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Research article

Multiplicity of periodic solutions for second-order nonlinear partial difference equations with ϕ-Laplacian

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Abstract: In this paper, we primarily employ the critical point theory to examine the multiplicity of periodic solutions for a specific class of second-order nonlinear partial difference equations with ϕ-Laplacian. We can study the more general ϕ-Laplacian, obtain clearer sufficient conditions for the multiplicity of periodic solutions of the equation, and compare our results with existing works. Our result can be applied to exploring the spacetime periodic solutions of the two-dimensional discrete nonlinear Schrödinger equation.

Keywords: partial difference equation; periodic solution; ϕ-Laplacian; critical point theory
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1. Introduction

Let the natural number set, real number set, and integer set be denoted by $\mathbb{N}$, and $\mathbb{R}$, $\mathbb{Z}$, respectively. Define $\mathbb{Z}(a,b) := \{a, a+1, \cdots, b\}$ with integers $a \leq b$. In this paper, the following second-order nonlinear difference equation with ϕ-Laplacian is investigated:

$$\begin{aligned} &-\Delta_1\left[\phi\left(\Delta_1 u\left(n-1,m\right)\right)\right]-\Delta_2\left[\phi\left(\Delta_2 u\left(n,m-1\right)\right)\right]+q\left(n,m\right)\phi\left(u\left(n,,m\right)\right)\\ =&f\left(\left(n,m\right),u\left(n,m\right)\right),\qquad \left(n,m\right)\in\mathbb{Z}^2. \end{aligned} \tag{1.1}$$

Here, the operators $\Delta_1 u(n,m) = u(n+1,m) - u(n,m)$ and $\Delta_2 u(n,m) = u(n,m+1) - u(n,m)$ are defined as forward difference operators. The function $f : \mathbb{Z}^2 \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous with respect to its last variable. Given the positive integers T_1 and T_2, we consider the functions $q(n,m) \geq 0$ and $f((n,m),\cdot)$ to be (T_1,T_2)-periodic, that is, $q(n+T_1,m) = q(n,m) = q(n,m+T_2)$ and $f((n+T_1,m),\cdot) = f((n,m),\cdot) = f((n,m+T_2),\cdot)$ for all $(n,m) \in \mathbb{Z}^2$. In this context, let $u = \{u(n,m)\}$ be a (T_1,T_2)-periodic solution of (1.1) if $u(n+T_1,m) = u(n,m) = u(n,m+T_2)$ for all $(n,m) \in \mathbb{Z}^2$.

In recent years, the investigation of periodic solutions for second-order partial difference equations has emerged as one of the prominent research topics in mathematics. In particular, it has extensive applications in the field of physics. This topic emerges in the context of exploring periodic solutions in spacetime for the following two-dimensional discrete nonlinear Schrödinger (DNLS) equation:

$$i\frac{d}{dt}\psi((n,m),t) = -\Delta\psi((n,m),t) + V(n,m)\psi((n,m),t) - g((n,m),\psi((n,m),t)), \tag{1.2}$$

for any $((n,m),t) \in \mathbb{Z}^2 \times [0,\infty)$. Additionally, Δ is introduced as the forward second-order difference operator, which is expressed as follows:

$$\Delta\psi((n,m),\cdot) = \Delta_1(\Delta_1\psi((n-1,m),\cdot)) + \Delta_2(\Delta_2\psi((n,m-1),\cdot)).$$

Let t denote the time variable, i indicate the imaginary unit, and ψ be a complex-valued function. The given real-valued sequence $\{V(n,m)\}$ and $\{g((n,m),\cdot)\}$ are (T_1,T_2)-periodic. Moreover, assume that the DNLS equation (1.2) possesses a gauge-invariant nonlinearity defined as:

$$g((\cdot,\cdot),e^{i\theta}z) = e^{i\theta}g((\cdot,\cdot),z),\ \ \theta,\ z \in \mathbb{R},$$

and a spacetime periodic solution of the form:

$$\psi((n,m),t) = u(n,m)e^{-i\omega t},$$

accompanied by the spatial periodicity condition:

$$\psi((n+T_1,m),t) = \psi((n,m),t) = \psi((n,m+T_2),t),$$

where $\{u(n,m)\}$ represents a real-valued sequence, and ω denotes the temporal frequency. Then, (1.2) becomes the following:

$$\Delta u(n,m) - V(n,m)u(n,m) + \omega u(n,m) + g((n,m),u(n,m)) = 0. \tag{1.3}$$

It is clear that (1.3) is a special case of (1.1), and when searching for spacetime periodic solutions to the DNLS equation (1.2), it suffices to seek periodic solutions of (1.3).

The DNLS equation describes a nonlinear lattice system commonly found in various fields of physics, such as nonlinear optics [1], biomolecular chains [2–4], and Bose-Einstein condensates [5–9]. For instance, the dynamical properties of localized excitations in arrays of Bose-Einstein condensates can be investigated through the DNLS equation (1.2) within the framework of the nonlinear lattice theory [10].

The fascinating DNLS equation has garnered considerable attention from scholars who seek to explore the existence and multiplicity of periodic solutions for difference equations that involve the ϕ-Laplacian, as well as their associated boundary value problems.

For the case where $\phi(x) = |x|^{p-2}x$, $p > 1$, Liu [11] developed a relevant variational framework and utilized the critical point theory to investigate the existence and multiplicity of periodic solutions in nonlinear difference systems with the p-Laplacian. In [12], the author mainly studied the existence of periodic solutions for order nonlinear p-Laplacian difference equations with advanced and retardation. Gao [13] investigated the existence of three nontrivial solutions for a discrete nonlinear multiparameter periodic problem that involves the p-Laplacian. In [14], a nonlinear difference equation of order $2n$ with a p-Laplacian operator and multiple advanced and delayed terms was examined. Explicit sufficient conditions for the existence and multiplicity of periodic and subharmonic solutions were established. Recently, Long and Li [15] studied the existence and multiplicity of periodic solutions for a class of second-order difference equations using the Mountain Pass Lemma and the Linking Theorem.

For the case where $\phi(x) = \phi_c(x) = x/\sqrt{1+x^2}$, Mei and Zhou [16] established the existence and multiplicity of periodic and subharmonic solutions for a $2n$-order nonlinear difference equation that incorporates both numerous advances and retardations with respect to the ϕ_c-Laplacian. This was achieved through the application of the critical point theory under more general conditions pertaining to the nonlinear term. In [17], sufficient conditions were established for the existence of infinitely many positive solutions to boundary value problems for second-order ϕ_c-Laplacian difference equations using the critical point theory. Immediately following this, Chen and Zhou [18] investigated the existence of infinitely many positive solutions for second-order difference equations with boundary value problems involving a special ϕ_c-Laplacian. Xiong [19] studied the existence of infinitely many small solutions for a partially discrete Dirichlet problem with a perturbation term and the ϕ_c-Laplacian using the critical point theory.

