epsilon$ and $\chi(\tau,\delta,\theta,\epsilon)=+\infty$ as $(\tau,\delta,\theta)\to(+\infty,+\infty,0)$. Therefore, there exist small enough positive constants τ, δ, and a sufficient large scalar θ such that $\chi(\tau,\delta,\theta,\epsilon)<1$.

Set $T=\frac{\ln(\mathrm{e}^{\gamma\tau}M/\epsilon)}{\gamma}$, $\iota=\lfloor\frac{T}{\tau}\rfloor$, $\tilde{T}=\iota\tau$, and $\lambda_0=\frac{\gamma}{p}\frac{\ln(\mathrm{e}^{\gamma\tau}/\epsilon)}{\ln(\mathrm{e}^{\gamma\tau}M/\epsilon)}$. Note that $\epsilon\in(0,1)$ and $M\ge1$. one has $0<\lambda_0\le\frac{\gamma}{p}<\lambda_1$. Moreover, it can be verified that

$$M\mathrm{e}^{-\gamma T+\gamma\tau}=\epsilon,\ \text{and}\ \mathrm{e}^{-p\lambda_0T+\gamma\tau}=\epsilon. \tag{3.37}$$

Using (3.34)–(3.37), we get

$$\begin{aligned}\mathbb{E}\left[|x(s_\ell+\tilde{T})|^p+\eta^p(s_\ell+\tilde{T})|\mathscr{F}_{s_\ell}\right]\le{}&\mathbb{E}\left[|y(s_\ell+\tilde{T})-x(s_\ell+\tilde{T})|^p|\mathscr{F}_{s_\ell}\right]+\mathbb{E}\left[|y(s_\ell+\tilde{T})|^p|\mathscr{F}_{s_\ell}\right]\\&+\mathbb{E}\left[\eta^p(s_\ell+\tilde{T})|\mathscr{F}_{s_\ell}\right]\end{aligned}$$

$$
\begin{aligned}
&\le \overline{\phi}^{\frac{p}{2}}(\tau,\delta,\theta,\tilde{T})(|x(s_\ell)|^p+\eta^p(s_\ell))+M\mathrm{e}^{-\gamma\tilde{T}}|x(s_\ell)|^p\\
&\quad+[\mathrm{e}^{-p\lambda_0\tilde{T}}+\overline{\psi}^{\frac{p}{2}}(\delta,\lambda_2,\lambda_0,\tilde{T})]\eta^p(s_\ell)+\overline{\psi}^{\frac{p}{2}}(\delta,\lambda_2,\lambda_0,\tilde{T})|x(s_\ell)|^p\\
&\le \chi(\tau,\delta,\theta,\epsilon)(|x(s_\ell)|^p+\eta^p(s_\ell))
\end{aligned}
$$

which means that $\mathbb{E}[|x(s_{\iota+\ell})|^p+\eta^p(s_{\iota+\ell})]\le\chi(\tau,\delta,\theta,\epsilon)\mathbb{E}[|x(s_\ell)|^p+\eta^p(s_\ell)]$. By Lemma 2.2, we conclude that the zero solution of the event-triggering stochastic systems (2.1)-(2.3)-(3.20)-(3.21) is almost surely exponentially stable. □

4. Stability verification and emulation control law design

In this section, we present Lyapunov-based conditions for verifying Assumption 2.2 in both the nonlinear and linear cases, respectively.

For the general nonlinear case, sufficient conditions guaranteeing the exponential moment stability of the closed-loop system (2.2) are stated as follows.

Proposition 4.1 (Stochastic control Lyapunov function (SCLF) condition)**.** *For each mode $i\in\mathcal{M}$, suppose there exists a stochastic control Lyapunov function $V_i\in C^2(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}_{\ge0})$ and constants $c_1>0$, $c_2>0$, $\mu>0$, $p\in(0,1)$, $q>p$, and η_i such that $c_1|x|^q\le V_i(x)\le c_2|x|^q$ and, for all $x\in\mathbb{R}^n$, $t\ge0$, one can choose a control $u=k_i(x)$ satisfying*

$$
\mathcal{L}V_i(t,x,k_i(x))+0.5c_{pq}\eta_i^2V_i(t,x)-c_{pq}\eta_i\mathcal{H}V_i(t,x,k_i(x))\le\mu V_i(t,x), \tag{4.1}
$$

