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Research article

Advanced modeling of dependent structures using the FGM-quadratic exponential bivariate distribution: Applications in computer and material sciences

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Abstract: Due to their ability to capture complex interactions between random variables, copula models are gaining increasing attention. When it comes to bivariate data modeling, one important area of statistical theory is the construction of families of distributions with specified marginals. An FGM-QEXD (bivariate quadratic exponential Farlie Gumbel Morgenstern distribution) is derived from the FGM copula and the new quadratic exponential marginal distribution, and is inspired by this. The statistical features of the FGM-QEXD are studied, encompassing: the conditional distribution, regression function, moment generating function, and correlation coefficient. Additionally, reliability measures were obtained, including the survival function, hazard rate function, mean residual life function, and vitality function. The model parameters are estimated via maximum likelihood (ML) and Bayesian methodologies. Furthermore, asymptotic confidence ranges for the model parameter are obtained. Monte Carlo simulation analysis is employed to evaluate the efficacy of both ML and Bayesian estimators. Two real-world datasets are employed to prove that FGM-QEXD is more flexible than the bivariate Weibull Farlie–Gumbel–Morgernstern (FGM), bivariate Lomax FGM, bivariate inverse Lomax FGM, bivariate Rayleigh FGM, bivariate Burr XII FGM, and bivariate Chen FGM distributions.

Keywords: Bayesian estimation; confidence intervals; FGM bivariate family; maximum likelihood estimation; simulation
Mathematics Subject Classification: 60B12, 62G30

1. Introduction

Modeling paired data and comprehending joint behaviour, covariance, and correlation are based on bivariate distributions. In fields such as reliability theory, risk analysis, bioinformatics, image processing, and machine learning, they are essential. For representing and analyzing the relationship between random variables (RVs), bivariate distributions are indispensable in many different domains. The creation of bivariate distributions requires a great deal of study. A simple method for generating a set of distributions with two variables using marginal values was proposed by Morgenstern [1]. A more comprehensive variant of Morgenstern's method, called the FGM family of distributions, was presented by Farlie [2]. A large number of bivariate distributions have been developed and studied by various scholars. As an illustration, Almetwally and Muhammed [3] suggested a novel bivariate Fréchet distribution that is dependent on FGM and Ali-Mikhail-Haq copula functions. A bivariate Weibull distribution was introduced, and some of its features were obtained by Almetwally et al. [4] using the FGM copula function. Lastly, a new family of bivariate continuous Lomax generators was shown to have various structural statistical features by Fayomi et al. [5]. Barakat et al. [6] introduced a novel statistical model, a bivariate Epanechnikov-Exponential distribution, founded on the FGM copula. Almetwally et al. [7] presented three novel bivariate models employing copula functions: the XLindley distribution with Frank, Gumbel, and Clayton copulas. Tovar et al. [8] presented a new bivariate probability distribution that is absolutely continuous. Taking into account the FGM copula and the unit-Weibull distribution. [9] introduced a novel bivariate iterated FGM distribution with Rayleigh marginals. For more details about iterated FGM, see [10]. Nelsen [11] describes the usage of a copula as a typical technique in statistics for creating bivariate distributions. To describe bivariate distributions with an explicit dependency structure, copulas can be used. Its purpose is to join bivariate distribution functions (DFs) that have uniform $[0,1]$ marginals. This is one technique to explore studies of bivariate distributions using copulas. The dependence between the two RVs will determine the copula function. In high-dimensional statistical contexts, copulas are helpful because they simplify the process of modeling and estimating the distribution of random vectors by calculating marginals and copulas independently. For given two marginal univariate distributions $F_x(x) = P(X \leq x)$ and $F_Y(y) = P(Y \leq y)$, a copula $C(u, v)$, and its probability density function (PDF), i.e., $c(u, v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$. Sklar [12] presented the joint cumulative density function (JCDF) and joint probability density function (JPDF), respectively, as follows:

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)), \tag{1.1}$$

and

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) c(F_X(x), F_Y(y)). \tag{1.2}$$

One of the most widespread and beneficial bivariate DF is the FGM model. The JCDF and JPDF of the FGM bivariate family are given, respectively by

$$F_{X,Y}(x, y) = F_X(x) F_Y(y) \left[1 + \theta \bar{F}_X(x) \bar{F}_Y(y)\right], \tag{1.3}$$

and

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \left[1 + \theta(2F_X(x) - 1)(2F_Y(y) - 1)\right], \quad -1 \leq \theta \leq 1, \tag{1.4}$$

where $\bar{F}_X(x)$ is the survival function (SF) (or the reliability function $R(x) = \bar{F}_X(x) = P(X > x)$) of $f_X(x)$. The FGM copula provides numerous benefits when modeling bivariate distributions. An important benefit of this copula is its versatility in representing a broad spectrum of dependence structures, ranging from complete independence to absolute reliance, see Gumbel [13]. Moreover, the FGM copula has the ability to handle asymmetrical dependence, which makes it suitable for modeling data with skewed or heavy-tailed distributions, see Morgenstern [1]. Furthermore, the FGM copula enables the creation of bivariate distributions that encompass a diverse set of marginals, including both continuous and discrete marginals . In addition, the FGM copula has a straightforward structure, making it computationally efficient and straightforward to implement in practical applications, see Joe [14]. The FGM distribution is a flexible and valuable family in applications as long as the correlation between the variables is not overly great. It can be used for arbitrary continuous marginals. Schucany et al. [15] showed that $-\frac{1}{3} \leq \rho \leq \frac{1}{3}$ if both $F_X(x)$ and $F_Y(y)$ are arbitrary continuous distributions with bounded nonzero variances. While several copulas, including Clayton, Gumbel, and Frank, are commonly utilized for modeling dependence structures, especially in the presence of significant or asymmetric tail dependence, the FGM copula is still a compelling and relevant choice in this analysis. The primary impetus for utilizing the FGM copula is its analytical manageability and closed-form simplicity, which enable the derivation of explicit expressions for essential statistical measures, including the moment-generating function, reliability functions, and regression curves. This property makes it especially suitable for investigating new marginals, where closed-form properties are crucial for both theoretical analysis and practical implementation. Moreover, the FGM copula offers interpretable dependence characterized by bounded correlation, which, while constrained in strength, is sufficient for numerous engineering and materials datasets that display moderate associations rather than extreme tail reliance. Moreover, families based on FGM exhibit a longstanding legacy of reliability and applied statistics, owing to their equilibrium between flexibility and mathematical convenience, rendering them a suitable foundation before advancing to more intricate copulas in subsequent research. The utilization of generalized distributions for data modeling remains prevalent today. Numerous academics have introduced novel generalizations for lifespan distributions applicable in diverse domains, including economics, actuarial science, medicine, and engineering. Zeghdoudi and Nedjar [16] developed the Gamma Lindley distribution to improve the analysis of several types of lifespan data. Eghwerido et al. [17] delineate a three-parameter class for lifetime Poisson processes within the Marshall-Olkin transformation family. Recently He et al. [18] developed an innovative methodology that integrates artificial neural networks with the Tweedie exponential dispersion process framework to adaptively calibrate the stochastic process model that most accurately represents the real degradation trend. Gemeay et al. [19] established a new universal two-parameter statistical distribution, which may be expressed as a combination of exponential and gamma distributions. Belil et al. [20] introduced a novel flexible two-parameter family of distributions, exemplified by an analysis of annual maximum flood data and survival periods of breast cancer patients. Furthermore, Beghriche et al. [21] presented a novel polynomial exponential distribution characterized by a single parameter. The resulting model, termed the polynomial exponential extended distribution, incorporates the original distribution as a specific instance and offers greater versatility for modeling various real data sources. Almetwally and Meraou [22] developed a sine extension of the exponential distribution. In recent years, numerous authors have suggested extensions and adaptations of the exponential distribution to augment its adaptability for modeling lifetime and reliability data. Transmuted exponential, generalized exponential, and odd-exponential families have been created to capture skewness and tail behavior more efficiently; for more details, see [23–25]. An innovative two-parameter quadratic exponential distribution (QEXD) was presented by Bousseba et al. [26], who also conducted an in-depth investigation into the statistical features and practical applications of this distribution. Additionally, they studied the important aspects of the distribution, including its asymptotic behaviour, moments, order statistics, and entropies. In addition to that, they outlined the concepts of fuzzy reliability, value at risk, mean excess function, restricted expected value function, tail value at risk, and tail variance. Let X be a continuous RV following the QEXD. Then the PDF can be expressed by

$$f_X(x) = \zeta(\alpha,\beta)\left(\alpha + \beta x + x^2\right)e^{-\beta x}, \tag{1.5}$$

where $\zeta(\alpha,\beta) = \frac{\beta^3}{K(\alpha,\beta)}$, $K(\alpha,\beta) = \beta^2 + \alpha\beta^2 + 2$, and $x, \alpha, \beta > 0$. The appropriate CDF, SF, and hazard rate function (HR) are as follows:

$$F_X(x) = 1 - e^{-\beta x}\left(1 + \frac{\beta^2 x^2 + (\beta^3 + 2\beta)x}{K(\alpha,\beta)}\right), \tag{1.6}$$

$$\bar{F}_X(x) = e^{-\beta x}\left(1 + \frac{\beta^2 x^2 + (\beta^3 + 2\beta)x}{K(\alpha,\beta)}\right), \tag{1.7}$$

and

$$HR_X(x) = \frac{\beta^3\left(\alpha + \beta x + x^2\right)}{\beta^2 x^2 + (\beta^3 + 2\beta)x + K(\alpha,\beta)}. \tag{1.8}$$

In this paper, the FGM copula provides a simple mechanism to introduce dependence between two RVs while preserving analytical tractability. When combined with QEXD marginals, the resulting FGM-QEXD model retains closed-form expressions for key distributional properties, including the joint density, reliability, and moment-generating functions. The quadratic form of the marginals introduces additional flexibility to capture skewness and kurtosis. At the same time, the FGM structure enables the modeling of weak-to-moderate positive or negative dependence through a single parameter θ. This combination yields a practical family of bivariate lifetime distributions that are suitable for applications in reliability, materials science, and survival. The proposed FGM-QEXD bivariate family is motivated by scenarios where two related lifetime variables exhibit weak-to-moderate dependence. Such cases are common in applied reliability and biomedical contexts. For instance, in mechanical engineering, stress strength models involve two correlated variables where the dependence is moderate rather than extreme. In computer and networked systems, correlated failure modes may arise when devices share common operating environments but still retain individual variability. In healthcare and biomedical studies, biomarkers or survival times from paired organs (such as kidneys or lungs) often display weak-to-moderate correlation, making highly restrictive dependence models less suitable. The FGM-QEXD model captures such dependence while remaining mathematically tractable, which provides a practical advantage over more complex alternatives such as Clayton or Gumbel copula models.

