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Research article

Lyapunov-type inequalities for self-adjoint differential equations of second order

Mohamed Jleli and Bessem Samet*

Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: bsamet@ksu.edu.sa.

Abstract: We present a general method that yields Patula, Hartman-Wintner, and Lyapunov-type inequalities for the second-order differential equation

$$-(\alpha(x)u'(x))' + \beta(x)u(x) = \gamma(x)u(x), \quad x \in I,$$

where I is an interval of $\mathbb{R}$, $\alpha \in C^1(I)$, $\alpha > 0, \beta, \gamma \in C(I)$, and $\beta \geq 0$. Applications cover the generalized radial Schrödinger equation and the modified Bessel equation and include an improved (sharpened) form of Bargmann's inequality.

Keywords: self-adjoint differential equations; Lyapunov-type inequalities; generalized radial Schrödinger's equation; modified Bessel's differential equation
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1. Introduction

We study the second-order divergence-form (self-adjoint) equation

$$-(\alpha(x)u'(x))' + \beta(x)u(x) = \gamma(x)u(x), \qquad x \in I, \tag{1.1}$$

on an interval $I \subset \mathbb{R}$, where $\alpha \in C^1(I)$ with $\alpha > 0$, and $\beta, \gamma \in C(I)$ with $\beta \geq 0$. Our aim is to derive Lyapunov-type inequalities for (1.1) under the assumption that $a_1, a_2 \in I$ are two *consecutive* zeros of a nontrivial solution u.

In the benchmark case $\alpha \equiv 1$ and $\beta \equiv 0$, (1.1) reduces to

$$-u''(x) = \gamma(x)u(x), \qquad x \in I, \tag{1.2}$$

for which Lyapunov's classical result [1] asserts: If $u \not\equiv 0$ satisfies $u(a_1) = u(a_2) = 0$ with $a_1 < a_2$, then

$$\int_{a_1}^{a_2} |\gamma(x)|\, dx > \frac{4}{a_2 - a_1}. \tag{1.3}$$

Several generalizations and extensions of the Lyapunov inequality (1.3) can be found in the literature. For example, by an application of the Sturm-Picone comparison theorem, Wintner [2] showed that if u solves (1.2) and $a_1 < a_2$ are consecutive zeros of u, then

$$\int_{a_1}^{a_2} \gamma^+(x)\,dx \;>\; \frac{4}{a_2 - a_1}, \tag{1.4}$$

where $\gamma^+(x) := \max\{0, \gamma(x)\}$. Hartman and Wintner [3] sharpened this by proving

$$\int_{a_1}^{a_2} (x - a_1)(a_2 - x)\,\gamma^+(x)\,dx \;>\; a_2 - a_1. \tag{1.5}$$

Under the same hypotheses, Patula [4] established the following subinterval inequalities:

$$\int_{a_1}^{c^*} \gamma^+(x)\,dx \;>\; \frac{1}{c^* - a_1}, \tag{1.6}$$

$$\int_{c^*}^{a_2} \gamma^+(x)\,dx \;>\; \frac{1}{a_2 - c^*}, \tag{1.7}$$

where $c^* \in (a_1, a_2)$ denotes any point at which $|u|$ attains its maximum on $[a_1, a_2]$.

In the special case $\beta \equiv 0$, Eq (1.1) reduces to

$$-\,(\alpha(x)u'(x))' = \gamma(x)u(x), \qquad x \in I. \tag{1.8}$$

Hartman [5] proved that if u solves (1.8) and $a_1 < a_2$ are consecutive zeros of u, then

$$\int_{a_1}^{a_2} \gamma^+(x)\,dx \;>\; 4\left(\int_{a_1}^{a_2} \frac{1}{\alpha(x)}\,dx\right)^{-1}. \tag{1.9}$$

In particular, taking $\alpha \equiv 1$ yields $\int_{a_1}^{a_2} \gamma^+(x)\,dx > 4/(a_2 - a_1)$, which recovers (1.4).

The Lyapunov inequality has been generalized in many directions, encompassing higher-order differential equations [6, 7], various nonlinear settings [8, 9], partial differential equations [10, 11], and fractional differential equations [12–14].

Lyapunov-type inequalities have a wide scope of applications-ranging from eigenvalue estimates and bounds on the number of bound states to stability criteria, disconjugacy, and disfocality. For comprehensive overviews, see the monograph [15].

In Section 2 we extend the classical inequalities–Wintner (1.4), Hartman-Wintner (1.5), Patula (1.6) and (1.7), and Hartman's weighted form (1.9)–to the self-adjoint Eq (1.1). Section 3 then specializes these results to the models (1.8), the generalized radial Schrödinger equation, and the modified Bessel equation. As a consequence, we obtain a sharpening of Bargmann's inequality [17] for the radial Schrödinger case.

2. Main results

Throughout this section we work under the standing hypotheses

$$\alpha \in C^1(I), \qquad \alpha > 0, \qquad \beta, \gamma \in C(I), \qquad \beta \geq 0.$$

Given $a_1, a_2 \in I$ with $a_1 < a_2$, we select functions $\varphi_{a_1}, \varphi_{a_2} \in C^2([a_1, a_2])$ satisfying:

(C1) $\{\varphi_{a_1}, \varphi_{a_2}\}$ is a fundamental system of solutions of

$$(\alpha(x)z'(x))' - \beta(x)z(x) = 0, \qquad a_1 < x < a_2. \tag{2.1}$$

(C2) $\varphi_{a_1}(a_1) = 0$ and $\varphi'_{a_1}(x) > 0$ for all $x \in [a_1, a_2]$.
(C3) $\varphi_{a_2}(a_2) = 0$ and $\varphi'_{a_2}(x) < 0$ for all $x \in [a_1, a_2]$.

The existence of φ_{a_1} and φ_{a_2} follows from the assumption $\beta \geq 0$ together with the maximum principle (see [16]); a direct construction avoiding the maximum principle is deferred to the remark below.

From **(C2)** and **(C3)** we have

$$\begin{aligned} \varphi_{a_1}(x) > 0, &\qquad a_1 < x \leq a_2, \\ \varphi_{a_2}(x) > 0, &\qquad a_1 \leq x < a_2, \\ W(\varphi_{a_2}, \varphi_{a_1})(x) > 0, &\qquad a_1 \leq x \leq a_2, \end{aligned}$$

where the (unweighted) Wronskian is

$$W(\varphi_{a_2}, \varphi_{a_1})(x) = \varphi_{a_2}(x)\,\varphi'_{a_1}(x) - \varphi'_{a_2}(x)\,\varphi_{a_1}(x).$$

Remark 2.1. For completeness, and to avoid invoking the maximum principle, we record a direct argument yielding the existence and the strict positivity/monotonicity of φ_{a_1} and φ_{a_2}.

Since $\alpha \in C^1(I)$ with $\alpha > 0$ and $\beta \in C(I)$, the initial value problem

$$\begin{cases} (\alpha(x)\varphi'_{a_1}(x))' - \beta(x)\varphi_{a_1}(x) = 0, \quad a_1 < x < a_2, \\ \varphi_{a_1}(a_1) = 0, \quad (\alpha\varphi'_{a_1})(a_1) = 1 \end{cases}$$

admits a unique solution $\varphi_{a_1} \in C^2([a_1, a_2])$, and $\varphi'_{a_1}(a_1) = 1/\alpha(a_1) > 0$.