where $\mathcal{L}V_i(t,x,u)=\frac{\partial V_i(t,x)}{\partial t}+\frac{\partial V_i(t,x)}{\partial x}f_i(t,x,u)+\frac{1}{2}\operatorname{tr}(g_i^\top(t,x,u)\frac{\partial^2V(t,x)}{\partial x^2}g_i(t,x,u))+\sum_{j\in\mathcal{M}}\pi_{ij}V_j(t,x)$, $\mathcal{H}V_i(t,x,u)=\frac{\partial V_i(t,x)}{\partial x}g_i(t,x,u)$, and $c_{pq}=1-\frac{q}{p}$. Then the closed-loop system (2.2) *is exponentially pth-moment stable, and its solution has the following estimate*

$$
\mathbb{E}|x(t)|^p\le(c_2/c_1)^{\frac{p}{q}}|x_0|^pe^{-\frac{p\mu}{q}(t-t_0)}. \tag{4.2}
$$

Proof. The proof follows similar arguments as those in Theorem 3 of [23] and is therefore omitted. □

It should be noted that Proposition 4.1 provides a constructive criterion: Any Lipschitz feedback $k_i(\cdot)$ that enforces the inequality (4.1) (e.g., via pointwise optimization or a quadratic program) guarantees Assumption 2.2. Consequently, the control input u can be systematically designed using the SCLF approach combined with quadratic programming. Furthermore, if the closed-loop system (2.2) possesses a strict-feedback structure, the stochastic backstepping technique can be employed as an alternative for constructing the controller.

For the linear case, the system is described by the Itô stochastic differential equation

$$
\mathrm{d}x(t)=\left[A_{\sigma(t)}x(t)+B_{\sigma(t)}u(t)\right]\mathrm{d}t+\left[C_{\sigma(t)}x(t)+D_{\sigma(t)}u(t)\right]\mathrm{d}w(t), \tag{4.3}
$$

where $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times n_u}$, $C\in\mathbb{R}^{n\times n}$, $D\in\mathbb{R}^{n\times n_u}$ are real matrices, and $u(t)$ is the control input.

We consider the mode-dependent state feedback control $u(t)=K_{\sigma(t)}x(t)$, where K_i, $i\in\mathcal{M}$ are constant matrices. Then, by applying Proposition 4.1, the almost sure stability criterion of the resulting closed-loop system can be established as stated below.

Proposition 4.2 (Stability criterion). *If for some prescribed scalars $p > 0$, $\mu > 0$, and $\eta_i \in \mathbb{R}$, $i \in \mathcal{M}$, there exist $n \times n$ positive definite matrices P_i, $i \in \mathcal{M}$, such that the following linear matrix inequalities (LMIs) hold for all $i \in \mathcal{M}$:*

$$P_i\bar{A}_i + \bar{A}_i^{\mathrm{T}}P_i + \bar{C}_i^{\mathrm{T}}P_i\bar{C}_i + \sum_{j=1}^{N}\pi_{ij}P_j + \left(\frac{c_p}{2}\eta_i^2 + \mu\right)P_i - c_p\eta_i(P_i\bar{C}_i + \bar{C}_i^{\mathrm{T}}P_i) \leq 0, \tag{4.4}$$

where $\bar{A}_i = A_i + B_iK_i$, $\bar{C}_i = C_i + D_iK_i$, and $c_p = 1 - \frac{p}{2}$, then system (4.3) *is almost surely exponentially stable, and its solution has the estimate* (4.2) *with $q = 2$, $c_1 = \min_{i\in\mathcal{M}}\lambda_{\min}(P_i)$, and $c_2 = \max_{i\in\mathcal{M}}\lambda_{\max}(P_i)$.*

Proof. The proof follows Proposition 4.1 and is therefore omitted. □

The following result shows that the aforementioned stability criterion can be used to determine almost surely exponentially stabilizing control laws.

Proposition 4.3 (Feedback controller design). *If for some prescribed scalars $p > 0$, $\mu > 0$, and $\eta_i \in \mathbb{R}$, $i \in \mathcal{M}$, there exist $n \times n$ positive definite matrices X_i, and $n_u \times n$ matrices $\bar{K}_i$, $i = 1, 2, \ldots, N$, such that*

$$\begin{bmatrix} \Psi_i(\mu, p) & X_iC_i^{\mathrm{T}} + \bar{K}_i^{\mathrm{T}}D_i^{\mathrm{T}} & \mathcal{Y}_i \\ * & -X_i & 0 \\ * & * & -\mathcal{X}_i \end{bmatrix} \leq 0, \ i \in \mathcal{M}, \tag{4.5}$$