Motivation

- A crucial factor in dependence modeling is not simply the adequacy of a copula in fitting the data, but also its manageability for theoretical analysis. Despite their prevalent application,

copulas like those of Clayton, Frank, and Gumbel frequently result in intricate expressions when addressing concomitants of order statistics and associated probabilistic characteristics. In contrast, the FGM copula maintains analytical simplicity, rendering it more adaptable for both theoretical derivations and practical implementations. The FGM-QEXD model, when integrated with QEXD marginals, maintains tractability while offering an enhanced empirical fit, evidenced by elevated log-likelihood values and reduced Akaike information criterion (AIC) scores in comparison with classical copulas and the FGM copula with different marginals. The combined benefits of analytical simplicity and empirical correctness drive the utilization of the FGM-QEXD copula as a reliable framework for modeling bivariate data.
- A novel, adaptable bivariate distribution with precise derivations of essential statistical functions (the moment generating function, conditional expectations, the vitality function, etc.).
- The QEXD is a logical extension of the exponential distribution, providing further shape freedom while maintaining simplicity, the capability to simulate growing, decreasing, or bathtub-shaped hazard functions, and analytically tractable forms of the PDF and CDF.
- The following advantages are enjoyed by the FGM-QEXD model as a consequence of coupling the FGM copula with QEXD marginals: more marginal flexibility due to the QEXD, a dependence system that is both lightweight and interpretable, and various functions' closed-form expressions: (moments, reliability measures, density, etc).
- The FGM-QEXD does better than competing models when applied to real-world datasets derived from computer science and materials data, including bivariate Lomax, bivariate inverse Lomax, bivariate Weibull, bivariate inverse Weibull, and versions of bivariate Rayleigh FGM distributions, providing a more accurate fit, more nuanced modeling of dependence, and more insightful interpretations.

The paper is structured as follows. The description of the suggested model is provided in Section 2. In Section 3, the FGM-QEXD's statistical features are also examined, including its conditional distribution, regression curve, moment-generating function, and correlation coefficient. The dependability measures, including the vitality function, the mean residual life, and the hazard function, are discussed in Section 4. Section 5 employs Bayesian approaches and maximum likelihood (ML) to estimate the model parameters. Parameters of the model are also associated with asymptotic confidence intervals (CIs). Using Monte Carlo simulations, the Bayesian and ML estimators were computed in Section 6. As an example, Section 7 analyzes two bivariate real-world data sets and finds them to be adequate. Finally, Section 8 concludes the work.

2. FGM-QEXD bivariate distribution

Let $X \sim QEXD(\alpha_1,\beta_1)$ and $Y \sim QEXD(\alpha_2,\beta_2)$. Thus, according to (1.3) and (1.6) the JCDF of bivariate QEXD based on the FGM copula, denoted by FGM-QEXD($\alpha_1,\beta_1,\alpha_2,\beta_2$), is given by

$$\begin{aligned} F_{X,Y}(x,y) &= \left(1 - e^{-\beta_1 x}\left(1 + \frac{\beta_1^2 x^2 + (\beta_1^3 + 2\beta_1)x}{K(\alpha_1,\beta_1)}\right)\right) \\ &\times \left(1 - e^{-\beta_2 y}\left(1 + \frac{\beta_2^2 y^2 + (\beta_2^3 + 2\beta_2)y}{K(\alpha_2,\beta_2)}\right)\right) \end{aligned}$$

$$\times \quad \left[1+\theta\left(e^{-\beta_1 x}\left(1+\frac{\beta_1^2x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}\right)\right)\right.$$
$$\times \quad \left.\left(e^{-\beta_2 y}\left(1+\frac{\beta_2^2y^2+(\beta_2^3+2\beta_2)y}{K(\alpha_2,\beta_2)}\right)\right)\right]. \tag{2.1}$$

Based on (1.4). The corresponding JPDF of FGM-QEXD($\alpha_1,\beta_1,\alpha_2,\beta_2$), is defined by

$$\begin{aligned} f_{X,Y}(x,y) &= \left(\zeta(\alpha_1,\beta_1)\left(\alpha_1+\beta_1x+x^2\right)e^{-\beta_1 x}\right)\left(\zeta(\alpha_2,\beta_2)\left(\alpha_2+\beta_2y+y^2\right)e^{-\beta_2 y}\right)\\ &\times \left[1+\theta\left(1-2\left(1-e^{-\beta_1 x}\left(1+\frac{\beta_1^2x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}\right)\right)\right)\right.\\ &\times \left.\left(1-2\left(1-e^{-\beta_2 y}\left(1+\frac{\beta_2^2y^2+(\beta_2^3+2\beta_2)y}{K(\alpha_2,\beta_2)}\right)\right)\right)\right], \end{aligned} \tag{2.2}$$

where $\zeta(\alpha_\ell,\beta_\ell)=\frac{\beta_\ell^3}{K(\alpha_\ell,\beta_\ell)}$, $K(\alpha_\ell,\beta_\ell)=\beta_\ell^2+\alpha_\ell\beta_\ell^2+2, \ell=1,2$. The JPDF (2.2) for a few chosen values of its parameters is shown in 3-D Figures 1 to 4. Given that it can handle a variety of data, the graphs in Figures 1 to 4 demonstrate how rich and widespread this family is. Several general trends can be observed from Figures 1 to 4: The overall peak of the surface is strongly influenced by the scale parameters β_1 and β_2. Smaller values of these parameters yield sharper peaks near the origin, while larger values produce flatter surfaces with heavier tails. The shape parameters α_1 and α_2 control skewness and kurtosis. Increasing α values makes the surface more skewed and shifts the density mass away from the origin, reflecting heavier-tailed behavior. This illustrates the flexibility of the FGM-QEXD family in capturing weak-to-moderate dependence structures while accommodating different marginal behaviors.

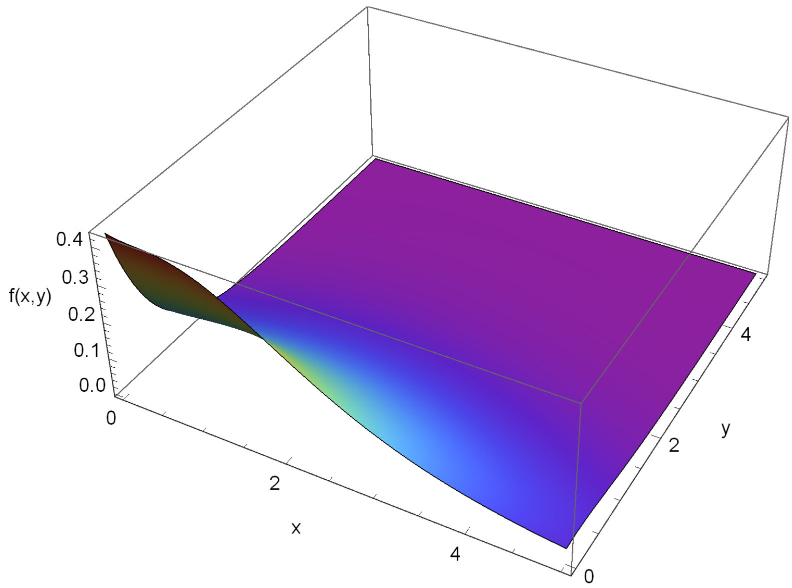

Figure 1. JPDF at $\alpha_1=1, \beta_1=1, \alpha_2=2, \beta_2=2, \theta=0.5$.

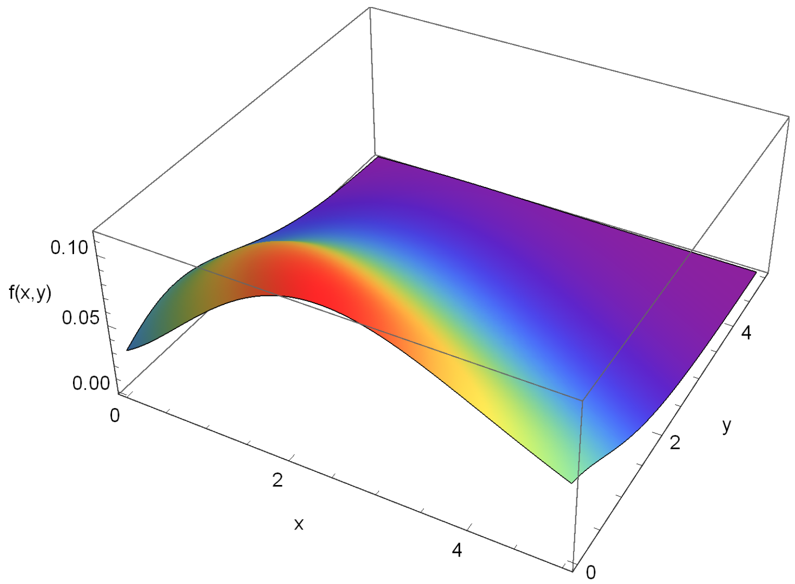

Figure 2. JPDF at $\alpha_1 = 0.5, \beta_1 = 0.9, \alpha_2 = 1, \beta_2 = 1.5, \theta = -0.5$.

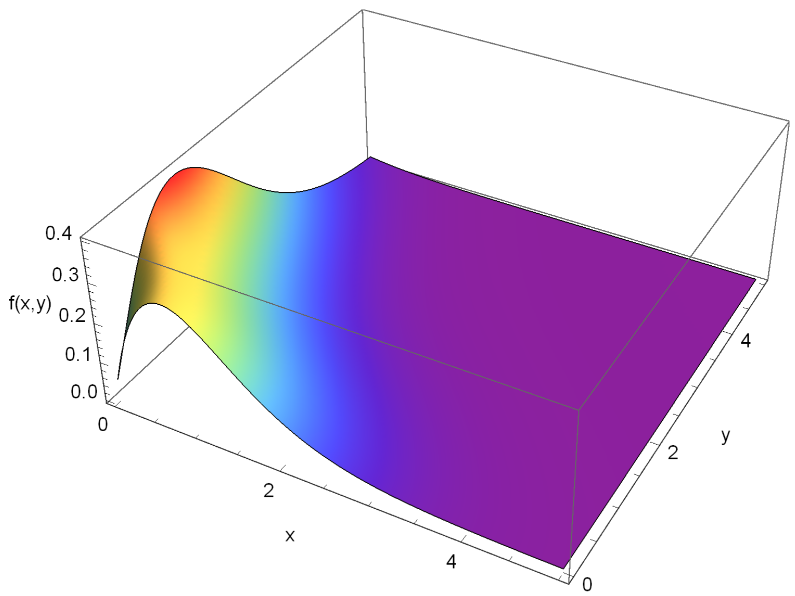

Figure 3. JPDF at $\alpha_1 = 1.5, \beta_1 = 2, \alpha_2 = 0.9, \beta_2 = 1.5, \theta = -0.9$.

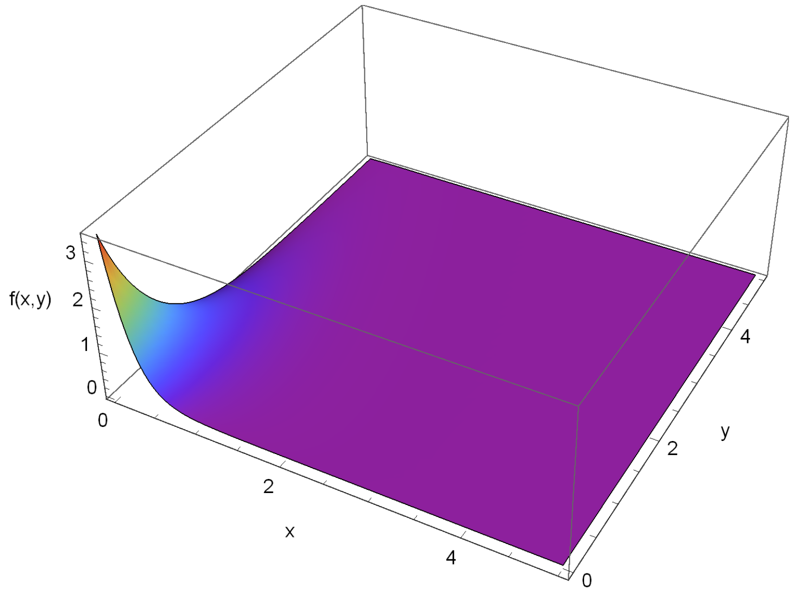

Figure 4. JPDF at $\alpha_1 = 2.5, \beta_1 = 3.5, \alpha_2 = 2, \beta_2 = 1.5, \theta = 0.9$.