We claim that φ_{a_1} satisfies **(C2)**.

Step 1. We prove that

$$\varphi'_{a_1}(t)\,\varphi_{a_1}(t) > 0, \quad \text{for all } t \in (a_1, a_2]. \tag{2.2}$$

For any solution φ of $(\alpha\varphi')' - \beta\varphi = 0$, one has

$$(\alpha\,\varphi\,\varphi')' = \alpha(\varphi')^2 + \beta\,\varphi^2.$$

Integrating from a_1 to $t \in (a_1, a_2]$ and using $\varphi_{a_1}(a_1) = 0$ gives

$$\alpha(t)\,\varphi'_{a_1}(t)\,\varphi_{a_1}(t) = \int_{a_1}^{t} (\alpha(\varphi'_{a_1})^2 + \beta\,\varphi_{a_1}^2)\,dx.$$

Since $\varphi'_{a_1}(a_1) = 1/\alpha(a_1) > 0$, by continuity there exists $\varepsilon > 0$ with $\varphi'_{a_1} > 0$ on $(a_1, a_1 + \varepsilon]$. Because $\alpha > 0$ and $\beta \geq 0$, the integrand $\alpha(\varphi'_{a_1})^2 + \beta\,\varphi_{a_1}^2$ is strictly positive on that subinterval; therefore, for every $t > a_1$, the right-hand side is > 0, and hence (2.2) holds.

Step 2. We show that

$$\varphi_{a_1}(t) > 0, \quad \text{for all } t \in (a_1, a_2]. \tag{2.3}$$

From $\varphi'_{a_1}(a_1) > 0$ and continuity, $\varphi'_{a_1} > 0$ on $(a_1, a_1 + \varepsilon]$ for some $\varepsilon > 0$ (small enough). Then (2.2) gives $\varphi_{a_1} > 0$ on $(a_1, a_1 + \varepsilon]$. If there were $\hat{x} \in (a_1, a_2]$ with $\varphi_{a_1}(\hat{x}) \leq 0$, then by the Intermediate Value Theorem there would exist $y \in (a_1, \hat{x}]$ with $\varphi_{a_1}(y) = 0$, contradicting (2.2). Thus, (2.3) holds.

Step 3. (monotonicity) Combining (2.2) and (2.3) yields $\varphi'_{a_1} > 0$ on $(a_1, a_2]$. Together with $\varphi'_{a_1}(a_1) = 1/\alpha(a_1) > 0$, we obtain

$$\varphi'_{a_1}(t) > 0, \quad \text{for all } t \in [a_1, a_2].$$

Hence φ_{a_1} satisfies **(C2)**.

Similarly, the initial value problem

$$\begin{cases} (\alpha(x)\varphi'_{a_2}(x))' - \beta(x)\varphi_{a_2}(x) = 0, & a_1 < x < a_2, \\ \varphi_{a_2}(a_2) = 0, \quad (\alpha\varphi'_{a_2})(a_2) = -1 \end{cases}$$

admits a unique solution $\varphi_{a_2} \in C^2([a_1, a_2])$, and, arguing as above (integrating from t to a_2), one gets $\varphi_{a_2} > 0$ on $[a_1, a_2)$ and $\varphi'_{a_2} < 0$ on $[a_1, a_2)$, i.e., **(C3)**.

Moreover, by **(C2)** and **(C3)**, we have

$$W(\varphi_{a_2}, \varphi_{a_1})(x) = \varphi_{a_2}(x)\,\varphi'_{a_1}(x) - \varphi_{a_1}(x)\,\varphi'_{a_2}(x) > 0$$

for all $x \in [a_1, a_2]$, which shows that φ_{a_1} and φ_{a_2} are linearly independent. Thus **(C1)** holds.

Remark 2.2. (On the role of $\beta \geq 0$) The hypothesis $\beta \geq 0$ is essential: When $\beta < 0$, the homogeneous equation may become oscillatory and obstruct **(C1)–(C3)**.

Indeed, let $\lambda > 0$, $\alpha \equiv 1$, and $\beta \equiv -\lambda^2 < 0$ on $I = [a_1, a_2]$ with length $L := a_2 - a_1$. Then

$$(\alpha z')' - \beta z = 0 \quad \Longleftrightarrow \quad z'' + \lambda^2 z = 0,$$

and the functions

$$\varphi_{a_1}(x) = \sin(\lambda(x - a_1)), \qquad \varphi_{a_2}(x) = \sin(\lambda(a_2 - x))$$

solve the equation with $\varphi_{a_1}(a_1) = 0$ and $\varphi_{a_2}(a_2) = 0$. However:

Failure of (C2) and (C3) for long intervals.

$$\varphi'_{a_1}(x) = \lambda\cos(\lambda(x - a_1)), \qquad \varphi'_{a_2}(x) = -\lambda\cos(\lambda(a_2 - x)).$$

If $L > \frac{\pi}{2\lambda}$, these derivatives change sign on $[a_1, a_2]$, so one cannot have $\varphi'_{a_1} > 0$ and $\varphi'_{a_2} < 0$ throughout.

Failure of (C1) at resonant lengths. If $L = \frac{k\pi}{\lambda}$ for some $k \in \mathbb{N}$, then

$$\varphi_{a_2}(x) = \sin(\lambda L - \lambda(x - a_1)) = (-1)^{k+1}\varphi_{a_1}(x),$$

so φ_{a_1} and φ_{a_2} are linearly dependent (Wronskian $\equiv 0$).

These obstructions underline why the nonnegativity condition $\beta \geq 0$ is pivotal for enforcing **(C1)–(C3)** in our framework.

Our first main result is the following theorem.

Theorem 2.1. *Assume that $u \in C^2(I)$ solves* (1.1)*, and let $a_1, a_2 \in I$ be two consecutive zeros of u. Let $c^* \in (a_1, a_2)$ be any point with $|u(c^*)| = \max_{a_1 \le x \le a_2} |u(x)|$. Then*

$$\int_{a_1}^{c^*} \varphi_{a_1}(x)\gamma^+(x)\,dx \;>\; \alpha(c^*)\varphi'_{a_1}(c^*), \tag{2.4}$$

$$\int_{c^*}^{a_2} \varphi_{a_2}(x)\gamma^+(x)\,dx \;>\; -\alpha(c^*)\varphi'_{a_2}(c^*). \tag{2.5}$$

Proof. Without restriction of the generality, we may assume that

$$u(x) > 0, \quad a_1 < x < a_2.$$

First, we claim that

$$\gamma^+ \not\equiv 0, \quad \text{on } (a_1, c^*), \tag{2.6}$$

and

$$\gamma^+ \not\equiv 0, \quad \text{on } (c^*, a_2). \tag{2.7}$$

Indeed, multiplying (1.1) by u, and integrating over $x \in (a_1, c^*)$, we obtain

$$-\int_{a_1}^{c^*} (\alpha(x)u'(x))'u(x)\,dx + \int_{a_1}^{c^*} \beta(x)u^2(x)\,dx = \int_{a_1}^{c^*} \gamma(x)u^2(x)\,dx. \tag{2.8}$$