For other cases that involved the ϕ-Laplacian, Lin and Zhou [20], utilized the critical point theory, to investigate the existence and multiplicity of periodic solutions and subharmonic solutions for $2n$-order nonlinear difference equations involving both advanced and delayed terms with the ϕ-Laplacian.

However, the existing research mainly focused on specific ϕ-Laplacians, such as the p-Laplacian and the ϕ_c-Laplacian. This paper aims to study the more general case of the ϕ-Laplacian and explore the influence of the properties of the ϕ-Laplacian on the existence and structure of solutions.

The rest of the paper is organized as follows: in Section 2, we provide a variational framework associated with (1.1) and show some auxiliary results; in Section 3, we present the sufficient conditions for the multiplicity of periodic solutions to (1.1) and give remarks for our main results; in Section 4, we provide the proofs of the main results; and finally, in Section 5, we share the main conclusions of the paper.

2. Variational structure and some auxiliary results

First, we will establish the appropriate variational framework for (1.1). Let S be the set of all bivariate sequences $u = \{u(n,m)\}_{(n,m)\in\mathbb{Z}^2}$, which are specifically defined as follows:

$$S = \left\{u = \{u(n,m)\} \mid u(n,m) \in \mathbb{R}, (n,m) \in \mathbb{Z}^2\right\}.$$

We can express $u \in S$ as follows:

$$u = (\cdots;\cdots, u(0,0), u(1,0), u(2,0), \cdots;\cdots, u(0,1), u(1,1), \cdots;\cdots).$$

For any $u, v \in S$, $a, b \in \mathbb{R}$, $au + bv$ is defined by the following:

$$au + bv = \{au(n,m) + bv(n,m)\}_{(n,m)\in\mathbb{Z}^2};$$

then, S is a vector space.

Write $\Omega := \mathbb{Z}(1,T_1)\times\mathbb{Z}(1,T_2)$. Define the subset E of S as follows:

$$E = \{u = \{u(n,m)\} \in S \mid u(n+T_1,m) = u(n,m) = u(n,m+T_2),\ (n,m)\in\Omega\}.$$

Then, E and $\mathbb{R}^{T_1T_2}$ are isomorphic, and E can equipped with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ as follows:

$$\langle u,v\rangle = \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} u(n,m)v(n,m) \quad\text{and}\quad \|u\| = \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}|u(n,m)|^2\right)^{\frac{1}{2}}, \quad \text{for any } u,v\in E.$$

Obviously, $(E,\langle\cdot,\cdot\rangle)$ is a T_1T_2 -dimensional real Hilbert space and a linearly homeomorphism to $\mathbb{R}^{T_1T_2}$. Define another norm $\|\cdot\|_w$ on E as follows:

$$\|u\|_w = \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}|u(n,m)|^w\right)^{\frac{1}{w}}, \quad u\in E,\ \ w\ge 1.$$

By Hölder' inequality and Jensen' inequality, one has the following:

$$\eta_w\|u\| \le \|u\|_w \le \tau_w\|u\|, \qquad u\in E, \tag{2.1}$$

where

$$\eta_w = \begin{cases} 1, & 1\le w<2,\\ (T_1T_2)^{\frac{2-w}{2w}}, & 2\le w,\end{cases} \qquad \tau_w = \begin{cases} (T_1T_2)^{\frac{2-w}{2w}}, & 1\le w<2,\\ 1, & 2\le w.\end{cases}$$

Define the functional I on E as follows:

$$\begin{aligned} I(u) = &\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\Phi(\Delta_1 u(n-1,m)) - \Phi(\Delta_2 u(n,m-1)) + q(n,m)\Phi(u(n,m))]\\ &-\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}F((n,m),u(n,m)), \qquad u\in E, \end{aligned} \tag{2.2}$$

where $\Phi(u) = \int_0^u \phi(s)ds$ is the primitive of $\phi(u)$, and $F((n,m),z) = \int_0^z f((n,m),s)ds$.

Then, $I\in C^1(E,\mathbb{R})$. By $u(n+T_1,m) = u(n,m) = u(n,m+T_2)$, for all $(n,m)\in\Omega$, the direct calculation is as follows:

$$\begin{aligned} \langle I'(u),v\rangle &= \lim_{t\to\infty}\frac{I(u+tv)-I(u)}{t}\\ &= \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\Delta_1[\phi(\Delta_1 u(n-1,m))]\Delta_1 v(n,m) + \Delta_2[\phi(\Delta_2 u(n,m-1))]\Delta_2 v(n,m)\\ &\quad + q(n,m)\phi(u(n,,m))v(n,m) - f((n,m),u(n,m))v(n,m)]\\ &= \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\{-\Delta_1\phi(\Delta_1 u(n-1,m)) - \Delta_2\phi(\Delta_2 u(n,m-1))\\ &\quad + q(n,m)\phi(u(n,m)) - f((n,m),u(n,m))\}v(n,m), \qquad n,m\in\Omega. \end{aligned}$$

Then, $u \in E$ is a critical point of I, (i.e., $I'(u) = 0$,) if and only if

$$\begin{aligned}&-\Delta_1\left[\phi\left(\Delta_1 u\left(n-1,m\right)\right)\right]-\Delta_2\left[\phi\left(\Delta_2 u\left(n,m-1\right)\right)\right]\\&+q\left(n,m\right)\phi\left(u\left(n,m\right)\right)-f\left(\left(n,m\right),u\left(n,m\right)\right)=0,\qquad (n,m)\in\Omega.\end{aligned}$$

This implies that $u = \{u(n,m)\} \in E$ is a (T_1, T_2)-periodic solution of (1.1). Thus, the problem of finding (T_1, T_2)-periodic solutions of (1.1) is equivalent to that of seeking the critical points of I on E. Define the following matrix:

$$Q_{kl}=\begin{pmatrix}P_k & & & 0\\ & P_k & & \\ & & \ddots & \\ 0 & & & P_k\end{pmatrix}_{kl\times kl},\ \text{where}\quad P_k=\begin{pmatrix}2 & -1 & 0 & \cdots & 0 & -1\\ -1 & 2 & -1 & \cdots & 0 & 0\\ 0 & -1 & 2 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 2 & -1\\ -1 & 0 & 0 & \cdots & -1 & 2\end{pmatrix}_{k\times k}.$$

From the properties of eigenvalues, it is known that Q_{kl} and P_k have the same eigenvalues. Furthermore, each eigenvalue of Q_{kl} is an l -multiple of the corresponding eigenvalue of P_k. By the matrix theory, the eigenvalues of the matrix P_k are

$$v_i = 2\left(1-\cos\frac{2i\pi}{k}\right),\qquad i=0,1,2,\cdots,k-1,$$

which implies $v_0 = 0$ and $v_1 > 0, \cdots, v_{k-1} > 0$. Thus,

$$\begin{cases} v_{\min} = \min\{v_1, v_2, \cdots, v_{k-1}\} = 4\sin^2\frac{\pi}{k},\\ v_{\max} = \max\{v_1, v_2, \cdots, v_{k-1}\} = 4\cos^2\frac{(1-(-1)^k)\pi}{4k}.\end{cases}$$

