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Research article

Traveling wave solutions to a one prey-two competing predators model with nonlocal delay

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Abstract: In this paper, we investigated the traveling wave solutions to a one prey-two competing predators model with nonlocal delay. First, we analyzed the stability of the positive equilibrium by using a Lyapunov function. Then, by examining the distribution of the roots of the characteristic equation, we ascertained the critical wave speed c^*. Finally, employing the cross iteration method and Schauder's fixed point theorem, we proved the existence of traveling wave solutions connecting the trivial equilibrium $(0, 0, 0)$ with the positive equilibrium (u^*, v^*, w^*) for wave speeds $c > c^*$. The incorporation of nonlocal delay into models featuring intra-specific and inter-specific competition significantly elevates computational complexity, thereby necessitating precise analytical estimates.

Keywords: one prey-two competing predators model; nonlocal delay; traveling wave solutions; Lyapunov function; cross-iteration
Mathematics Subject Classification: 34C37, 35C07

1. Introduction

Prey-predator interaction is one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1, 2]. In particular, the dynamics of these interactions can be affected by inter-specific or intra-specific competition among species, and these competitive pressures can affect the population dynamics and stability of prey-predator systems [3–5]. In [3], Long, Wang, and Li considered the following prey-predator model with inter-specific and intra-specific competition:

$$\begin{cases} \frac{dx}{dt} = x(1 - x) - \frac{xy}{b_1+x} - \frac{xz}{b_2+x}, \\ \frac{dy}{dt} = \frac{a_1 xy}{b_1+x} - k_1 y - m_1 yz - h_1 y^2, \\ \frac{dz}{dt} = \frac{a_2 xz}{b_2+x} - k_2 z - m_2 yz - h_2 z^2, \end{cases} \tag{1.1}$$

where $x(t)$, $y(t)$, and $z(t)$ denote the population densities of a prey species and two competing predator species at time t, respectively. The constants $a_1, a_2, b_1, b_2, k_1, k_2, m_1, m_2, h_1, h_2$ are positive. For more details on the background of this system, see [3].

The fact that the spatial distribution of the population is heterogeneous and given that the trend of things at a given moment in time depends not only on the present but may also depend on the past, some researchers have added diffusion and time delay to the model [6–8]. But the population individuals are mobile and their spatial positions change over time. To address this, Britton [9, 10] proposed that the time delay term should be combined with a spatially weighted average, which is called the time-space time delay or the nonlocal delay. It is usually expressed in convolution form.

Given that the introduction of nonlocal delay can enhance the precision of models, many researchers have investigated the impact of nonlocal delay on the dynamics of ecological models [11–13].

As a class of solutions with spatial translation invariance, traveling wave solutions can account for the oscillations of solutions and the finite-speed propagation of disturbances caused by nonlocal delay. Li et al. [14] employed Schauder's fixed point theorem and an iteration scheme to demonstrate the existence of traveling wave solutions for the following reaction-diffusion competition-cooperation model with nonlocal delay and stage-structure:

$$\begin{cases} \frac{\partial v_1(x,t)}{\partial t} = d_1 \frac{\partial^2 v_1(x,t)}{\partial x^2} + \alpha_1 u_1(x,t) - \gamma_1 v_1(x,t) - \alpha_1 (g_1 \circledast u_1)(x,t), \\ \frac{\partial u_1(x,t)}{\partial t} = D_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + \alpha_1 (g_1 \circledast u_1)(x,t) - a_1 u_1^2(x,t) - b_1 u_1(x,t) u_2(x,t), \\ \frac{\partial v_2(x,t)}{\partial t} = d_2 \frac{\partial^2 v_2(x,t)}{\partial x^2} + \alpha_2 u_2(x,t) - \gamma_2 v_2(x,t) - \alpha_2 (g_2 \circledast u_2)(x,t), \\ \frac{\partial u_2(x,t)}{\partial t} = D_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + \alpha_2 (g_2 \circledast u_2)(x,t) + b_2 u_1(x,t) u_2(x,t) - a_2 u_2^2(x,t), \end{cases}$$

for $t > 0$, $x \in \mathbb{R}$, and $(g_i \circledast u_i)(x,t)$ is the nonlocal delay specially defined by

$$(g_i \circledast u_i)(x,t) = \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{-\gamma_i s} g_i(y,s) u_i(x-y, t-s) dy ds, \ i = 1, 2,$$

where the specific forms of the kernels $g_1(x,t)$ and $g_2(x,t)$ are found in [14]. $v_i(x,t)$, $u_i(x,t)$ represent the densities of the immature and mature populations of two species at location x and time t, respectively. d_i and $D_i (i = 1, 2)$ are the diffusion of the immature and mature members of two species, α_1 and α_2 represent the birth rates, γ_1 and γ_2 are immature death rates, and a_1 and a_2 are the rate of inter-specific competition among two mature species. b_1 and b_2 are the rate of competition and cooperation between the two mature species, respectively. All the parameters are positive constants.

In [15], Li and Huang considered the following reaction-diffusion prey-predator system with nonlocal delay:

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + u_1 (r_1 - a_1 u_2 - b_1 u_1), & (x,t) \in \Omega \times (0, +\infty), \\ \frac{\partial u_2}{\partial t} = d_2 \Delta u_2 + u_2 \left(r_2 - b_2 u_2 - a_2 u_3 + a_3 \int_\Omega \int_{-\infty}^t K_1(t-s, x, y) u_1(y,s) ds dy \right), & (x,t) \in \Omega \times (0, +\infty), \\ \frac{\partial u_3}{\partial t} = d_3 \Delta u_3 + u_3 \left(-\alpha - b_3 u_3 + a_4 \int_\Omega \int_{-\infty}^t K_2(t-s, x, y) u_2(y,s) ds dy \right), & (x,t) \in \Omega \times (0, +\infty), \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial u_3}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0, +\infty), \\ u_{i0}(x, \theta) = \phi_i(x, \theta) \geq 0 (i = 1, 2, 3), & (x,t) \in \overline{\Omega} \times (-\infty, 0], \end{cases} \tag{1.2}$$

where Ω is the bounded domain in R^N ($N \geq 1$ is an integer) with a smooth boundary $\partial\Omega$. u_1 represents the density of the prey; u_2 is for the density of the prey, and the same for the predator; u_3 represents the density of the predator. The initial functions $u_{i0}(x,\theta)(i = 1,2,3)$ are nonnegative bounded and Hölder continuous. $\int_\Omega \int_{-\infty}^t K_1(t-s,x,y)u_1(y,s)dsdy$ represents a time delay due to gestation. All parameters are positive constants. Li and Huang employed the method of Lyapunov functions to study the stability of positive equilibrium of system (1.2), and the existence of traveling wave solutions was proved by constructing upper-lower solutions. For more details on the background of this system, see [15].

Motivated by [3, 14, 15], in this paper, we consider the following one prey-two competing predators model with nonlocal delay:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta u + u(1-u) - \frac{u(g_1*v)}{b_1+u} - \frac{u(g_2*w)}{b_2+u}, & (x,t) \in \Omega \times (0,+\infty), \\ \frac{\partial v}{\partial t} = d_2\Delta v + \alpha_1(g_3*v) + \frac{a_1u(g_1*v)}{b_1+u} - m_1vw - h_1v^2, & (x,t) \in \Omega \times (0,+\infty), \\ \frac{\partial w}{\partial t} = d_3\Delta w + \alpha_2(g_4*w) + \frac{a_2u(g_2*w)}{b_2+u} - m_2vw - h_2w^2, & (x,t) \in \Omega \times (0,+\infty), \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ (u,v,w)(x,t) = (\phi,\varphi,\psi)(x,t) \geq 0, & (x,t) \in \overline{\Omega} \times (-\infty,0], \end{cases} \tag{1.3}$$

where $u(x,t)$, $v(x,t)$, and $w(x,t)$ denote the population densities of a prey species and two competing predator species at location x and time t, respectively. Ω is a bounded domain in R^N ($N \geq 1$ is an integer) with a smooth boundary $\partial\Omega$. The boundary conditions indicate that the populations do not move across the boundary $\partial\Omega$. The initial conditions reflect the historical state of the populations, ensuring continuity from past to present. Δ is a Laplace operator, $d_1,\ d_2,\ d_3 > 0$ are the diffusion coefficients. Both predator species prey on the same resource, with their consumption rates following Holling type II functional responses. Here, a_1 and a_2 denote the searching efficiencies, while b_1 and b_2 represent the half-saturation constants. The birth rates of two predator species are α_1 and α_2. The strengths of inter-specific competition between the two predator species are quantified by m_1 and m_2, and the intra-specific competition within each predator species is measured by h_1 and h_2. η is the unit outer normal vector on the boundary. All the above coefficients have been normalized.

In system (1.3), $(g_i * v)(x,t)$ $(i = 1,3)$, $(g_j * w)(x,t)$ $(j = 2,4)$ are the nonlocal delay defined by

$$(g_i * v)(x,t) = \int_{-\infty}^t \int_\Omega g_i(x-y,t-s)v(y,s)dyds = \int_0^{+\infty} \int_\Omega g_i(y,s)v(x-y,t-s)dyds,$$

$$(g_j * w)(x,t) = \int_{-\infty}^t \int_\Omega g_j(x-y,t-s)w(y,s)dyds = \int_0^{+\infty} \int_\Omega g_j(y,s)w(x-y,t-s)dyds,$$

where the kernel $g_n(x,t)$ $(n = 1,2,3,4)$ satisfies

$$g_n(x,t) = \hat{g}_n(x,t)k_n(t), \quad x \in \Omega,\ k_n(t) \geq 0,$$

$$g_n(x,t) = g_n(-x,t), \qquad \int_\Omega \hat{g}_n(y,t)dy = 1\ for\ t \geq 0,$$

$$\int_0^\infty k_n(t)dt = 1, \quad tk_n(t) \in L^1((0,\infty);R),$$

and $\forall c \geq 0, \lambda \geq 0$,

$$\int_0^{+\infty} \int_\Omega g_n(y,s)e^{-\lambda(y+cs)}dyds < +\infty,\ y \in \Omega \subset R^N$$

holds. $\hat{g}_n(x,t)$ are nonnegative functions which are continuous in $x \in \overline{\Omega}$ for each $t \in [0,+\infty)$ and measurable in $t \in [0,+\infty)$ for each pair $x \in \overline{\Omega}$. In this paper, $g_1 * v$ and $g_2 * w$ denote effective predation pressures, and $g_3 * v$ and $g_4 * w$ denote the number born at location y and time $t - s$ that are still alive now at location x and time t, respectively.

The analysis of system (1.3) is nontrivial. The interplay of nonlocal time delay with intra-specific and inter-specific competition not only increases computational complexity but also requires more precise estimates.

System (1.3) has a unique positive equilibrium $E^* = (u^*, v^*, w^*)$ if and only if the following conditions $(H1)$ or $(H2)$ hold.

$(H1)$ $\frac{m_1 m_2}{h_1 h_2} = 1$. If

(a) $[m_2\alpha_1(R_1 + 1) - h_1\alpha_2(R_2 + 1)][m_2\alpha_1(\widetilde{R}_1 + 1) - h_1\alpha_2(\widetilde{R}_2 + 1)] < 0$, where $R_k = \frac{a_k}{\alpha_k(1+2b_k)}$, $\widetilde{R}_k = \frac{a_k}{\alpha_k(1+b_k)}$ $(k = 1, 2)$,

(b) $\Theta = \frac{m_1}{b_1+u^*} - \frac{h_1}{b_2+u^*} \neq 0$,

(c) $m_1(1 - u^*) - \frac{\theta_i(u^*)}{b_2+u^*} > 0$, $\frac{\theta_i(u^*)}{b_1+u^*} - h_1(1 - u^*) > 0$, where $\theta_i(u^*) = \frac{a_i u^*}{b_i+u^*} + \alpha_i$ $(i = 1, 2)$

hold, then system (1.3) has a unique solution $u^* \in (\frac{1}{2}, 1)$, and

$$v^* = \frac{m_1(1 - u^*) - \frac{\theta_i(u^*)}{b_2+u^*}}{\Theta}, \quad w^* = \frac{\frac{\theta_i(u^*)}{b_1+u^*} - h_1(1 - u^*)}{\Theta}.$$

$(H2)$ $\frac{m_1 m_2}{h_1 h_2} \neq 1$. If

(a) $\Phi(\frac{1}{2})\Phi(1) < 0$, where $\Phi(x) = (m_1 m_2 - h_1 h_2)G_3(x) - G_4(x)$,

(b) $m_1 m_2 - h_1 h_2 > 0,\ 2(b_1 + b_2) \geq 1,\ \hat{P}_3 \leq 0,\ \hat{P}_2 \leq 0,\ \hat{P}_1 \leq 0$, or
$m_1 m_2 - h_1 h_2 < 0,\ 2(b_1 + b_2) \leq 1,\ \hat{P}_3 \geq 0,\ \hat{P}_2 \geq 0,\ \hat{P}_1 \geq 0$

hold, then system (1.3) has a unique solution $u^* \in (\frac{1}{2}, 1)$, and

$$v^* = \frac{\bar{P}_2 u^{*2} + \bar{P}_1 u^* + \bar{P}_0}{(m_1 m_2 - h_1 h_2)(b_1 + u^*)(b_2 + u^*)}, \quad w^* = \frac{P_2 u^{*2} + P_1 u^* + P_0}{(m_1 m_2 - h_1 h_2)(b_1 + u^*)(b_2 + u^*)}.$$

Among them, the specific form of $G_3(x), G_4(x), \hat{P}_i (i = 1, 2, 3), \bar{P}_j$, and $P_j (j = 0, 1, 2)$ can be found in [3].

The article is organized as follows: In Section 2, by employing a Lyapunov function, we investigate the globally asymptotic stability of the positive equilibrium E^* for (1.3). In Section 3, we give an eigenvalue problem that will be needed in subsequent sections. It is here that we define boundary conditions and the critical wave speed c^* of travelling wave solutions. In Section 4, by selecting appropriate kernel functions, we exploit Schauder's fixed point theorem to establish the existence of traveling wave solutions connecting the trivial equilibrium with the positive equilibrium. In the Appendix, we construct and verify upper-lower solutions.

2. Stability analysis

In this section, we study the globally asymptotic stability of the positive equilibrium E^* for (1.3) by using the Lyapunov function method.

Theorem 2.1. *Assuming* $(u, v, w) \in [C([0, T] \times \overline{\Omega}) \bigcap C^{1,2}([0, T] \times \Omega)]^3$ $(T > 0)$ *is a solution of system (1.3) with initial value* $(\phi, \varphi, \psi)(x, t) \geq 0$, *then* $(0, 0, 0) \leq (u, v, w) \leq (M_1, M_2, M_3)$, $(x, t) \in \overline{\Omega} \times [0, +\infty)$, *where*

$$M_1 = \max\left\{1, \|\phi(x, t)\|_{L^\infty(\overline{\Omega}\times[0,+\infty))}\right\},$$

$$M_2 = \max\left\{\frac{\alpha_1 b_1 + (\alpha_1 + a_1)M_1}{h_1(b_1 + M_1)}, \|\varphi(x,t)\|_{L^\infty\left(\overline{\Omega}\times[0,+\infty)\right)}\right\},$$
$$M_3 = \max\left\{\frac{\alpha_2 b_2 + (\alpha_2 + a_2)M_1}{h_2(b_2 + M_1)}, \|\psi(x,t)\|_{L^\infty\left(\overline{\Omega}\times[0,+\infty)\right)}\right\}. \tag{2.1}$$

Proof. According to the maximum principle [16], if $(\phi,\varphi,\psi)(x,t) > 0$ for all $x \in \overline{\Omega}$, $t \le 0$, we have $(u,v,w)(x,t) > 0$ for $t > 0$. So

$$\frac{\partial u}{\partial t} - d_1\Delta u = u(1-u) - \frac{u(g_1 * v)}{b_1 + u} - \frac{u(g_2 * w)}{b_2 + u} \le u(1-u),$$

and by the comparison principle [16], we deduce that $u \le \max\left\{1, \|\phi(x,t)\|_{L^\infty\left(\overline{\Omega}\times[0,+\infty)\right)}\right\} \triangleq M_1$.