On the other hand, using that $u'(c^*) = 0$ and $u(a_1) = 0$, we obtain

$$\begin{aligned} -\int_{a_1}^{c^*} (\alpha(x)u'(x))'u(x)\,dx &= -[\alpha(x)u'(x)u(x)]_{x=a_1}^{c^*} + \int_{a_1}^{c^*} \alpha(x)[u'(x)]^2\,dx \\ &= \int_{a_1}^{c^*} \alpha(x)[u'(x)]^2\,dx. \end{aligned}$$

Since a_1, a_2 are two consecutive zeros of u and $\alpha > 0$, then

$$-\int_{a_1}^{c^*} (\alpha(x)u'(x))'u(x)\,dx = \int_{a_1}^{c^*} \alpha(x)[u'(x)]^2\,dx > 0,$$

which implies by (2.8) that

$$\int_{a_1}^{c^*} \gamma(x)u^2(x)\,dx > 0.$$

This proves (2.6). Similarly, multiplying (1.1) by u, and integrating over $x \in (c^*, a_2)$, we obtain

$$\int_{c^*}^{a_2} \gamma(x)u^2(x)\,dx > 0,$$

which proves (2.7).

Next, multiplying (1.1) by φ_{a_1}, and integrating over $x \in (a_1, c^*)$, we obtain

$$-\int_{a_1}^{c^*} (\alpha(x)u'(x))'\varphi_{a_1}(x)\,dx + \int_{a_1}^{c^*} \beta(x)u(x)\varphi_{a_1}(x)\,dx = \int_{a_1}^{c^*} \varphi_{a_1}(x)\gamma(x)u(x)\,dx. \tag{2.9}$$

Writing

$$\gamma(x) = \gamma^+(x) - \gamma^-(x),$$

where

$$\gamma^-(x) = \max\{0, -\gamma(x)\},$$

using (2.6) and (2.9), we obtain

$$\begin{aligned}
&- \int_{a_1}^{c^*} (\alpha(x)u'(x))'\varphi_{a_1}(x)\,dx + \int_{a_1}^{c^*} \beta(x)u(x)\varphi_{a_1}(x)\,dx \\
&\leq - \int_{a_1}^{c^*} (\alpha(x)u'(x))'\varphi_{a_1}(x)\,dx + \int_{a_1}^{c^*} \beta(x)u(x)\varphi_{a_1}(x)\,dx + \int_{a_1}^{c^*} \varphi_{a_1}(x)\gamma^-(x)u(x)\,dx \\
&= \int_{a_1}^{c^*} \varphi_{a_1}(x)\gamma^+(x)u(x)\,dx \\
&< u(c^*) \int_{a_1}^{c^*} \varphi_{a_1}(x)\gamma^+(x)\,dx,
\end{aligned}$$

that is,

$$- \int_{a_1}^{c^*} (\alpha(x)u'(x))'\varphi_{a_1}(x)\,dx + \int_{a_1}^{c^*} \beta(x)u(x)\varphi_{a_1}(x)\,dx < u(c^*) \int_{a_1}^{c^*} \varphi_{a_1}(x)\gamma^+(x)\,dx. \tag{2.10}$$

On the other hand, integrating by parts, and using that $u(a_1) = u'(c^*) = \varphi_{a_1}(a_1) = 0$, we obtain

$$\begin{aligned}
- \int_{a_1}^{c^*} (\alpha(x)u'(x))'\varphi_{a_1}(x)\,dx &= -[\alpha(x)u'(x)\varphi_{a_1}(x)]_{x=a_1}^{c^*} + \int_{a_1}^{c^*} u'(x)(\alpha(x)\varphi'_{a_1}(x))\,dx \\
&= -[\alpha(x)u'(x)\varphi_{a_1}(x)]_{x=a_1}^{c^*} + [u(x)\alpha(x)\varphi'_{a_1}(x)]_{x=a_1}^{c^*} \\
&\quad - \int_{a_1}^{c^*} u(x)(\alpha(x)\varphi'_{a_1}(x))'\,dx \\
&= u(c^*)\alpha(c^*)\varphi'_{a_1}(c^*) - \int_{a_1}^{c^*} u(x)(\alpha(x)\varphi'_{a_1}(x))'\,dx.
\end{aligned}$$

Then, by (2.10), we obtain

$$u(c^*)\alpha(c^*)\varphi'_{a_1}(c^*) - \int_{a_1}^{c^*} u(x)\left[(\alpha(x)\varphi'_{a_1}(x))' - \beta(x)\varphi_{a_1}(x)\right]\,dx < u(c^*) \int_{a_1}^{c^*} \varphi_{a_1}(x)\gamma^+(x)\,dx.$$

Furthermore, by **(C1)**, we know that

$$(\alpha(x)\varphi'_{a_1}(x))' - \beta(x)\varphi_{a_1}(x) = 0, \quad a_1 < x < c^*.$$

Consequently, we deduce that

$$u(c^*) \int_{a_1}^{c^*} \varphi_{a_1}(x)\gamma^+(x)\,dx > u(c^*)\alpha(c^*)\varphi'_{a_1}(c^*),$$

which yields (2.4).

Similarly, multiplying (1.1) by φ_{a_2}, and integrating over $x \in (c^*, a_2)$, we obtain

$$-\int_{c^*}^{a_2} (\alpha(x)u'(x))'\varphi_{a_2}(x)\,dx + \int_{c^*}^{a_2} \beta(x)u(x)\varphi_{a_2}(x)\,dx = \int_{c^*}^{a_2} \varphi_{a_2}(x)\gamma(x)u(x)\,dx,$$

which implies by (2.7) that

$$-\int_{c^*}^{a_2} (\alpha(x)u'(x))'\varphi_{a_2}(x)\,dx + \int_{c^*}^{a_2} \beta(x)u(x)\varphi_{a_2}(x)\,dx < u(c^*)\int_{c^*}^{a_2} \varphi_{a_2}\gamma^+(x)\,dx.$$

Then, integrating by parts, and using that $u(a_2) = u'(c^*) = \varphi_{a_2}(a_2) = 0$, we obtain

$$-u(c^*)\alpha(c^*)\varphi'_{a_2}(c^*) - \int_{c^*}^{a_2} u(x)\left[(\alpha(x)\varphi'_{a_2}(x))' - \beta(x)\varphi_{a_2}(x)\right]dx < u(c^*)\int_{c^*}^{a_2} \varphi_{a_2}(x)\gamma^+(x)\,dx.$$