3. Statistical properties of FGM-QEXD

A number of important analytical features of the FGM-QEXD distribution are examined here: conditional distribution, regression function, moment generating function (MGF), and correlation coefficient.

3.1. Conditional distributions based FGM-QEXD

The conditional PDF of Y given X is given by

$$\begin{aligned} f_{Y|X}(y|x) &= \left(\zeta(\alpha_2,\beta_2)\left(\alpha_2+\beta_2 y+y^2\right)e^{-\beta_2 y}\right) \\ &\times \left[1+\theta\left(1-2\left(1-e^{-\beta_1 x}\left(1+\frac{\beta_1^2 x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}\right)\right)\right)\right. \\ &\times \left.\left(1-2\left(1-e^{-\beta_2 y}\left(1+\frac{\beta_2^2 y^2+(\beta_2^3+2\beta_2)y}{K(\alpha_2,\beta_2)}\right)\right)\right)\right]. \end{aligned}$$

The conditional CDF of Y given X is

$$\begin{aligned} F_{Y|X}(y \mid x) &= \left(1-e^{-\beta_2 y}\left(1+\frac{\beta_2^2 y^2+(\beta_2^3+2\beta_2)y}{K(\alpha_2,\beta_2)}\right)\right) \\ &\times \left[1+\theta\left(e^{-\beta_2 y}\left(1+\frac{\beta_2^2 x^2+(\beta_2^3+2\beta_2)y}{K(\alpha_2,\beta_2)}\right)\right)\right. \\ &\times \left.\left(1-2\left(1-e^{-\beta_1 x}\left(1+\frac{\beta_1^2 x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}\right)\right)\right)\right]. \end{aligned}$$

Consequently, for the FGM-QEXD$(\alpha_1,\beta_1,\alpha_2,\beta_2)$, the regression curve for Y given $X = x$ is

$$\mathbb{E}[Y \mid X = x] = \mu_Y + \theta\,(1-2F_X(x))\int_0^\infty y\, f_Y(y)\,(1-2F_Y(y))\,dy$$

$$= \frac{(\alpha_2+2)\beta_2^2+6}{\beta_2\left(\alpha_2\beta_2+\beta_2^2+2\right)} + \theta\left(1-2\left(1-e^{-\beta_1 x}\left(1+\frac{\beta_1^2x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}\right)\right)\right)$$
$$\times \int_0^\infty y\, f_Y(y)\,(1-2F_Y(y))\; dy$$
$$= \frac{(\alpha_2+2)\beta_2^2+6}{\beta_2\left(\alpha_2\beta_2+\beta_2^2+2\right)} + \theta\left(1-2\left(1-e^{-\beta_1 x}\left(1+\frac{\beta_1^2x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}\right)\right)\right)$$
$$\times -\frac{(2\alpha_2(\alpha_2+3)+3)\beta_2^4+3(6\alpha_2+5)\beta_2^2+15}{4\beta_2\left((\alpha_2+1)\beta_2^2+2\right)^2}.$$

3.2. Moment generating function for FGM-QEXD

The moment generating function (MGF) of the bivariate FGM-QEXD distribution can be expressed as

$$M_{X,Y}(t_1,t_2) \;=\; M_X(t_1)\,M_Y(t_2) \;+\; \theta\, I(t_1,t_2),$$

where $M_X(t_1)$ and $M_Y(t_2)$ are the marginal MGFs of X and Y, respectively, and $I(t_1,t_2)$ is the copula based correction term. For $t_1<\beta_1$ and $t_2<\beta_2$, the marginal MGFs take the closed forms

$$M_X(t_1) = \frac{\beta_1^3\left[-\alpha_1(t_1-\beta_1)^2+\beta_1(t_1-\beta_1)-2\right]}{((\alpha_1+1)\beta_1^2+2)(t_1-\beta_1)^3},$$

$$M_Y(t_2) = \frac{\beta_2^3\left[-\alpha_2(t_2-\beta_2)^2+\beta_2(t_2-\beta_2)-2\right]}{((\alpha_2+1)\beta_2^2+2)(t_2-\beta_2)^3}.$$

The adjustment integral is given by

$$I(t_1,t_2) = \int_0^\infty\int_0^\infty e^{t_1x+t_2y} f_X(x)f_Y(y)\,(1-2F_X(x))(1-2F_Y(y))\,dx\,dy.$$

After simplification, the final closed form is

$$I(t_1,t_2) \;=\; \frac{t_1t_2\beta_1^3\beta_2^3}{\Delta_{t_1,t_2}}\mathcal{P}_1(t_1;\alpha_1,\beta_1)\,\mathcal{P}_2(t_2;\alpha_2,\beta_2),$$

where

$$\Delta_{t_1,t_2} = (t_1-2\beta_1)^5(t_1-\beta_1)^3(2+\alpha_1\beta_1+\beta_1^2)(2+(1+\alpha_1)\beta_1^2)$$
$$\times (t_2-2\beta_2)^5(t_2-\beta_2)^3(2+\alpha_2\beta_2+\beta_2^2)(2+(1+\alpha_2)\beta_2^2),$$

and the polynomials $\mathcal{P}_1$ and $\mathcal{P}_2$ are given by

$$\mathcal{P}_1(t_1;\alpha_1,\beta_1) = t_1^6\alpha_1(2+(1+\alpha_1)\beta_1^2)$$
$$+\,8\beta_1^4\Big(15+3(5+6\alpha_1)\beta_1^2+(3+2\alpha_1(3+\alpha_1))\beta_1^4\Big)$$
$$+\,4t_1\beta_1^3\Big(-60-3(25+36\alpha_1)\beta_1^2-2(9+\alpha_1(21+8\alpha_1))\beta_1^4\Big)$$

$$+ 2t_1^2\beta_1^2\Big(80 + 14(10 + 19\alpha_1)\beta_1^2 + (41 + 2\alpha_1(59 + 26\alpha_1))\beta_1^4\Big)$$
$$+ t_1^4\Big(4 + 4(6 + 31\alpha_1)\beta_1^2 + (11 + \alpha_1(66 + 41\alpha_1))\beta_1^4\Big)$$
$$- 2t_1^3\beta_1\Big(20 + 4(15 + 43\alpha_1)\beta_1^2 + (22 + \alpha_1(85 + 44\alpha_1))\beta_1^4\Big)$$
$$- t_1^5\beta_1\Big(2 + \beta_1^2 + \alpha_1(24 + (13 + 10\alpha_1)\beta_1^2)\Big),$$

$$\begin{aligned}\mathcal{P}_2(t_2;\alpha_2,\beta_2) = {} & t_2^6\alpha_2(2 + (1 + \alpha_2)\beta_2^2)\\
& + 8\beta_2^4\Big(15 + 3(5 + 6\alpha_2)\beta_2^2 + (3 + 2\alpha_2(3 + \alpha_2))\beta_2^4\Big)\\
& + 4t_2\beta_2^3\Big(-60 - 3(25 + 36\alpha_2)\beta_2^2 - 2(9 + \alpha_2(21 + 8\alpha_2))\beta_2^4\Big)\\
& + 2t_2^2\beta_2^2\Big(80 + 14(10 + 19\alpha_2)\beta_2^2 + (41 + 2\alpha_2(59 + 26\alpha_2))\beta_2^4\Big)\\
& + t_2^4\Big(4 + 4(6 + 31\alpha_2)\beta_2^2 + (11 + \alpha_2(66 + 41\alpha_2))\beta_2^4\Big)\\
& - 2t_2^3\beta_2\Big(20 + 4(15 + 43\alpha_2)\beta_2^2 + (22 + \alpha_2(85 + 44\alpha_2))\beta_2^4\Big)\\
& - t_2^5\beta_2\Big(2 + \beta_2^2 + \alpha_2(24 + (13 + 10\alpha_2)\beta_2^2)\Big).\end{aligned}$$

3.3. Correlation coefficient of FGM-QEXD

Let $X \sim QEXD(\alpha_1,\beta_1)$ and $Y \sim QEXD(\alpha_2,\beta_2)$. Then the Pearson correlation coefficient of X and Y is given by

$$\begin{aligned}\rho(x,y) &= \theta\frac{\int_0^\infty\int_0^\infty xyf_X(x)f_Y(y)(2F_X(x)-1)(2F_Y(y)-1)dxdy}{\sigma_x\sigma_y}\\
&= \theta\frac{\frac{\left((2\alpha_1(\alpha_1+3)+3)\beta_1^4+3(6\alpha_1+5)\beta_1^2+15\right)\left((2\alpha_2(\alpha_2+3)+3)\beta_2^4+3(6\alpha_2+5)\beta_2^2+15\right)}{16\beta_1\beta_2\left((\alpha_1+1)\beta_1^2+2\right)^2\left((\alpha_2+1)\beta_2^2+2\right)^2}}{\sigma_x\sigma_y},\end{aligned} \tag{3.1}$$

where

$$\sigma_x = \sqrt{Var(x)} = \sqrt{\frac{\beta_1\left(-(\alpha_1+2)\beta_1^2 + 2(\alpha_1+3)\beta_1 - 6\right) + 24}{\beta_1^2\left(\alpha_1\beta_1 + \beta_1^2 + 2\right)}}$$

and

$$\sigma_y = \sqrt{Var(Y)} = \sqrt{\frac{\beta_2\left(-(\alpha_2+2)\beta_2^2 + 2(\alpha_2+3)\beta_2 - 6\right) + 24}{\beta_2^2\left(\alpha_2\beta_2 + \beta_2^2 + 2\right)}}.$$

The two Figures 5 and 6 visualize the correlation coefficient based on FGM-QEXD bivariate distribution under different parameter setting.

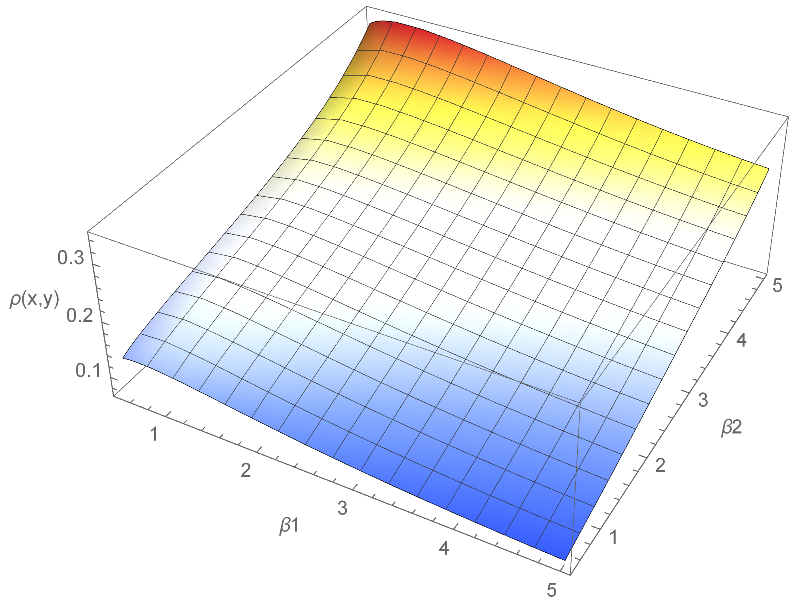

Figure 5. $\rho(x, y)$ at $\alpha_1 = 7.5$, $\alpha_2 = 6$, $\theta = 0.99$.