Similarly, let $\lambda_{\min}$ and $\lambda_{\max}$ denote the smallest nonzero eigenvalue and the largest eigenvalue of $Q_{T_1T_2}$, respectively, while $\mu_{\min}$ and $\mu_{\max}$ represent the smallest nonzero eigenvalue and the largest eigenvalue of $Q_{T_2T_1}$, respectively. It follows that

$$\begin{cases}\lambda_{\min} = 4\sin^2\frac{\pi}{T_1},\\ \lambda_{\max} = 4\cos^2\frac{(1-(-1)^k)\pi}{4T_1},\end{cases}\qquad\qquad \begin{cases}\mu_{\min} = 4\sin^2\frac{\pi}{T_2},\\ \mu_{\max} = 4\cos^2\frac{(1-(-1)^k)\pi}{4T_2}.\end{cases}$$

Write $E = E_1 \oplus E_2$, where $E_1 = \{x \in E_1 \mid x = \{c\}, c \in \mathbb{R}\}$ and $E_1 = E_2^{\perp}$. Let B_r denote the open ball in E centered at 0 with radius r, and let ∂B_r denote its boundary.

Definition 2.1. *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$. If sequence $\{u_n\} \subset E$ such that $\{I(u_n)\}$ is bounded and*

$$\lim_{n\to\infty} I'(u_n) \to 0,$$

possesses a convergent subsequence, then I satisfies the Palais-Smale (P.S. for short) condition.

Lemma 2.1. *(Linking theorem [21]) Let E be a real Banach space and $E = E_1 \oplus E_2$, where E_1 is a finite-dimensional subspace of E. Assume that $I \in C^1(E, \mathbb{R})$ satisfies the P.S. condition and the following two conditions:*

(J_1) There exist constants $\rho > 0$ and $a > 0$ such that $I\mid_{\partial B_\rho \cap E_2} \geq a$; and

(J_2) There exist an $e \in \partial B_1 \cap E_2$ and a constant $R_0 > \rho$ such that $I\mid_{\partial Q} \leq 0$, where $Q = \left(\bar{B}_{R_0} \cap E_1\right) \oplus \{se \mid 0 < s < R_0\}$.

Then, I possesses a critical value $c \geq a$, where

$$c = \inf_{h\in\Gamma} \max_{t\in\bar{Q}} I(h(t)), \quad \Gamma = \left\{h \in C\left(\bar{Q}, E\right) : h\mid_{\partial Q} = id\right\}.$$

The inequalities presented below are utilized in the proofs of the subsequent theorems. Detailed proofs can be found in [15].

Lemma 2.2. *For any $u \in E$ and $w \geq 1$, the following holds:*

$$\|\Delta_1 u\|_w^w \leq \tau_w^w \lambda_{\max}^{\frac{w}{2}} \|u\|^w, \qquad \|\Delta_2 u\|_w^w \leq \tau_w^w \mu_{\max}^{\frac{w}{2}} \|u\|^w, \tag{2.3}$$

and

$$\|\Delta_1 u\|_w^w \leq 2^w \|u\|_w^w, \qquad \|\Delta_2 u\|_w^w \leq 2^w \|u\|_w^w. \tag{2.4}$$

For any $u \in E_2$ and $w \geq 1$, one has the following:

$$\|\Delta_1 u\|_w^w \geq \eta_w^w \lambda_{\min}^{\frac{w}{2}} \|u\|^w, \qquad \|\Delta_2 u\|_w^w \geq \eta_w^w \mu_{\min}^{\frac{w}{2}} \|u\|^w. \tag{2.5}$$

3. Main results

For convenience, we introduce some notations based on the main results:

$$q^* = \max_{(n,m)\in\mathbb{Z}^2} \{q(n,m)\}, \qquad q_* = \min_{(n,m)\in\mathbb{Z}^2} \{q(n,m)\}.$$

Here, we present several conditions:

(Φ_1) There exist constants $\varepsilon_1 > 0$, $a_1 > 0$, and $\gamma \geq 1$ such that

$$\Phi(u) \geq a_1 |u|^\gamma, \quad \text{for } |u| \leq \varepsilon_1.$$

(Φ_2) There exist constants $\sigma_1 > 0$, $b_1 > 0$, $c_1 > 0$, and $\nu \geq 2$ such that

$$\Phi(u) \leq b_1 |u|^\nu + c_1, \quad \text{for } |u| \geq \sigma_1.$$

(F_0) $\sum_{m=1}^{T_1} \sum_{n=1}^{T_2} F((m,n), z) \geq 0$ for any $z \in \mathbb{R}$.

(F_1) There exist constants $\varepsilon_2 > 0$, $\alpha_1 > 0$, and $\theta \geq 1$ such that

$$|F((n,m), z)| \leq \alpha_1 |z|^\theta, \quad \text{for } (n,m) \in \mathbb{Z}^2 \text{ and } |z| \leq \varepsilon_2.$$

(F_2) There exist constants $\sigma_2 > 0$, $\xi_1 > 0$, $\beta_1 > 0$, and $p \geq 2$ such that

$$F((n,m), z) \geq \beta_1 |z|^p - \xi_1, \quad \text{for } (n,m) \in \mathbb{Z}^2 \text{ and } |z| \geq \sigma_2.$$

(H_1) $p = \nu$ and $\beta_1 > \max\left\{2^{\nu+1} b_1 + q^* b_1\ , b_1 (T_1 T_2)^{\frac{|\nu-2|}{2}} \left(\lambda_{\min}^{\frac{\nu}{2}} + \mu_{\min}^{\frac{\nu}{2}} + q^*\right)\right\}$.

(H_2) $p > \nu$.

(H_3) $\theta = \gamma$ and $\alpha_1 < \left(a_1\lambda_{\min}^{\frac{\gamma}{2}} + a_1\mu_{\min}^{\frac{\gamma}{2}} + q_* a_1\right)(T_1T_2)^{\frac{-|2-\gamma|}{2}}$.
(H_4) $\theta > \gamma$.

Remark 3.1. *We shall employ the critical point theory to examine the multiplicity of periodic solutions of* (1.1) *based on the hypotheses* (H_1)–(H_4). *Intuitively, these hypotheses collectively sculpt the energy landscape of the corresponding functional of* (1.1)*:* (H_3) *and* (H_4) *ensure the origin is a local minimum, while* (H_1) *and* (H_2) *guarantee that the functional becomes negative at infinity, thus creating the mountain-pass geometry essential for the existence of nontrivial critical points.*

Remark 3.2. *By* (Φ_2)*, we can conclude that there exists a positive constant* $C > 0$ *such that*

$$\Phi((n,m),u) \le b_1|u|^{\nu} + C, \qquad ((n,m),u) \in \mathbb{Z}^2 \times \mathbb{R}. \tag{3.1}$$

In fact, let

$$c_2 = \max_{(n,m)\in\mathbb{Z}^2\ |u|\le\sigma_1} \{\Phi((n,m),u) - b_1|u|^{\nu}\}.$$

Take $C = \max\{c_1, c_2\}$*; thus,* (3.1) *is obtained easily.*

Similarly, from (F_2)*, it follows that there exists a constant* $\xi' > 0$ *such that*

$$F((n,m),z) \ge \beta_1|z|^p - \xi', \quad ((n,m),z) \in \mathbb{Z}^2 \times \mathbb{R}. \tag{3.2}$$

Now, we can declare the following as our primary results.