Again

$$\frac{\partial v}{\partial t} - d_2\Delta v = \alpha_1(g_3 * v) + \frac{a_1 u(g_1 * v)}{b_1 + u} - m_1 vw - h_1 v^2 \le v(\alpha_1 + \frac{a_1 u}{b_1 + u} - h_1 v),$$

and by the comparison principle,

$$v \le \max\left\{\frac{\alpha_1 b_1 + (\alpha_1 + a_1)M_1}{h_1(b_1 + M_1)}, \|\varphi(x,t)\|_{L^\infty\left(\overline{\Omega}\times[0,+\infty)\right)}\right\} \triangleq M_2.$$

Similarly,

$$w \le \max\left\{\frac{\alpha_2 b_2 + (\alpha_2 + a_2)M_1}{h_2(b_2 + M_1)}, \|\psi(x,t)\|_{L^\infty\left(\overline{\Omega}\times[0,+\infty)\right)}\right\} \triangleq M_3.$$

Hence the solution $(u,v,w)(x,t)$ of system (1.3) is uniformly bounded on $\overline{\Omega}\times[0,+\infty)$. □

Further we discuss the globally asymptotic stability of the positive equilibrium E^* (u^*, v^*, w^*).

Lemma 2.1. *[17, 18] Assume $a, b > 0$, $\phi, \varphi \in C^1([a,+\infty))$, $\varphi(t) \ge 0$, $\phi(t)$ has a lower bound, and if $\phi'(t) \le -b\varphi(t)$ and there exists a constant $R > 0$ such that $\varphi'(t)$ has an upper bound for every $t \ge a$, then $\lim_{t\to\infty}\varphi(t) = 0$.*

Theorem 2.2. *Assume that $(H1)$ or $(H2)$ are satisfied, and $\frac{M_2}{b_1+u^*} + \frac{M_3}{b_2+u^*} < 1$ holds. Then the positive equilibrium E^* (u^*, v^*, w^*) for system (1.3) is globally asymptotically stable.*

Proof. System (1.3) can be rewritten as

$$\begin{aligned}
\frac{\partial u}{\partial t} =& d_1\Delta u + u\left[-(u-u^*) - \frac{g_1 * (v-v^*)}{b_1+u^*} - \frac{g_2 * (w-w^*)}{b_2+u^*} + \frac{(g_1 * v)(u-u^*)}{(b_1+u)(b_1+u^*)} + \frac{(g_2 * w)(u-u^*)}{(b_2+u)(b_2+u^*)}\right],\\
\frac{\partial v}{\partial t} =& d_2\Delta v + \frac{\alpha_1 v[g_3 * (v-v^*)]}{v^*} + \frac{a_1 u^* v[g_1 * (v-v^*)]}{v^*(b_1+u^*)} - \frac{\alpha_1(g_3 * v)(v-v^*)}{v^*} - \frac{a_1 u^*(g_1 * v)(v-v^*)}{v^*(b_1+u^*)}\\
&+ \frac{a_1 b_1(g_1 * v)(u-u^*)}{(b_1+u)(b_1+u^*)} - m_1 v(w-w^*) - h_1 v(v-v^*),\\
\frac{\partial w}{\partial t} =& d_3\Delta w + \frac{\alpha_2 w[g_4 * (w-w^*)]}{w^*} + \frac{a_2 u^* w[g_2 * (w-w^*)]}{w^*(b_2+u^*)} - \frac{\alpha_2(g_4 * w)(w-w^*)}{w^*}
\end{aligned}$$

$$-\frac{a_2u^*(g_2*w)(w-w^*)}{w^*(b_2+u^*)}+\frac{a_2b_2(g_2*w)(u-u^*)}{(b_2+u)(b_2+u^*)}-m_2w(v-v^*)-h_2w(w-w^*).$$

Let (u,v,w) be a positive solution for system (1.3), and define the Lyapunov function

$$V_1(t)=\int_\Omega(u-u^*-u^*\ln\frac{u}{u^*})dx+\rho_2\int_\Omega(v-v^*-v^*\ln\frac{v}{v^*})dx+\rho_3\int_\Omega(w-w^*-w^*\ln\frac{w}{w^*})dx,$$

where $\rho_i(i=2,3)$ are positive constants to be determined.

Then,

$$\begin{aligned}\frac{dV_1(t)}{dt}=&\int_\Omega\frac{\partial u}{\partial t}(1-\frac{u^*}{u})dx+\rho_2\int_\Omega\frac{\partial v}{\partial t}(1-\frac{v^*}{v})dx+\rho_3\int_\Omega\frac{\partial w}{\partial t}(1-\frac{w^*}{w})dx\\
=&-d_1\int_\Omega\frac{u^*}{u^2}|\nabla u|^2dx-\rho_2d_2\int_\Omega\frac{v^*}{v^2}|\nabla v|^2dx-\rho_3d_3\int_\Omega\frac{w^*}{w^2}|\nabla w|^2dx\\
&+\int_\Omega(u-u^*)\left[-(u-u^*)-\frac{g_1*(v-v^*)}{b_1+u^*}-\frac{g_2*(w-w^*)}{b_2+u^*}+\frac{(g_1*v)(u-u^*)}{(b_1+u)(b_1+u^*)}+\frac{(g_2*w)(u-u^*)}{(b_2+u)(b_2+u^*)}\right]dx\\
&+\rho_2\int_\Omega\frac{v-v^*}{v}\left[\frac{\alpha_1v[g_3*(v-v^*)]}{v^*}+\frac{a_1u^*v[g_1*(v-v^*)]}{v^*(b_1+u^*)}-\frac{\alpha_1(g_3*v)(v-v^*)}{v^*}\right.\\
&\left.-\frac{a_1u^*(g_1*v)(v-v^*)}{v^*(b_1+u^*)}+\frac{a_1b_1(g_1*v)(u-u^*)}{(b_1+u)(b_1+u^*)}-m_1v(w-w^*)-h_1v(v-v^*)\right]dx\\
&+\rho_3\int_\Omega\frac{w-w^*}{w}\left[\frac{\alpha_2w[g_4*(w-w^*)]}{w^*}+\frac{a_2u^*w[g_2*(w-w^*)]}{w^*(b_2+u^*)}-\frac{\alpha_2(g_4*w)(w-w^*)}{w^*}\right.\\
&\left.-\frac{a_2u^*(g_2*w)(w-w^*)}{w^*(b_2+u^*)}+\frac{a_2b_2(g_2*w)(u-u^*)}{(b_2+u)(b_2+u^*)}-m_2w(v-v^*)-h_2w(w-w^*)\right]dx.\end{aligned}$$

From (1.3) and applying the inequality $ab\le\frac{1}{2}\kappa a^2+\frac{1}{2\kappa}b^2$ $(\kappa>0)$, we obtain

$$\begin{aligned}\frac{dV_1(t)}{dt}\le&-d_1\int_\Omega\frac{u^*}{u^2}|\nabla u|^2dx-\rho_2d_2\int_\Omega\frac{v^*}{v^2}|\nabla v|^2dx-\rho_3d_3\int_\Omega\frac{w^*}{w^2}|\nabla w|^2dx\\
&-\int_\Omega L_u(u-u^*)^2dx-\rho_2\int_\Omega L_v(v-v^*)^2dx-\rho_3\int_\Omega L_w(w-w^*)^2dx\\
&+\int_\Omega\int_\Omega\int_{-\infty}^t g_1(x-y,t-s)\left[\frac{\kappa_1}{2}(v(y,s)-v^*)^2+\frac{1}{2\kappa_1}(u(x,t)-u^*)^2\right]dsdydx\\
&+\int_\Omega\int_\Omega\int_{-\infty}^t g_2(x-y,t-s)\left[\frac{\kappa_2}{2}(w(y,s)-w^*)^2+\frac{1}{2\kappa_2}(u(x,t)-u^*)^2\right]dsdydx\\
&+\rho_2\alpha_1\int_\Omega\int_\Omega\int_{-\infty}^t g_3(x-y,t-s)\left[\frac{\kappa_3}{2}(v(y,s)-v^*)^2+\frac{1}{2\kappa_3}(v(x,t)-v^*)^2\right]dsdydx\\
&+\rho_2a_1u^*\int_\Omega\int_\Omega\int_{-\infty}^t g_1(x-y,t-s)\left[\frac{\kappa_3}{2}(v(y,s)-v^*)^2+\frac{1}{2\kappa_3}(v(x,t)-v^*)^2\right]dsdydx\\
&+\rho_2a_1b_1M_2\int_\Omega\left[\frac{\kappa_4}{2}(v(x,t)-v^*)^2+\frac{1}{2\kappa_4}(u(x,t)-u^*)^2\right]dx\\
&+(\rho_2m_1+\rho_3m_2)\int_\Omega\left[\frac{\kappa_5}{2}(v(x,t)-v^*)^2+\frac{1}{2\kappa_5}(w(x,t)-w^*)^2\right]dx\end{aligned}$$

$$+\rho_3\alpha_2\int_\Omega\int_\Omega\int_{-\infty}^t g_4(x-y,t-s)\left[\frac{\kappa_6}{2}(w(y,s)-w^*)^2+\frac{1}{2\kappa_6}(w(x,t)-w^*)^2\right]dsdydx$$
$$+\rho_3 a_2 u^*\int_\Omega\int_\Omega\int_{-\infty}^t g_2(x-y,t-s)\left[\frac{\kappa_6}{2}(w(y,s)-w^*)^2+\frac{1}{2\kappa_6}(w(x,t)-w^*)^2\right]dsdydx$$
$$+\rho_3 a_2 b_2 M_3\int_\Omega\left[\frac{\kappa_7}{2}(w(x,t)-w^*)^2+\frac{1}{2\kappa_7}(u(x,t)-u^*)^2\right]dx,$$

where

$$L_u=1-\frac{g_1*v}{(b_1+u)(b_1+u^*)}-\frac{g_2*w}{(b_2+u)(b_2+u^*)}>1-\frac{M_2}{(b_1+u^*)}-\frac{M_3}{(b_2+u^*)}>0,$$
$$L_v=\frac{\alpha_1(g_3*v)}{vv^*}+\frac{a_1u^*(g_1*v)}{vv^*(b_1+u^*)}+h_1>0,$$
$$L_w=\frac{\alpha_2(g_4*w)}{ww^*}+\frac{a_1u^*(g_2*w)}{ww^*(b_2+u^*)}+h_2>0.$$

By the properties of the kernel function, we have

$$\begin{aligned}\frac{dV_1(t)}{dt}\leq&-d_1\int_\Omega\frac{u^*}{u^2}|\nabla u|^2dx-\rho_2d_2\int_\Omega\frac{v^*}{v^2}|\nabla v|^2dx-\rho_3d_3\int_\Omega\frac{w^*}{w^2}|\nabla w|^2dx\\
&-\left[L_u-\frac{1}{2\kappa_1}-\frac{1}{2\kappa_2}-\frac{\rho_2a_1b_1M_2}{2\kappa_4}-\frac{\rho_3a_2b_2M_3}{2\kappa_7}\right]\int_\Omega(u(x,t)-u^*)^2dx\\
&-\left[\rho_2\left(L_v-\frac{a_1u^*+\alpha_1}{2\kappa_3}-\frac{a_1b_1M_2\kappa_4}{2}\right)-\frac{\rho_2m_1+\rho_3m_2}{2}\kappa_5\right]\int_\Omega(v(x,t)-v^*)^2dx\\
&-\left[\rho_3\left(L_w-\frac{a_2u^*+\alpha_2}{2\kappa_6}-\frac{a_2b_2M_3\kappa_7}{2}\right)-\frac{\rho_2m_1+\rho_3m_2}{2\kappa_5}\right]\int_\Omega(w(x,t)-w^*)^2dx\\
&+\frac{\kappa_1+\rho_2a_1u^*\kappa_3}{2}\int_\Omega\int_\Omega\int_0^\infty g_1(y,s)(v(x-y,t-s)-v^*)^2dsdydx\\
&+\frac{\kappa_2+\rho_3a_2u^*\kappa_6}{2}\int_\Omega\int_\Omega\int_0^\infty g_2(y,s)(w(x-y,t-s)-w^*)^2dsdydx\\
&+\frac{\rho_2\alpha_1\kappa_3}{2}\int_\Omega\int_\Omega\int_0^\infty g_3(y,s)(v(x-y,t-s)-v^*)^2dsdydx\\
&+\frac{\rho_3\alpha_2\kappa_6}{2}\int_\Omega\int_\Omega\int_0^\infty g_4(y,s)(w(x-y,t-s)-w^*)^2dsdydx.\end{aligned}$$

Let

$$\begin{aligned}V(t)=&V_1(t)+\frac{\kappa_1+\rho_2a_1u^*\kappa_3}{2}\int_\Omega\int_\Omega\int_0^\infty\int_{t-s}^t g_1(y,s)(v(x-y,r)-v^*)^2drdsdydx\\
&+\frac{\kappa_2+\rho_3a_2u^*\kappa_6}{2}\int_\Omega\int_\Omega\int_0^\infty\int_{t-s}^t g_2(y,s)(w(x-y,r)-w^*)^2drdsdydx\\
&+\frac{\rho_2\alpha_1\kappa_3}{2}\int_\Omega\int_\Omega\int_0^\infty\int_{t-s}^t g_3(y,s)(v(x-y,r)-v^*)^2drdsdydx\\
&+\frac{\rho_3\alpha_2\kappa_6}{2}\int_\Omega\int_\Omega\int_0^\infty\int_{t-s}^t g_4(y,s)(w(x-y,r)-w^*)^2drdsdydx,\end{aligned}$$

and then

$$\begin{aligned}\frac{dV(t)}{dt} \leq & -d_1\int_\Omega \frac{u^*}{u^2}|\nabla u|^2dx-\rho_2 d_2\int_\Omega \frac{v^*}{v^2}|\nabla v|^2dx-\rho_3 d_3\int_\Omega \frac{w^*}{w^2}|\nabla w|^2dx\\ & -\left[L_u-\frac{1}{2\kappa_1}-\frac{1}{2\kappa_2}-\frac{\rho_2 a_1 b_1 M_2}{2\kappa_4}-\frac{\rho_3 a_2 b_2 M_3}{2\kappa_7}\right]\int_\Omega (u(x,t)-u^*)^2dx\\ & -\left[\rho_2\left(L_v-\frac{a_1u^*+\alpha_1}{2\kappa_3}-\frac{a_1b_1M_2\kappa_4}{2}\right)-\frac{\rho_2m_1+\rho_3m_2}{2}\kappa_5\right]\int_\Omega (v(x,t)-v^*)^2dx\\ & -\left[\rho_3\left(L_w-\frac{a_2u^*+\alpha_2}{2\kappa_6}-\frac{a_2b_2M_3\kappa_7}{2}\right)-\frac{\rho_2m_1+\rho_3m_2}{2\kappa_5}\right]\int_\Omega (w(x,t)-w^*)^2dx\\ & +\frac{\kappa_1+\rho_2a_1u^*\kappa_3}{2}\int_\Omega\int_\Omega\int_0^\infty g_1(y,s)(v(x-y,t)-v^*)^2dsdydx\\ & +\frac{\kappa_2+\rho_3a_2u^*\kappa_6}{2}\int_\Omega\int_\Omega\int_0^\infty g_2(y,s)(w(x-y,t)-w^*)^2dsdydx\\ & +\frac{\rho_2\alpha_1\kappa_3}{2}\int_\Omega\int_\Omega\int_0^\infty g_3(y,s)(v(x-y,t)-v^*)^2dsdydx\\ & +\frac{\rho_3\alpha_2\kappa_6}{2}\int_\Omega\int_\Omega\int_0^\infty g_4(y,s)(w(x-y,t)-w^*)^2dsdydx.\end{aligned}$$