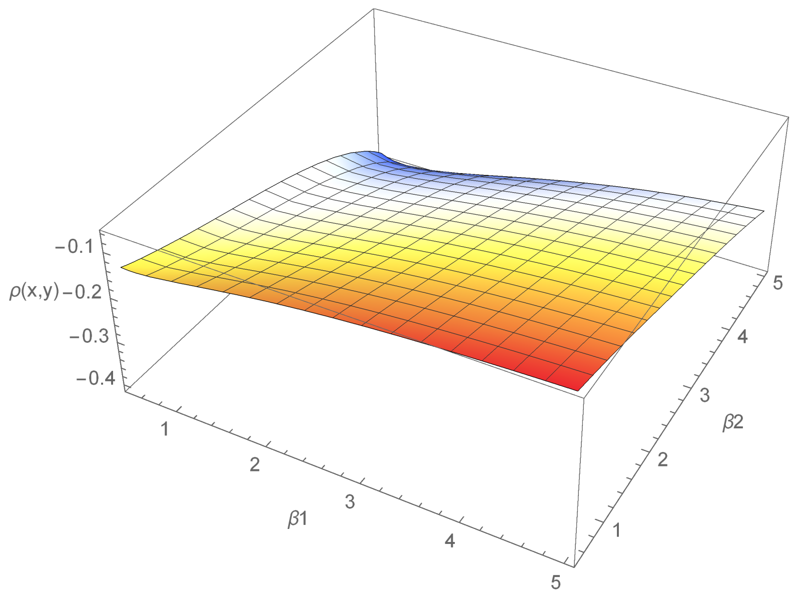

Figure 6. $\rho(x, y)$ at $\alpha_1 = 8$, $\alpha_2 = 10$, $\theta = -0.99$.

4. Reliability measures

Reliability theory plays a pivotal role in many applied fields, including engineering, medical statistics, and risk analysis (see [27]). A thorough understanding of the underlying failure behavior of systems or components often requires more than just basic probability functions. In this section, several key reliability measures derived from the bivariate FGM-QEXD model are investigated, namely the mean residual life (MRL) function, the vitality function, and the HR.

4.1. Mean residual life in FGM-QEXD

The Mean Residual Life (MRL) function denotes the anticipated residual lifespan contingent upon survival to a specific moment. It is extensively utilized in survival analysis and reliability engineering to forecast the duration a component or system is anticipated to operate after having already endured a specified period. Within the framework of the FGM-QEXD model, the MRL function elucidates the influence of reliance and marginal factors on expected remaining life. A unit's average life after surviving for a given amount of time t is called the MRL. A distribution with a finite mean can be fully determined by the MRL function, which is similar to the PDF or characteristic function; this is done using an inversion formula (cf. Mansour et al. [28] and Guess and Proschan [29]). The MRL is useful for both parametric and nonparametric models. Life insurance prices and benefits are determined by actuaries using MRL. Researchers use MRL to examine survival studies in the biomedical setting. The MRL was first proposed by Shanbag and Kotz [30] for vector-valued RVs as

$$m(x,y) = \left(m_1(x,y), m_2(x,y)\right), \tag{4.1}$$

where

$$m_1(x,y) = E(X - x|X \geq x, Y \geq y)$$

and

$$m_2(x,y) = E(Y - y|X \geq x, Y \geq y).$$

The expressions for $m_1(x,y)$ and $m_2(x,y)$ in FGM-QEXD(α_1,β_1, α_2,β_2), are obtained as

$$\begin{aligned} m_1(x,y) &= \frac{1}{\bar{F}(x,y)} \int_x^\infty \int_y^\infty (s-x) f_{X,Y}(s,t)\, dt\, ds, \\ &= \frac{\bar{F}(y)A(x) + \theta B(x)C(y)}{\bar{F}(x,y)}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} A(x) &= \int_x^\infty (s-x) f_X(s) ds \\ &= \frac{e^{\beta_1(-x)} \left(\beta_1^2 \left(\alpha_1 + x^2 + 2\right) + \beta_1^3 x + 4\beta_1 x + 6\right)}{\beta_1 \left((\alpha_1+1)\beta_1^2 + 2\right)}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} B(x) &= \int_x^\infty (s-x) f_X(s)(1 - 2F_X(s)))ds \\ &= \frac{e^{-2x\beta_1}}{4\beta_1 \left(2 + (1+\alpha_1)\beta_1^2\right)^2} \Big(33 + 50x\beta_1 - 4e^{x\beta_1}(2 + (1+\alpha_1)\beta_1^2)(6 + 4x\beta_1 + (2 + x^2 + \alpha_1)\beta^2 + x\beta_1^3) \\ &+ \beta_1^2(25 + 14\alpha_1 + 2x^4\beta_1^2 + (5 + 2\alpha_1(3+\alpha_1))\beta_1^2 + 4x^3\beta_1(3+\beta_1^2) \\ &+ 2x\beta_1(17 + 6\alpha_1 + (3+2\alpha)_1\beta_1^2) + 2x^2(17 + (9+2\alpha_1)\beta_1^2 + \beta_1^4))\Big), \end{aligned} \tag{4.4}$$

and

$$C(y) = \int_y^\infty f_Y(t)(1 - 2F_Y(t))dt$$

$$= \frac{1}{\left((\alpha_2+1)\beta_2^2+2\right)^2}$$
$$\times \; e^{-2\beta_2 y}\left(\beta_2^2\left(\alpha_2+y^2+1\right)+\beta_2^3 y+2\beta_2 y+2\right)$$
$$\times \; \left(\beta_2^2+\beta_2^2(\alpha_2+y(\beta_2+y))-\left((\alpha_2+1)\beta_2^2+2\right)e^{\beta_2 y}+2\beta_2 y+2\right). \tag{4.5}$$

Now, From (4.3)–(4.5) in (4.2), $m_1(x,y)$ is obtained. The same solution approach is applied for $m_2(x,y)$.

4.2. *Vitality function*

The Vitality function quantifies a system's resilience by measuring its ability to endure stress or harsh operating circumstances. This function elucidates a system's behavior under cumulative stress for reliability modeling and serves as a tool for evaluating robustness. Integrating the Vitality function within the FGM-QEXD framework elucidates the impact of interdependence among components on the system's resilience. The vitality function is a helpful tool for modelling life time data. It was thoroughly explored by Kupka and Loo [31] in relation to their research on the ageing process. This idea was applied by Kotz and Shanbhag [32] to produce multiple life time distribution characterizations. The vitality function offers a more direct assessment of the failure pattern since it is expressed in terms of an increased average life span, whereas the hazard rate represents the chance of sudden death within a life span (cf. Mansour et al. [28]). The vitality function linked to a non-negative RV X is defined as $m(x) = E(X|X > x)$. The bivariate vitality function of random vector (X,Y) is defined on a positive domain as a binomial vector as

$$V(x,y) = (V_1(x,y), V_2(x,y)), \tag{4.6}$$

where

$$V_1(x,y) = E(X|X \geq x, Y \geq y)$$

and

$$V_2(x,y) = E(Y|X \geq x, Y \geq y).$$

For more details, see Sankaran and Nair [33]. Moreover, $V_i(x,y)$ is related to $m_i(x,y)$ by

$$V_i(x,y) = (x, i=1)(y, i=2) + m_i(x,y), \quad i = 1,2. \tag{4.7}$$

Here $V_2(x,y)$ computes the expected lifetime to the first component as the sum of current age x and the average lifetime remaining to it, assuming the second component has survived past age y. $V_2(x,y)$ has a similar interpretation.

4.3. *Bivariate reliability function of FGM-QEXD distribution*

The Bivariate Hazard function extends the notion of failure rate to a two-dimensional context (see [34]). It delineates the immediate risk of failure for a system comprising two interconnected lifespans. This is especially significant in reliability studies including the interaction of numerous components or stress factors, enabling practitioners to assess the impact of reliance (via θ) on joint failure risks. The bivariate reliability function $R_{X,Y}(x,y)$ for the FGM-QEXD distribution is defined as the probability that $X > x$ and $Y > y$

$$R(x,y) = P(X > x, Y > y) = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x,y).$$

Incorporating the marginal (1.6) and JCDF (2.1) into the notion of reliability function as

$$\begin{aligned}R(x,y) &= 1-\left(1-e^{-\beta_1 x}\left(1+\frac{\beta_1^2x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}\right)\right)\\&\quad-\left(1-e^{-\beta_2 y}\left(1+\frac{\beta_2^2y^2+(\beta_2^3+2\beta_2)y}{K(\alpha_2,\beta_2)}\right)\right)\\&\quad+F_{X,Y}(x,y)\\&=e^{-\beta_1 x}\left(1+\frac{\beta_1^2x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}\right)\\&\quad+e^{-\beta_2 y}\left(1+\frac{\beta_2^2y^2+(\beta_2^3+2\beta_2)y}{K(\alpha_2,\beta_2)}\right)-1\\&\quad+F_{X,Y}(x,y).\end{aligned}$$

Combining terms and using the joint CDF from equation (2.1), the reliability function becomes

$$\begin{aligned}R(x,y) &= e^{-\beta_1 x}\left(1+\frac{\beta_1^2x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}\right)\\&\quad\times e^{-\beta_2 y}\left(1+\frac{\beta_2^2y^2+(\beta_2^3+2\beta_2)y}{K(\alpha_2,\beta_2)}\right)\\&\quad\times\left[1+\theta\left(1-e^{-\beta_1 x}\left(1+\frac{\beta_1^2x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}\right)\right)\right.\\&\quad\times\left.\left(1-e^{-\beta_2 y}\left(1+\frac{\beta_2^2y^2+(\beta_2^3+2\beta_2)y}{K(\alpha_2,\beta_2)}\right)\right)\right].\end{aligned}$$

4.4. Bivariate hazard function of FGM-QEXD distribution

The bivariate HR $h_{X,Y}(x,y)$ is defined as the ratio of the JPDF to the joint reliability function

$$\begin{aligned}HR(x,y) &= \frac{f_{X,Y}(x,y)}{R(x,y)}\\&=\frac{\zeta(\alpha_1,\beta_1)(\alpha_1+\beta_1 x+x^2)}{1+\frac{\beta_1^2x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}}\\&\quad\times\frac{\zeta(\alpha_2,\beta_2)(\alpha_2+\beta_2 y+y^2)}{1+\frac{\beta_2^2y^2+(\beta_2^3+2\beta_2)y}{K(\alpha_2,\beta_2)}}\\&\quad\times\left[\frac{1+\theta\,(1-2(1-T(x)))\,(1-2(1-T(y)))}{1+\theta(1-T(X))(1-T(Y))}\right],\end{aligned}$$

where

$$\begin{aligned}T(x)&=e^{-\beta_1 x}\left(1+\frac{\beta_1^2x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}\right),\\T(y)&=e^{-\beta_2 y}\left(1+\frac{\beta_2^2y^2+(\beta_2^3+2\beta_2)y}{K(\alpha_2,\beta_2)}\right).\end{aligned}$$

Figures 7 and 8 depicts 3D plots of the joint HR of a FGM-QEXD for various parameter values. The HR surface increases with time, reflecting the rising risk of failure as components age. Higher α values accentuate this increase, producing sharper hazard growth, while larger β values moderate the rate of risk escalation. The dependence parameter θ tilts the hazard surface: positive θ indicates simultaneous higher risks across both components (clustering of failures), whereas negative θ shows divergence, with high risk in one component accompanied by lower risk in the other.