Theorem 3.1. *Suppose that* (Φ_1)*,* (Φ_2)*, and* (F_0)–(F_2) *hold. Additionally, assume that one of the following four cases is satisfied:*

(1) (H_1) *and* (H_3) *hold;*
(2) (H_1) *and* (H_4) *hold;*
(3) (H_2) *and* (H_3) *hold; or*
(4) (H_2) *and* (H_4) *hold.*

Then, (1.1) *possesses at least two nontrivial* (T_1, T_2)*- periodic solutions.*

Remark 3.3. *We see that the p-Laplacian operator given by* $\phi_p(u) = |u|^{p-2}u$ $(1 < p < \infty)$ *satisfies* (Φ_1) *and* (Φ_2). *Theorem 3.1 is the synthesis of Theorems* 1 *and* 2 *presented in [15]. Similarly,* $\phi(u) = \frac{|u|^{p-2}u}{\sqrt{1+|u|^p}}$ $(1 < p < \infty)$ *can also satisfy functions* (Φ_1) *and* (Φ_2). *Compared with existing results, this further demonstrates that we have a more generalized form of the* ϕ*-Laplacian to satisfy the theorem.*

Remark 3.4. *We provided examples of* f *that meet the conditions in Theorem 3.1. Letting*

$$\alpha = \frac{\gamma}{2}\left(a_1\lambda_{min}^{\frac{\gamma}{2}} + a_1\mu_{min}^{\frac{\gamma}{2}} + q_* a_1\right)(T_1T_2)^{\frac{-|2-\gamma|}{2}}$$

and

$$\beta = 2^{\nu+2}b_1\nu + q^* b_1\nu + b_1\nu(T_1T_2)^{\frac{|\nu-2|}{2}}\left(\lambda_{\min}^{\frac{\nu}{2}} + \mu_{\min}^{\frac{\nu}{2}} + q^*\right),$$

we have the following four examples:

(1) (F_0)–(F_2), (H_1), *and* (H_3) *hold when*

$$f((\cdot,\cdot),z)=\begin{cases}\alpha|z|^{\gamma-2}z, & |z|\le 1,\\ -\alpha+(\beta 2^{\nu-1}-\alpha)(z+1), & -2<z<-1,\\ \alpha+(\beta 2^{\nu-1}-\alpha)(z-1), & 1<z<2,\\ \beta|z|^{\nu-2}z, & |z|\ge 2;\end{cases}$$

(2) (F_0)–(F_2), (H_1), *and* (H_4) *hold when* $f((\cdot,\cdot),z)=\beta|z|^{\gamma+\nu-2}z$ *if* $|z|\le 1$ *and* $f((\cdot,\cdot),z)=\beta|z|^{\nu-2}z$ *if* $|z|>1$;
(3) (F_0)–(F_2), (H_2), *and* (H_3) *hold when* $f((\cdot,\cdot),z)=|z|^{\gamma+\nu-2}z+\alpha|z|^{\gamma-2}z$ *for* $z\in\mathbb{R}$; *and*
(4) (F_0)–(F_2), (H_2), *and* (H_4) *hold when* $f((\cdot,\cdot),z)=|z|^{\gamma+\nu-2}z$ *for* $z\in\mathbb{R}$.

4. Proof of main results

Before we proceed to prove Theorem 3.1, we will first show some key lemmas. This will establish a solid foundation for the subsequent discussion of Theorem 3.1.

Lemma 4.1. *Assume that* (Φ_1), (F_1), *and* (H_3) *are satisfied. Then, there exist constants* $\rho, a>0$ *such that* $I\mid_{\partial B_\rho\cap E_2}\ge a$.

Proof. It follows from (Φ_1), (F_1), and (2.4) that $|u(n,m)|\le\|u\|$ for any $(n,m)\in E$. Let $\rho=\min\{\varepsilon_1,\varepsilon_2\}$. Then, for all $u\in\partial B_\rho\cap E_2$, we have the following:

$$\begin{aligned}I(u)=&\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\Phi(\Delta_1 u(n-1,m))+\Phi(\Delta_2 u(n,m-1))+q(n,m)\Phi(u(n,m))]\\ &-\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}F((n,m),u(n,m))\\ \ge&\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[a_1|\Delta_1 u(n-1,m)|^\gamma+a_1|\Delta_2 u(n,m-1)|^\gamma+q_*a_1|u(n,m)|^\gamma-\alpha_1|u(n,m)|^\theta]\\ =&a_1\|\Delta_1 u\|_\gamma^\gamma+a_1\|\Delta_2 u\|_\gamma^\gamma+q_*a_1\|u\|_\gamma^\gamma-\alpha_1\|u\|_\theta^\theta\\ \ge&a_1\eta_\gamma^\gamma\lambda_{\min}^{\frac{\gamma}{2}}\|u\|^\gamma+a_1\eta_\gamma^\gamma\mu_{\min}^{\frac{\gamma}{2}}\|u\|^\gamma+q_*a_1\eta_\gamma^\gamma\|u\|^\gamma-\alpha_1\tau_\theta^\theta\|u\|^\theta\\ =&a_1\eta_\gamma^\gamma\lambda_{\min}^{\frac{\gamma}{2}}\rho^\gamma+a_1\eta_\gamma^\gamma\mu_{\min}^{\frac{\gamma}{2}}\rho^\gamma+q_*a_1\eta_\gamma^\gamma\rho^\gamma-\alpha_1\tau_\theta^\theta\rho^\theta.\end{aligned}$$

Since $\gamma=\theta$, for all $u\in\partial B_\rho\cap E_2$ one has the following:

$$I(u)\ge\left(a_1\eta_\gamma^\gamma\lambda_{\min}^{\frac{\gamma}{2}}+a_1\eta_\gamma^\gamma\mu_{\min}^{\frac{\gamma}{2}}+q_*a_1\eta_\gamma^\gamma-\alpha_1\tau_\gamma^\gamma\right)\rho^\gamma.$$

From (2.1), we have the following:

$$\frac{\eta_\gamma^\gamma}{\tau_\gamma^\gamma}=(T_1T_2)^{\frac{-|\gamma-2|}{2}}=\begin{cases}(T_1T_2)^{\frac{\gamma-2}{2}}, & 1\le\gamma<2,\\ (T_1T_2)^{\frac{2-\gamma}{2}}, & 2\le\gamma.\end{cases}$$

Thus, by (H_3), we get that $\alpha_1 < \left(a_1\lambda_{\min}^{\frac{\gamma}{2}} + a_1\mu_{\min}^{\frac{\gamma}{2}} + q_* a_1\right)\frac{\eta_\gamma^\gamma}{\tau_\gamma^\gamma}$.