Since

$$\int_\Omega\int_\Omega\int_0^\infty g(y,s)(v(x-y,t)-v^*)^2dsdydx=\int_\Omega\int_\Omega\int_0^\infty g(y,s)(v(x-y,t)-v^*)^2dsdxdy=\int_\Omega (v(x-y,t)-v^*)^2dy,$$

then

$$\begin{aligned}\frac{dV(t)}{dt} \leq & -d_1\int_\Omega \frac{u^*}{u^2}|\nabla u|^2dx-\rho_2 d_2\int_\Omega \frac{v^*}{v^2}|\nabla v|^2dx-\rho_3 d_3\int_\Omega \frac{w^*}{w^2}|\nabla w|^2dx\\ & -\left[L_u-\frac{1}{2\kappa_1}-\frac{1}{2\kappa_2}-\frac{\rho_2 a_1 b_1 M_2}{2\kappa_4}-\frac{\rho_3 a_2 b_2 M_3}{2\kappa_7}\right]\cdot\int_\Omega (u(x,t)-u^*)^2dx\\ & -\left\{\rho_2\left[L_v-\frac{a_1u^*+\alpha_1}{2}\left(\kappa_3+\frac{1}{\kappa_3}\right)-\frac{a_1b_1M_2\kappa_4}{2}\right]-\frac{\rho_2m_1+\rho_3m_2}{2}\kappa_5-\frac{\kappa_1}{2}\right\}\cdot\int_\Omega (v(x,t)-v^*)^2dx\\ & -\left\{\rho_3\left[L_w-\frac{a_2u^*+\alpha_2}{2}\left(\kappa_6+\frac{1}{\kappa_6}\right)-\frac{a_2b_2M_3\kappa_7}{2}\right]-\frac{\rho_2m_1+\rho_3m_2}{2\kappa_5}-\frac{\kappa_2}{2}\right\}\cdot\int_\Omega (w(x,t)-w^*)^2dx. \qquad (2.2)\end{aligned}$$

For any $T>0$, integrating (2.2) over $[0,T]$, we have

$$\begin{aligned}& d_1u^*\left\|\frac{|\nabla u|}{u^2}\right\|^2_{L^2(\Omega_T)}+\rho_2d_2v^*\left\|\frac{|\nabla v|}{v^2}\right\|^2_{L^2(\Omega_T)}+\rho_3d_3w^*\left\|\frac{|\nabla w|}{w^2}\right\|^2_{L^2(\Omega_T)}\\ & +L_u\|u-u^*\|^2_{L^2(\Omega_T)}+\rho_2L_v\|v-v^*\|^2_{L^2(\Omega_T)}+\rho_3L_w\|w-w^*\|^2_{L^2(\Omega_T)}\\ \leq & V(0)+\left(\frac{1}{2\kappa_1}+\frac{1}{2\kappa_2}+\frac{\rho_2a_1b_1M_2}{2\kappa_4}+\frac{\rho_3a_2b_2M_3}{2\kappa_7}\right)\|u-u^*\|^2_{L^2(\Omega_T)}\\ & +\left[\rho_2\left(\frac{a_1u^*+\alpha_1}{2}\left(\kappa_3+\frac{1}{\kappa_3}\right)+\frac{a_1b_1M_2\kappa_4}{2}\right)+\frac{\rho_2m_1+\rho_3m_2}{2}\kappa_5+\frac{\kappa_1}{2}\right]\|v-v^*\|^2_{L^2(\Omega_T)}\end{aligned}$$

$$+\left[\rho_3\left(\frac{a_2u^*+\alpha_2}{2}\left(\kappa_6+\frac{1}{\kappa_6}\right)+\frac{a_2b_2M_3\kappa_7}{2}\right)+\frac{\rho_2m_1+\rho_3m_2}{2\kappa_5}+\frac{\kappa_2}{2}\right]\|w-w^*\|^2_{L^2(\Omega_T)}. \quad (2.3)$$

Choose

$$\kappa_1=\kappa_4=\frac{1+\rho_2a_1b_1M_2}{L_u},\quad \kappa_2=\kappa_7=\frac{1+\rho_3a_2b_2M_3}{L_u},\quad \kappa_5=\frac{1}{\rho_2m_1+\rho_3m_2},$$
$$\kappa_3+\frac{1}{\kappa_3}=\frac{1}{\rho_2(a_1u^*+\alpha_1)},\quad \kappa_6+\frac{1}{\kappa_6}=\frac{1}{\rho_3(a_2u^*+\alpha_2)},$$

and it is derived from (2.3) that

$$d_1u^*\left\|\frac{|\nabla u|}{u^2}\right\|^2_{L^2(\Omega_T)}+\rho_2d_2v^*\left\|\frac{|\nabla v|}{v^2}\right\|^2_{L^2(\Omega_T)}+\rho_3d_3w^*\left\|\frac{|\nabla w|}{w^2}\right\|^2_{L^2(\Omega_T)}+\rho_2L_v\|v-v^*\|^2_{L^2(\Omega_T)}+\rho_3L_w\|w-w^*\|^2_{L^2(\Omega_T)}$$
$$\le V(0)+\left[1+\frac{(1+v_2a_1b_1M_2)^2}{2L_u}\right]\|v-v^*\|^2_{L^2(\Omega_T)}+\frac{1}{2}\left[1+\frac{(1+\rho_3a_2b_2M_3)^2}{L_u}+(\rho_2m_1+\rho_3m_2)^2\right]\|w-w^*\|^2_{L^2(\Omega_T)}.$$

We choose the appropriate $\rho_i>0 (i=2,3)$ such that

$$\rho_2L_v>1+\frac{(1+\rho_2a_1b_1M_2)^2}{2L_u},\quad \rho_3L_w>\frac{1}{2}\left[1+\frac{(1+\rho_3a_2b_2M_3)^2}{L_u}+(\rho_2m_1+\rho_3m_2)^2\right]$$

hold.

Therefore, we obtain

$$\left\|\frac{|\nabla v|}{v^2}\right\|_{L^2(\Omega_T)}\le C_1,\quad \left\|\frac{|\nabla w|}{w^2}\right\|_{L^2(\Omega_T)}\le C_2, \quad (2.4)$$

and

$$\|v-v^*\|_{L^2(\Omega_T)}\le C_3,\quad \|w-w^*\|_{L^2(\Omega_T)}\le C_4, \quad (2.5)$$

for some constants $C_i (i=1,2,3,4)$ independent of T.

Similarly, choose

$$\kappa_1=\kappa_4=\frac{L_u}{1+\rho_2a_1b_1M_2},\quad \kappa_2=\kappa_7=\frac{L_u}{1+\rho_3a_2b_2M_3},\quad \kappa_5=\frac{1}{\rho_2m_1+\rho_3m_2},$$
$$\kappa_3+\frac{1}{\kappa_3}=\frac{1}{\rho_2(a_1u^*+\alpha_1)},\quad \kappa_6+\frac{1}{\kappa_6}=\frac{1}{\rho_3(a_2u^*+\alpha_2)},$$

and we can obtain

$$\left\|\frac{|\nabla u|}{u^2}\right\|_{L^2(\Omega_T)}\le C_5,\quad \|u-u^*\|_{L^2(\Omega_T)}\le C_6, \quad (2.6)$$

for some positive constants C_5 and C_6 independent of $T>0$.

Using the conditions of Theorem 2.1 and (2.2), we show that there exists a positive constant δ such that

$$\frac{dV(t)}{dt}\le-\delta\int_\Omega[(u-u^*)^2+(v-v^*)^2+(w-w^*)^2]dx\le 0. \quad (2.7)$$

By integration of (2.7), and from (2.1), (2.4)–(2.6), it is easily seen that

$$\frac{d}{dt}\int_\Omega[(u-u^*)^2+(v-v^*)^2+(w-w^*)^2]dx$$

has an upper bound. Then, using Lemma 2.1 and (2.7), we see that

$$\|u(\cdot,t)-u^*\|_{L^2(\Omega)}\to 0,\quad \|v(\cdot,t)-v^*\|_{L^2(\Omega)}\to 0,\quad \|w(\cdot,t)-w^*\|_{L^2(\Omega)}\to 0. \tag{2.8}$$

By Sobolev embedding theorems, there exists a constant $C>0$ such that $\forall\,\chi\in W_2^1(\Omega)$, we have

$$\|\chi(x,t)\|_{L^\infty(\Omega)}\le C\|\chi\|^{\frac{1}{2}}_{W_2^1(\Omega)}\|\chi\|^{\frac{1}{2}}_{L^2(\Omega)}. \tag{2.9}$$

It follows from (2.1), (2.4)–(2.6), (2.8), and (2.9) that

$$\|u(\cdot,t)-u^*\|_{L^\infty(\Omega)}\to 0,\quad \|v(\cdot,t)-v^*\|_{L^\infty(\Omega)}\to 0.\quad \|w(\cdot,t)-w^*\|_{L^\infty(\Omega)}\to 0.$$

Namely, (u,v,w) converges uniformly to (u^*,v^*,w^*). Using the fact that $V(u,v,w)$ is decreasing for t, one can derive that (u^*,v^*,w^*) is globally asymptotically stable. □

3. Eigenvalue problem

In this section, we present some preliminary results that will be needed for the subsequent sections. A traveling wave solution of system (1.3) takes the special form

$$(u,v,w)(x,t)=(u,v,w)(x\cdot\nu+ct),$$

where $\nu\in R^N$ denotes a unit propagation direction vector, with $x\cdot\nu$ representing the standard inner product in R^N. For a wave speed $c>0$, define the traveling wave coordinate $t=x\cdot\nu+ct$. Then system (1.3) has a traveling wave solution $(u(t),v(t),w(t))$ connecting the trivial equilibrium $(0,0,0)$ with the positive equilibrium (u^*,v^*,w^*) if and only if the following system

$$\begin{cases} d_1u''(t)-cu'(t)+u(t)(1-u(t))-\dfrac{u(t)(g_1*v)(t)}{b_1+u(t)}-\dfrac{u(t)(g_2*w)(t)}{b_2+u(t)}=0,\\ d_2v''(t)-cv'(t)+\alpha_1(g_3*v)(t)+\dfrac{a_1u(t)(g_1*v)(t)}{b_1+u(t)}-m_1v(t)w(t)-h_1v^2(t)=0,\\ d_3w''(t)-cw'(t)+\alpha_2(g_4*w)(t)+\dfrac{a_2u(t)(g_2*w)(t)}{b_2+u(t)}-m_2v(t)w(t)-h_2w^2(t)=0, \end{cases} \tag{3.1}$$

has a solution that satisfies the following asymptotic boundary conditions:

$$\lim_{t\to-\infty}(u,v,w)(t)=(0,0,0),\qquad \lim_{t\to+\infty}(u,v,w)(t)=(u^*,v^*,w^*), \tag{3.2}$$

where

$$(g_i*\varphi)(t)=\int_0^{+\infty}\int_\Omega g_i(y,s)\varphi(t-y\cdot\nu-cs)dyds\ (i=1,2,3,4). \tag{3.3}$$

As $t\to\infty$, the linearized system (3.1) at $(0,0,0)$ yields

$$d_1u''-cu'+u=0,\quad d_2v''-cv'+\alpha_1(g_3*v)=0,\quad d_3w''-cw'+\alpha_2(g_4*w)=0.$$

Let the solutions to the above equations all be $e^{\lambda t}$. Then

$$\begin{cases} \Delta_1(\lambda,c):=d_1\lambda^2-c\lambda+1,\\ \Delta_2(\lambda,c):=d_2\lambda^2-c\lambda+\alpha_1\int_0^{+\infty}\int_\Omega g_3(y,s)e^{-\lambda(y\cdot\nu+cs)}dyds,\\ \Delta_3(\lambda,c):=d_3\lambda^2-c\lambda+\alpha_2\int_0^{+\infty}\int_\Omega g_4(y,s)e^{-\lambda(y\cdot\nu+cs)}dyds, \end{cases} \tag{3.4}$$

where $y\in\Omega\subset R^N$.

We easily obtain the following lemma.

Lemma 3.1. *If there exist* $c_i^* > 0, \lambda_i^* > 0$ $(i = 2, 3)$*, then the following results:*

(i) $\Delta_i(\lambda_i^*, c_i^*) = 0$, $\frac{\partial \Delta_i(\lambda,c)}{\partial \lambda} \mid_{(\lambda_i^*, c_i^*)} = 0$;

(ii) *for every* $0 < c < c_i^*$, $\Delta_i(\lambda, c) > 0$ *for all* $\lambda > 0$;

(iii) *for every* $c > c_i^*$, $\Delta_i(\lambda, c) = 0$ *has two positive real roots* λ_i, $\widetilde{\lambda}_i$ *satisfying* $0 < \lambda_i < \lambda_i^* < \widetilde{\lambda}_i$, *and*

$$\Delta_i(\lambda, c) \begin{cases} > 0, & \forall\, \lambda \in (0, \lambda_i) \cup (\widetilde{\lambda}_i, +\infty), \\ < 0, & \forall\, \lambda \in (\lambda_i, \widetilde{\lambda}_i) \end{cases}$$

hold.

Proof. For $\forall c > 0, \lambda > 0$, we have $\Delta_i(0, c) = \alpha_i > 0$, $\Delta_i(+\infty, c) \to +\infty (i = 2, 3)$, and for $j = 3, 4$,

$$\begin{aligned} \frac{\partial \Delta_i(\lambda, c)}{\partial \lambda} \mid_{\lambda=0} &= 2d_i\lambda - c + \alpha_i \int_0^{+\infty} \int_\Omega g_j(y, s)[-(y \cdot \nu + cs)e^{-\lambda(y \cdot \nu + cs)}]dyds \mid_{\lambda=0} \\ &= -c - \alpha_i \int_0^{+\infty} \int_\Omega g_j(y, s)(y \cdot \nu + cs)dyds. \end{aligned}$$

Due to $g_j(y, s) = g_j(-y, s)$, we have $\int_\Omega y g_j(y, s)dy = 0$, so

$$\frac{\partial \Delta_i(\lambda, c)}{\partial \lambda} \mid_{\lambda=0} = -c - c\alpha_i \int_0^{+\infty} \int_\Omega s g_j(y, s)dyds < 0,$$

$$\frac{\partial^2 \Delta_i(\lambda, c)}{\partial \lambda^2} = 2d_i + \alpha_i \int_0^{+\infty} \int_\Omega (y \cdot \nu + cs)^2 g_j(y, s)e^{-\lambda(y \cdot \nu + cs)}dyds > 0.$$

Therefore, $\Delta_i(\lambda, c)$ is a convex function.

The strict convexity of $\Delta_i(\lambda, c)$ ensures that the derivative $\frac{\partial \Delta_i}{\partial \lambda}$ is strictly increasing, so that the derivative gradually rises from a negative value, eventually tending toward $+\infty$, and it must equal zero at some point $\lambda_i^* > 0$, which is the unique minimum point.