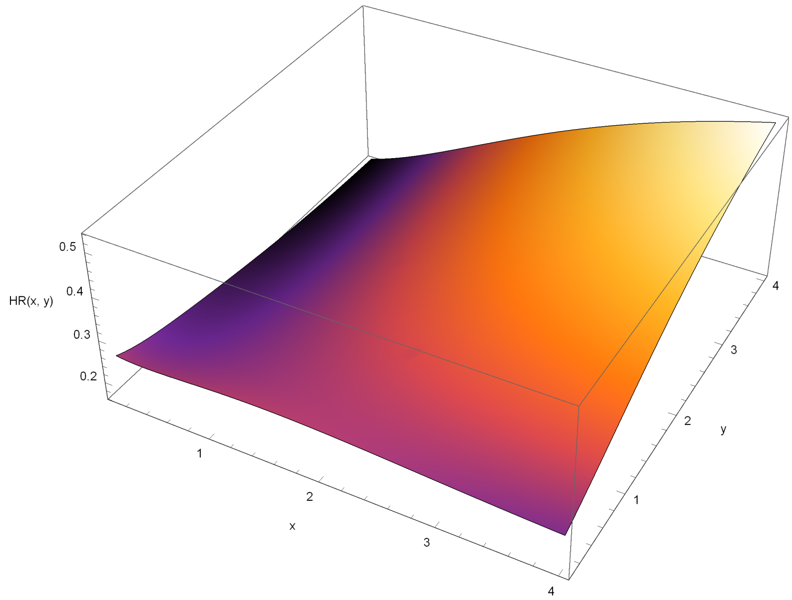

Figure 7. Bivariate HR at $\alpha_1 = 1.5, \beta_1 = 1, \alpha_2 = 2, \beta_2 = 1.2, \theta = 0.5$.

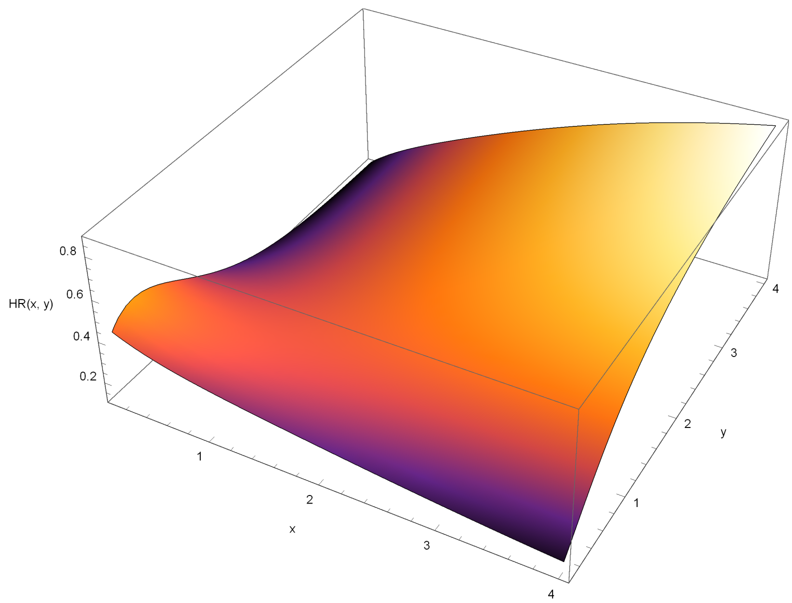

Figure 8. Bivariate HR at $\alpha_1 = 3, \beta_1 = 0.9, \alpha_2 = 0.5, \beta_2 = 2, \theta = 0.9$.

5. Methods of estimation

In this section, three estimation methods are discussed for estimating the unknown parameters of the FGM-QEXD: the ML and Bayesian estimation. Moreover, asymptotic confidence intervals are constructed using the Fisher information matrix (FIM) for the model's parameters.

5.1. The ML estimation

When it comes to statistics, the ML method is both essential and frequently utilized. One can obtain parameter estimates with desirable statistical properties, such as consistency, asymptotic unbiasedness, efficiency, and asymptotic normality, by employing the ML approach (cf. [6]). To acquire the parameter estimates using the ML approach, one must compute the parameter estimates that maximize the likelihood of the sample data. You can use the PDF from Eq (2.2) to get the log likelihood function lnL as

$$L(\alpha_1,\beta_1,\alpha_2,\beta_2,\theta) = \sum_{i=1}^{n} \ln f_{X,Y}(x_i, y_i),$$

where

$$\begin{aligned} f_{X,Y}(x,y) &= \left(\zeta(\alpha_1,\beta_1)(\alpha_1+\beta_1 x + x^2)e^{-\beta_1 x}\right)\left(\zeta(\alpha_2,\beta_2)(\alpha_2+\beta_2 y + y^2)e^{-\beta_2 y}\right) \\ &\quad \times \left[1+\theta A(x)B(y)\right], \\ A(x) &= 1-2\left(1-e^{-\beta_1 x}\left(1+\frac{\beta_1^2x^2+(\beta_1^3+2\beta_1)x}{K(\alpha_1,\beta_1)}\right)\right), \end{aligned}$$

and

$$B(y) = 1-2\left(1-e^{-\beta_2 y}\left(1+\frac{\beta_2^2y^2+(\beta_2^3+2\beta_2)y}{K(\alpha_2,\beta_2)}\right)\right).$$

Thus

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=1}^{n} \frac{A(x_i)B(y_i)}{1+\theta A(x_i)B(y_i)}.$$

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha_1} &= \sum_{i=1}^{n}\left[\frac{1}{\alpha_1+\beta_1 x_i + x_i^2} + \frac{\partial \ln\zeta(\alpha_1,\beta_1)}{\partial \alpha_1} + \frac{\theta\frac{\partial A(x_i)}{\partial \alpha_1}B(y_i)}{1+\theta A(x_i)B(y_i)}\right] \\ \frac{\partial \ell}{\partial \beta_1} &= \sum_{i=1}^{n}\left[\frac{x_i}{\alpha_1+\beta_1 x_i + x_i^2} - x_i + \frac{\partial \ln\zeta(\alpha_1,\beta_1)}{\partial \beta_1} + \frac{\theta\frac{\partial A(x_i)}{\partial \beta_1}B(y_i)}{1+\theta A(x_i)B(y_i)}\right], \end{aligned}$$

where

$$\begin{aligned} \frac{\partial \ln\zeta(\alpha_1,\beta_1)}{\partial \alpha_1} &= -\frac{\beta_1^2}{K(\alpha_1,\beta_1)} \\ \frac{\partial \ln\zeta(\alpha_1,\beta_1)}{\partial \beta_1} &= \frac{3}{\beta_1} - \frac{2\beta_1+\alpha_1 2\beta_1}{K(\alpha_1,\beta_1)}. \end{aligned}$$

These are symmetric to the derivatives with respect to α_1 and β_1, with x replaced by y and all subscripts 1 replaced with 2.

5.2. Asymptotic confidence intervals

To generate asymptotic confidence intervals (CIs) for the unknown parameters in Θ, the FIM is often utilized, relying on the asymptotic normality of the ML estimates (for additional information regarding other applications of FIM, see to [35] and [6]). Assuming certain regularity conditions, the ML estimates $\widehat{\Theta}$ follow a normal distribution. The distribution of the estimator $\widehat{\Theta} = (\widehat{\alpha_1}, \widehat{\beta_1}, \widehat{\alpha_2}, \widehat{\beta_2}, \widehat{\theta})$ approaches a normal distribution with a mean τ and a covariance matrix equal to the inverse of the FIM, represented as $\boldsymbol{I}^{-1}(\Theta)$. The negative anticipated values of the second-order derivatives of $\ln L$, make up the FIM.

Let $\Theta = (\alpha_1, \beta_1, \alpha_2, \beta_2, \theta)^\top$ be the parameter vector. The FIM $\mathcal{I}(\Theta)$ is given by

$$\mathcal{I}(\Theta) = -E\left[\frac{\partial^2 \ell(\Theta)}{\partial\Theta\partial\Theta^\top}\right],$$

where $\ell(\Theta)$ is the log-likelihood function. The observed FIM evaluated at the ML $\hat{\Theta}$

$$\mathcal{I}_O(\hat{\Theta}) = -\left.\frac{\partial^2 \ell(\Theta)}{\partial\Theta\partial\Theta^\top}\right|_{\Theta=\hat{\Theta}},$$

where $\mathcal{I}_O(\hat{\Theta})$ is a negative Hessian matrix of the $\ln L$ evaluated at the ML $\hat{\Theta}$. It quantifies the curvature of the log-likelihood function at the estimated parameters, providing a measure of how ”sharp” the likelihood peak is. The required second derivatives (diagonal elements) are

$$\frac{\partial^2 \ell}{\partial \alpha_1^2} = \sum_{i=1}^{n}\left[-\frac{1}{(\alpha_1+\beta_1 x_i + x_i^2)^2} + \frac{\partial^2 \ln\zeta(\alpha_1,\beta_1)}{\partial\alpha_1^2} - \frac{\left(\theta\frac{\partial A(x_i)}{\partial\alpha_1}B(y_i)\right)^2}{(1+\theta A(x_i)B(y_i))^2}\right],$$

$$\frac{\partial^2 \ell}{\partial \beta_1^2} = \sum_{i=1}^{n}\left[-\frac{x_i^2}{(\alpha_1+\beta_1 x_i + x_i^2)^2} + \frac{\partial^2 \ln\zeta(\alpha_1,\beta_1)}{\partial\beta_1^2} - \frac{\left(\theta\frac{\partial A(x_i)}{\partial\beta_1}B(y_i)\right)^2}{(1+\theta A(x_i)B(y_i))^2}\right],$$

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\sum_{i=1}^{n}\frac{(A(x_i)B(y_i))^2}{(1+\theta A(x_i)B(y_i))^2}.$$

with similar expressions for α_2 and β_2. Also, the off-diagonal elements are

$$\frac{\partial^2 \ell}{\partial \alpha_1\partial\beta_1} = \sum_{i=1}^{n}\left[-\frac{x_i}{(\alpha_1+\beta_1 x_i + x_i^2)^2} + \frac{\partial^2 \ln\zeta(\alpha_1,\beta_1)}{\partial\alpha_1\partial\beta_1} - \frac{\theta^2\frac{\partial A(x_i)}{\partial\alpha_1}\frac{\partial A(x_i)}{\partial\beta_1}B(y_i)^2}{(1+\theta A(x_i)B(y_i))^2}\right],$$

$$\frac{\partial^2 \ell}{\partial \alpha_1\partial\theta} = -\sum_{i=1}^{n}\frac{\frac{\partial A(x_i)}{\partial\alpha_1}B(y_i)}{(1+\theta A(x_i)B(y_i))^2},$$

$$\frac{\partial^2 \ell}{\partial \beta_1\partial\theta} = -\sum_{i=1}^{n}\frac{\frac{\partial A(x_i)}{\partial\beta_1}B(y_i)}{(1+\theta A(x_i)B(y_i))^2},$$

with similar expressions for other parameter pairs. Under regularity conditions, the ML $\hat{\Theta}$ has asymptotic distribution

$$\sqrt{n}(\hat{\Theta}-\Theta_0) \xrightarrow{d} N_5(\mathbf{0}, \mathcal{I}^{-1}(\Theta_0)),$$

where Θ_0 is the true parameter vector. For each parameter Θ_j, the $100(1-\alpha)\%$ asymptotic CI is

$$\hat{\Theta}_j \pm z_{\alpha/2}\sqrt{[\mathcal{I}_O^{-1}(\hat{\Theta})]_{jj}},$$

where $z_{\alpha/2}$ is the $(1-\alpha/2)$-quantile of the standard normal distribution.