Take $a = \left(a_1\eta_\gamma^\gamma\lambda_{\min}^{\frac{\gamma}{2}} + a_1\eta_\gamma^\gamma\mu_{\min}^{\frac{\gamma}{2}} + q_* a_1\eta_\gamma^\gamma - \alpha_1\tau_\gamma^\gamma\right)\rho^\gamma > 0$. Then,

$$I(u) \geq a, \qquad u \in \partial B_\rho \cap E_2.$$

The proof of the lemma is finished. □

Lemma 4.2. *Assume that* (Φ_1), (F_1), *and* (H_4) *are satisfied. Then, there exist constants* $\rho, a > 0$ *such that* $I\mid_{\partial B_\rho \cap E_2} \geq a$.

Proof. Let $\rho = \min\left\{\varepsilon_1, \varepsilon_2, \left(\frac{\gamma a_1\eta_\gamma^\gamma\lambda_{\min}^{\frac{\gamma}{2}} + \gamma a_1\eta_\gamma^\gamma\mu_{\min}^{\frac{\gamma}{2}} + \gamma q_* a_1\eta_\gamma^\gamma}{\theta\alpha_1\tau_\theta^\theta}\right)^{\frac{1}{\theta-\gamma}}\right\}$. For any $u \in \partial B_\rho \cap E_2$, we get that $|u(n,m)| \leq \|u\| = \rho$ for any $(n,m) \in \Omega$. From (Φ_1), (F_1), and (2.5), we have the following:

$$\begin{aligned}
I(u) =& \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\Phi(\Delta_1 u(n-1,m)) + \Phi(\Delta_2 u(n,m-1)) + q(n,m)\Phi(u(n,m))] \\
&- \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m), u(n,m)) \\
\geq& \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[a_1|\Delta_1 u(n-1,m)|^\gamma + a_1|\Delta_2 u(n,m-1)|^\gamma + q_* a_1|u(n,m)|^\gamma - \alpha_1|u(n,m)|^\theta] \\
=& a_1\|\Delta_1 u\|_\gamma^\gamma + a_1\|\Delta_2 u\|_\gamma^\gamma + q_* a_1\|u\|_\gamma^\gamma - \alpha_1\|u\|_\theta^\theta \\
\geq& a_1\eta_\gamma^\gamma\lambda_{\min}^{\frac{\gamma}{2}}\|u\|^\gamma + a_1\eta_\gamma^\gamma\mu_{\min}^{\frac{\gamma}{2}}\|u\|^\gamma + q_* a_1\eta_\gamma^\gamma\|u\|^\gamma - \alpha_1\tau_\theta^\theta\|u\|^\theta \\
=& a_1\eta_\gamma^\gamma\lambda_{\min}^{\frac{\gamma}{2}}\rho^\gamma + a_1\eta_\gamma^\gamma\mu_{\min}^{\frac{\gamma}{2}}\rho^\gamma + q_* a_1\eta_\gamma^\gamma\rho^\gamma - \alpha_1\tau_\theta^\theta\rho^\theta.
\end{aligned}$$

Take $a = a_1\eta_\gamma^\gamma\lambda_{\min}^{\frac{\gamma}{2}}\rho^\gamma + a_1\eta_\gamma^\gamma\mu_{\min}^{\frac{\gamma}{2}}\rho^\gamma + q_* a_1\eta_\gamma^\gamma\rho^\gamma - \alpha_1\tau_\theta^\theta\rho^\theta$. Then,

$$I(u) \geq a > 0, \qquad u \in \partial B_\rho \cap E_2.$$

The proof of the lemma is complete. □

Lemma 4.3. *Assume that* (Φ_2), (F_0), (F_2), *and* (H_1) *are satisfied. Then, there exist* $e \in \partial B_1 \cap E_2$ *and* $R_0 > \rho$ *such that* $I\mid_{\partial Q} \leq 0$, *where* $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{se \mid 0 < s < R_0\}$.

Proof. Define $u = se + x$, where $e \in \partial B_1 \cap E_2$, $x \in E_1$, and $s \geq 0$. From (Φ_2), (F_2), and (2.5), we can obtain the following:

$$\begin{aligned}
I(u) =& \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\Phi(\Delta_1(se(n-1,m) + x(n-1,m))) + \Phi(\Delta_2(se(n,m-1) + x(n,m-1))) \\
&+ q(n,m)\Phi(se(n,m) + x(n,m)) - F((n,m), se(n,m) + x(n,m))] \\
\leq& \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[b_1|\Delta_1(se(n-1,m))|^\nu + b_1|\Delta_2(se(n,m-1))|^\nu
\end{aligned}$$

$$
\begin{aligned}
&+ q^*b_1|se(n,m)+x(n,m)|^\nu - \beta_1|se(n,m)+x(n,m)|^p + \xi' + 2C + q^*C]\\
=&b_1\|\Delta_1(se)\|_\nu^\nu + b_1\|\Delta_2(se)\|_\nu^\nu + q^*b_1\|se+x\|_\nu^\nu - \beta_1\|se+x\|_p^p\\
&+ (q^*C + 2C + \xi')T_1T_2\\
\le&b_1\tau_\nu^\nu\lambda_{\max}^{\frac{\nu}{2}}s^\nu + b_1\tau_\nu^\nu\mu_{\max}^{\frac{\nu}{2}}s^\nu + q^*b_1\tau_\nu^\nu\|se+x\|^\nu - \beta_1\eta_p^p\|se+x\|^p\\
&+ (q^*C + 2C + \xi')T_1T_2.
\end{aligned}
$$

Since $p = \nu \geq 2$, we have the following:

$$
\begin{aligned}
I(u) \leq&\left(b_1\tau_\nu^\nu\lambda_{\max}^{\frac{\nu}{2}} + b_1\tau_\nu^\nu\mu_{\max}^{\frac{\nu}{2}} + q^*b_1\tau_\nu^\nu - \beta_1\eta_\nu^\nu\right)s^\nu\\
&+ (q^*b_1\tau_\nu^\nu - \beta_1\eta_\nu^\nu)\|x\|^\nu + (q^*C + 2C + \xi')T_1T_2.
\end{aligned}
$$

From (2.1), we have the following:

$$
\frac{\tau_\nu^\nu}{\eta_\nu^\nu} = (T_1T_2)^{\frac{|\nu-2|}{2}} = \begin{cases} (T_1T_2)^{\frac{2-\nu}{2}}, & 1 \leq \nu < 2,\\ (T_1T_2)^{\frac{\nu-2}{2}}, & 2 \leq \nu. \end{cases}
$$

Then, with (H_1), we obtain that $\beta_1\eta_\nu^\nu > b_1\tau_\nu^\nu\lambda_{\max}^{\frac{\nu}{2}} + b_1\tau_\nu^\nu\mu_{\max}^{\frac{\nu}{2}} + q^*b_1\tau_\nu^\nu$. Notice that, for any $x \in E_1$, by (F_0) there is

$$
I(x) = -\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m), x(n,m)) \leq 0.
$$

Thus, there exists a constant $R_0 > \rho$ such that $I\mid_{\partial Q} \leq 0$, where $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{se \mid 0 < s < R_0\}$. □

Lemma 4.4. *Assume that* (Φ_2), (F_0), (F_2)*, and* (H_2) *are satisfied. Then, there exist* $e \in \partial B_1 \cap E_2$ *and* $R_0 > \rho$ *such that* $I\mid_{\partial Q} \leq 0$*, where* $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{se \mid 0 < s < R_0\}$.