Moreover,

$$\Delta_i(\lambda, 0) = d_i\lambda^2 + \alpha_i \int_0^{+\infty} \int_\Omega g_j(y, s)e^{-\lambda(y \cdot \nu + cs)}dyds > 0,$$

$$\frac{\partial \Delta_i(\lambda, c)}{\partial c} = -\lambda - \alpha_i\lambda \int_0^{+\infty} \int_\Omega s g_j(y, s)e^{-\lambda(y \cdot \nu + cs)}dyds < 0,$$

for all $\lambda > 0$, and the value $\Delta_i(\lambda_i^*, c)$ of $\Delta_i(\lambda, c)$ at λ_i^* changes with c. If c increases, $\Delta_i(\lambda, c)$ decreases, and then $\Delta_i(\lambda_i^*, c)$ decreases. Therefore, there exists $c_i^* > 0$, $\lambda_i^* > 0$ such that

$$\Delta_i(\lambda_i^*, c^*) = 0, \quad \frac{\partial \Delta_i(\lambda, c)}{\partial \lambda} \mid_{(\lambda_i^*, c_i^*)} = 0.$$

For $0 < c < c_i^*$, minimum value $\Delta_i(\lambda_i^*, c) > 0$, so that $\Delta_i(\lambda, c) > 0$ for all $\lambda > 0$; for $c > c_i^*$, according to the convexity and monotonic decrease of function $\Delta_i(\lambda, c)$ with respect to $c > 0$, and $\Delta_i(\lambda_i^*, c^*) = 0$, combined with the theorem of existence of zeros, we can see that $\Delta_i(\lambda, c)$ has two positive real roots λ_i, $\widetilde{\lambda}_i$ satisfying $0 < \lambda_i < \lambda_i^* < \widetilde{\lambda}_i$, and

$$\Delta_i(\lambda, c) \begin{cases} > 0, & \forall\, \lambda \in (0, \lambda_i) \cup (\widetilde{\lambda}_i, +\infty), \\ < 0, & \forall\, \lambda \in (\lambda_i, \widetilde{\lambda}_i). \end{cases}$$

□

Similarly, we easily obtain the following lemma.

Lemma 3.2. *There exist* $c_1^* > 0, \lambda_1^* > 0$ *satisfying* $\Delta_1(\lambda_1^*, c_1^*) = 0$, $\frac{\partial \Delta_1(\lambda,c)}{\partial \lambda}\mid_{(\lambda_1^*,c_1^*)} = 0$. *Then the following results:*

(i) *for every* $0 < c < c_1^*$, $\Delta_1(\lambda, c) = 0$ *has no real root, and* $\Delta_1(\lambda, c) > 0$ *for all* $\lambda > 0$*;*

(ii) *for every* $c > c_1^*$, $\Delta_1(\lambda, c) = 0$ *has two positive real roots* $\lambda_1, \widetilde{\lambda}_1$ *satisfying* $0 < \lambda_1 < \lambda_1^* < \widetilde{\lambda}_1$, *and*

$$\Delta_1(\lambda, c)\begin{cases} > 0, & \forall\, \lambda \in (0, \lambda_1) \cup (\widetilde{\lambda}_1, +\infty), \\ < 0, & \forall\, \lambda \in (\lambda_1, \widetilde{\lambda}_1) \end{cases}$$

hold.

4. Existence of traveling wave solutions

In this section, we will prove the existence of traveling wave solutions for (1.3) by employing a combination of the cross iteration method and Schauder's fixed point theorem. To achieve this, we choose the following kernel function:

$$g_i(x,t) = \frac{1}{\tau_i} e^{-\frac{t}{\tau_i}} \frac{1}{\sqrt{4\pi d_2 t}} e^{-\frac{\|x\|^2}{4d_2 t}}\ (i = 1, 3),\ x \in \Omega \subset R^N,$$

$$g_i(x,t) = \frac{1}{\tau_i} e^{-\frac{t}{\tau_i}} \frac{1}{\sqrt{4\pi d_3 t}} e^{-\frac{\|x\|^2}{4d_3 t}}\ (i = 2, 4),\ x \in \Omega \subset R^N,$$

where $\frac{1}{\tau_i} e^{-\frac{t}{\tau_i}}$ denotes the time delay effects of biological processes (predation, reproduction), $\tau_i > 0$ is the time-scale parameter, and $\frac{1}{\sqrt{4\pi d_j t}} e^{-\frac{\|x\|^2}{4d_j t}}$ $(j = 2, 3)$ describes the random diffusion of species in space. Then

$$\begin{aligned} (g_i * v)(t) &= \int_0^{+\infty} \frac{1}{\tau_i} e^{-\frac{s}{\tau_i}} \int_\Omega \frac{1}{\sqrt{4\pi d_2 s}} e^{-\frac{\|y\|^2}{4d_2 s}} v(t - y\cdot\nu - cs)dyds,\ (i = 1, 3), \\ (g_i * w)(t) &= \int_0^{+\infty} \frac{1}{\tau_i} e^{-\frac{s}{\tau_i}} \int_\Omega \frac{1}{\sqrt{4\pi d_3 s}} e^{-\frac{\|y\|^2}{4d_3 s}} w(t - y\cdot\nu - cs)dyds,\ (i = 2, 4). \end{aligned} \tag{4.1}$$

Let

$$\begin{aligned} f_1(u(t), v(t), w(t)) &= u(t)(1 - u(t)) - \frac{u(t)(g_1 * v)(t)}{b_1 + u(t)} - \frac{u(t)(g_2 * w)(t)}{b_2 + u(t)}, \\ f_2(u(t), v(t), w(t)) &= \alpha_1(g_3 * v)(t) + \frac{a_1 u(t)(g_1 * v)(t)}{b_1 + u(t)} - m_1 v(t)w(t) - h_1 v^2(t), \\ f_3(u(t), v(t), w(t)) &= \alpha_2(g_4 * w)(t) + \frac{a_2 u(t)(g_2 * w)(t)}{b_2 + u(t)} - m_2 v(t)w(t) - h_2 w^2(t). \end{aligned}$$

Suppose f_1, f_2, f_3 satisfy the following hypotheses:
$(A1)$ $f_i(0,0,0) = f_i(u^*, v^*, w^*) = 0, \quad i = 1, 2, 3.$
$(A2)$ For $0 \le u_1(t), u_2(t) \le M_1, 0 \le v_1(t), v_2(t) \le M_2, 0 \le w_1(t), w_2(t) \le M_3, t \in R$, there exists $L > 0$ such that

$$|f_1(u_1, v_1, w_1) - f_1(u_2, v_2, w_2)| + |f_2(u_1, v_1, w_1) - f_2(u_2, v_2, w_2)|$$

$$+|f_3(u_1,v_1,w_1)-f_3(u_2,v_2,w_2)| \le L(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|).$$

It is easy to obtain that system (3.1) satisfies the partial quasi-monotonicity conditions (PQM). Namely, there exist three positive constants $\beta_1,\beta_2,\beta_3 > 0$ such that

$$\begin{aligned}
&f_1(u_1,v_1,w_1)-f_1(u_2,v_1,w_1)+\beta_1(u_1-u_2)\ge 0,\\
&f_1(u_1,v_1,w_1)-f_1(u_1,v_2,w_2)\le 0,\\
&f_2(u_1,v_1,w_1)-f_2(u_2,v_2,w_1)+\beta_2(v_1-v_2)\ge 0,\\
&f_2(u_1,v_1,w_1)-f_2(u_1,v_1,w_2)\le 0,\\
&f_3(u_1,v_1,w_1)-f_3(u_2,v_2,w_2)+\beta_3(w_1-w_2)\ge 0
\end{aligned} \tag{4.2}$$

hold.

Let

$$W=\{(u,v,w)\in C(R,R^3)\mid (0,0,0)\le (u(t),v(t),w(t))\le (M_1,M_2,M_3), t\in R\}.$$

Define the operator $H=(H_1,H_2,H_3): W\to C(R,R^3)$ by

$$\begin{aligned}
H_1(u,v,w)(t)&=\beta_1 u(t)+f_1(u(t),v(t).w(t)),\\
H_2(u,v,w)(t)&=\beta_2 v(t)+f_2(u(t),v(t),w(t)),\\
H_3(u,v,w)(t)&=\beta_3 w(t)+f_3(u(t),v(t),w(t)),
\end{aligned}$$

where

$$\beta_1\ge 2M_1+\frac{1}{b_1}M_2+\frac{1}{b_2}M_3-1,\quad \beta_2\ge 2h_1M_2+m_1M_3,\quad \beta_3\ge m_2M_2+(2h_2+m_2)M_3, \tag{4.3}$$

and then (3.1) can be rewritten as

$$\begin{cases}
d_1u''(t)-cu'(t)-\beta_1u(t)+H_1(u,v,w)(t)=0,\\
d_2v''(t)-cv'(t)-\beta_2v(t)+H_2(u,v,w)(t)=0,\\
d_3w''(t)-cw'(t)-\beta_3w(t)+H_3(u,v,w)(t)=0.
\end{cases} \tag{4.4}$$

Define the operator $F=(F_1,F_2,F_3): W\to C(R,R^3)$ by

$$\begin{aligned}
F_1(u,v,w)(t)&=\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left(\int_{-\infty}^{t} e^{\lambda_{11}(t-s)}H_1(u,v,w)(s)ds+\int_{t}^{+\infty} e^{\lambda_{12}(t-s)}H_1(u,v,w)(s)ds\right),\\
F_2(u,v,w)(t)&=\frac{1}{d_2(\lambda_{22}-\lambda_{21})}\left(\int_{-\infty}^{t} e^{\lambda_{21}(t-s)}H_2(u,v,w)(s)ds+\int_{t}^{+\infty} e^{\lambda_{22}(t-s)}H_2(u,v,w)(s)ds\right),\\
F_3(u,v,w)(t)&=\frac{1}{d_3(\lambda_{32}-\lambda_{31})}\left(\int_{-\infty}^{t} e^{\lambda_{31}(t-s)}H_3(u,v,w)(s)ds+\int_{t}^{+\infty} e^{\lambda_{32}(t-s)}H_3(u,v,w)(s)ds\right),
\end{aligned}$$

where

$$\lambda_{11}=\frac{c-\sqrt{c^2+4\beta_1d_1}}{2d_1}<0,\qquad \lambda_{21}=\frac{c-\sqrt{c^2+4\beta_2d_2}}{2d_2}<0,\quad \lambda_{31}=\frac{c-\sqrt{c^2+4\beta_3d_3}}{2d_3}<0,$$

$$\lambda_{12} = \frac{c + \sqrt{c^2 + 4\beta_1 d_1}}{2d_1} > 0, \qquad \lambda_{22} = \frac{c + \sqrt{c^2 + 4\beta_2 d_2}}{2d_2} > 0, \quad \lambda_{32} = \frac{c + \sqrt{c^2 + 4\beta_3 d_3}}{2d_3} > 0.$$

Then $F_i(u, v, w)$ satisfy

$$\begin{cases} d_1 F_1''(u, v, w)(t) - cF_1'(u, v, w)(t) - \beta_1 F_1(u, v, w)(t) + H_1(u, v, w)(t) = 0, \\ d_2 F_2''(u, v, w)(t) - cF_2'(u, v, w)(t) - \beta_2 F_2(u, v, w)(t) + H_2(u, v, w)(t) = 0, \\ d_3 F_3''(u, v, w)(t) - cF_3'(u, v, w)(t) - \beta_3 F_3(u, v, w)(t) + H_3(u, v, w)(t) = 0. \end{cases} \tag{4.5}$$

It is obvious that a fixed point of F in W is a solution of (4.4), which is a traveling wave solution of (1.3) connecting $(0, 0, 0)$ and (u^*, v^*, w^*) if it satisfies (3.2). Hence, the next step is to prove the existence of a fixed point of F in W.

Let $\mu \in (0, \min\{-\lambda_{11}, \lambda_{12}, -\lambda_{21}, \lambda_{22}, -\lambda_{31}, \lambda_{32}\})$. Define the Banach space

$$B_\mu(R, R^3) = \left\{(u, v, w)(t) \in C(R, R^3) : \| (u, v, w)(t) \|_\mu < \infty\right\},$$

and exponentially weighted norm

$$\| (u, v, w)(t) \|_\mu = \sup_{t \in R} | (u, v, w)(t) | e^{-\mu t}.$$

It is easy to show that W is a bounded closed convex subset of $B_\mu(R, R^3)$.

The operators H_i and $F_i (i = 1, 2, 3)$ have the following properties. For convenience, we let $\mathbf{\Phi}_1(t) = (u_1, v_1, w_1)(t)$, $\mathbf{\Phi}_2(t) = (u_2, v_2, w_2)(t)$.

Lemma 4.1. *Assume that (4.3) is satisfied. Then for $t \in R$ with $(0, 0, 0) \le \mathbf{\Phi}_2(t) \le \mathbf{\Phi}_1(t) \le (M_1, M_2, M_3)$, the following:*

$$\begin{aligned} &H_1(u_1, v_1, w_1)(t) \ge H_1(u_2, v_1, w_1)(t),\ H_1(u_1, v_1, w_1)(t) \le H_1(u_1, v_2, w_2)(t), \\ &H_2(u_1, v_1, w_1)(t) \ge H_2(u_2, v_2, w_1)(t),\ H_2(u_1, v_1, w_1)(t) \le H_2(u_1, v_1, w_2)(t), \\ &H_3(u_1, v_1, w_1)(t) \ge H_3(u_2, v_2, w_2)(t) \end{aligned}$$

hold.

Proof. Let $K_i(u) = \frac{u}{b_i + u} (i = 1, 2)$, $K_i'(u) = \frac{b_i}{(b_i+u)^2} > 0$. Then $K_i(u)$ is increasing on $[0, +\infty)$. For $0 \le u_2 \le u_1 \le M_1$, by the Lagrange mean value theorem, there exists $\xi > 0$ satisfying $u_2 \le \xi \le u_1$ such that

$$0 \le K_i(u_1) - K_i(u_2) = K_i'(\xi)(u_1 - u_2) = \frac{b_i}{(b_i + u)^2}(u_1 - u_2) \le \frac{1}{b_i}(u_1 - u_2).$$

By the definition of $H = (H_1, H_2, H_3)$, we have

$$\begin{aligned} &H_1(u_1, v_1, w_1)(t) - H_1(u_2, v_1, w_1)(t) = f_1(u_1, v_1, w_1)(t) + \beta_1 u_1(t) - H_1(u_2, v_1, w_1)(t) - \beta_1 u_2(t) \\ =&u_1(t) - u_2(t) - (u_1^2(t) - u_2^2(t)) - \left(\frac{u_1(t)}{b_1 + u_1(t)} - \frac{u_2(t)}{b_1 + u_2(t)}\right)(g_1 * v_1)(t) \\ &- \left(\frac{u_1(t)}{b_2 + u_1(t)} - \frac{u_2(t)}{b_2 + u_2(t)}\right)(g_2 * w_1)(t) + \beta_1(u_1(t) - u_2(t)) \end{aligned}$$

$$
\begin{aligned}
&\geq u_1(t)-u_2(t)-(u_1(t)+u_2(t))(u_1(t)-u_2(t))-\frac{1}{b_1}(u_1(t)-u_2(t))(g_1*v_1)(t)\\
&\quad -\frac{1}{b_2}(u_1(t)-u_2(t))(g_2*w_1(t))+\beta_1(u_1(t)-u_2(t))\\
&\geq(1-2M_1-\frac{1}{b_1}M_2-b_2M_3+\beta_1)(\mathbf{\Phi}_1(t)-\mathbf{\Phi}_2(t))\geq 0,\\
&\quad H_1(u_1,v_1,w_1)(t)-H_1(u_1,v_2,w_2)(t)=-\frac{u_1(t)}{b_1+u_1(t)}[g_1*(v_1-v_2)](t)-\frac{u_1(t)}{b_2+u_1(t)}[g_2*(w_1-w_2)](t)<0,\\
&\quad H_2(u_1,v_1,w_1)(t)-H_2(u_2,v_2,w_1)(t)=f_2(u_1,v_1,w_1)(t)+\beta_2v_1(t)-H_2(u_2,v_2,w_1)(t)-\beta_2v_2(t)\\
&=\alpha_1[g_3*(v_1-v_2)](t)+a_1\left[\frac{u_1(t)(g_1*v_1)(t)}{b_1+u_1(t)}-\frac{u_2(t)(g_1*v_2)(t)}{b_1+u_2(t)}\right]\\
&\quad -m_1w_1(t)(v_1(t)-v_2(t))-h_1(v_1^2(t)-v_2^2(t))+\beta_2(v_1(t)-v_2(t))\\
&\geq(\beta_2-2h_1M_2-m_1M_3)(\mathbf{\Phi}_1(t)-\mathbf{\Phi}_2(t))\geq 0,\\
&\quad H_2(u_1,v_1,w_1)(t)-H_2(u_1,v_1,w_2)(t)=-m_1v_1(t)(w_1(t)-w_2(t))\leq 0,\\
&\quad H_3(u_1,v_1,w_1)(t)-H_3(u_2,v_2,w_2)(t)=f_3(u_1,v_1,w_1)(t)+\beta_3w_1(t)-H_3(u_2,v_2,w_2)(t)-\beta_3w_2(t)\\
&=\alpha_2[g_4*(w_1-w_2)](t)+a_2\left[\frac{u_1(t)(g_2*w_1)(t)}{b_2+u_1(t)}-\frac{u_2(t)(g_2*w_2)(t)}{b_2+u_2(t)}\right]\\
&\quad -m_2\left(v_1(t)w_1(t)-v_2(t)w_2(t)\right)-h_2\left(w_1^2(t)-w_2^2(t)\right)+\beta_3(w_1(t)-w_2(t))\\
&\geq(\beta_3-m_2M_2-2h_2M_3-m_2M_3)(\mathbf{\Phi}_1(t)-\mathbf{\Phi}_2(t))\geq 0.
\end{aligned}
$$

□

Similarly, we have

Lemma 4.2. *Assume that (4.3) is satisfied. Then for $t\in R$ with $(0,0,0)\leq\mathbf{\Phi}_2(t)\leq\mathbf{\Phi}_1(t)\leq(M_1,M_2,M_3)$, the following:*

$$
\begin{aligned}
&F_1(u_1,v_1,w_1)(t)\geq F_1(u_2,v_1,w_1)(t),\quad F_1(u_1,v_1,w_1)(t)\leq F_1(u_1,v_2,w_2)(t),\\
&F_2(u_1,v_1,w_1)(t)\geq F_2(u_2,v_2,w_1)(t),\quad F_2(u_1,v_1,w_1)(t)\leq F_2(u_1,v_1,w_2)(t),\\
&F_3(u_1,v_1,w_1)(t)\geq F_3(u_2,v_2,w_2)(t)
\end{aligned}
$$

hold.