5.3. Bayesian estimation

A potent tool for estimating unknown parameters from observable data is the Bayesian estimate technique. It is possible to revise a hypothesis's probability using newly acquired information and the Bayes Theorem, a notion from probability theory. This method offers some benefits over the conventional ML approach due to the fact that it estimates with prior knowledge in mind. Additionally, it can determine the level of uncertainty associated with each parameter. Careful attention is needed when choosing a prior PDF and hyperparameter. A potent tool for estimating unknown parameters from observable data is the Bayesian estimate technique. It is possible to revise a hypothesis's probability using newly acquired information and the Bayes Theorem, a notion from probability theory. This method offers some benefits over the conventional ML approach due to the fact that it estimates with prior knowledge in mind. Additionally, it can determine the level of uncertainty associated with each parameter. Based on assumptions about the data, a reasonable prior PDF and hyperparameter values need to be chosen. Let $\Theta = (\alpha_1, \beta_1, \alpha_2, \beta_2, \theta)$ be the parameter vector of the FGM-QEXD distribution. Given observed data $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^n$, the posterior distribution is

$$\pi(\Theta|\mathcal{D}) \propto \mathcal{L}(\mathcal{D}|\Theta) \times \pi(\Theta),$$

where $\mathcal{L}(\mathcal{D}|\Theta) = \prod_{i=1}^n f_{X,Y}(x_i, y_i|\Theta)$ is the likelihood function, $\pi(\Theta)$ is the prior distribution

The likelihood function is given by:

$$\begin{aligned}\mathcal{L}(\mathcal{D}|\Theta) = \prod_{i=1}^{n} & \left[\zeta(\alpha_1,\beta_1)(\alpha_1+\beta_1 x_i + x_i^2)e^{-\beta_1 x_i}\right] \\ & \times \left[\zeta(\alpha_2,\beta_2)(\alpha_2+\beta_2 y_i + y_i^2)e^{-\beta_2 y_i}\right] \\ & \times \left[1+\theta A(x_i)B(y_i)\right].\end{aligned}$$

The following independent priors are recommended.

$$\alpha_\ell \sim \text{Gamma}(a_{\alpha_\ell}, b_{\alpha_\ell}), \quad \beta_\ell \sim \text{Gamma}(a_{\beta_\ell}, b_{\beta_\ell}), \quad \theta \sim \text{Uniform}(-1,1)., \ell = 1,2$$

The joint prior density is:

$$\pi(\Theta) = \pi(\alpha_1)\pi(\beta_1)\pi(\alpha_2)\pi(\beta_2)\pi(\theta).$$

Therefore, the posterior distribution is:

$$\begin{aligned}\pi(\Theta|\mathcal{D}) \propto & \left[\prod_{i=1}^{n} f_{X,Y}(x_i, y_i|\Theta)\right] \\ & \times \alpha_1^{a_{\alpha_1}-1} e^{-b_{\alpha_1}\alpha_1} \times \beta_1^{a_{\beta_1}-1} e^{-b_{\beta_1}\beta_1} \\ & \times \alpha_2^{a_{\alpha_2}-1} e^{-b_{\alpha_2}\alpha_2} \times \beta_2^{a_{\beta_2}-1} e^{-b_{\beta_2}\beta_2}\end{aligned}$$

$$\times \mathbb{I}_{[-1,1]}(\theta).$$

The full conditional distributions for each parameter are

$$\pi(\alpha_1|\cdot) \propto \left[\prod_{i=1}^{n} f_{X,Y}(x_i, y_i|\Theta)\right] \times \alpha_1^{a_{\alpha_1}-1} e^{-b_{\alpha_1}\alpha_1},$$

$$\pi(\beta_1|\cdot) \propto \left[\prod_{i=1}^{n} f_{X,Y}(x_i, y_i|\Theta)\right] \times \beta_1^{a_{\beta_1}-1} e^{-b_{\beta_1}\beta_1},$$

$$\pi(\alpha_2|\cdot) \propto \left[\prod_{i=1}^{n} f_{X,Y}(x_i, y_i|\Theta)\right] \times \alpha_2^{a_{\alpha_2}-1} e^{-b_{\alpha_2}\alpha_2},$$

$$\pi(\beta_2|\cdot) \propto \left[\prod_{i=1}^{n} f_{X,Y}(x_i, y_i|\Theta)\right] \times \beta_2^{a_{\beta_2}-1} e^{-b_{\beta_2}\beta_2},$$

$$\pi(\theta|\cdot) \propto \left[\prod_{i=1}^{n} (1 + \theta A(x_i)B(y_i))\right] \times \mathbb{I}_{[-1,1]}(\theta)$$

6. Simulation

This section investigates the efficiency of the proposed estimation procedure for the parameters of the bivariate distribution FGM-QEXD. The parameters are estimated using two approaches: ML and Bayesian estimation. A simulation study is conducted to assess the performance of both methods under different conditions. The study focuses on estimating the copula and marginal parameters, with an emphasis on assessing the accuracy and robustness of the estimates across various sample sizes.

The simulation was implemented in R version 4.2.2, employing the packages, copula, Kendall, stats 4,'rstan','brms', and 'rjags' for the generation of random samples and the estimation of parameters. For each scenario, a total of 1,000 independent random samples were generated from the proposed bivariate distribution. The true values of the distribution parameters used in the simulation are as follows:

- In Table 1, $\theta = 0.4$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 2$, $\beta_2 = 4$.
- In Table 2, $\theta = -0.4$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 2$, $\beta_2 = 4$.
- In Table 3, $\theta = 0.5$, $\alpha_1 = 2$, $\alpha_2 = 1.5$, $\beta_1 = 1.5$, $\beta_2 = 1$.
- In Table 4, $\theta = 0.9$, $\alpha_1 = 4$, $\alpha_2 = 3$, $\beta_1 = 3$, $\beta_2 = 2$.

Table 1. ML and Bayesian estimates with their Bias, and MSE for $\theta = 0.4$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 2, \beta_2 = 4$ across different sample sizes.

	ML			Bayesian		
n	Estimation	Bias	MSE	Estimation	Bias	MSE
50	$\hat{\theta} = 0.388$	-0.012	0.0065	$\hat{\theta} = 0.3950$	-0.0050	0.0023
	$\hat{\alpha}_1 = 1.023$	0.023	0.0758	$\hat{\alpha}_1 = 1.0280$	0.0280	0.0041
	$\hat{\alpha}_2 = 2.006$	0.006	0.0885	$\hat{\alpha}_2 = 1.9600$	-0.0400	0.0050
	$\hat{\beta}_1 = 2.011$	0.011	0.0621	$\hat{\beta}_1 = 2.0300$	0.0300	0.0064
	$\hat{\beta}_2 = 3.981$	0.019	0.0704	$\hat{\beta}_2 = 3.9800$	0.0200	0.0018
100	$\hat{\theta} = 0.396$	-0.004	0.0032	$\hat{\theta} = 0.3980$	-0.0020	0.0015
	$\hat{\alpha}_1 = 1.012$	0.012	0.0425	$\hat{\alpha}_1 = 1.0120$	0.0120	0.0021
	$\hat{\alpha}_2 = 2.001$	0.001	0.0486	$\hat{\alpha}_2 = 1.9950$	0.0050	0.0013
	$\hat{\beta}_1 = 1.995$	0.005	0.0363	$\hat{\beta}_1 = 2.0150$	0.0150	0.0027
	$\hat{\beta}_2 = 3.993$	0.007	0.0387	$\hat{\beta}_2 = 3.9950$	0.0050	0.0007
150	$\hat{\theta} = 0.399$	-0.001	0.0021	$\hat{\theta} = 0.4010$	0.0010	0.0011
	$\hat{\alpha}_1 = 1.005$	0.005	0.0284	$\hat{\alpha}_1 = 1.0050$	0.0050	0.0012
	$\hat{\alpha}_2 = 2.003$	0.003	0.0311	$\hat{\alpha}_2 = 2.0000$	0.0000	0.0008
	$\hat{\beta}_1 = 2.001$	0.001	0.0226	$\hat{\beta}_1 = 2.0080$	0.0080	0.0019
	$\hat{\beta}_2 = 3.998$	0.002	0.0253	$\hat{\beta}_2 = 3.9970$	0.0030	0.0005
200	$\hat{\theta} = 0.400$	0.0001	0.0015	$\hat{\theta} = 0.4005$	0.0005	0.0008
	$\hat{\alpha}_1 = 1.002$	0.002	0.0183	$\hat{\alpha}_1 = 1.0030$	0.0030	0.0009
	$\hat{\alpha}_2 = 2.001$	0.001	0.0227	$\hat{\alpha}_2 = 2.0010$	0.0010	0.0007
	$\hat{\beta}_1 = 2.000$	0.000	0.0149	$\hat{\beta}_1 = 2.0040$	0.0040	0.0014
	$\hat{\beta}_2 = 3.999$	0.001	0.0175	$\hat{\beta}_2 = 3.9990$	0.0010	0.0004

Table 2. ML and Bayesian estimates with their Bias, and MSE for $\theta = -0.4$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 2, \beta_2 = 4$ across different sample sizes.

	ML			Bayesian		
n	Estimation	Bias	MSE	Estimation	Bias	MSE
50	$\hat{\theta} = -0.387$	0.013	0.0068	$\hat{\theta} = -0.3921$	0.0079	0.0035
	$\hat{\alpha}_1 = 1.018$	0.018	0.0746	$\hat{\alpha}_1 = 1.0415$	0.0415	0.0057
	$\hat{\alpha}_2 = 2.001$	0.011	0.0853	$\hat{\alpha}_2 = 1.9742$	0.0258	0.0061
	$\hat{\beta}_1 = 2.007$	0.007	0.0609	$\hat{\beta}_1 = 2.0653$	0.0653	0.0083
	$\hat{\beta}_2 = 3.987$	0.013	0.0717	$\hat{\beta}_2 = 3.9576$	0.0424	0.0078
100	$\hat{\theta} = -0.398$	0.002	0.0031	$\hat{\theta} = -0.3967$	0.0033	0.0021
	$\hat{\alpha}_1 = 1.008$	0.008	0.0394	$\hat{\alpha}_1 = 1.0210$	0.0210	0.0026
	$\hat{\alpha}_2 = 2.003$	0.003	0.0449	$\hat{\alpha}_2 = 1.9882$	0.0118	0.0029
	$\hat{\beta}_1 = 1.996$	0.004	0.0331	$\hat{\beta}_1 = 2.0235$	0.0235	0.0042
	$\hat{\beta}_2 = 3.995$	0.005	0.0364	$\hat{\beta}_2 = 3.9731$	0.0269	0.0043
150	$\hat{\theta} = -0.399$	0.001	0.0021	$\hat{\theta} = -0.3993$	0.0007	0.0014
	$\hat{\alpha}_1 = 1.004$	0.004	0.0272	$\hat{\alpha}_1 = 1.0097$	0.0097	0.0014
	$\hat{\alpha}_2 = 2.002$	0.002	0.0301	$\hat{\alpha}_2 = 1.9937$	0.0063	0.0017
	$\hat{\beta}_1 = 1.998$	0.002	0.0215	$\hat{\beta}_1 = 2.0105$	0.0105	0.0023
	$\hat{\beta}_2 = 3.999$	0.001	0.0257	$\hat{\beta}_2 = 3.9812$	0.0188	0.0029
200	$\hat{\theta} = -0.400$	0.000	0.0014	$\hat{\theta} = -0.4008$	-0.0008	0.0011
	$\hat{\alpha}_1 = 1.002$	0.002	0.0192	$\hat{\alpha}_1 = 1.0049$	0.0049	0.0010
	$\hat{\alpha}_2 = 2.001$	0.001	0.0223	$\hat{\alpha}_2 = 1.9982$	0.0018	0.0010
	$\hat{\beta}_1 = 2.001$	0.001	0.0141	$\hat{\beta}_1 = 2.0052$	0.0052	0.0016
	$\hat{\beta}_2 = 3.999$	0.001	0.0185	$\hat{\beta}_2 = 3.9903$	0.0097	0.0016

Table 3. ML and Bayesian estimates with their Bias, and MSE for $\theta = 0.5$, $\alpha_1 = 2$, $\alpha_2 = 1.5$, $\beta_1 = 1.5$, $\beta_2 = 1$ across different sample sizes.