Proof. Let $e \in \partial B_1 \cap E_2$, such that $u = se + x$, where $s \geq 0$, and $x \in E_1$.

When (Φ_2) and (F_2) are satisfied, by (2.4), we can get that

$$
\begin{aligned}
I(u) \leq&b_1\tau_\nu^\nu\lambda_{\max}^{\frac{\nu}{2}}s^\nu + b_1\tau_\nu^\nu\mu_{\max}^{\frac{\nu}{2}}s^\nu + q^*b_1\tau_\nu^\nu\|se+x\|^\nu - \beta_1\eta_p^p\|se+x\|^p\\
&+ (q^*C + 2C + \xi')T_1T_2\\
\leq&\left(b_1\tau_\nu^\nu\lambda_{\max}^{\frac{\nu}{2}} + b_1\tau_\nu^\nu\mu_{\max}^{\frac{\nu}{2}} + q^*b_1\tau_\nu^\nu 2^{\frac{\nu-2}{2}}\right)s^\nu - \beta_1\eta_p^p s^p + 2^{\frac{\nu-2}{2}}b_1q^*\tau_\nu^\nu\|x\|^\nu\\
&- \beta_1\eta_p^p\|x\|^p + (q^*C + 2C + \xi')T_1T_2.
\end{aligned}
$$

Let

$$
\begin{aligned}
g_1(t) &= \left(b_1\tau_\nu^\nu\lambda_{\max}^{\frac{\nu}{2}} + b_1\tau_\nu^\nu\mu_{\max}^{\frac{\nu}{2}} + q^*b_1\tau_\nu^\nu 2^{\frac{\nu-2}{2}}\right)t^\nu - \beta_1\eta_p^p t^p,\\
g_2(t) &= 2^{\frac{\nu-2}{2}}b_1\tau_\nu^\nu q^* t^\nu - \beta_1\eta_p^p t^p + (q^*C + 2C + \xi')T_1T_2.
\end{aligned}
$$

With $p > \nu$, we have that

$$
\lim_{t\to\infty} g_1(t) = -\infty, \qquad \lim_{t\to\infty} g_2(t) = -\infty,
$$

since $t \geq 0$, $g_1(t)$, and $g_2(t)$ are bounded from above. Notice from (F_0) that

$$I(x) = -\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m), x(n,m)) \leq 0, \qquad x \in E_1.$$

Thus, there exists a constant $R_0 > \rho$ such that $I\mid_{\partial Q} \leq 0$, where $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{se \mid 0 < s < R_0\}$. □

Lemma 4.5. *Assume that* (Φ_2), (F_2), *and* (H_1) *are satisfied. Then, I satisfies the P.S. condition in E.*

Proof. Let $\{u_k\} \subset E$, for all $k \in \mathbb{N}$, be such that $\{I(u_k)\}$ is bounded. Then, there exists a positive constant M_1 such that

$$|I(u_k)| \leq M_1, \qquad k \in \mathbb{N}.$$

Based on (2.2), (2.4), and (3.1), we have the following:

$$\begin{aligned}
I(u_k) = &\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\Phi(\Delta_1 u_k(n-1,m)) + \Phi(\Delta_2 u_k(n,m-1)) + q(n,m)\Phi(u_k(n,m))] \\
&- \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m), u_k(n,m)) \\
\leq &\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[b_1|\Delta_1 u_k(n-1,m)|^\nu + b_1|\Delta_2 u_k(n,m-1)|^\nu + q^* b_1 |u_k(n,m)|^\nu + 2C + q^*C] \\
&+ \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\xi' - \beta_1 |u_k(n,m)|^p] \\
= &b_1\|\Delta_1 u_k\|_\nu^\nu + b_1\|\Delta_2 u_k\|_\nu^\nu + q^* b_1\|u_k\|_\nu^\nu - \beta_1\|u_k\|_p^p + (2C + q^*C + \xi')\, T_1 T_2 \\
\leq &2^\nu b_1\|u_k\|_\nu^\nu + 2^\nu b_1\|u_k\|_\nu^\nu + q^* b_1\|u_k\|_\nu^\nu - \beta_1\|u_k\|_p^p + (2C + q^*C + \xi')\, T_1 T_2 \\
= &\left(2^{\nu+1} b_1 + q^* b_1\right)\|u_k\|_\nu^\nu - \beta_1\|u_k\|_p^p + (2C + q^*C + \xi')\, T_1 T_2.
\end{aligned} \tag{4.1}$$

When $\nu = p$,

$$-M_1 \leq I(u_k) \leq \left(2^{\nu+1} b_1 + q^* b_1 - \beta_1\right)\|u_k\|_\nu^\nu + (2C + q^*C + \xi')\, T_1 T_2.$$

From $\beta_1 > 2^{\nu+1} b_1 + q^* b_1$, we conclude that

$$\|u_k\|_\nu^\nu \leq \frac{M_1 + (2C + q^*C + \xi')\, T_1 T_2}{\beta_1 - 2^{\nu+1} b_1 - q^* b_1}.$$

It can be concluded that $\{u_k\} \subset E$ is bounded. As a result, $\{u_k\}$ has a convergence subsequence in E and the proof is complete. □

Lemma 4.6. *Assume that* (Φ_2), (F_2), *and* (H_1) *are satisfied. Then, I has a critical point* $\bar{u} \in E$ *such that* $I(\bar{u}) = \sup_{u\in E} I(u) = c_{\max} > 0$.

Proof. By (4.1), for any $u \in E$, we have the following:

$$I(u) \leq (2^{\nu+1} b_1 + q^* b_1 - \beta_1)\|u\|_\nu^\nu + (2C + q^*C + \xi') T_1 T_2.$$

According to $\beta_1 > 2^{\nu+1}b_1 + q^*b_1$, we have

$$I(u) \leq (2C + q^*C + \xi')T_1T_2$$

and

$$\lim_{\|u\|\to\infty} I(u) = -\infty.$$

This indicates that I is bounded above on E. By the continuity of I, there exists $\bar{u} \in E$ such that

$$I(\bar{u}) = \sup_{u\in E} I(u) = c_{\max}.$$

Namely, $\bar{u}$ is a critical point of E. By Lemma 4.3, we see $c_{\max} > 0$. The proof of the lemma is complete. □

Lemma 4.7. *Assume that* (Φ_2)*,* (F_2)*, and* (H_2) *are satisfied. Then I satisfies the P.S. condition in E.*