where $\Psi_i(\mu, p) = A_iX_i + X_iA_i^{\mathrm{T}} + B_i\bar{K}_i + \bar{K}_i^{\mathrm{T}}B_i^{\mathrm{T}} - c_p\eta_i(C_iX_i + X_iC_i^{\mathrm{T}} + D_i\bar{K}_i + \bar{K}_i^{\mathrm{T}}D_i^{\mathrm{T}}) + (0.5c_p\eta_i^2 + \mu + \pi_{ii})X_i$, $\mathcal{Y}_i = \begin{bmatrix} \sqrt{\pi_{i1}}X_i & \ldots & \sqrt{\pi_{i(i-1)}}X_i & \sqrt{\pi_{i(i+1)}}X_i & \ldots & \sqrt{\pi_{iN}}X_i \end{bmatrix}$, and $\mathcal{X}_i = \mathrm{diag}\{X_1, \ldots, X_{i-1}, X_{i+1}\ldots, X_N\}$. Then the feedback control $u(t) = K_{\sigma(t)}x(t)$ with $K_i = \bar{K}_iX_i^{-1}$ is almost surely exponentially stabilized system (4.3)*, and the solution $x(t)$ satisfies* (4.2) *with $q = 2$, $c_1 = \frac{1}{\max_{i\in\mathcal{M}}\lambda_{\max}(X_i)}$, and $c_2 = \frac{1}{\min_{i\in\mathcal{M}}\lambda_{\min}(X_i)}$.*

Proof. By using $P = X^{-1}$ and the Schur complement, it can be proven that (4.5) is equivalent to (4.4). □

Remark 4.1. It should be noted that once the scalar parameters $p > 0$, $\mu > 0$, and $\eta_i \in \mathbb{R}$ are specified, the matrix inequalities (4.4) and (4.5) become LMIs. In practice, we employ a two-step iterative procedure: First, the scalar parameters μ, p, and η_i are fixed within reasonable ranges, and then the resulting conditions reduce to LMIs in the matrix variables P_i or X_i, which can be efficiently solved using standard LMI solvers. By scanning the scalar parameters, feasible solutions to the coupled inequalities can be obtained.

5. Numerical examples

Example 5.1 (Stabilization by noise). Consider the Markovian jump system (4.3) with the parameters:

$$A_1 = \begin{bmatrix} 0.5 & 1 \\ 0 & -1.5 \end{bmatrix}, \ A_2 = \begin{bmatrix} -1.5 & 0 \\ 1 & 0.5 \end{bmatrix}, \ \Pi = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ B_i = C_i = 0, i = 1, 2,$$

and the switching signal $\{\sigma(t) \in \mathcal{M} \triangleq \{1,2\}\}_{t\geq 0}$. Figure 2a shows that the sample path trajectory of the open-loop system is unstable. Moreover, the controller acts only on the diffusion term. The existing results of the event-triggered control, e.g., [7,8,20], are not applicable to such systems. By solving the linear matrix inequalities (4.5) with the choice $p = 10^{-5}$, $\mu = 0.5$, $\eta_1 = 1.8$, and $\eta_2 = -1.9$, we have the gain matrices: $K_1 = \begin{bmatrix} 1.1074 & 0.6873 \end{bmatrix}$, $K_2 = \begin{bmatrix} -0.7256 & -1.0928 \end{bmatrix}$, and the parameters $c_1 = 3.42\times 10^{-4}$ and $c_2 = 2.40 \times 10^{-3}$. Therefore, Theorem 3.1 (resp. Theorem 3.2) ensures the existence of a periodic event-triggering controller (2.3) with a static ETM (3.1) (resp. a dynamic ETM (3.20)–(3.21)) that stabilizes the stochastic hybrid system (2.1) in the almost sure sense. In fact, one can verify that the conditions of Theorem 3.2 hold with $\tau = 10^{-324}$, $\delta = 10^{-162}$, $L_1 = 1.8251$, $L_2 = 1.3088$, $\theta = 10^{155}$, $\lambda_1 = 0.25$, $\lambda_2 = 1$, and $\epsilon = 1 - 10^{-5}$. Nevertheless, it should be emphasized that the comparison system method for designing event-triggering parameters is rather conservative. Indeed, when the parameters of ETM (3.20)–(3.21) are instead chosen as $\tau = 0.001$, $\delta = 0.01$, $\theta = 1$, $\lambda_1 = 0.5$, $\lambda_2 = 0.1$, with initial conditions $x_0 = (1, -1.5)^\top$ and $\sigma_0 = 1$, the simulation results in Figure 2 demonstrate that both the sample-path solution of the closed-loop system and the trajectory of the auxiliary variable $\eta(t)$ still converge to the origin.