Since (by **(C1)**)

$$(\alpha(x)\varphi'_{a_2}(x))' - \beta(x)\varphi_{a_2}(x) = 0, \quad c^* < x < a_2,$$

it holds that

$$u(c^*)\int_{c^*}^{a_2} \varphi_{a_2}(x)\gamma^+(x)\,dx > -u(c^*)\alpha(c^*)\varphi'_{a_2}(c^*),$$

which yields (2.5). This completes the proof of Theorem 2.1. □

Remark 2.3. Note that the choice of functions φ_{a_1} and φ_{a_2} satisfying **(C1)–(C3)** is not unique. However, the obtained inequalities (2.4) and (2.5) are independent of any choice. Indeed, let us assume that $\{\overline{\varphi_{a_1}}, \overline{\varphi_{a_2}}\}$ is another fundamental set of solutions of the homogeneous differential Eq (2.1), and $\overline{\varphi_{a_1}}, \overline{\varphi_{a_2}}$ satisfy **(C2)** and **(C3)**. Then,

$$\overline{\varphi_{a_1}} = \mu_1\varphi_{a_1} + \mu_2\varphi_{a_2}, \quad \overline{\varphi_{a_2}} = \lambda_1\varphi_{a_1} + \lambda_2\varphi_{a_2}$$

for some $(\mu_1, \mu_2), (\lambda_1, \lambda_2) \in \mathbb{R}^2$. Since $\varphi_{a_1}(a_1) = \overline{\varphi_{a_1}}(a_1) = 0$, then $\mu_2\varphi_{a_2}(a_1) = 0$. On the other hand, by **(C3)**, we have $\varphi_{a_2}(a_1) > \varphi_{a_2}(a_2) = 0$, which yields $\mu_2 = 0$, $\overline{\varphi_{a_1}} = \mu_1\varphi_{a_1}$, and $\mu_1 > 0$. Similarly, since $\varphi_{a_2}(a_2) = \overline{\varphi_{a_2}}(a_2) = 0$, then $\lambda_1\varphi_{a_1}(a_2) = 0$. Furthermore, by **(C2)**, we have $\varphi_{a_1}(a_2) > \varphi_{a_1}(a_1) = 0$, which yields $\lambda_1 = 0$, $\overline{\varphi_{a_2}} = \lambda_2\varphi_{a_2}$, and $\lambda_2 > 0$. Consequently,

$$\int_{a_1}^{c^*} \overline{\varphi_{a_1}}(x)\gamma^+(x)\,dx > \alpha(c^*)\overline{\varphi_{a_1}}'(c^*)$$

is equivalent to (2.4). Similarly,

$$\int_{c^*}^{a_2} \overline{\varphi_{a_2}}(x)\gamma^+(x)\,dx > -\alpha(c^*)\overline{\varphi_{a_2}}'(c^*)$$

is equivalent to (2.5).

Next, using Theorem 2.1, we obtain the following Patula-type inequalities.

Theorem 2.2. *Assume that $u \in C^2(I)$ solves (1.1), and let $a_1, a_2 \in I$ be two consecutive zeros of u. Let $c^* \in (a_1, a_2)$ be any point with $|u(c^*)| = \max_{a_1 \le x \le a_2} |u(x)|$. Then*

$$\int_{a_1}^{c^*} \gamma^+(x)\,dx > \frac{\alpha(c^*)\varphi'_{a_1}(c^*)}{\varphi_{a_1}(c^*)}, \tag{2.11}$$

$$\int_{c^*}^{a_2} \gamma^+(x)\,dx > -\frac{\alpha(c^*)\varphi'_{a_2}(c^*)}{\varphi_{a_2}(c^*)}. \tag{2.12}$$

Proof. By **(C2)**, we have

$$\int_{a_1}^{c^*} \varphi_{a_1}(x)\gamma^+(x)\,dx \le \varphi_{a_1}(c^*)\int_{a_1}^{c^*} \gamma^+(x)\,dx.$$

Since $\varphi_{a_1}(c^*) > 0$, using (2.4) and the above inequality, we obtain (2.11).

Similarly, by **(C3)**, we obtain

$$\int_{c^*}^{a_2} \varphi_{a_2}(x)\gamma^+(x)\,dx \le \varphi_{a_2}(c^*)\int_{c^*}^{a_2} \gamma^+(x)\,dx.$$

Since $\varphi_{a_2}(c^*) > 0$, using (2.5) and the above inequality, we obtain (2.12). □

Our next main result is the following Hartman-Wintner-type inequality.

Theorem 2.3. *Assume that $u \in C^2(I)$ is a solution to* (1.1)*, and $a_1, a_2 \in I$ are two consecutive zeros of u. Then, the following inequality holds:*

$$\int_{a_1}^{a_2} \varphi_{a_1}(x)\varphi_{a_2}(x)\gamma^+(x)\,dx > \min_{a_1\le s\le a_2}\left[\alpha(s)W(\varphi_{a_2},\varphi_{a_1})(s)\right]. \tag{2.13}$$

Proof. Let $c^* \in (a_1, a_2)$ be such that $|u(c^*)| = \max_{a_1\le x\le a_2} |u(x)|$. We have

$$\int_{a_1}^{a_2} \varphi_{a_1}(x)\varphi_{a_2}(x)\gamma^+(x)\,dx = \int_{a_1}^{c^*} \varphi_{a_1}(x)\varphi_{a_2}(x)\gamma^+(x)\,dx + \int_{c^*}^{a_2} \varphi_{a_1}(x)\varphi_{a_2}(x)\gamma^+(x)\,dx. \tag{2.14}$$

Since φ_{a_2} is a decreasing function (by **(C3)**), we have

$$\int_{a_1}^{c^*} \varphi_{a_1}(x)\varphi_{a_2}(x)\gamma^+(x)\,dx \ge \varphi_{a_2}(c^*)\int_{a_1}^{c^*} \varphi_{a_1}(x)\gamma^+(x)\,dx,$$

which implies by (2.4) that

$$\int_{a_1}^{c^*} \varphi_{a_1}(x)\varphi_{a_2}(x)\gamma^+(x)\,dx > \alpha(c^*)\varphi_{a_2}(c^*)\varphi'_{a_1}(c^*). \tag{2.15}$$

Similarly, since φ_{a_1} is an increasing function (by **(C2)**), we have

$$\int_{c^*}^{a_2} \varphi_{a_1}(x)\varphi_{a_2}(x)\gamma^+(x)\,dx \ge \varphi_{a_1}(c^*)\int_{c^*}^{a_2} \varphi_{a_2}(x)\gamma^+(x)\,dx,$$

which implies by (2.5) that

$$\int_{c^*}^{a_2} \varphi_{a_1}(x)\varphi_{a_2}(x)\gamma^+(x)\,dx > -\alpha(c^*)\varphi'_{a_2}(c^*)\varphi_{a_1}(c^*). \tag{2.16}$$

Hence, by (2.14)–(2.16), we obtain

$$\begin{aligned}\int_{a_1}^{a_2} \varphi_{a_1}(x)\varphi_{a_2}(x)\gamma^+(x)\,dx &> \alpha(c^*)\varphi_{a_2}(c^*)\varphi'_{a_1}(c^*) - \alpha(c^*)\varphi'_{a_2}(c^*)\varphi_{a_1}(c^*)\\ &= \alpha(c^*)\left[\varphi_{a_2}(c^*)\varphi'_{a_1}(c^*) - \varphi'_{a_2}(c^*)\varphi_{a_1}(c^*)\right]\\ &= \alpha(c^*)W(\varphi_{a_2},\varphi_{a_1})(c^*)\\ &\ge \min_{a_1\le s\le a_2}\left[\alpha(s)W(\varphi_{a_2},\varphi_{a_1})(s)\right],\end{aligned}$$

which proves (2.13). □

Using Theorem 2.3, we obtain the following result.