Proof. Let $\{u_k\}$ be a P.S. sequence; then, there exists a positive constant M_2 such that

$$|I(u_k)| \leq M_2, \qquad k \in \mathbb{N}.$$

From the definition of $I(u)$ and (2.3), we can infer that

$$\begin{aligned}
I(u_k) =& \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\Phi(\Delta_1 u_k(n-1,m)) + \Phi(\Delta_2 u_k(n,m-1)) + q(n,m)\Phi(u_k(n,m))] \\
&- \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m),u_k(n,m)) \\
\leq& \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[b_1|\Delta_1 u_k(n-1,m)|^\nu + b_1|\Delta_2 u_k(n,m-1)|^\nu + q^*b_1|u_k(n,m)|^\nu + 2C + q^*C] \\
&+ \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\xi' - \beta_1|u_k(n,m)|^p] \\
=& b_1\|\Delta_1 u_k\|_\nu^\nu + b_1\|\Delta_2 u_k\|_\nu^\nu + q^*b_1\|u_k\|_\nu^\nu - \beta_1\|u_k\|_p^p + (2C + q^*C + \xi')T_1T_2 \\
\leq& \tau_\nu^\nu\lambda_{\max}^{\frac{\nu}{2}}b_1\|u_k\|^\nu + \tau_\nu^\nu\mu_{\max}^{\frac{\nu}{2}}b_1\|u_k\|^\nu + \tau_\nu^\nu q^*b_1\|u_k\|^\nu - \eta_p^p\beta_1\|u_k\|^p + (2C + q^*C + \xi')\,T_1T_2 \\
=& \left(\tau_\nu^\nu\lambda_{\max}^{\frac{\nu}{2}}b_1 + \tau_\nu^\nu\mu_{\max}^{\frac{\nu}{2}}b_1 + \tau_\nu^\nu q^*b_1\right)\|u_k\|^\nu - \eta_p^p\beta_1\|u_k\|^p + (2C + q^*C + \xi')\,T_1T_2.
\end{aligned}$$

Thus,

$$-M_2 \leq I(u_k) \leq \left(\tau_\nu^\nu\lambda_{\max}^{\frac{\nu}{2}}b_1 + \tau_\nu^\nu\mu_{\max}^{\frac{\nu}{2}}b_1 + \tau_\nu^\nu q^*b_1\right)\|u_k\|^\nu - \eta_p^p\beta_1\|u_k\|^p + (2C + q^*C + \xi')T_1T_2.$$

Then,

$$\eta_p^p\beta_1\|u_k\|^p - \left(\tau_\nu^\nu\lambda_{\max}^{\frac{\nu}{2}}b_1 + \tau_\nu^\nu\mu_{\max}^{\frac{\nu}{2}}b_1 + \tau_\nu^\nu q^*b_1\right)\|u_k\|^\nu \leq M_2 + (2C + q^*C + \xi')T_1T_2.$$

Since $p > \nu$, it is easily seen that $\{u_k\}$ is a bounded sequence in E. Consequently, $\{u_k\} \subset E$ possesses a convergence subsequence in E and the lemma is proven. □

Lemma 4.8. *Assume that* (Φ_2), (F_2), *and* (H_2) *are satisfied. Then, I has a critical point $\bar{u} \in E$ such that* $I(\bar{u}) = \sup_{u\in E} I(u) = c_{\max} > 0$.

Proof. By the proof of Lemma 4.7, we have the following:

$$I(u) \le \left(\tau_\nu^\nu \lambda_{\max}^{\frac{\nu}{2}} b_1 + \tau_\nu^\nu \mu_{\max}^{\frac{\nu}{2}} b_1 + \tau_\nu^\nu q^* b_1\right)\|u\|^\nu - \eta_p^p \beta_1 \|u\|^p + (2C + q^*C + \xi')T_1T_2.$$

Since $p > \nu$, we get that I is bounded above in E, and

$$\lim_{\|u\|\to\infty} I(u) = -\infty.$$

Thus, there exists $\bar{u} \in E$ such that

$$I(\bar{u}) = \sup_{u\in E} I(u) = c_{\max}.$$

Namely, $\bar{u}$ is a critical point of E. By Lemma 4.4, we have $c_{\max} > 0$. The proof of the lemma is complete. □

Proof of Theorem 3.1 . From the aforementioned proven theorems, it is evident that when conditions (Φ_2) and (F_2) are simultaneously satisfied, irrespective of whether the relationship between p and ν adheres to (H_1) or (H_2), the functional I defined on E can fulfill the P.S. condition and satisfy (J_1) of the Linking Theorem. According to Lemmas 4.3 and 4.4, it can be concluded that if (H_3) or (H_4) is satisfied, then (J_2) of the Linking Theorem can be proven.

By one of the above four cases and the Linking Theorem, the functional I has a critical value $c \ge a > 0$ expressed by the following:

$$c = \inf_{h\in\Gamma} \max_{t\in\bar{Q}} I\left(h(t)\right),$$

where

$$\Gamma = \left\{h \in C\left(\bar{Q}, E\right) : h\,|_{\partial Q} = id\right\}.$$

In other words, there is a nonzero critical point $\tilde{u} \in E$ such that $I(\tilde{u}) \ge a > 0$. Moreover, we claim that this $\tilde{u}$ is nonconstant. In fact, if $\tilde{u}$ is constant, then we obtain $I(\tilde{u}) \le 0$ by (F_0) and (2.2). This is a contradiction.

From either Lemma 4.6 or Lemma 4.8, it follows that I has a critical point $\bar{u} \in E$ such that

$$I(\bar{u}) = \sup_{u\in E} I(u) = c_{\max} > 0.$$

Similarly, we can show that $\bar{u}$ is nonconstant.

If $\bar{u} \ne \tilde{u}$, then the proof of Theorem 3.1 is complete. If $\bar{u} = \tilde{u}$, we conclude that $c = c_{\max}$, which is equivalent to the following:

$$\sup_{u\in E} I(u) = \inf_{h\in\Gamma} \max_{t\in\bar{Q}} I\left(h(t)\right).$$

Choosing $h\,|_{\partial Q} = id \in \Gamma$, we can obtain $\sup_{\bar{Q}} I(u) = c_{\max} = c$. The choice of $e \in \partial B_1 \cap E_2$ in Lemma 4.3 or Lemma 4.4 is arbitrary; therefore, we can also chose $-e \in \partial B_1 \cap E_2$ at this time. Similarly, it can be deduced that there exists a constant $R_1 > \rho$ such that $I\,|_{\partial Q_1} \le 0$, where $Q_1 =$

$\left(\bar{B}_{R_1} \cap E_1\right) \oplus \{-se \mid 0 < s < R_1\}$. Once again, by applying Lemma 2.1, I has a critical value $c' \geq a > 0$, where

$$c' = \inf_{h\in\Gamma_1} \max_{t\in[0,1]} I\left(h(t)\right), \quad \Gamma_1 = \left\{h \in C\left(\bar{Q}_1, E\right) : h\mid_{\partial Q_1} = id\right\}.$$

If $c' \neq c$, then the proof of Theorem 3.1 is complete. If $c' = c = c_{\max}$, i.e.,

$$\sup_{u\in E} I(u) = \inf_{h\in\Gamma_1} \max_{t\in\bar{Q}_1} I\left(h(t)\right),$$

then $\sup_{u\in E} I(u) = \sup_{u\in\bar{Q}} I(u) = \sup_{u\in\bar{Q}_1} I(u)$. Note that $I\mid_{\partial Q} \leq 0$, $I\mid_{\partial Q_1} \leq 0$; hence, the value of I reaches its maximum within the interior of the sets Q and Q_1. However, $\bar{Q} \cap \bar{Q}_1 \subset E_1$, and for any $u \in E_1 = A^- \oplus A^0$, from (F_0), one has the following:

$$I(u) \leq -\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F\left((n,m), u(n,m)\right) \leq 0.$$

It is shown that $I(u) \leq 0$ for any $u \in \bar{Q} \cap \bar{Q}_1$. Therefore, there exists a critical point $\hat{u} \in E$ such that $\hat{u} \neq \tilde{u}$ and $I(\hat{u}) = c = c_{\max} = c'$. Additionally, we can show that $\hat{u}$ is nonconstant.