Lemma 4.3. *$F=(F_1,F_2,F_3):W\to C(R,R^3)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(R,R^3)$.*

Proof. Note that for $i=1,2,3,4,\ j=2,3$,

$$
\begin{aligned}
&\int_0^{+\infty}\int_\Omega g_i(s,y)e^{\mu|y\cdot\nu+cs|}dyds\leq\int_0^{+\infty}\frac{1}{\tau_i}e^{\frac{s}{\tau_i}}\int_\Omega\frac{1}{\sqrt{4\pi d_js}}e^{-\frac{\|y\|^2}{4d_js}}e^{\mu(|y\cdot\nu|+cs)}dyds\\
=&\int_0^{+\infty}\frac{1}{\tau_i}e^{-\frac{s}{\tau_i}+\mu cs+d_j\mu^2s}\int_\Omega\frac{1}{\sqrt{4\pi d_js}}e^{-\frac{(\|y\|-2d_js\mu\|\nu\|)^2}{4d_js}}dyds=\int_0^{+\infty}\frac{1}{\tau_i}e^{-\frac{1+\mu c\tau_i+d_j\mu^2\tau_i}{\tau_i}s}ds=\frac{1}{1+\mu c\tau_i+d_j\tau_i\mu^2}.
\end{aligned}
$$

Therefore

$$
|H_1(u_1,v_1,w_1)(t)-H_1(u_2,v_2,w_2)(t)|\,e^{-\mu|t|}\leq|f_1(u_1,v_1,w_1)(t)-f_1(u_2,v_2,w_2)(t)|\,e^{-\mu|t|}+\beta_1|\mathbf{\Phi}_1-\mathbf{\Phi}_2|_\mu
$$

$$
\begin{aligned}
&= \bigg| u_1(t) - u_2(t) - (u_1^2(t) - u_2^2(t)) + \frac{u_2(t)}{b_1 + u_2(t)}(g_1 * v_2)(t) - \frac{u_1(t)}{b_1 + u_1(t)}(g_1 * v_1)(t) \\
&\quad + \frac{u_2(t)}{b_2 + u_2(t)}(g_2 * w_2)(t) - \frac{u_1(t)}{b_2 + u_1(t)}(g_2 * w_1)(t) \bigg| e^{-\mu|t|} + \beta_1 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu \\
&\le \bigg\{ |u_1(t) - u_2(t)| + |u_1(t) + u_2(t)||u_1(t) - u_2(t)| + \frac{u_2(t)}{b_1 + u_2(t)}(g_1 * |v_2 - v_1|)(t) \\
&\quad + |g_1 * v_1|(t) \left| \frac{u_2(t)}{b_1 + u_2(t)} - \frac{u_1(t)}{b_1 + u_1(t)} \right| + \frac{u_2(t)}{b_2 + u_2(t)}(g_2 * |w_2 - w_1|)(t) \\
&\quad + |g_2 * w_1|(t) \left| \frac{u_2(t)}{b_2 + u_2(t)} - \frac{u_1(t)}{b_2 + u_1(t)} \right| \bigg\} e^{-\mu|t|} + \beta_1 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu \\
&\le (2M_1 + 1)|\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu + M_1 \int_0^{+\infty} \int_\Omega g_1(s, y) e^{\mu|y \cdot \nu + cs|} dy ds |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu + \frac{1}{b_1} M_2 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu \\
&\quad + M_1 \int_0^{+\infty} \int_\Omega g_2(s, y) e^{\mu|y \cdot \nu + cs|} dy ds |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu + \frac{1}{b_2} M_3 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu + \beta_1 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu \\
&\le \vartheta_1 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu,
\end{aligned}
$$

$$
\begin{aligned}
&|H_2(u_1, v_1, w_1)(t) - H_2(u_2, v_2, w_2)(t)| e^{-\mu|t|} \le |f_2(u_1, v_1, w_1)(t) - f_2(u_2, v_2, w_2)(t)| e^{-\mu|t|} + \beta_2 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu \\
&= \bigg| \alpha_1 [g_3 * (v_1 - v_2)](t) + a_1 \left[\frac{u_1(t)}{b_1 + u_1(t)}(g_1 * v_1)(t) - \frac{u_2(t)}{b_1 + u_2(t)}(g_1 * v_2)(t) \right] \\
&\quad + m_1(v_2(t) w_2(t) - v_1(t) w_1(t)) + h_1(v_2^2(t) - v_1^2(t)) \bigg| e^{-\mu|t|} + \beta_2 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu \\
&\le \alpha_1 \int_0^{+\infty} \int_\Omega g_3(s, y) e^{\mu|y \cdot \nu + cs|} dy ds |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu + a_1 M_1 \int_0^{+\infty} \int_\Omega g_1(s, y) e^{\mu|y \cdot \nu + cs|} dy ds |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu \\
&\quad + a_1 b_1 M_2 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu + m_1(M_2 + M_3)|\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu + 2h_1 M_2 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu + \beta_2 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu \\
&\le \vartheta_2 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu.
\end{aligned}
$$

Similarly, we have $|H_3(u_1, v_1, w_1)(t) - H_3(u_2, v_2, w_2)(t)| e^{-\mu|t|} \le \vartheta_3 |\mathbf{\Phi}_1 - \mathbf{\Phi}_2|_\mu$, where

$$
\begin{aligned}
\vartheta_1 &= \left(2 + \frac{1}{1 + \mu c \tau_1 + d_2 \mu^2 \tau_1} + \frac{1}{1 + \mu c \tau_2 + d_3 \mu^2 \tau_2} \right) M_1 + \frac{1}{b_1} M_2 + \frac{1}{b_2} M_3 + 1 + \beta_1, \\
\vartheta_2 &= \frac{\alpha_1}{1 + \mu c \tau_3 + d_2 \mu^2 \tau_3} + \frac{a_1}{1 + \mu c \tau_1 + d_2 \mu^2 \tau_1} M_1 + (a_1 b_1 + m_1 + 2h_1) M_2 + m_1 M_3 + \beta_2, \\
\vartheta_3 &= \frac{\alpha_2}{1 + \mu c \tau_4 + d_3 \mu^2 \tau_4} + \frac{a_2}{1 + \mu c \tau_2 + d_3 \mu^2 \tau_2} M_1 + (a_2 b_2 + m_2 + 2h_2) M_2 + m_2 M_3 + \beta_3.
\end{aligned}
$$

Then for $t \ge 0$, we have

$$
\begin{aligned}
&|F_1(u_1, v_1, w_1)(t) - F_1(u_2, v_2, w_2)(t)| e^{-\mu|t|} \\
=& \frac{1}{d_1(\lambda_{12} - \lambda_{11})} \left[\int_{-\infty}^t e^{\lambda_{11}(t-s)} |H_1(u_1, v_1, w_1)(s) - H_1(u_2, v_2, w_2)(s)| ds \right. \\
&\left. + \int_t^{+\infty} e^{\lambda_{12}(t-s)} |H_1(u_1, v_1, w_1)(s) - H_1(u_2, v_2, w_2)(s)| ds \right] e^{-\mu|t|}
\end{aligned}
$$

$$\leq \frac{\vartheta_1 e^{-\mu|t|}}{d_1(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{t} e^{\lambda_{11}(t-s)}e^{-\mu|s|}ds + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)}e^{-\mu|s|}ds\right]|\mathbf{\Phi}_1-\mathbf{\Phi}_2|_\mu$$
$$= \frac{\vartheta_1 e^{-\mu|t|}}{d_1(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{0} e^{\lambda_{11}t}e^{-(\lambda_{11}+\mu)s}ds + \int_{0}^{t} e^{\lambda_{11}t}e^{(\mu-\lambda_{11})s}ds + \int_{t}^{+\infty} e^{\lambda_{12}t}e^{(\mu-\lambda_{12})s}ds\right]|\mathbf{\Phi}_1-\mathbf{\Phi}_2|_\mu$$
$$= \frac{\vartheta_1}{d_1(\lambda_{12}-\lambda_{11})}\left[\frac{2\mu}{\lambda_{11}^2-\mu^2}e^{(\lambda_{11}-\mu)t} + \frac{\lambda_{11}-\lambda_{21}}{(\mu-\lambda_{11})(\lambda_{21}-\mu)}\right]|\mathbf{\Phi}_1-\mathbf{\Phi}_2|_\mu$$
$$\leq \frac{\vartheta_1}{d_1(\lambda_{12}-\lambda_{11})}\left[\frac{2\mu}{\lambda_{11}^2-\mu^2} + \frac{\lambda_{11}-\lambda_{21}}{(\mu-\lambda_{11})(\lambda_{21}-\mu)}\right]|\mathbf{\Phi}_1-\mathbf{\Phi}_2|_\mu.$$

For $t < 0$, a similar inequality holds as above. Thus, we proved that $F_1 : W \to C(R, R^3)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(R, R^3)$.

Following analogous reasoning, we obtain that $(F_2, F_3) : W \to C(R, R^3)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(R, R^3)$. □

Next, we give the definition of upper and lower solutions.

Definition 1. The continuous functions $(\underline{u}, \underline{v}, \underline{w})$ and $(\bar{u}, \bar{v}, \bar{w})$ on R are called a pair of lower and upper solutions of system (3.1) if they satisfy the following conditions:

(i) $0 \leq \underline{u}(t) \leq \bar{u}(t), 0 \leq \underline{v}(t) \leq \bar{v}(t), 0 \leq \underline{w}(t) \leq \bar{w}(t)$ for $t \in R$.

(ii) There exists a finite number set $\mathbb{D} \subset R$ such that

(a) $\bar{u}$, $\underline{u}$, $\bar{v}$, $\underline{v}$, $\bar{w}$, $\underline{w}$ are in $C^3(R \setminus \mathbb{D})$.

(b) The right and left limits of $\underline{u}'$, $\bar{u}'$, $\underline{v}'$, $\bar{v}'$, $\underline{w}'$, $\bar{w}'$ all exist at each $t \in \mathbb{D}$ and satisfy

$$\bar{u}'(t-) \geq \bar{u}'(t+), \qquad \bar{v}'(t-) \geq \bar{v}'(t+), \qquad \bar{w}'(t-) \geq \bar{w}'(t+),$$
$$\underline{u}'(t-) \leq \underline{u}'(t+), \qquad \underline{v}'(t-) \leq \underline{v}'(t+), \qquad \underline{w}'(t-) \leq \underline{w}'(t+).$$

(iii) At $\pm\infty$, the first and second derivatives of $\bar{u}$, $\underline{u}$, $\bar{v}$, $\underline{v}$, $\bar{w}$, $\underline{w}$ have at most exponential growth.

(iv) For every continuous function (u, v, w) with $\underline{u} \leq u \leq \bar{u}$, $\underline{v} \leq v \leq \bar{v}$, $\underline{w} \leq w \leq \bar{w}$,

$$\begin{cases} d_1\bar{u}''(t) - c\bar{u}'(t) + \bar{u}(t)(1-\bar{u}(t)) - \dfrac{\bar{u}(t)(g_1 * \underline{v})(t)}{b_1 + \bar{u}(t)} - \dfrac{\bar{u}(t)(g_2 * \underline{w})(t)}{b_2 + \bar{u}(t)} \leq 0, \\ d_2\bar{v}''(t) - c\bar{v}'(t) + \alpha_1(g_3 * \bar{v})(t) + \dfrac{a_1\bar{u}(t)(g_1 * \bar{v})(t)}{b_1 + \bar{u}(t)} - m_1\bar{v}(t)\underline{w}(t) - h_1\bar{v}^2(t) \leq 0, \\ d_3\bar{w}''(t) - c\bar{w}'(t) + \alpha_2(g_4 * \bar{w})(t) + \dfrac{a_2\bar{u}(t)(g_2 * \bar{w})(t)}{b_2 + \bar{u}(t)} - m_2\bar{v}(t)\bar{w}(t) - h_2\bar{w}^2(t) \leq 0, \\ d_1\underline{u}''(t) - c\underline{u}'(t) + \underline{u}(t)(1-\underline{u}(t)) - \dfrac{\underline{u}(t)(g_1 * \bar{v})(t)}{b_1 + \underline{u}(t)} - \dfrac{\underline{u}(t)(g_2 * \bar{w})(t)}{b_2 + \underline{u}(t)} \geq 0, \\ d_2\underline{v}''(t) - c\underline{v}'(t) + \alpha_1(g_3 * \underline{v})(t) + \dfrac{a_1\underline{u}(t)(g_1 * \underline{v})(t)}{b_1 + \underline{u}(t)} - m_1\underline{v}(t)\bar{w}(t) - h_1\underline{v}^2(t) \geq 0, \\ d_3\underline{w}''(t) - c\underline{w}'(t) + \alpha_2(g_4 * \underline{w})(t) + \dfrac{a_2\underline{u}(t)(g_2 * \underline{w})(t)}{b_2 + \underline{u}(t)} - m_2\underline{v}(t)\underline{w}(t) - h_2\underline{w}^2(t) \geq 0, \end{cases} \quad \forall\ t \in R \setminus \mathbb{D}.$$

Assume that a pair of upper-lower solutions $(\bar{u}, \bar{v}, \bar{w})$ and $(\underline{u}, \underline{v}, \underline{w})$ is given such that

$(P1)$ $(0,0,0) \leq (\underline{u}(t), \underline{v}(t), \underline{w}(t)) \leq (\bar{u}(t), \bar{v}(t), \bar{w}(t)) \leq (M_1, M_2, M_3)$, $t \in R$.

$(P2)$ $\lim_{t\to-\infty}(\bar{u}(t), \bar{v}(t), \bar{w}(t)) = (0,0,0)$, $\lim_{t\to+\infty}(\underline{u}(t), \underline{v}(t), \underline{w}(t)) = \lim_{t\to+\infty}(\bar{u}, \bar{v}, \bar{w}) = (u^*, v^*, w^*)$.