Theorem 2.4. *Assume that* $u \in C^2(I)$ *is a solution to* (1.1)*, and* $a_1, a_2 \in I$ *are two consecutive zeros of* u*. Then, the following inequality holds:*

$$\int_{a_1}^{a_2} \gamma^+(x)\,dx > \frac{\min\limits_{a_1\le s\le a_2}[\alpha(s)W(\varphi_{a_2},\varphi_{a_1})(s)]}{\max\limits_{a_1\le t\le a_2}[\varphi_{a_1}(t)\varphi_{a_2}(t)]}. \tag{2.17}$$

Proof. We have

$$\int_{a_1}^{a_2} \varphi_{a_1}(x)\varphi_{a_2}(x)\gamma^+(x)\,dx \le \max_{a_1\le t\le a_2}[\varphi_{a_1}(t)\varphi_{a_2}(t)] \int_{a_1}^{a_2} \gamma^+(x)\,dx.$$

Then, (2.17) follows from (2.13) and the above inequality. □

3. Applications

In this section, we study some special cases of (1.1).

3.1. The differential Eq (1.8)

We consider the second-order differential Eq (1.8), where $\alpha \in C^1(I)$, $\alpha > 0$, and $\gamma \in C(I)$. This differential equation is a special case of (1.1) with $\beta = 0$.

Let $a_1, a_2 \in I$ with $a_1 < a_2$. We introduce the functions φ_{a_1} and φ_{a_2} defined by

$$\varphi_{a_1}(x) = \int_{a_1}^{x} \frac{1}{\alpha(s)}\,ds, \quad a_1 \le x \le a_2,$$

and

$$\varphi_{a_2}(x) = \int_{x}^{a_2} \frac{1}{\alpha(s)}\,ds, \quad a_1 \le x \le a_2.$$

Differentiating under the integral sign gives

$$\varphi'_{a_1}(x) = \frac{1}{\alpha(x)}, \qquad \varphi'_{a_2}(x) = -\frac{1}{\alpha(x)}.$$

Hence

$$\alpha\varphi'_{a_1} \equiv 1, \quad \alpha\varphi'_{a_2} \equiv -1,$$

so $(\alpha\varphi'_{a_i})'(x) = 0$ for $i = 1, 2$, i.e., each φ_{a_i} solves the homogeneous differential equation

$$(\alpha(x)z'(x))' = 0, \quad a_1 < x < a_2. \tag{3.1}$$

Moreover,

$$\varphi_{a_1}(a_1) = 0, \;\; \varphi_{a_2}(a_2) = 0, \qquad \varphi'_{a_1} = \frac{1}{\alpha} > 0, \;\; \varphi'_{a_2} = -\frac{1}{\alpha} < 0,$$

so $\varphi_{a_1}, \varphi_{a_2} > 0$ on (a_1, a_2) and are strictly increasing/decreasing, respectively. The Wronskian is

$$\begin{aligned} W(\varphi_{a_2},\varphi_{a_1})(x) &= \varphi_{a_2}(x)\varphi'_{a_1}(x) - \varphi_{a_1}(x)\varphi'_{a_2}(x) \\ &= \frac{\varphi_{a_1}(x)+\varphi_{a_2}(x)}{\alpha(x)}, \end{aligned}$$

that is,

$$W(\varphi_{a_2},\varphi_{a_1})(x)=\frac{1}{\alpha(x)}\int_{a_1}^{a_2}\frac{1}{\alpha(s)}\,ds>0. \tag{3.2}$$

Hence $\{\varphi_{a_1},\varphi_{a_2}\}$ is a fundamental set of solutions of (3.1). Consequently, the functions φ_{a_1} and φ_{a_2} satisfy **(C1)–(C3)**.

Applying Theorem 2.2, we obtain the following Patula-type inequalities.

Corollary 3.1. *Let $\alpha\in C^1(I)$, $\alpha>0$, and $\gamma\in C(I)$. Assume that $u\in C^2(I)$ is a solution to* (1.8)*, and $a_1,a_2\in I$ are two consecutive zeros of u. Let $c^*\in(a_1,a_2)$ be any point with $|u(c^*)|=\max_{a_1\le x\le a_2}|u(x)|$. Then*

$$\begin{aligned}\int_{a_1}^{c^*}\gamma^+(x)\,dx &> \left(\int_{a_1}^{c^*}\frac{1}{\alpha(x)}\,dx\right)^{-1},\\ \int_{c^*}^{a_2}\gamma^+(x)\,dx &> \left(\int_{c^*}^{a_2}\frac{1}{\alpha(x)}\,dx\right)^{-1}.\end{aligned}$$

Applying Theorem 2.3, we obtain the following Hartman-Wintner-type inequality.

Corollary 3.2. *Let $\alpha\in C^1(I)$, $\alpha>0$, and $\gamma\in C(I)$. Assume that $u\in C^2(I)$ is a solution to* (1.8)*, and $a_1,a_2\in I$ are two consecutive zeros of u. Then, the following inequality holds:*

$$\int_{a_1}^{a_2}\left(\int_{a_1}^{x}\frac{1}{\alpha(s)}\,ds\right)\left(\int_{x}^{a_2}\frac{1}{\alpha(s)}\,ds\right)\gamma^+(x)\,dx>\int_{a_1}^{a_2}\frac{1}{\alpha(x)}\,dx.$$

Applying Theorem 2.4, we obtain the following Hartman inequality [5].

Corollary 3.3. *Let $\alpha\in C^1(I)$, $\alpha>0$, and $\gamma\in C(I)$. Assume that $u\in C^2(I)$ is a solution to* (1.8)*, and $a_1,a_2\in I$ are two consecutive zeros of u. Then,* (1.9) *holds.*

Proof. By (2.17) and (3.2), we have

$$\begin{aligned}\int_{a_1}^{a_2}\gamma^+(x)\,dx &> \frac{\min\limits_{a_1\le\tau\le a_2}[\alpha(\tau)W(\varphi_{a_2},\varphi_{a_1})(\tau)]}{\max\limits_{a_1\le t\le a_2}[\varphi_{a_1}(t)\varphi_{a_2}(t)]}\\ &=\left(\max_{a_1\le t\le a_2}[\varphi_{a_1}(t)\varphi_{a_2}(t)]\right)^{-1}\int_{a_1}^{a_2}\frac{1}{\alpha(x)}\,dx.\end{aligned} \tag{3.3}$$

On the other hand, we have

$$\begin{aligned}\max_{a_1\le t\le a_2}[\varphi_{a_1}(t)\varphi_{a_2}(t)] &= \max_{a_1\le t\le a_2}\left(\int_{a_1}^{t}\frac{1}{\alpha(s)}\,ds\right)\left(\int_{t}^{a_2}\frac{1}{\alpha(s)}\,ds\right)\\ &=\max_{0\le r\le b} r(b-r)\\ &=\frac{b^2}{4},\end{aligned}$$

where

$$b=\int_{a_1}^{a_2}\frac{1}{\alpha(s)}\,ds.$$

Then, by (3.3), we obtain

$$\int_{a_1}^{a_2} \gamma^+(x)\,dx > \frac{4}{b}$$
$$= 4\left(\int_{a_1}^{a_2} \frac{1}{\alpha(s)}\,ds\right)^{-1},$$

which proves (1.9). □

3.2. Generalized radial Schrödinger's equation

We consider the following generalized radial Schrödinger equation

$$-\left(\frac{1}{x^k}u'(x)\right)' + \frac{(m-k)}{x^{k+2}}u(x) = \gamma(x)u(x), \quad x > 0, \tag{3.4}$$

where $m \geq k \geq 0$, and $\gamma \in C((0,\infty))$. Clearly, (3.4) is a special case of (1.1) with

$$\alpha(x) = \frac{1}{x^k}, \quad \beta(x) = \frac{m-k}{x^{k+2}}, \quad I = (0,\infty).$$