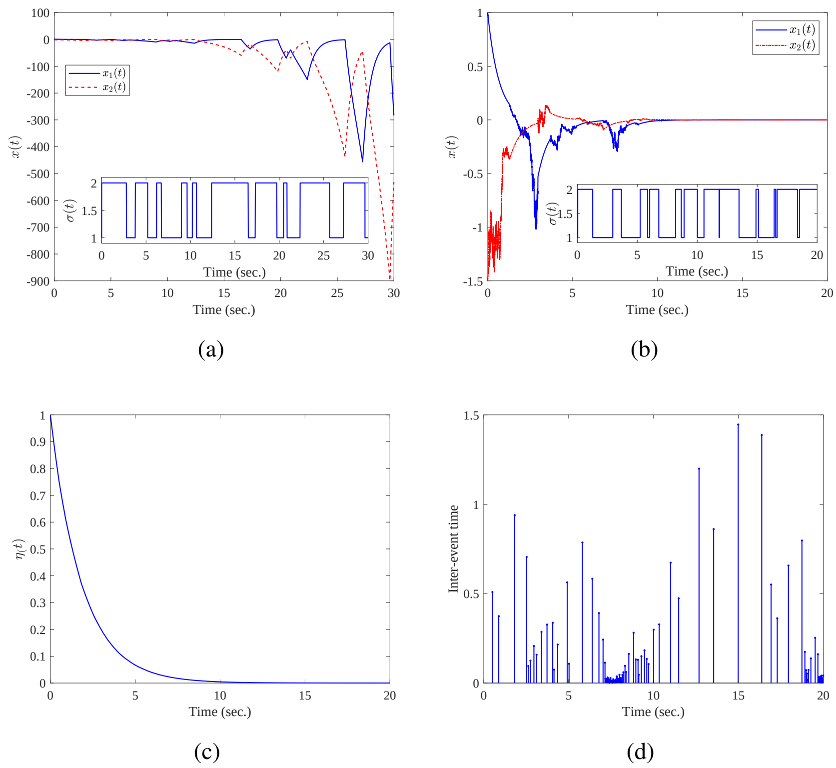

Figure 2. Simulation for Example 5.1: (a) State response of the open-loop system; (b) state response of the closed-loop system; (c) state response of $\eta(t)$; and (d) inter-event times.

Example 5.2 (Disturbance by noise)**.** Consider a single-link robot arm [24] described by the following dynamic equation:

$$J_{\sigma(t)}\ddot{q}(t) + M_{\sigma(t)}gL\sin(q(t)) + D(t)\dot{q}(t) = u(t) \tag{5.1}$$

where $q(t)$ and $u(t)$ are the angle position of the arm and the control input, respectively. $M_{\sigma(t)}$ is the mass of the payload, $J_{\sigma(t)}$ is the moment of inertia, g is the acceleration of gravity, L is the length of the arm, and $D(t)$ is the coefficient of viscous friction interfered by white noise, satisfying $D(t) = D_0 + D_w\xi(t)$, where the nominal value $D_0 = 2$, D_w is the noise intensity, and $\xi(t)$ is Gaussian while noise that satisfies $\int_0^t \xi(s)ds = w(t)$, $t \geq 0$. Moreover, the behavior of $\sigma(t)$ is modeled as Markov chain with three different states with generator: $\Pi = \begin{bmatrix} -3 & 0.5 & 2.5 \\ 1 & -2 & 1 \\ 0.7 & 0.3 & -1 \end{bmatrix}$. The parameters $g = 9.81$, $L = 0.5$, $J_i = M_i$, $i = 1, 2, 3$, and the values of J_1, J_2, J_3 are given by 1, 5, and 10, respectively.