	ML			Bayesian		
n	Estimation	Bias	MSE	Estimation	Bias	MSE
50	$\hat{\theta} = 0.487$	-0.013	0.0032	$\hat{\theta} = 0.4876$	-0.0124	0.0051
	$\hat{\alpha}_1 = 2.015$	0.015	0.0814	$\hat{\alpha}_1 = 2.0312$	0.0312	0.0047
	$\hat{\alpha}_2 = 1.523$	0.023	0.0662	$\hat{\alpha}_2 = 1.4784$	0.0216	0.0027
	$\hat{\beta}_1 = 1.478$	0.022	0.0491	$\hat{\beta}_1 = 1.5215$	0.0215	0.0029
	$\hat{\beta}_2 = 1.012$	0.012	0.0196	$\hat{\beta}_2 = 0.9828$	0.0172	0.0018
100	$\hat{\theta} = 0.492$	-0.008	0.0023	$\hat{\theta} = 0.4971$	-0.0029	0.0029
	$\hat{\alpha}_1 = 1.996$	0.004	0.0451	$\hat{\alpha}_1 = 2.0145$	0.0145	0.0019
	$\hat{\alpha}_2 = 1.508$	0.008	0.0376	$\hat{\alpha}_2 = 1.4938$	0.0062	0.0012
	$\hat{\beta}_1 = 1.491$	0.009	0.0328	$\hat{\beta}_1 = 1.5110$	0.0110	0.0015
	$\hat{\beta}_2 = 1.006$	0.006	0.0102	$\hat{\beta}_2 = 0.9921$	0.0079	0.0009
150	$\hat{\theta} = 0.497$	-0.003	0.0014	$\hat{\theta} = 0.5008$	0.0008	0.0021
	$\hat{\alpha}_1 = 2.008$	0.008	0.0291	$\hat{\alpha}_1 = 2.0063$	0.0063	0.0011
	$\hat{\alpha}_2 = 1.502$	0.002	0.0213	$\hat{\alpha}_2 = 1.4982$	0.0018	0.0007
	$\hat{\beta}_1 = 1.496$	0.004	0.0198	$\hat{\beta}_1 = 1.5071$	0.0071	0.0009
	$\hat{\beta}_2 = 1.004$	0.004	0.0074	$\hat{\beta}_2 = 0.9983$	0.0017	0.0005
200	$\hat{\theta} = 0.499$	-0.001	0.0011	$\hat{\theta} = 0.5012$	0.0012	0.0017
	$\hat{\alpha}_1 = 2.004$	0.004	0.0179	$\hat{\alpha}_1 = 2.0027$	0.0027	0.0008
	$\hat{\alpha}_2 = 1.502$	0.002	0.0147	$\hat{\alpha}_2 = 1.4993$	0.0007	0.0004
	$\hat{\beta}_1 = 1.498$	0.002	0.0126	$\hat{\beta}_1 = 1.5045$	0.0045	0.0006
	$\hat{\beta}_2 = 1.001$	0.001	0.0046	$\hat{\beta}_2 = 0.9992$	0.0008	0.0003

Table 4. ML and Bayesian estimates with their Bias, and MSE for $\theta = 0.9$, $\alpha_1 = 4$, $\alpha_2 = 3$, $\beta_1 = 3, \beta_2 = 2$ across different sample sizes.

	ML			Bayesian		
n	Estimation	Bias	MSE	Estimation	Bias	MSE
50	$\hat{\theta} = 0.925$	0.025	0.0181	$\hat{\theta} = 0.918$	0.018	0.0199
	$\hat{\alpha}_1 = 4.012$	0.012	0.0036	$\hat{\alpha}_1 = 4.036$	0.036	0.0041
	$\hat{\alpha}_2 = 3.018$	0.018	0.0049	$\hat{\alpha}_2 = 3.041$	0.041	0.0053
	$\hat{\beta}_1 = 2.979$	-0.021	0.1440	$\hat{\beta}_1 = 2.961$	-0.039	0.1504
	$\hat{\beta}_2 = 1.983$	-0.017	0.1012	$\hat{\beta}_2 = 2.008$	0.008	0.0965
100	$\hat{\theta} = 0.914$	0.014	0.0103	$\hat{\theta} = 0.911$	0.011	0.0114
	$\hat{\alpha}_1 = 4.008$	0.008	0.0017	$\hat{\alpha}_1 = 4.021$	0.021	0.0022
	$\hat{\alpha}_2 = 3.012$	0.012	0.0025	$\hat{\alpha}_2 = 3.026$	0.026	0.0030
	$\hat{\beta}_1 = 2.990$	-0.010	0.0692	$\hat{\beta}_1 = 2.972$	-0.028	0.0721
	$\hat{\beta}_2 = 1.991$	-0.009	0.0548	$\hat{\beta}_2 = 2.004$	0.004	0.0523
150	$\hat{\theta} = 0.907$	0.007	0.0056	$\hat{\theta} = 0.906$	0.006	0.0059
	$\hat{\alpha}_1 = 4.004$	0.004	0.0010	$\hat{\alpha}_1 = 4.011$	0.011	0.0015
	$\hat{\alpha}_2 = 3.006$	0.006	0.0014	$\hat{\alpha}_2 = 3.014$	0.014	0.0018
	$\hat{\beta}_1 = 2.994$	-0.006	0.0411	$\hat{\beta}_1 = 2.986$	-0.014	0.0430
	$\hat{\beta}_2 = 1.995$	-0.005	0.0322	$\hat{\beta}_2 = 2.001$	0.001	0.0315
200	$\hat{\theta} = 0.903$	0.003	0.0031	$\hat{\theta} = 0.902$	0.002	0.0034
	$\hat{\alpha}_1 = 4.002$	0.002	0.0007	$\hat{\alpha}_1 = 4.007$	0.007	0.0010
	$\hat{\alpha}_2 = 3.003$	0.003	0.0009	$\hat{\alpha}_2 = 3.008$	0.008	0.0011
	$\hat{\beta}_1 = 2.996$	-0.004	0.0265	$\hat{\beta}_1 = 2.991$	-0.009	0.0272
	$\hat{\beta}_2 = 1.997$	-0.003	0.0210	$\hat{\beta}_2 = 2.000$	0.000	0.0201

Performance metrics, including Bias and mean squared error (MSE), were calculated to evaluate the accuracy and efficiency of the estimators. The results indicate that both estimation methods perform well, with the accuracy of the parameter estimates improving as the sample size increases, thus enhancing their precision and reliability.

In the simulation study, the parameter of the FGM copula, denoted by θ, was assigned the values 0.4, -0.4, 0.5, and 0.9. The simulation was conducted for varying sample sizes, specifically $n = 50, 100, 150$, and 200. For each scenario, the simulation was replicated 1,000 times using a Monte Carlo approach. The resulting estimates, along with their associated Bias and MSE, are presented in Tables 1–4. From the results provided in these tables, the following conclusions can be drawn:

- The simulation results demonstrate that both the MLs and Bayesian estimates of the unknown parameters perform robustly across various scenarios. In particular, the estimates are characterized by relatively low levels of Bias and MSE, which indicates that the ML and Bayesian methods provide accurate and consistent parameter estimations. This performance improves further as the sample size increases, with Bias and MSE decreasing significantly, demonstrating the efficiency of the estimators. These results confirm that the proposed estimation approach is reliable and robust, especially in the context of the bivariate model based on the FGM copula and quadratic exponential marginals.

- The simulation results further demonstrate that as the sample size increases, the estimated values of the model parameters converge toward their true (nominal) values. This behavior highlights the consistency of the estimation methods, where estimators become more accurate as the sample size grows. The observed reduction in both Bias and MSE across simulations with larger sample sizes provides strong empirical support for this convergence. These findings align with the theoretical expectation that, under regularity conditions, MLs are asymptotically unbiased and efficient. Consequently, the results validate the effectiveness of the proposed estimation procedures in real-world applications.

The following algorithm outlines the procedure for computing the ML estimates of the parameters of the FGM-QEXD model, including initialization, likelihood specification, optimization, and post-processing steps.

Algorithm 1 ML estimation for FGM-QEXD parameters $(\alpha_1, \beta_1, \alpha_2, \beta_2, \theta)$

Input: Data $\{(x_i, y_i)\}_{i=1}^n$

Output: ML estimations $\hat{\Theta} = (\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\theta})$, SEs, Bias, MSE

Initialization:

(1) Fit univariate QEXD margins separately on $\{x_i\}$ and $\{y_i\}$ to obtain $(\alpha_1^{(0)}, \beta_1^{(0)})$, $(\alpha_2^{(0)}, \beta_2^{(0)})$.

(2) Compute empirical Kendall?s $\hat{\tau}_K$ from $\{(x_i, y_i)\}$. Set the initial value of θ as

$$\theta^{(0)} = \begin{cases} -1, & \text{if } \frac{9}{2}\hat{\tau}_K < -1, \\ \frac{9}{2}\hat{\tau}_K, & \text{if } -1 \leq \frac{9}{2}\hat{\tau}_K \leq 1, \\ 1, & \text{if } \frac{9}{2}\hat{\tau}_K > 1. \end{cases}$$

Likelihood: $\ell(\Theta) = \sum_{i=1}^n \log f_{X,Y}(x_i, y_i \mid \Theta)$.

Optimization:

(1) Use L-BFGS-B with bounds $(\alpha_\ell, \beta_\ell > 0;\ \theta \in (-1, 1))$.

(2) Provide analytic score from Sec. 5.1.

(3) Convergence: stop if relative change in ℓ and $\Theta < 10^{-8}$ or $\|\nabla \ell\| < 10^{-6}$.

Post-processing:

(1) *Real data application:* Compute observed Fisher information $I_O(\hat{\Theta}) = -\partial^2 \ell(\Theta)/\partial\Theta\partial\Theta^\top|_{\Theta=\hat{\Theta}}$; report the function of interest.

(2) *Simulation study:* Repeat the procedure R times (e.g. $R = 1000$). For each parameter ϕ with true value ϕ_0:

$$\text{Bias}(\phi) = \frac{1}{R}\sum_{r=1}^{R}(\hat{\phi}^{(r)} - \phi_0), \quad \text{MSE}(\phi) = \frac{1}{R}\sum_{r=1}^{R}(\hat{\phi}^{(r)} - \phi_0)^2.$$

Algorithm 2 Bayesian estimation via Metropolis-within-Gibbs for FGM-QEXD

Input: Data $\{(x_i, y_i)\}_{i=1}^n$; hyperparameters for Gamma priors.