For $0 < a_1 < a_2$, we introduce the functions φ_{a_1} and φ_{a_2} defined by

$$\varphi_{a_1}(x) = x^{\alpha_1}\left(x^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right), \quad a_1 \leq x \leq a_2,$$

and

$$\varphi_{a_2}(x) = x^{\alpha_1}\left(a_2^{\alpha_2-\alpha_1} - x^{\alpha_2-\alpha_1}\right), \quad a_1 \leq x \leq a_2,$$

where

$$\alpha_1 = \frac{k+1-\sqrt{(k+1)^2+4(m-k)}}{2},$$

and

$$\alpha_2 = \frac{k+1+\sqrt{(k+1)^2+4(m-k)}}{2}.$$

It can be easily seen that

$$\alpha_1 \leq 0 < \alpha_2.$$

The reader can easily check that $\{\varphi_{a_1}, \varphi_{a_2}\}$ is a fundamental set of solutions of the homogeneous differential equation

$$\left(\frac{1}{x^k}z'(x)\right)' - \frac{(m-k)}{x^{k+2}}z(x) = 0, \quad a_1 < x < a_2.$$

On the other hand, we have

$$\varphi_{a_1}(a_1) = 0, \quad \varphi'_{a_1}(x) = x^{\alpha_1-1}\left(\alpha_2 x^{\alpha_2-\alpha_1} - \alpha_1 a_1^{\alpha_2-\alpha_1}\right) > 0, \quad a_1 \leq x \leq a_2,$$

and

$$\varphi_{a_2}(a_2) = 0, \quad \varphi'_{a_2}(x) = x^{\alpha_1-1}\left(\alpha_1 a_2^{\alpha_2-\alpha_1} - \alpha_2 x^{\alpha_2-\alpha_1}\right) < 0, \quad a_1 \leq x \leq a_2,$$

which show that the functions φ_{a_1} and φ_{a_2} satisfy **(C1)–(C3)**. Furthermore, an elementary calculation shows that

$$W(\varphi_{a_2}, \varphi_{a_1})(x) = (\alpha_2 - \alpha_1)\left(a_2^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)x^k, \quad a_1 \leq x \leq a_2.$$

Applying Theorem 2.2, we obtain the following Patula-type inequalities.

Corollary 3.4. *Let $m \geq k \geq 0$, and $\gamma \in C((0,\infty))$. Assume that $u \in C^2((0,\infty))$ is a solution to* (3.4)*, and $a_1, a_2 > 0$ are two consecutive zeros of u. Let $c^* \in (a_1, a_2)$ be any point with $|u(c^*)| = \max_{a_1 \leq x \leq a_2} |u(x)|$. Then*

$$\begin{aligned} \int_{a_1}^{c^*} \gamma^+(x)\,dx &> \frac{\alpha_2 c^{*\alpha_2-\alpha_1} - \alpha_1 a_1^{\alpha_2-\alpha_1}}{c^{*k+1}\left(c^{*\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)}, \\ \int_{c^*}^{a_2} \gamma^+(x)\,dx &> \frac{\alpha_2 c^{*\alpha_2-\alpha_1} - \alpha_1 a_2^{\alpha_2-\alpha_1}}{c^{*k+1}\left(a_2^{\alpha_2-\alpha_1} - c^{*\alpha_2-\alpha_1}\right)}. \end{aligned}$$

Next, applying Theorem 2.3, we obtain the following Hartman-Wintner-type inequality.

Corollary 3.5. *Let $m \geq k \geq 0$, and $\gamma \in C((0,\infty))$. Assume that $u \in C^2((0,\infty))$ is a solution to* (3.4)*, and $a_1, a_2 > 0$ are two consecutive zeros of u. Then, the following inequality holds:*

$$\int_{a_1}^{a_2} x^{2\alpha_1}\left(x^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)\left(a_2^{\alpha_2-\alpha_1} - x^{\alpha_2-\alpha_1}\right)\gamma^+(x)\,dx > (\alpha_2-\alpha_1)\left(a_2^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right). \tag{3.5}$$

Using Corollary 3.5, we obtain the following result.

Corollary 3.6. *Let $m \geq k \geq 0$, and $\gamma \in C((0,\infty))$. Assume that $u \in C^2((0,\infty))$ is a solution to* (3.4)*, and $a_1, a_2 > 0$ are two consecutive zeros of u. Then, the following inequality holds:*

$$\int_{a_1}^{a_2} x^{2\alpha_1}\gamma^+(x)\,dx > \frac{4(\alpha_2-\alpha_1)}{a_2^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}}. \tag{3.6}$$

Proof. By (3.5), we have

$$\begin{aligned} &(\alpha_2-\alpha_1)\left(a_2^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right) \\ &< \int_{a_1}^{a_2} x^{2\alpha_1}\left(x^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)\left(a_2^{\alpha_2-\alpha_1} - x^{\alpha_2-\alpha_1}\right)\gamma^+(x)\,dx \\ &\leq \max_{a_1 \leq s \leq a_2}\left[\left(s^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)\left(a_2^{\alpha_2-\alpha_1} - s^{\alpha_2-\alpha_1}\right)\right]\int_{a_1}^{a_2} x^{2\alpha_1}\gamma^+(x)\,dx, \end{aligned}$$

which yields

$$\int_{a_1}^{a_2} x^{2\alpha_1}\gamma^+(x)\,dx > \frac{(\alpha_2-\alpha_1)\left(a_2^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)}{\max_{a_1 \leq s \leq a_2}\left[\left(s^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)\left(a_2^{\alpha_2-\alpha_1} - s^{\alpha_2-\alpha_1}\right)\right]}. \tag{3.7}$$

On the other hand, we have

$$\begin{aligned} \max_{a_1 \leq s \leq a_2}\left[\left(s^{\alpha_2-\alpha_1} - a_2^{\alpha_2-\alpha_1}\right)\left(a_2^{\alpha_2-\alpha_1} - s^{\alpha_2-\alpha_1}\right)\right] &= \max_{A \leq X \leq B}(X-A)(B-X) \\ &= \frac{(B-A)^2}{4}, \end{aligned}$$

where $A = a_1^{\alpha_2-\alpha_1}$ and $B = a_2^{\alpha_2-\alpha_1}$. Then, by (3.7), we obtain

$$\begin{aligned} \int_{a_1}^{a_2} x^{2\alpha_1}\gamma^+(x)\,dx &> \frac{4(\alpha_2-\alpha_1)(B-A)}{(B-A)^2} \\ &= \frac{4(\alpha_2-\alpha_1)}{B-A}, \end{aligned}$$

which proves (3.6). □

Using again Corollary 3.5, we obtain the following result.