Define a profile set

$$\Gamma((\underline{u},\underline{v},\underline{w}),(\bar{u},\bar{v},\bar{w}))=\begin{cases}(i)\ (\underline{u},\underline{v},\underline{w})(t)\leq(u,v,w)(t)\leq(\bar{u},\bar{v},\bar{w})(t),\\ (ii)\ (u,v,w)(t)\ \textit{is nondecreasing in}\ t\in R.\end{cases}$$

It is obvious that $\Gamma((\underline{u},\underline{v},\underline{w}),(\bar{u},\bar{v},\bar{w}))$ is nonempty, closed, convex, and bounded. We have the following results.

Lemma 4.4. $F(\Gamma)\subset\Gamma$.

Lemma 4.5. $F:\Gamma\to\Gamma$ *is compact with respect to the decay norm* $|\cdot|_\mu$.

Combining Lemmas 4.4 and 4.5, we obtain that F is compact continuous on Γ, and F has a fixed point on Γ by applying Schauder's fixed point theorem. Further, W also has a fixed point on Γ since $\Gamma\subset W$. We have the following theorem.

Theorem 4.1. *Suppose that there is a pair of upper-lower solutions* $(\bar{u},\bar{v},\bar{w})$ *and* $(\underline{u},\underline{v},\underline{w})$ *for (3.1) satisfying (P1) and (P2). Then (3.1) has a traveling wave solution connecting* $(0,0,0)$ *with* (u^*,v^*,w^*).

5. Conclusions

This paper investigates the traveling wave solutions of a one prey-two competing predators system (1.3) with nonlocal delay, where the delays for predators take into account gestation and migration. First, by constructing a Lyapunov function, we demonstrate the global asymptotic stability of the positive equilibrium E^* (see Theorem 2.2). This result indicates that when there is intra-specific and inter-specific competition between predators and prey, if system (1.3) has a unique positive equilibrium E^*, this solution is globally asymptotically stable. From a biological perspective, this means that predators and prey can coexist in the long term, with population densities eventually stabilizing near the positive equilibrium E^*. Second, by analyzing the distribution of the roots of the characteristic equation $\triangle_i(\lambda,c)(i=1,2,3)$, we determine the critical wave speed $c^*=\max\{c_1^*,c_2^*,c_3^*\}$. Then, employing Schauder's fixed point theorem and the cross iteration method, we define a compact operator $F=(F_1,F_2,F_3)$ in a Banach space. By verifying its PQM and compactness (see Lemmas 4.3–4.5), we transform the existence of traveling wave solutions into a fixed point problem on $\Gamma\subset W$ (see Theorem 4.1).

In prey-predators competition systems, the introduction of nonlocal delays significantly elevates computational complexity, requires precise analytical estimates, and can more realistically reflect the spatial migration and historical dependence of biological populations (such as predation delays and reproductive cycles), overcoming the limitations of traditional local delay models that neglect spatial heterogeneity. However, this study relies on specific kernel functions to analyze the effects of nonlocal delay, which may not fully represent the complex memory and migration patterns in ecological processes. Future research could explore more general kernel functions to more accurately characterize the spatial and temporal effects of nonlocal delay.

Author contributions

Qi Liu: methodology, theoretical analysis, implementation, core conclusion formulation, and initial draft composition; Yujuan Jiao: manuscript review and editing, critical insights for interpretation enhancement, publication standards compliance, and clarity and coherence improvement. All authors reviewed and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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Appendix

In this section, we further verify the existence of traveling wave solutions by constructing a pair of upper-lower solutions. Substituting (4.1) into (3.4) yields the following:

$$\begin{cases}\Delta_1(\lambda,c)=d_1\lambda^2-c\lambda+1,\\ \Delta_2(\lambda,c)=d_2\lambda^2-c\lambda+\frac{\alpha_1}{1+c\tau_i\lambda-d_2\tau_i\lambda^2},\\ \Delta_3(\lambda,c)=d_3\lambda^2-c\lambda+\frac{\alpha_2}{1+c\tau_i\lambda-d_3\tau_i\lambda^2}.\end{cases} \tag{A.1}$$

First, we have the following lemma.

Lemma A.1. *Assume*

$$\frac{(2u^*-1)^2}{4}\geq\frac{u^*M_2}{b_1+u^*}+\frac{u^*M_3}{b_2+u^*} \tag{A.2}$$

holds, and there exists $\varepsilon_1 \in (0, \frac{2u^*-1}{2})$ *such that*

$$-\varepsilon_1^2 + (2u^* - 1)\varepsilon_1 + \frac{u^* v^*}{b_1 + u^*} - \frac{(u^* - \varepsilon_1)M_2}{b_1 + u^* - \varepsilon_1} + \frac{u^* w^*}{b_2 + u^*} - \frac{(u^* - \varepsilon_1)M_3}{b_2 + u^* - \varepsilon_1} > \varepsilon_0,$$

where $\varepsilon_0 > 0$ *is a constant.*

Proof. Let

$$\begin{aligned} g_1(\varepsilon_1) &= -\varepsilon_1^2 + (2u^* - 1)\varepsilon_1, \\ g_2(\varepsilon_1) &= -\frac{u^* v^*}{b_1 + u^*} + \frac{(u^* - \varepsilon_1)M_2}{b_1 + u^* - \varepsilon_1}, \\ g_3(\varepsilon_1) &= -\frac{u^* w^*}{b_2 + u^*} + \frac{(u^* - \varepsilon_1)M_3}{b_2 + u^* - \varepsilon_1}. \end{aligned}$$

Direct calculation yields

$$\begin{aligned} g_1(0) = g_1(2u^* - 1) &= 0, \\ \max\{g_1(\varepsilon_1)\} = g_1\left(\frac{2u^* - 1}{2}\right) &= \frac{(2u^* - 1)^2}{4}, \\ \max\{g_2(\varepsilon_1)\} \le g_2(0) &\le \frac{u^* M_2}{b_1 + u^*}, \\ \max\{g_3(\varepsilon_1)\} \le g_3(0) &\le \frac{u^* M_3}{b_2 + u^*}. \end{aligned}$$

By (A.2) there exists ε_1^* such that

$$0 < \varepsilon_1^* < \frac{2u^* - 1}{2},$$

and

$$g_1(\varepsilon_1) \ge g_2(\varepsilon_1) + g_3(\varepsilon_1) \ \ for \ \ \varepsilon_1^* \le \varepsilon_1 < \frac{2u^* - 1}{2}.$$

□

Take $c > c^* := \max\left\{c_1^*, c_2^*, c_3^*\right\}$. There exists $\varepsilon_2 > v^*,\ \varepsilon_3 > w^*, \eta \in \left(1, \min\left\{2, \frac{\widetilde{\lambda}_i}{\lambda_j}, \frac{\lambda_i + \lambda_i}{\lambda_i}\right\}\right)$ $(i, j = 1, 2, 3$ *and* $i \ne j)$ such that $\Delta_i(\eta\lambda_i, c) < 0$ and

$$\begin{cases} h_1\varepsilon_1 - m_1 w^* - \frac{a_1 u^*}{b_1 + u^*} > 0, & h_2\varepsilon_3 + m_2\varepsilon_2 - \frac{a_2 u^*}{b_2 + u^*} > 0, \\ 1 + c\lambda_2\tau_1 - d_1\tau_1{\lambda_2}^2 > 0, & 1 + c\eta\lambda_2\tau_1 - d_1\tau_1\eta{\lambda_2}^2 > 0, \\ 1 + c\lambda_3\tau_2 - d_2\tau_2{\lambda_3}^2 > 0, & 1 + c\eta\lambda_3\tau_2 - d_2\tau_2\eta{\lambda_3}^2 > 0 \end{cases} \tag{A.3}$$

hold, and M_1 satisfies

$$M_1 < \min\left\{\frac{2h_1 - \alpha_1}{a_1}, \frac{2h_2 - \alpha_2}{a_2}\right\}. \tag{A.4}$$

We define the non-negative, bounded, continuous functions $\bar{u}(t), \bar{v}(t), \bar{w}(t)$ and $\underline{u}(t), \underline{v}(t), \underline{w}(t)$ on R as follows:

$$\bar{u}(t) = \begin{cases} e^{\lambda_1 t}, & \forall\, t \le t_1, \\ u^* + u^* e^{-\lambda t}, & \forall\, t > t_1, \end{cases} \qquad \underline{u}(t) = \begin{cases} e^{\lambda_1 t} - q e^{\eta\lambda_1 t}, & \forall\, t \le t_4, \\ u^* - \varepsilon_1 e^{-\lambda t}, & \forall\, t > t_4, \end{cases}$$

$$\bar{v}(t)=\begin{cases} e^{\lambda_2 t}+qe^{\eta\lambda_2 t}, & \forall\, t\le t_2,\\ v^*+v^*e^{-\lambda t}, & \forall\, t>t_2,\end{cases}\qquad \underline{v}(t)=\begin{cases} e^{\lambda_2 t}-qe^{\eta\lambda_2 t}, & \forall\, t\le t_5,\\ v^*-\varepsilon_2 e^{-\lambda t}, & \forall\, t>t_5,\end{cases}$$

$$\bar{w}(t)=\begin{cases} e^{\lambda_3 t}+qe^{\eta\lambda_3 t}, & \forall\, t\le t_3,\\ w^*+w^*e^{-\lambda t}, & \forall\, t>t_3,\end{cases}\qquad \underline{w}(t)=\begin{cases} e^{\lambda_3 t}-qe^{\eta\lambda_3 t}, & \forall\, t\le t_6,\\ w^*-\varepsilon_3 e^{-\lambda t}, & \forall\, t>t_6,\end{cases}$$

where $\lambda>0$ is small enough, $q>1$ is large enough, and

$$t_1>0>\max\{t_2,t_3,t_4,t_5,t_6\}.$$

Lemma A.2. *Assume that (A.3) and (A.4) hold. Then $(\bar{u}(t),\bar{v}(t),\bar{w}(t))$ is an upper solution of (3.1).*

Proof. For $\bar{u}(t)$, by the definitions of λ_1 and $\bar{u}(t)$, we have

$$\bar{u}'(t_1-)=\lambda_1 e^{\lambda_1 t}>-\lambda u^*e^{-\lambda t}=\bar{u}'(t_1+).$$

(i) If $t\le t_1$, $\bar{u}(t)=e^{\lambda_1 t}$, then

$$\begin{aligned}&d_1\bar{u}''(t)-c\bar{u}'(t)+\bar{u}(t)(1-\bar{u}(t))-\frac{\bar{u}(t)(g_1*\underline{v})(t)}{b_1+\bar{u}(t)}-\frac{\bar{u}(t)(g_2*\underline{w})(t)}{b_2+\bar{u}(t)}\\ \le& d_1\bar{u}''(t)-c\bar{u}'(t)+\bar{u}(t)=e^{\lambda_1 t}\Delta_1(\lambda_1,c)=0.\end{aligned}$$

(ii) If $t>t_1$, $\bar{u}(t)=u^*+u^*e^{-\lambda t}$, then

$$\begin{aligned}&d_1\bar{u}''(t)-c\bar{u}'(t)+\bar{u}(t)(1-\bar{u}(t))-\frac{\bar{u}(t)(g_1*\underline{v})(t)}{b_1+\bar{u}(t)}-\frac{\bar{u}(t)(g_2*\underline{w})(t)}{b_2+\bar{u}(t)}\\ \le& d_1\lambda^2u^*e^{-\lambda t}+c\lambda u^*e^{-\lambda t}-(u^*+u^*e^{-\lambda t})^2\\ \le& P_1(\lambda)-u^{*2}:=Q_1(\lambda),\end{aligned}$$

where $P_1(\lambda):=d_1\lambda^2u^*e^{-\lambda t}+c\lambda u^*e^{-\lambda t}$, and then $P_1(0)=0$ implies that $Q_1(0)=-u^{*2}<0$. Consequently, for sufficiently small λ, it follows that $Q_1(\lambda)<0$.

For $\bar{v}(t)$, from the assumptions on $q,\ \eta,\ \lambda$ and the definitions of $\lambda_2,\ \bar{v}(t)$, it follows that

$$\bar{v}'(t_2-)=\lambda_2 e^{\lambda_2 t}+q\eta\lambda_2 e^{\eta\lambda_2 t}>-\lambda v^*e^{-\lambda t}=\bar{v}'(t_2+).$$

(i) If $t\le t_2<0$, since $\bar{v}(t)=e^{\lambda_2 t}+qe^{\eta\lambda_2 t}$, then for $i=1,3$,

$$\begin{aligned}g_i*\bar{v}=&\int_0^{+\infty}\frac{1}{\tau_i}e^{-\frac{s}{\tau_i}}\int_\Omega\frac{1}{\sqrt{4\pi d_2 s}}e^{-\frac{\|y\|^2}{4d_2 s}}(e^{\lambda_2(t-y\cdot\nu-cs)}+qe^{\eta\lambda_2(t-y\cdot\nu-cs)})dyds\\ =&\int_0^{+\infty}\frac{1}{\tau_i}e^{\lambda_2 t}e^{-\frac{s}{\tau_i}-c\lambda_2 s+d_2 s\lambda_2^2}\int_\Omega\frac{1}{\sqrt{4\pi d_2 s}}e^{-\frac{(\|y\|+2d_2 s\lambda_2\|\nu\|)^2}{4d_2 s}}dyds\\ &+\int_0^{+\infty}\frac{q}{\tau_i}e^{\eta\lambda_2 t}e^{-\frac{s}{\tau_i}-c\eta\lambda_2 s+d_2 s\eta^2\lambda_2^2}\int_\Omega\frac{1}{\sqrt{4\pi d_2 s}}e^{-\frac{(\|y\|+2d_2 s\eta\lambda_2\|\nu\|)^2}{4d_2 s}}dyds\\ =&e^{\lambda_2 t}\int_0^{+\infty}\frac{1}{\tau_i}e^{-\frac{1+c\lambda_2\tau_i-d_2\tau_i\lambda_2^2}{\tau_i}s}ds+qe^{\eta\lambda_2 t}\int_0^{+\infty}\frac{1}{\tau_i}e^{-\frac{1+c\eta\lambda_2\tau_i-d_2\eta^2\lambda_2^2\tau_i}{\tau_i}s}ds\\ =&\frac{e^{\lambda_2 t}}{1+c\tau_i\lambda_2-d_2\tau_i\lambda_2^2}+\frac{qe^{\eta\lambda_2 t}}{1+c\tau_i\eta\lambda_2-d_2\tau_i\eta^2\lambda_2^2},\end{aligned}\tag{A.5}$$

thus

$$\begin{aligned}
&d_2\bar{v}''(t) - c\bar{v}'(t) + \alpha_1(g_3 * \bar{v})(t) + \frac{a_1\bar{u}(t)(g_1 * \bar{v})(t)}{b_1 + \bar{u}(t)} - m_1\bar{v}(t)\underline{w}(t) - h_1\bar{v}^2(t) \\
\le & d_2\lambda_2^2 e^{\lambda_2 t} + d_2 q\eta^2\lambda_2^2 e^{\eta\lambda_2 t} - c\lambda_2 e^{\lambda_2 t} - cq\eta\lambda_2 e^{\eta\lambda_2 t} + \frac{\alpha_1 e^{\lambda_2 t}}{1 + c\tau_3\lambda_2 - d_2\tau_3{\lambda_2}^2} \\
&+ \frac{\alpha_1 q e^{\eta\lambda_2 t}}{1 + c\tau_3\eta\lambda_2 - d_2\tau_3\eta^2{\lambda_2}^2} + a_1 e^{\lambda_1 t}\left(\frac{e^{\lambda_2 t}}{1 + c\tau_1\lambda_2 - d_2\tau_1{\lambda_2}^2} + \frac{q e^{\eta\lambda_2 t}}{1 + c\tau_1\eta\lambda_2 - d_2\tau_1\eta^2{\lambda_2}^2}\right) \\
\le & e^{\eta\lambda_2 t}\left[q\Delta_2(\eta\lambda_2, c) + a_1 e^{(\frac{\lambda_1+\lambda_2}{\lambda_2} - \eta)\lambda_2 t} + a_1 q e^{\lambda_1 t}\right] \\
\le & e^{\eta\lambda_2 t}\left[q(\Delta_2(\eta\lambda_2, c) + a_1 e^{\lambda_1 t}) + a_1\right],
\end{aligned}$$

given $\Delta_2(\eta\lambda_2, c) < 0$, since $t_2 < 0$ and $q > 1$ is large enough such that

$$q(\Delta_2(\eta\lambda_2, c) + a_1 e^{\lambda_1 t}) + a_1 < 0.$$

(ii) If $t > t_2$, since $\bar{v}(t) = v^* + v^* e^{-\lambda t}$, then for $i = 1, 3$,

$$\begin{aligned}
g_i * \bar{v} = & \int_0^{+\infty} \frac{1}{\tau_i} e^{-\frac{s}{\tau_i}} \int_\Omega \frac{1}{\sqrt{4\pi d_2 s}} e^{-\frac{\|y\|^2}{4d_2 s}} (v^* + v^* e^{-\lambda(t - y\cdot\nu - cs)}) dy ds \\
= & v^* + \int_0^{+\infty} \frac{v^*}{\tau_i} e^{-\lambda t} e^{-\frac{s}{\tau_i} + \lambda cs + d_2 s\lambda^2} \int_\Omega \frac{1}{\sqrt{4\pi d_2 s}} e^{-\frac{(\|y\| - 2d_2 s\lambda\|\nu\|)^2}{4d_2 s}} dy ds \\
= & v^* + v^* e^{-\lambda t} \int_0^{+\infty} \frac{1}{\tau_i} e^{-\frac{1 - c\tau_i\lambda - d_2\tau_i\lambda^2}{\tau_i} s} ds \\
= & v^* + \frac{v^* e^{-\lambda t}}{1 - c\tau_i\lambda - d_2\tau_i\lambda^2}.
\end{aligned} \tag{A.6}$$