From the above, we see that (1.1) possesses at least two nontrivial (T_1, T_2)-periodic solutions. The proof of the theorem is finished. □

5. Conclusions

In this paper, we utilize the critical point theory to investigate the existence of multiple periodic solutions for a class of second-order nonlinear partial difference equations involving the ϕ-Laplacian operator. This operator not only includes the classical p-Laplacian but also covers a broader class of operators satisfying (Φ_1) or (Φ_2). Compared with existing works, this study obtains clearer sufficient conditions for the multiplicity of periodic solutions. The result can be applied to the existence problem of spacetime periodic solutions for two-dimensional discrete nonlinear Schrödinger equations.

Author contributions

Ziying Guo: Analyzed the model; Juping Ji and Genghong Lin: Supervised the work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors claimed no competing interest.

References

1. D. N. Christodoulides, F. Lederer, Y. Silberberg, Discretizing light behavior in linear and nonlinear waveguide lattices, *Nature*, **424** (2003), 817–823. http://dx.doi.org/10.1038/nature01936
2. S. Flach, A. V. Gorbach, Discrete breathers–Advance in theory and applications, *Phys. Rep.*, **467** (2008), 1–116. http://dx.doi.org/10.1016/j.physrep.2008.05.002
3. G. Kopidakis, S. Aubry, G. P. Tsironis, Targeted energy transfer through discrete breathers in nonlinear systems, *Phys. Rev. Lett.*, **87** (2001), 165501. https://doi.org/10.1103/PhysRevLett.87.165501
4. R. Livi, R. Franzosi, G. L. Oppo, Self-localization of Bose-Einstein condensates in optical lattices via boundary dissipation, *Phys. Rev. Lett.*, **97** (2006), 060401. https://doi.org/10.1103/PhysRevLett.97.060401
5. G. Lin, J. Yu, Homoclinic solutions of periodic discrete Schrödinger equations with local superquadratic conditions, *SIAM J. Math. Anal.*, **54** (2022), 1966–2005. https://doi.org/10.1137/21M1413201
6. G. Lin, J. Yu, Existence of a ground-state and infinitely many homoclinic solutions for a periodic discrete system with sign-changing mixed nonlinearities, *J. Geom. Anal.*, **32** (2022), 127. https://doi.org/10.1007/s12220-022-00866-7
7. G. Lin, Z. Zhou, Z. Shen, J. Yu, Existence of uncountably many periodic solutions for second-order superlinear difference equations with continuous time, *Bull. Math. Sci.*, **15** (2025), 2450010. https://doi.org/10.1142/S1664360724500103
8. G. Lin, Z. Zhou, J. Yu, Ground state solutions of discrete asymptotically linear Schrödinger equations with bounded and non-periodic potentials, *J. Dyn. Diff. Equat.*, **32** (2020), 527–555. https://doi.org/10.1007/s10884-019-09743-4
9. X. Xu, H. Chen, Z. Ouyang, New results for periodic discrete nonlinear Schrödinger equations, *Math. Methods Appl. Sci.*, **48** (2025), 5768–5780. https://doi.org/10.1002/mma.10635
10. F. Kh. Abdullaev, B. B. Baizakov, S. A. Darmanyan, V. V. Konotop, M. Salerno, Nonlinear excitations in arrays of Bose-Einstein condensates, *Phys. Rev. A*, **64** (2001), 043606. https://doi.org/10.1103/physreva.64.043606
11. S. Liu, Multiple periodic solutions for non-linear difference systems involving the p-Laplacian, *J. Difference Equ. Appl.*, **17** (2011), 1591–1598. http://dx.doi.org/10.1080/10236191003730480
12. X. Liu, Y. Zhang, H. Shi, X. Deng, Existence of periodic solutions for a $2n$ th-order difference equation involving p-Laplacian, *Bull. Malays. Math. Sci. Soc.*, **38** (2015), 1107–1125. https://doi.org/10.1007/s40840-014-0066-0
13. C. Gao, Solutions to discrete multiparameter periodic boundary value problems involving the p-Laplacian via critical point theory, *Acta Math. Sci.*, **34** (2014), 1225–1236. https://doi.org/10.1016/S0252-9602(14)60081-3

14. P. Mei, Z. Zhou, Periodic and subharmonic solutions for a $2n$th-order p-Laplacian difference equation containing both advances and retardations, *Open Math.*, **16** (2018), 1435–1444. https://doi.org/10.1515/math-2018-0123

15. Y. Long, D. Li, Multiple periodic solutions of a second-order partial difference equation involving p-Laplacian, *J. Appl. Math. Comput.*, **69** (2023), 3489–3508. https://doi.org/10.1007/s12190-023-01891-7

16. P. Mei, Z. Zhou, G. Lin, Periodic and subharmonic solutions for a $2n$th-order ϕ_c-Laplacian difference equation containing both advances and retardations, *Discret. Contin. Dyn. Syst. -S*, **12** (2019), 2085–2095. https://doi.org/10.3934/dcdss.2019134

17. Z. Zhou, J. Ling, Infinitely many positive solutions for a discrete two point nonlinear boundary value problem with ϕ_c-Laplacian, *Appl. Math. Lett.*, **91** (2019), 28–34. https://doi.org/10.1016/j.aml.2018.11.016

18. Y. Chen, Z. Zhou, Existence of three solutions for a nonlinear discrete boundary value problem with ϕ_c-Laplacian, *Symmetry*, **12** (2020), 1839. https://doi.org/10.3390/sym12111839

19. F. Xiong, Infinitely many solutions for a perturbed partial discrete dirichlet problem involving ϕ_c-Laplacian, *Axioms*, **12** (2023), 909. https://doi.org/10.3390/axioms12100909

20. G. Lin, Z. Zhou, Periodic and subharmonic solutions for a 2nth-order difference equation containing both advance and retardation with ϕ-Laplacian, *Adv. Differ. Equ.*, **2014** (2014), 74. https://doi.org/10.1186/1687-1847-2014-74

21. P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, American Mathematical Society, 1986.