Corollary 3.7. *Let $m \geq k \geq 0$, and $\gamma \in C((0,\infty))$. Assume that $u \in C^2((0,\infty))$ is a solution to* (3.4)*, and $a_1, a_2 > 0$ are two consecutive zeros of u. Then, the following inequality holds:*

$$\int_{a_1}^{a_2} x^{\alpha_1+\alpha_2}\gamma^+(x)\,dx > \frac{(\alpha_2-\alpha_1)\left(a_2^{\frac{\alpha_2-\alpha_1}{2}} + a_2^{\frac{\alpha_2-\alpha_1}{2}}\right)}{a_2^{\frac{\alpha_2-\alpha_1}{2}} - a_1^{\frac{\alpha_2-\alpha_1}{2}}}. \tag{3.8}$$

Proof. By (3.5), we have

$$\begin{aligned}
&(\alpha_2-\alpha_1)\left(a_2^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)\\
&< \int_{a_1}^{a_2} x^{2\alpha_1}\left(x^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)\left(a_2^{\alpha_2-\alpha_1} - x^{\alpha_2-\alpha_1}\right)\gamma^+(x)\,dx\\
&= \int_{a_1}^{a_2} \frac{\left(x^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)\left(a_2^{\alpha_2-\alpha_1} - x^{\alpha_2-\alpha_1}\right)}{x^{\alpha_2-\alpha_1}} x^{\alpha_1+\alpha_2}\gamma^+(x)\,dx\\
&\leq \max_{a_1\leq s\leq a_2} \frac{\left(s^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)\left(a_2^{\alpha_2-\alpha_1} - s^{\alpha_2-\alpha_1}\right)}{s^{\alpha_2-\alpha_1}} \int_{a_1}^{a_2} x^{\alpha_1+\alpha_2}\gamma^+(x)\,dx,
\end{aligned}$$

which yields

$$\begin{aligned}
&\int_{a_1}^{a_2} x^{\alpha_1+\alpha_2}\gamma^+(x)\,dx\\
&> \left[\max_{a_1\leq s\leq a_2} \frac{\left(s^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)\left(a_2^{\alpha_2-\alpha_1} - s^{\alpha_2-\alpha_1}\right)}{s^{\alpha_2-\alpha_1}}\right]^{-1} (\alpha_2-\alpha_1)\left(a_2^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right).
\end{aligned} \tag{3.9}$$

Furthermore, we have

$$\max_{a_1\leq s\leq q_2} \frac{\left(s^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)\left(a_2^{\alpha_2-\alpha_1} - s^{\alpha_2-\alpha_1}\right)}{s^{\alpha_2-\alpha_1}} = \max_{A\leq X\leq B} \frac{(X-A)(B-X)}{X},$$

where $A = a_1^{\alpha_2-\alpha_1}$ and $B = a_2^{\alpha_2-\alpha_1}$. Letting

$$L(X) = \frac{(X-A)(B-X)}{X}, \quad A \leq X \leq B,$$

and differentiating L, we obtain

$$L'(X) = \frac{\left(\sqrt{AB} - X\right)\left(\sqrt{AB} + X\right)}{X^2}.$$

This shows that

$$\max_{A\leq X\leq B} L(X) = L\left(\sqrt{AB}\right) = \left(\sqrt{B} - \sqrt{A}\right)^2.$$

Consequently, we obtain

$$\max_{a_1\leq s\leq a_2} \frac{\left(s^{\alpha_2-\alpha_1} - a_1^{\alpha_2-\alpha_1}\right)\left(a_2^{\alpha_2-\alpha_1} - s^{\alpha_2-\alpha_1}\right)}{s^{\alpha_2-\alpha_1}} = \left(\sqrt{B} - \sqrt{A}\right)^2,$$

which implies by (3.9) that

$$\int_{a_1}^{a_2} x^{\alpha_1+\alpha_2}\gamma^+(x)\,dx > \frac{(\alpha_2-\alpha_1)(B-A)}{\left(\sqrt{B}-\sqrt{A}\right)^2} = \frac{(\alpha_2-\alpha_1)\left(\sqrt{B}+\sqrt{A}\right)}{\sqrt{B}-\sqrt{A}},$$

which proves (3.8). □

Consider now the radial Schrödinger equation

$$-u''(x)+\frac{\ell(\ell+1)}{x^2}u(x)=\gamma(x)u(x), \quad x>0, \tag{3.10}$$

where $\ell \geq 0$ is the angular momentum. Then, (3.10) is a special case of (3.4) with

$$k=0, \quad m=\ell(\ell+1).$$

In this case, we have

$$\alpha_1=\frac{1-\sqrt{1+4\ell(\ell+1)}}{2}=-\ell,$$

and

$$\alpha_2=\frac{1+\sqrt{1+4\ell(\ell+1)}}{2}=\ell+1.$$

From Corollary 3.5, we deduce the following Hartman-Wintner-type inequality for (3.10).

Corollary 3.8. *Let* $\ell \geq 0$ *and* $\gamma \in C((0,\infty))$*. Assume that* $u \in C^2((0,\infty))$ *is a solution to* (3.10)*, and* $a_1, a_2 > 0$ *are two consecutive zeros of* u*. Then, the following inequality holds:*

$$\int_{a_1}^{a_2} x^{-2\ell}\left(x^{2\ell+1}-a_1^{2\ell+1}\right)\left(a_2^{2\ell+1}-x^{2\ell+1}\right)\gamma^+(x)\,dx > (2\ell+1)\left(a_2^{2\ell+1}-a_1^{2\ell+1}\right).$$

From Corollary 3.6, we deduce the following result.

Corollary 3.9. *Let* $\ell \geq 0$ *and* $\gamma \in C((0,\infty))$*. Assume that* $u \in C^2((0,\infty))$ *is a solution to* (3.10)*, and* $a_1, a_2 > 0$ *are two consecutive zeros of* u*. Then, the following inequality holds:*

$$\int_{a_1}^{a_2} x^{-2\ell}\gamma^+(x)\,dx > \frac{4(2\ell+1)}{a_2^{2\ell+1}-a_1^{2\ell+1}}.$$

From Corollary 3.7, we deduce the following result.