The discussion will be divided into the following two parts:

(1) $t > t_1 > t_2$, $\bar{u}(t) = u^* + u^* e^{-\lambda t}$, and then

$$\begin{aligned}
&d_2\bar{v}''(t) - c\bar{v}'(t) + \alpha_1(g_3 * \bar{v})(t) + \frac{a_1\bar{u}(t)(g_1 * \bar{v})(t)}{b_1 + \bar{u}(t)} - m_1\bar{v}(t)\underline{w}(t) - h_1\bar{v}^2(t) \\
\le & d_2\lambda^2 v^* e^{-\lambda t} + c\lambda v^* e^{-\lambda t} + \alpha_1 v^*\left(1 + \frac{e^{-\lambda t}}{1 - c\tau_3\lambda - d_2\tau_3\lambda^2}\right) + a_1 M_1 v^*\left(1 + \frac{e^{-\lambda t}}{1 - c\tau_1\lambda - d_2\tau_1\lambda^2}\right) - h_1(v^* + v^* e^{-\lambda t})^2 \\
\le & v^*(P_2(\lambda) + \alpha_1 + a_1 M_1 - 4h_1 e^{-\lambda t}) := Q_2(\lambda),
\end{aligned}$$

where $P_2(\lambda) = e^{-\lambda t}\left(d_2\lambda^2 + c\lambda + \frac{\alpha_1}{1 - c\tau_3\lambda - d_2\tau_3\lambda^2} + \frac{a_1 M_1}{1 - c\tau_1\lambda - d_2\tau_1\lambda^2}\right)$. Then $P_2(0) = \alpha_1 + a_1 M_1$ implies that $Q_2(0) = 2v^*(\alpha_1 + a_1 M_1 - 2h_1) < 0$ by (A.4). Consequently, for sufficiently small λ, it follows that $Q_2(\lambda) < 0$.

(2) $t_1 > t > t_2$, $\bar{u}(t) = e^{\lambda_1 t}$, and similar to (1), we get

$$d_2\bar{v}''(t) - c\bar{v}'(t) + \alpha_1(g_3 * \bar{v})(t) + \frac{a_1\bar{u}(t)(g_1 * \bar{v})(t)}{b_1 + \bar{u}(t)} - m_1\bar{v}(t)\underline{w}(t) - h_1\bar{v}^2(t) \le 0.$$

For $\bar{w}(t)$, from the assumptions on $q,\ \eta,\ \lambda$ and the definitions of $\lambda_3,\ \bar{w}(t)$, it follows that

$$\bar{w}'(t_3-) = \lambda_3 e^{\lambda_3 t} + q\eta\lambda_3 e^{\eta\lambda_3 t} > -\lambda w^* e^{-\lambda t} = \bar{w}'(t_3+).$$

(i) If $t \le t_3 < 0$, $\bar{w}(t) = e^{\lambda_3 t} + qe^{\eta\lambda_3 t}$, and similar to (A.5), for $i = 2, 4$,

$$g_i * \bar{w} = \frac{e^{\lambda_3 t}}{1 + c\tau_i\lambda_3 - d_3\tau_i{\lambda_3}^2} + \frac{qe^{\eta\lambda_3 t}}{1 + c\tau_i\eta\lambda_3 - d_3\tau_i\eta^2{\lambda_3}^2},$$

thus

$$\begin{aligned}
& d_3\bar{w}''(t) - c\bar{w}'(t) + \alpha_2(g_4 * \bar{w})(t) + \frac{a_2\bar{u}(t)(g_2 * \bar{w})(t)}{b_2 + \bar{u}(t)} - m_2\bar{v}(t)\bar{w}(t) - h_2\bar{w}^2(t) \\
\le & d_3\lambda_3^2 e^{\lambda_3 t} + d_3 q\eta^2\lambda_3^2 e^{\eta\lambda_3 t} - c\lambda_3 e^{\lambda_3 t} - cq\eta\lambda_3 e^{\eta\lambda_3 t} + \frac{\alpha_2 e^{\lambda_3 t}}{1 + c\tau_4\lambda_3 - d_3\tau_4{\lambda_3}^2} \\
& + \frac{\alpha_2 qe^{\eta\lambda_3 t}}{1 + c\tau_4\eta\lambda_3 - d_3\tau_4\eta^2{\lambda_3}^2} + a_2 e^{\lambda_1 t}\left(\frac{e^{\lambda_3 t}}{1 + c\tau_2\lambda_3 - d_3\tau_2{\lambda_3}^2} + \frac{qe^{\eta\lambda_3 t}}{1 + c\tau_2\eta\lambda_3 - d_3\tau_2\eta^2{\lambda_3}^2}\right) \\
\le & e^{\eta\lambda_3 t}\left[q\Delta_3(\eta\lambda_3, c) + a_2 e^{(\frac{\lambda_1+\lambda_3}{\lambda_3} - \eta)\lambda_3 t} + a_2 qe^{\lambda_3 t}\right] \\
\le & e^{\eta\lambda_3 t}\left[q(\Delta_3(\eta\lambda_3, c) + a_2 e^{\lambda_3 t}) + a_2\right],
\end{aligned}$$

given $\Delta_3(\eta\lambda_3, c) < 0$, since $t_3 < 0$ and $q > 1$ is large enough such that

$$q(\Delta_3(\eta\lambda_3, c) + a_2 e^{\lambda_3 t}) + a_2 < 0.$$

(ii) If $t > t_3$, $\bar{w}(t) = w^* + w^* e^{-\lambda t}$, and similar to (A.6), for $i = 2, 4$,

$$g_i * \bar{w} = w^* + \frac{w^* e^{-\lambda t}}{1 - c\tau_i\lambda - d_3\tau_i\lambda^2}.$$

The discussion will be divided into the following two parts:

(1) $t > t_1 > t_3$, $\bar{u}(t) = u^* + u^* e^{-\lambda t}$, and then

$$\begin{aligned}
& d_3\bar{w}''(t) - c\bar{w}'(t) + \alpha_2(g_4 * \bar{w})(t) + \frac{a_2\bar{u}(t)(g_2 * \bar{w})(t)}{b_2 + \bar{u}(t)} - m_2\bar{v}(t)\bar{w}(t) - h_2\bar{w}^2(t) \\
\le & d_3\lambda^2 w^* e^{-\lambda t} + c\lambda w^* e^{-\lambda t} + \alpha_2 w^*\left(1 + \frac{e^{-\lambda t}}{1 - c\tau_4\lambda - d_3\tau_4\lambda^2}\right) \\
& + \frac{a_2 w^*}{b_2 + u^*}(u^* + u^* e^{-\lambda t})\left(1 + \frac{e^{-\lambda t}}{1 - c\tau_2\lambda - d_3\tau_2\lambda^2}\right) - (m_2 w^* v^* + h_2 w^{*2})(1 + e^{-\lambda t})^2 \\
\le & P_3(\lambda) + \left[\frac{a_2 u^* w^*}{b_2 + u^*}\left(\frac{e^{-\lambda t}}{1 - c\tau_2\lambda - d_3\tau_2\lambda^2} - 1\right) - \alpha_2 w^*\right](1 + e^{-\lambda t}) := Q_3(\lambda),
\end{aligned}$$

where $P_3(\lambda) := w^* e^{-\lambda t}\left(d_3\lambda^2 + c\lambda + \frac{\alpha_2}{1 - c\tau_4\lambda - d_3\tau_4\lambda^2} - \alpha_2\right)$. Then $P_3(0) = 0$ implies that $Q_3(0) = -2\alpha_2 w^* < 0$. Consequently, for sufficiently small λ, it follows that $Q_3(\lambda) < 0$.

(2) $t_1 > t > t_3$, $\bar{u}(t) = e^{\lambda_1 t}$, and then

$$\begin{aligned}
& d_3\bar{w}''(t) - c\bar{w}'(t) + \alpha_2(g_4 * \bar{w})(t) + \frac{a_2\bar{u}(t)(g_2 * \bar{w})(t)}{b_2 + \bar{u}(t)} - m_2\bar{v}(t)\bar{w}(t) - h_2\bar{w}^2(t) \\
\le & d_3\lambda^2 w^* e^{-\lambda t} + c\lambda w^* e^{-\lambda t} + \alpha_2 w^*\left(1 + \frac{e^{-\lambda t}}{1 - c\tau_4\lambda - d_3\tau_4\lambda^2}\right) + a_2 M_1 w^*\left(1 + \frac{e^{-\lambda t}}{1 - c\tau_2\lambda - d_3\tau_2\lambda^2}\right) - h_2(w^* + w^* e^{-\lambda t})^2
\end{aligned}$$

$$\leq v^* \left(P_4(\lambda) + \alpha_1 + a_2 M_1 - 4h_2 e^{-\lambda t}\right) := Q_4(\lambda),$$

where $P_4(\lambda) = e^{-\lambda t}\left(d_3\lambda^2 + c\lambda + \frac{\alpha_2}{1-c\tau_4\lambda-d_3\tau_4\lambda^2} + \frac{a_2M_1}{1-c\tau_2\lambda-d_3\tau_2\lambda^2}\right)$. Then $P_4(0) = \alpha_1 + a_2M_1$ implies that $Q_4(0) = 2w^*(\alpha_2 + a_2M_1 - 2h_2) < 0$ for (A.4). Consequently, for sufficiently small λ, it follows that $Q_4(\lambda) < 0$.

To summarize, $(\bar{u}(t), \bar{v}(t), \bar{w}(t))$ is an upper solution of (3.1). □

Lemma A.3. *Assume that (A.2) and (A.3) hold. Then $(\underline{u}(t), \underline{v}(t), \underline{w}(t))$ is a lower solution of (3.1).*

Proof. For $\underline{u}(t)$, from the assumptions on q, η, λ and the definitions of λ_1, $\underline{u}(t)$, it follows that

$$\underline{u}'(t_4-) = \lambda_1 e^{\lambda_1 t} - q\eta\lambda_1 e^{\eta\lambda_1 t} < \varepsilon_1 \lambda e^{-\lambda t} = \underline{u}'(t_4+).$$

(i) If $t \leq t_4$, then $\underline{u}(t) = e^{\lambda_1 t} - qe^{\eta\lambda_1 t} < 1$. Therefore, we have

$$\begin{aligned}
&d_1\underline{u}''(t) - c\underline{u}'(t) + \underline{u}(t)(1 - \underline{u}(t)) - \frac{\underline{u}(t)(g_1 * \bar{v})(t)}{b_1 + \underline{u}(t)} - \frac{\underline{u}(t)(g_2 * \bar{w})(t)}{b_2 + \underline{u}(t)}\\
\geq& d_1\lambda_1^2 e^{\lambda_1 t} - d_1 q\eta^2{\lambda_1}^2 e^{\eta\lambda_1 t} - c\lambda_1 e^{\lambda_1 t} + cq\eta\lambda_1 e^{\eta\lambda_1 t} + e^{\lambda_1 t} - qe^{\eta\lambda_1 t}\\
&- (e^{\lambda_1 t} - qe^{\eta\lambda_1 t})^2 - e^{\lambda_1 t}(e^{\lambda_2 t} + qe^{\eta\lambda_2 t}) - e^{\lambda_1 t}(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})\\
\geq& - e^{\eta\lambda_1 t}\left[q(\Delta_1(\eta\lambda_1, c) + e^{(\lambda_1+\eta(\lambda_2-\lambda_1))t} + e^{(\lambda_1+\eta(\lambda_3-\lambda_1))t}) + e^{(2-\eta)\lambda_1 t} + e^{(\frac{\lambda_2+\lambda_1}{\lambda_1}-\eta)\lambda_1 t} + e^{(\frac{\lambda_3+\lambda_1}{\lambda_1}-\eta)\lambda_1 t}\right]\\
\geq& - e^{\eta\lambda_1 t}[q(\Delta_1(\eta\lambda_1, c) + 2e^{\lambda_1 t}) + 3e^{\lambda_1 t}],
\end{aligned}$$

given $\Delta_1(\eta\lambda_1, c) < 0$, since $t_4 < 0$ and $q > 1$ is large enough such that

$$q(\Delta_1(\eta\lambda_1, c) + 2e^{\lambda_1 t}) + 3e^{\lambda_1 t} < 0.$$

(ii) If $t > t_4$, then

$$\begin{aligned}
&d_1\underline{u}''(t) - c\underline{u}'(t) + \underline{u}(t)(1 - \underline{u}(t)) - \frac{\underline{u}(t)(g_1 * \bar{v})(t)}{b_1 + \underline{u}(t)} - \frac{\underline{u}(t)(g_2 * \bar{w})(t)}{b_2 + \underline{u}(t)}\\
\geq& - d_1\lambda^2\varepsilon_1 e^{-\lambda t} - c\lambda\varepsilon_1 e^{-\lambda t} + u^* - u^{*2} - \varepsilon_1 e^{-\lambda t} + 2u^*\varepsilon_1 e^{-\lambda t} - \varepsilon_1^2 e^{-2\lambda t}\\
&- \frac{(u^* - \varepsilon_1 e^{-\lambda t})(g_1 * \bar{v})(t)}{b_1 + u^* - \varepsilon_1 e^{-\lambda t}} - \frac{(u^* - \varepsilon_1 e^{-\lambda t})(g_2 * \bar{w})(t)}{b_2 + u^* - \varepsilon_1 e^{-\lambda t}}\\
=&\varepsilon_1 e^{-\lambda t} P_5(\lambda) + (2u^* - 1)\varepsilon_1 e^{-\lambda t} - \varepsilon_1^2 e^{-2\lambda t} + \frac{u^* v^*}{b_1 + u^*} - \frac{(u^* - \varepsilon_1 e^{-\lambda t})(g_1 * \bar{v})(t)}{b_1 + u^* - \varepsilon_1 e^{-\lambda t}}\\
&+ \frac{u^* w^*}{b_2 + u^*} - \frac{(u^* - \varepsilon_1 e^{-\lambda t})(g_2 * \bar{w})(t)}{b_2 + u^* - \varepsilon_1 e^{-\lambda t}} := Q_5(\lambda),
\end{aligned}$$

where $P_5(\lambda) := -d_1\lambda^2 - c\lambda$. Then $P_5(0) = 0$. On the other hand, let

$$\begin{aligned}
I(\lambda, t) :=&(2u^* - 1)\varepsilon_1 e^{-\lambda t} - \varepsilon_1^2 e^{-2\lambda t} + \frac{u^* v^*}{b_1 + u^*} - \frac{(u^* - \varepsilon_1 e^{-\lambda t})(g_1 * \bar{v})(t)}{b_1 + u^* - \varepsilon_1 e^{-\lambda t}} + \frac{u^* w^*}{b_2 + u^*} - \frac{(u^* - \varepsilon_1 e^{-\lambda t})(g_2 * \bar{w})(t)}{b_2 + u^* - \varepsilon_1 e^{-\lambda t}}\\
\geq&(2u^* - 1)\varepsilon_1 e^{-\lambda t} - \varepsilon_1^2 e^{-2\lambda t} + \frac{u^* v^*}{b_1 + u^*} - \frac{(u^* - \varepsilon_1 e^{-\lambda t})M_2}{b_1 + u^* - \varepsilon_1 e^{-\lambda t}} + \frac{u^* w^*}{b_2 + u^*} - \frac{(u^* - \varepsilon_1 e^{-\lambda t})M_3}{b_2 + u^* - \varepsilon_1 e^{-\lambda t}}.
\end{aligned}$$

By Lemma A.1

$$I(\lambda,0) \geq (2u^*-1)\varepsilon_1 - \varepsilon_1^2 + \frac{u^*v^*}{b_1+u^*} - \frac{(u^*-\varepsilon_1)M_2}{b_1+u^*-\varepsilon_1} + \frac{u^*w^*}{b_2+u^*} - \frac{(u^*-\varepsilon_1)M_3}{b_2+u^*-\varepsilon_1} > \varepsilon_0 > 0.$$

We can choose $\delta_1 > 0$ such that $\delta^* := \varepsilon_1 + \delta_1$ satisfies

$$(2u^*-1)\delta - \delta^2 + \frac{u^*v^*}{b_1+u^*} - \frac{(u^*-\delta)M_2}{b_1+u^*-\delta} + \frac{u^*w^*}{b_2+u^*} - \frac{(u^*-\delta)M_3}{b_2+u^*-\delta} > \frac{\varepsilon_0}{2} > 0,$$

for $\delta \in [\varepsilon_1, \delta^*]$.