By introducing the variable changes: $x_1(t) = q(t)$, $x_2(t) = \dot{q}(t)$, system (5.1) can be written as system (2.1) with $f_i(t, x, u) = \begin{bmatrix} x_2 \\ -\frac{M_i gL}{J_i}\sin(x_1) - \frac{D_0}{J_i}x_2 + \frac{1}{J_i}u \end{bmatrix}$ and $g_i(t, x, u) = \begin{bmatrix} 0 \\ -\frac{D_w}{J_i}x_2 \end{bmatrix}$. Clearly, Assumption 2.1 holds with $L_1 = 5.3109$. Choose $u(t) = K_{\sigma(t)}x_1(t)$ in which $K_1 = -0.13$, $K_2 = -0.54$, and $K_3 = -0.65$, then it can be verified that Assumption 2.2 holds. By applying Theorem 3.2 with the choices $\tau = 10^{-325}$, $\delta = 10^{-165}$, $L_1 = 5.3109$, $L_2 = 0.65$, $\theta = 10^{155}$, $\epsilon = 1 - 10^{-6}$, $\lambda_1 = 2.5 \times 10^5$, and $\lambda_2 = 1$, it can be verified that inequality (3.23) holds. Therefore, there exists a periodic event-triggered control law designed in (2.3) with the dynamic ETM (3.20)–(3.21) such that system (5.1) can be almost surely stabilized. On the other hand, when the parameters are chosen as $\tau = 0.01$, $\delta = 0.2$, $\theta = 1$, $\lambda_1 = 0.1$, and $\lambda_2 = 1$, the simulation results are shown in Figure 3, indicating that the trajectories of system (5.1) still converge to the origin. This demonstrates that the theoretical conditions derived in Theorem 3.2 are conservative, and in practice, much less restrictive parameter choices are sufficient to guarantee stabilization.

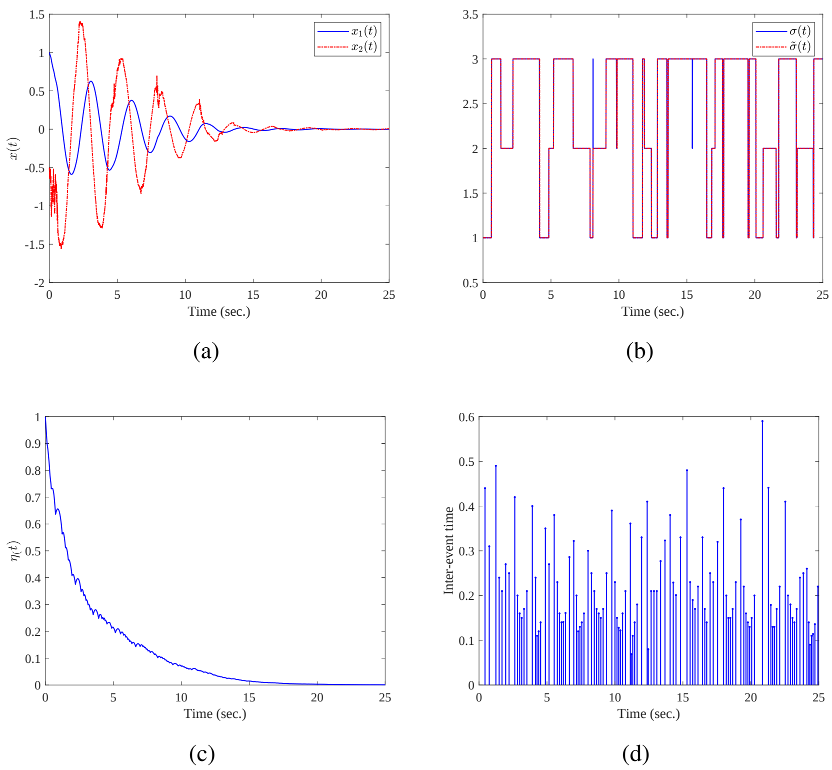

Figure 3. Simulation for Example 5.2: (a) state response of $x(t)$; (b) switching signals; (c) state response of $\eta(t)$; (d) inter-event times.

6. Conclusions

This paper has shown that there exist static and dynamic periodic event-triggered asynchronous controllers such that the hybrid stochastic system driven by the continuous-time Markov chain and Brownian motion is almost surely exponentially stable as long as the corresponding continuous-time feedback control system is almost surely exponentially stable, and the sampling period and event-triggering parameters are small enough. Furthermore, the proposed PETC strategies can incorporate the positive effects of stochastic noises and can be used to solve stochastic stabilization problems in the presence of sampled measurements. Several promising directions for future research are worth pursuing, including (i) extending the proposed framework to broader classes of stochastic systems, such as impulsive systems [25], (ii) developing approaches to reduce conservatism by exploiting local Lipschitz properties, and (iii) integrating event-triggered strategies with learning-based control methods.

Author contributions

Shixian Luo: Methodology, conceptualization, validation, writing – original draft; Linna Wei: Methodology, writing – review & editing; Zi-Peng Wang: Methodology, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Zi-Peng Wang is an editorial board member for AIMS Mathematics and was not involved in the editorial review and the decision to publish this article.

The authors declare that there are no conflicts of interest.

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