Corollary 3.10. *Let* $\ell \geq 0$ *and* $\gamma \in C((0,\infty))$*. Assume that* $u \in C^2((0,\infty))$ *is a solution to* (3.10)*, and* $a_1, a_2 > 0$ *are two consecutive zeros of* u*. Then, the following inequality holds:*

$$\int_{a_1}^{a_2} x\gamma^+(x)\,dx > \frac{(2\ell+1)\left(a_2^{\frac{2\ell+1}{2}}+a_1^{\frac{2\ell+1}{2}}\right)}{a_2^{\frac{2\ell+1}{2}}-a_1^{\frac{2\ell+1}{2}}}. \tag{3.11}$$

Remark 3.1. From a result due to Bargmann [17], under the assumptions of Corollary 3.10, we have

$$\int_{a_1}^{a_2} x|\gamma(x)|\,dx > 2\ell + 1. \tag{3.12}$$

Clearly, our obtained inequality (3.11) improves (3.12). Indeed, we have

$$2\ell + 1 < \frac{(2\ell+1)\left(a_2^{\frac{2\ell+1}{2}} + a_1^{\frac{2\ell+1}{2}}\right)}{a_2^{\frac{2\ell+1}{2}} - a_1^{\frac{2\ell+1}{2}}} < \int_{a_1}^{a_2} x\gamma^+(x)\,dx \leq \int_{a_1}^{a_2} x|\gamma(x)|\,dx.$$

3.3. The modified Bessel differential equation

We consider the modified Bessel differential equation

$$-(xu'(x))' + xu(x) = \gamma(x)u(x), \quad x > 0. \tag{3.13}$$

The above differential equation is a special case of (1.1) with

$$\alpha(x) = \beta(x) = x, \quad I = (0, \infty).$$

For $0 < a_1 < a_2$, we consider the homogeneous Bessel differential equation

$$(xz'(x))' - xz(x) = 0, \quad a_1 < x < a_2. \tag{3.14}$$

It is well-known (see, e.g., Abramowitz and Stegun [18] and DLMF [19]) that I_0 and K_0 are two linearly independent solutions to (3.14), where I_0 (resp. K_0) is the modified Bessel function of the first kind of order zero (resp. the modified Bessel function of the second kind of order zero). We recall below some useful properties of I_0 and K_0 (see [18] for more details):

- The special functions I_0 and K_0 have the following integral representations:

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} \cosh(x\cos\theta)\,d\theta, \quad K_0(x) = \int_0^{\infty} \cos(x\sinh\theta)\,d\theta, \quad x > 0.$$

- For all $x > 0$, we have

$$I_0(x) > 0, \quad I_0'(x) > 0.$$

- For all $x > 0$, we have

$$K_0(x) > 0, \quad K_0'(x) < 0.$$

- For all $x > 0$, we have

$$W(K_0, I_0)(x) = \frac{1}{x}. \tag{3.15}$$

Now, we introduce the functions φ_{a_1} and φ_{a_2} defined by

$$\varphi_{a_1}(x) = K_0(a_1)I_0(x) - I_0(a_1)K_0(x), \quad a_1 \leq x \leq a_2,$$

and

$$\varphi_{a_2}(x) = I_0(a_2)K_0(x) - K_0(a_2)I_0(x), \quad a_1 \leq x \leq a_2.$$

Using the above properties of I_0 and K_0, it can be easily seen that $\{\varphi_{a_1}, \varphi_{a_2}\}$ is a fundamental set of solutions of the homogeneous Bessel differential Eq (3.14). On the other hand, we have

$$\varphi_{a_1}(a_1) = 0, \quad \varphi'_{a_1}(x) = K_0(a_1)I'_0(x) - I_0(a_1)K'_0(x) > 0, \quad a_1 \leq x \leq a_2,$$

and

$$\varphi_{a_2}(a_2) = 0, \quad \varphi'_{a_2}(x) = I_0(a_2)K'_0(x) - K_0(a_2)I'_0(x) < 0, \quad a_1 \leq x \leq a_2,$$

which show that the functions φ_{a_1} and φ_{a_2} satisfy **(C1)–(C3)**. Furthermore, using (3.15), we obtain

$$W(\varphi_{a_2}, \varphi_{a_1})(x) = \frac{I_0(a_2)K_0(a_1) - I_0(a_1)K_0(a_2)}{x}, \quad a_1 \leq x \leq a_2.$$

Then, applying Theorem 2.2, we obtain the following Patula-type inequalities.

Corollary 3.11. *Let $\gamma \in C((0,\infty))$. Assume that $u \in C^2(I)$ is a solution to* (3.13)*, and $a_1, a_2 > 0$ are two consecutive zeros of u. Let $c^* \in (a_1, a_2)$ be any point with $|u(c^*)| = \max_{a_1 \leq x \leq a_2} |u(x)|$. Then*

$$\begin{aligned} \int_{a_1}^{c^*} \gamma^+(x)\,dx &> \frac{c^*\left(K_0(a_1)I'_0(c^*) - I_0(a_1)K'_0(c^*)\right)}{K_0(a_1)I_0(c^*) - I_0(a_1)K_0(c^*)}, \\ \int_{c^*}^{a_2} \gamma^+(x)\,dx &> \frac{c^*\left(K_0(a_2)I'_0(c^*) - I_0(a_2)K'_0(c^*)\right)}{I_0(a_2)K_0(c^*) - K_0(a_2)I_0(c^*)}. \end{aligned}$$

Remark 3.2. Note that

$$K_0(a_1)I_0(c^*) - I_0(a_1)K_0(c^*) > 0.$$

Indeed, since I_0 is strictly increasing and K_0 is strictly decreasing on $(0,\infty)$, and both functions are positive there, we obtain

$$\begin{aligned} K_0(a_1)I_0(c^*) &> K_0(a_1)I_0(a_1) \\ &> K_0(c^*)I_0(a_1). \end{aligned}$$

Hence the desired inequality follows. Similarly, one shows that

$$I_0(a_2)K_0(c^*) - K_0(a_2)I_0(c^*) > 0.$$

Applying Theorem 2.3, we obtain the following Hartman-Wintner-type inequality.

Corollary 3.12. *Let $\gamma \in C((0,\infty))$. Assume that $u \in C^2(I)$ is a solution to* (3.13)*, and $a_1, a_2 > 0$ are two consecutive zeros of u. Then, the following inequality holds:*

$$\begin{aligned} &\int_{a_1}^{a_2} \left(K_0(a_1)I_0(x) - I_0(a_1)K_0(x)\right)\left(I_0(a_2)K_0(x) - K_0(a_2)I_0(x)\right)\gamma^+(x)\,dx \\ &> I_0(a_2)K_0(a_1) - I_0(a_1)K_0(a_2). \end{aligned}$$

4. Conclusions

We developed a unified framework for Lyapunov-type inequalities associated with the second-order differential equation

$$-(\alpha u')' + \beta u = \gamma u \quad (x \in I),$$

under the standing hypotheses $\alpha \in C^1(I)$ with $\alpha > 0$ and $\beta \geq 0$. The approach uses boundary-adapted fundamental solutions φ_{a_1} and φ_{a_2} with strict monotonicity obtained via a direct ODE identity, together with the interior maximizer c^* between consecutive zeros. Within this setting we extend the classical bounds (1.4)–(1.7) and (1.9) to the general operator $L = -(\alpha u')' + \beta u$. In the special case $\beta \equiv 0$ we recover (1.9), and if, moreover $\alpha \equiv 1$, this reduces to (1.4).

The scope of the method is illustrated on two model families: generalized radial Schrödinger equations–where we obtain a refinement of Bargmann's inequality–and the modified Bessel equation, for which the constants can be expressed explicitly in terms of the modified Bessel functions.

We also show that $\beta \geq 0$ is genuinely needed, since $\beta < 0$ may induce oscillatory behavior that destroys the required monotonicity and even linear independence of the boundary-adapted solutions.

Natural directions for further study include sign-changing β, alternative boundary conditions, non-self-adjoint perturbations, discrete and fractional analogues, and higher-dimensional radial reductions.

Author contributions

Jleli Mohamed: Investigation, formal analysis; Samet Bessem: Investigation, writing-review and editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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