If $t \in (t_4, 0]$, noting that $\varepsilon_1 e^{-\lambda t}$ is decreasing on $(t_4, 0]$, for sufficiently small λ such that $\varepsilon_1 \leq \varepsilon_1 e^{-\lambda t} < \varepsilon_1 e^{-\lambda t_4} = \varepsilon_1 + \delta_1 = \delta^*$, then $I(\lambda, t) > 0$.

If $t > 0$, we have

$$\begin{aligned} I(\lambda,t) \geq & (2u^*-1)\varepsilon_1 e^{-\lambda t} - \varepsilon_1^2 e^{-2\lambda t} + \frac{u^*v^*}{b_1+u^*} - \frac{(u^*-\varepsilon_1 e^{-\lambda t})}{b_1+u^*-\varepsilon_1 e^{-\lambda t}}\left(v^* + \frac{v^* e^{-\lambda t}}{1-c\tau_1\lambda - d_2\tau_1\lambda^2}\right) \\ & + \frac{u^*w^*}{b_2+u^*} - \frac{(u^*-\varepsilon_1 e^{-\lambda t})}{b_2+u^*-\varepsilon_1 e^{-\lambda t}}\left(w^* + \frac{w^* e^{-\lambda t}}{1-c\tau_2\lambda - d_3\tau_2\lambda^2}\right) \\ \geq & (2u^*-1)\varepsilon_1 e^{-\lambda t} - \varepsilon_1^2 e^{-\lambda t} + \frac{u^*v^*}{b_1+u^*} - \frac{u^*}{b_1+u^*}\left(v^* + \frac{v^* e^{-\lambda t}}{1-c\tau_1\lambda - d_2\tau_1\lambda^2}\right) \\ & + \frac{u^*w^*}{b_2+u^*} - \frac{u^*}{b_2+u^*}\left(w^* + \frac{w^* e^{-\lambda t}}{1-c\tau_2\lambda - d_3\tau_2\lambda^2}\right) \\ = & e^{-\lambda t}\left[(2u^*-1)\varepsilon_1 - \varepsilon_1^2 - \frac{u^*v^*}{b_1+u^*}\frac{1}{1-c\tau_1\lambda - d_2\tau_1\lambda^2} - \frac{u^*w^*}{b_2+u^*}\frac{1}{1-c\tau_2\lambda - d_3\tau_2\lambda^2}\right]. \end{aligned}$$

Since

$$\max\{(2u^*-1)\varepsilon_1 - \varepsilon_1^2\} = \frac{(2u^*-1)^2}{4} \geq \frac{u^*M_2}{b_1+u^*} + \frac{u^*M_3}{b_2+u^*},$$

then there exists $0 < \varepsilon_1^{**} < \frac{2u^*-1}{2}$ such that

$$(2u^*-1)\varepsilon_1 - \varepsilon_1^2 - \frac{u^*v^*}{b_1+u^*}\frac{1}{1-c\tau_1\lambda - d_2\tau_1\lambda^2} - \frac{u^*w^*}{b_2+u^*}\frac{1}{1-c\tau_2\lambda - d_3\tau_2\lambda^2} > 0,$$

for $\varepsilon_1 \in (\varepsilon_1^{**}, \frac{2u^*-1}{2})$.

Therefore, taking $\varepsilon_1' = \max\{\varepsilon_1^*, \varepsilon_1^{**}\}$, we have $I(\lambda, t) \geq 0$ for $\varepsilon_1 \in (\varepsilon_1', \frac{2u^*-1}{2})$, that is, for sufficiently small λ such that $Q_5(\lambda) \geq 0$.

For $\underline{v}(t)$, from the assumptions on q, η, λ and the definitions of λ_2, $\underline{v}(t)$, it follows that

$$\underline{v}'(t_5-) = \lambda_2 e^{\lambda_2 t} - q\eta\lambda_2 e^{\eta\lambda_2 t} < \varepsilon_2 e^{-\lambda t} = \underline{v}'(t_5+).$$

(i) If $t \leq t_5$, $\underline{v}(t) = e^{\lambda_2 t} - qe^{\eta\lambda_2 t}$, $\bar{w}(t) \leq e^{\lambda_3 t} + qe^{\eta\lambda_3 t}$, and similar to (A.5), for $i = 1, 3$,

$$g_i * \underline{v} = \frac{e^{\lambda_2 t}}{1 + c\tau_i\lambda_2 - d_2\tau_i{\lambda_2}^2} - \frac{qe^{\eta\lambda_2 t}}{1 + c\tau_i\eta\lambda_2 - d_2\tau_i\eta^2{\lambda_2}^2},$$

thus

$$
\begin{aligned}
&d_2\underline{v}''(t) - c\underline{v}'(t) + \alpha_1(g_3 * \underline{v})(t) + \frac{a_1\underline{u}(t)(g_1 * \underline{v})(t)}{b_1 + \underline{u}(t)} - m_1\underline{v}(t)\bar{w}(t) - h_1\underline{v}^2(t) \\
\geq & D_2\lambda_2^2 e^{\lambda_2 t} - d_2 q\eta^2\lambda_2^2 e^{\eta\lambda_2 t} - c\lambda_2 e^{\lambda_2 t} + cq\eta\lambda_2 e^{\eta\lambda_2 t} + \frac{\alpha_1 e^{\lambda_2 t}}{1 + c\tau_3\lambda_2 - d_2\tau_3{\lambda_2}^2} \\
& - \frac{\alpha_1 q e^{\eta\lambda_2 t}}{1 + c\tau_3\eta\lambda_2 - d_2\tau_3\eta^2{\lambda_2}^2} - m_1(e^{\lambda_2 t} - qe^{\eta\lambda_2 t})(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}) - h_1(e^{\lambda_2 t} - qe^{\eta\lambda_2 t})^2 \\
\geq & - e^{\eta\lambda_2 t}\left[q(\Delta_2(\eta\lambda_2, c) + m_1 e^{(\lambda_2 + \eta(\lambda_3 - \lambda_2))t}) + m_1 e^{(\frac{\lambda_2+\lambda_3}{\lambda_2} - \eta)\lambda_2 t} + h_1 e^{(2-\eta)\lambda_2 t}\right] \\
\geq & - e^{\eta\lambda_2 t}\left[q(\Delta_2(\eta\lambda_2, c) + m_1) + m_1 + h_1\right],
\end{aligned}
$$

given $\Delta_2(\eta\lambda_2, c) < 0$, since $t_5 < 0$ and $q > 1$ is large enough such that

$$
q(\Delta_2(\eta\lambda_2, c) + m_1) + m_1 + h_1 < 0.
$$

(ii) If $t > t_5$, $\underline{v}(t) = v^* - \varepsilon_2 e^{-\lambda t}$, $\bar{w}(t) \leq w^* + we^{-\lambda t}$, and similar to (A.6),

$$
g_3 * \underline{v} = v^* - \frac{\varepsilon_2 e^{-\lambda t}}{1 - c\tau_3\lambda - d_2\tau_3\lambda^2},
$$

thus

$$
\begin{aligned}
&d_2\underline{v}''(t) - c\underline{v}'(t) + \alpha_1(g_3 * \underline{v})(t) + \frac{a_1\underline{u}(t)(g_1 * \underline{v})(t)}{b_1 + \underline{u}(t)} - m_1\underline{v}(t)\bar{w}(t) - h_1\underline{v}^2(t) \\
\geq & - d_2\lambda^2\varepsilon_2 e^{-\lambda t} - c\lambda\varepsilon_2 e^{-\lambda t} + \alpha_1\left(v^* - \frac{\varepsilon_2 e^{-\lambda t}}{1 - c\tau_3\lambda - d_2\tau_3\lambda^2}\right) \\
& - m_1(v^* - \varepsilon_2 e^{-\lambda t})(w^* + we^{-\lambda t}) - h_1(v^* - \varepsilon_2 e^{-\lambda t})^2 \\
= & \varepsilon_2 e^{-\lambda t} P_6(\lambda) + (v^* - \varepsilon_2 e^{-\lambda t})\left[(h_1\varepsilon_2 - m_1 w^*)e^{-\lambda t} - \frac{a_1 u^*}{b_1 + u^*}\right] := Q_6(\lambda),
\end{aligned}
$$

where $P_6(\lambda) := -d_2\lambda^2 - c\lambda - \frac{\alpha_1}{1 - c\tau_3\lambda - d_2\tau_3\lambda^2} + \alpha_1$. Then $P_6(0) = 0$ implies that $Q_6(0) > 0$ by (A.3). Consequently, for sufficiently small λ, it follows that $Q_6(\lambda) > 0$.

For $\underline{w}(t)$, from the assumptions on $q,\ \eta,\ \lambda$ and the definitions of $\lambda_3,\ \underline{w}(t)$, it follows that

$$
\underline{w}'(t_6-) = \lambda_3 e^{\lambda_3 t} - q\eta\lambda_3 e^{\eta\lambda_3 t} < \varepsilon_3 e^{-\lambda t} = \underline{w}'(t_6+).
$$

(i) If $t \leq t_6$, $\underline{w}(t) = e^{\lambda_3 t} - qe^{\eta\lambda_3 t}$, $\underline{v}(t) = e^{\lambda_2 t} - qe^{\eta\lambda_2 t}$, and similar to (A.5), for $i = 2, 4$,

$$
g_i * \underline{w} = \frac{e^{\lambda_3 t}}{1 + c\tau_i\lambda_3 - d_3\tau_i{\lambda_3}^2} - \frac{qe^{\eta\lambda_3 t}}{1 + c\tau_i\eta\lambda_3 - d_3\tau_i\eta^2{\lambda_3}^2},
$$

thus

$$
d_3\underline{w}''(t) - c\underline{w}'(t) + \alpha_2(g_4 * \underline{w})(t) + \frac{a_2\underline{u}(t)(g_2 * \underline{w})(t)}{b_2 + \underline{u}(t)} - m_2\underline{v}(t)\underline{w}(t) - h_2\underline{w}^2(t)
$$

$$
\begin{aligned}
&\geq d_3\lambda_3^2 e^{\lambda_3 t} - d_3 q\eta^2\lambda_3^2 e^{\eta\lambda_3 t} - c\lambda_3 e^{\lambda_3 t} + cq\eta\lambda_3 e^{\eta\lambda_3 t} + \frac{\alpha_2 e^{\lambda_3 t}}{1 + c\tau_4\lambda_3 - d_3\tau_4{\lambda_3}^2} \\
&\quad - \frac{\alpha_2 q e^{\eta\lambda_3 t}}{1 + c\tau_4\eta\lambda_3 - d_3\tau_4\eta^2{\lambda_3}^2} - m_2(e^{\lambda_2 t} - qe^{\eta\lambda_2 t})(e^{\lambda_3 t} - qe^{\eta\lambda_3 t}) - h_2(e^{\lambda_3 t} - qe^{\eta\lambda_3 t})^2 \\
&\geq - e^{\eta\lambda_3 t}\left[q\Delta_3(\eta\lambda_3, c) + m_2 e^{(\frac{\lambda_2+\lambda_3}{\lambda_3} - \eta)\lambda_3 t} + h_2 e^{(2-\eta)\lambda_3 t}\right] \\
&\geq - e^{\eta\lambda_3 t}\left[q\Delta_3(\eta\lambda_3, c) + m_2 + h_2\right],
\end{aligned}
$$

given $\Delta_3(\eta\lambda_3, c) < 0$, since $t_6 < 0$ and $q > 1$ is large enough such that

$$
q\Delta_3(\eta\lambda_3, c) + m_2 + h_2 < 0.
$$

(ii) If $t > t_6$, $\underline{w}(t) = w^* - \varepsilon_3 e^{-\lambda t}$, $\underline{v}(t) = v^* - \varepsilon_2 e^{-\lambda t}$, and similar to (A.6),

$$
g_4 * \underline{w} = w^* - \frac{\varepsilon_3 e^{-\lambda t}}{1 - c\tau_4\lambda - d_3\tau_4\lambda^2},
$$

thus

$$
\begin{aligned}
&d_3\underline{w}''(t) - c\underline{w}'(t) + \alpha_2(g_4 * \underline{w})(t) + \frac{a_2\underline{u}(t)(g_2 * \underline{w})(t)}{b_2 + \underline{u}(t)} - m_2\underline{v}(t)\underline{w}(t) - h_2\underline{w}^2(t) \\
&\geq - d_3\lambda^2\varepsilon_3 e^{-\lambda t} - c\lambda\varepsilon_3 e^{-\lambda t} + \alpha_2\left(w^* - \frac{\varepsilon_3 e^{-\lambda t}}{1 - c\tau_4\lambda - d_3\tau_4\lambda^2}\right) \\
&\quad - m_2(v^* - \varepsilon_2 e^{-\lambda t})(w^* - \varepsilon_3 e^{-\lambda t}) - h_2(w^* - \varepsilon_3 e^{-\lambda t})^2 \\
&= \varepsilon_3 e^{-\lambda t} P_7(\lambda) + (w^* - \varepsilon_3 e^{-\lambda t})\left[(h_2\varepsilon_3 + m_2\varepsilon_2)e^{-\lambda t} - \frac{a_2 u^*}{b_2 + u^*}\right] := Q_7(\lambda),
\end{aligned}
$$

where $P_7(\lambda) := -d_3\lambda^2 - c\lambda - \frac{\alpha_2}{1 - c\tau_4\lambda - d_3\tau_4\lambda^2} + \alpha_2$. Then $P_7(0) = 0$ implies that $Q_7(0) > 0$ by (A.3). Consequently, for sufficiently small λ, it follows that $Q_7(\lambda) > 0$.

To summarize, $(\underline{u}(t), \underline{v}(t), \underline{w}(t))$ is a lower solution of (3.1). □