Output: Posterior draws $\{\Theta^{(t)}\}$ and summaries (Bias, MSE)

Prior:

$$\alpha_\ell \sim \text{Gamma}(a_{\alpha\ell}, b_{\alpha\ell}),$$
$$\beta_\ell \sim \text{Gamma}(a_{\beta\ell}, b_{\beta\ell}),$$
$$\theta \sim \text{Uniform}(-1, 1).$$

Initialization:

(1) Use ML estimates $\hat{\Theta}$ or marginal-based initials for $(\alpha_1, \beta_1, \alpha_2, \beta_2)$.
(2) For θ, compute empirical Kendall?s $\hat{\tau}_K$ and set

$$\theta^{(0)} = \begin{cases} -1, & \text{if } \frac{9}{2}\hat{\tau}_K < -1, \\ \frac{9}{2}\hat{\tau}_K, & \text{if } -1 \leq \frac{9}{2}\hat{\tau}_K \leq 1, \\ 1, & \text{if } \frac{9}{2}\hat{\tau}_K > 1. \end{cases}$$

(3) Transform to unconstrained scale: $\tilde{\alpha}_\ell = \log \alpha_\ell$, $\tilde{\beta}_\ell = \log \beta_\ell$, $\tilde{\theta} = \tanh^{-1}(\theta^{(0)})$.

MCMC: For $t = 1, \ldots, T$:

(1) For each α_ℓ,β_ℓ: propose on log-scale $\log \phi^* \sim \mathcal{N}(\log \phi^{(t-1)}, s_\phi^2)$, accept with probability $\min\{1, \pi(\Theta^*|D)/\pi(\Theta^{(t-1)}|D)\}$.
(2) For θ: propose $\tilde{\theta}^* \sim \mathcal{N}(\tilde{\theta}^{(t-1)}, s_\theta^2)$, set $\theta^* = \tanh(\tilde{\theta}^*)$, accept with same rule.

Post-processing:

(1) *Real data application:* Discard the first B iterations (burn-in, e.g. $B = 2000$). Retain $T - B$ draws. For each parameter ϕ, compute posterior function of interest.
(2) *Simulation study:* For each replication $r = 1, \ldots, R$, take posterior means $\tilde{\Theta}^{(r)}$ as point estimates. For each parameter ϕ with true value ϕ_0:

$$\text{Bias}(\phi) = \frac{1}{R}\sum_{r=1}^{R}(\tilde{\phi}^{(r)} - \phi_0), \quad \text{MSE}(\phi) = \frac{1}{R}\sum_{r=1}^{R}(\tilde{\phi}^{(r)} - \phi_0)^2.$$

7. Real data application

Here, the suggested bivariate FGM-QEXD distribution is applied to two real-world datasets from distinct scientific fields, computer science and iron materials, to demonstrate its practical utility and versatility. The purpose of this evaluation is to see how accurately the FGM-QEXD model represents the marginal behaviors and underlying dependence structure in real-world data. The model selection criteria, which include the Akaike Information Criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (AICc), Hannan-Quinn information criterion (HQIC), and consistent Akaike information Criterion (CAIC), are calculated using ML to fit the distribution and evaluate the model's performance. To visually confirm the appropriateness of the fit, graphical diagnostics are also created, including contour plots, scatter plots, joint density surfaces, and marginal density overlays.

7.1. Computer science data

This subsection includes analyses of a real-world data set. The data set relates to $n = 50$ simulated simple computer series systems consisting of a processor and a memory. The data was gathered and analyzed based on Oliveira et al. [36]. The data set contains $n = 50$ simulated rudimentary computer systems with processors and memory. An operating computer will be able to operate when both parts are working properly (the processors and memory). Assume the system is nearing the end of its lifecycle. The degeneration advances rapidly in a short period of time Ahmad et al. [37]. In a short time (in hours), the degeneration advances rapidly. In the case of the first component, a deadly shock can destroy either it or the second component at random, due to the system's greater vulnerability to shocks. The FGM-QEXD is fit to the processor lifetime and memory lifetime separately. As an illustration of the data, A basic statistical analysis (joint scatter plot, JPDF contour, JPDF surcace, empirical copula, and fitting marginal of x and Y) is shown in Figure 9. The ML estimates of the FGM-QEXD parameters as $\hat{\alpha}_1 = 1.353$, $\hat{\beta}_1 = 1.186$, $\hat{\alpha}_2 = 2.825$, $\hat{\beta}_2 = 1.097$, and $\hat{\theta} = 0.376$, respectively. Table 5 clearly shows that, in terms of data fit, the FGM-QEXD model performs better than the bivariate FGM-Lomax distribution (FGM-LD), the bivariate FGM-Inverse Lomax distribution (FGM-ILD), the bivariate FGM-Weibull distribution (FGM-WD), the bivariate FGM-Inverse Weibull distribution (FGM-IWD), the bivariate FGM-Burr XII distribution (FGM-BXD), and the bivariate FGM-Rayleigh distribution (FGM-RD).

Table 5. Goodness of fit for computer datasets.

	$-\ln L$	AIC	AICc	BIC	HQIC	CAIC
FGM-QEXD	149.571	309.142	310.506	318.702	312.783	323.702
FGM-LD	151.412	312.823	314.187	322.383	316.464	327.383
FGM-ILD	165.826	341.653	343.016	351.213	345.293	356.213
FGM-WD	149.915	309.829	311.193	319.389	313.47	324.389
FGM-IWD	188.443	386.886	388.249	396.446	390.526	401.446
FGM-BXD	160.873	331.74	333.109	341.305	335.386	346.305
FGM-RD	194.211	394.423	394.945	400.159	396.607	403.159

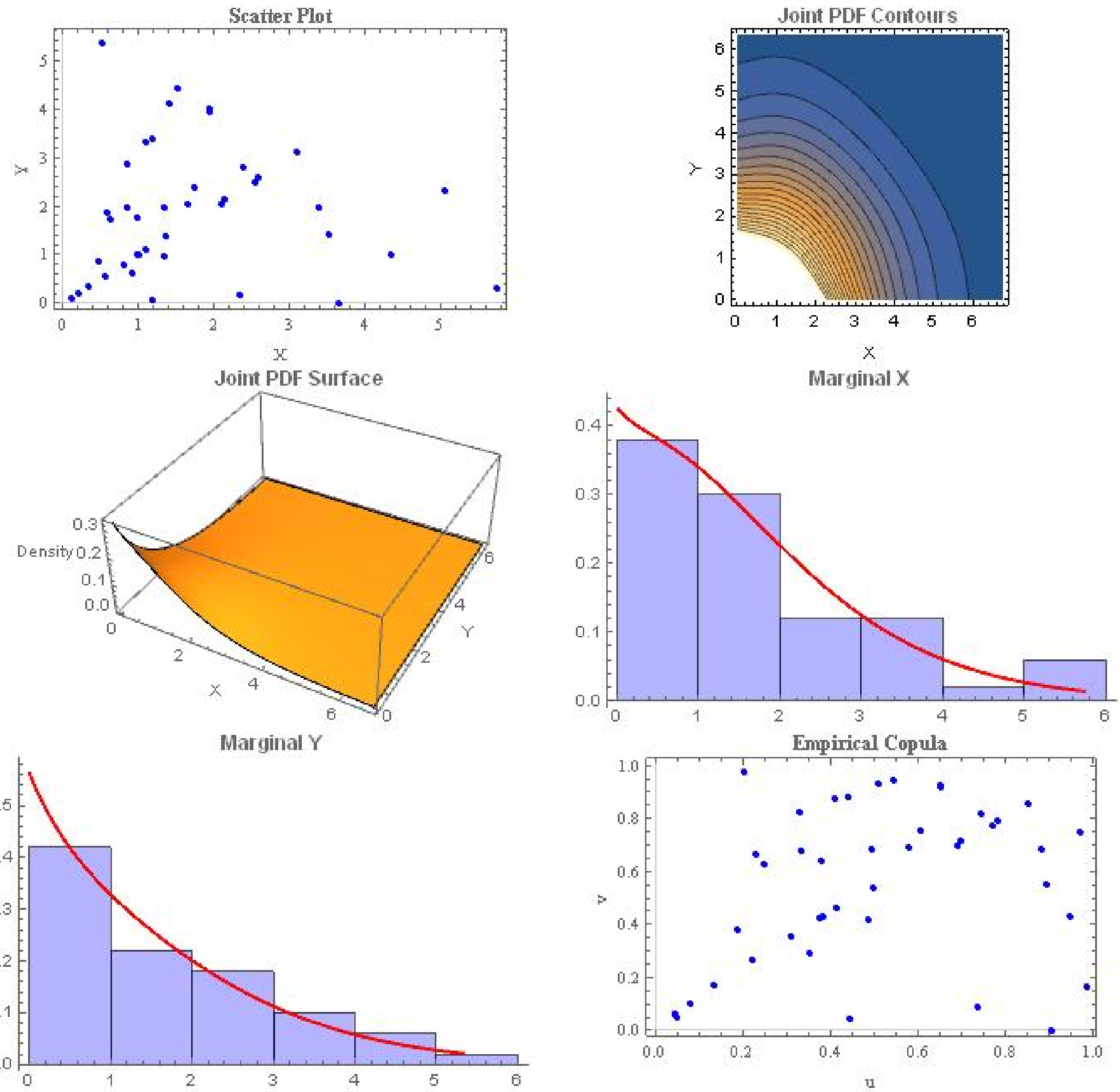

Figure 9. Statistical representation of computer science data.

7.2. *Iron material jobs data*

Dasgupta [38] was the source of the data. A perforation procedure is used for activities involving iron sheets. In particular, an L-shaped rectangular sheet measuring 100 mm by 150 mm is used, with four holes drilled into it, two on each arm. This is done quickly via piercing, which involves employing a 100-ton press that operates at 250 strokes per hour. Two holes are punched at the same time during every operation. Miniature or light-duty truck chassis cannot be completed without these punctured L-shaped iron sheets. After the metal sheet is pierced, a ridge, or burr, is formed around each hole. The metal granules that make up the burr are unevenly raised around the rim of the hole as a result of the high pressure employed in piercing distorting the contact surface. The applied piercing load and the properties of the metal grain determine the burr size. Skrotzki et al. [39] found that composition, melting point, cooling rate, thermal and constitutional undercooling, and convection all affect the grain structure and texture of metal. Finally, the burr is eliminated by chamfering with a drill. The burr measurements for the datasets were obtained using a dial gauge with a minimum resolution of 20 microns (μm), equivalent to 0.02 millimeters. The initial dataset consists of 50 observations of burr measurements, measured in millimeters. The hole diameter is recorded as 12 mm, while the sheet thickness is measured at 3.15 mm. The second set of data similarly includes 50 observations, but this time the sheet thickness is 2 mm and the hole width is 9 mm. For every set, one hole is chosen, turned to face a certain direction, and its diameter is measured from there. Two separate computers are linked to these two databases. For more details on the iron material jobs data, see [7]. X: 0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16. Y: 0.06, 0.12, 0.14, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.22, 0.14, 0.06, 0.04, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.04, 0.14, 0.26, 0.18, 0.16. As an illustration of the data, A basic statistical analysis (joint scatter plot, JPDF contour, JPDF surcace, empirical copula, and fitting marginal of x and Y) is shown in Figure 10. The ML estimates of the FGM-QEXD parameters as $\hat{\alpha}_1 = 0$, $\hat{\beta}_1 = 12.217$, $\hat{\alpha}_2 = 6.4 \times 10^{-8}$, $\hat{\beta}_2 = 11.341, \textit{and } \hat{\theta} = 0.753$, respectively. Table 6 clearly shows that, in terms of data fit, the FGM-QEXD model performs better than the bivariate FGM-LD, bivariate FGM-IWD, bivariate FGM-BXD, bivariate FGM-RD, bivariate FGM-Chen distribution (FGM-CD).

Table 6. Goodness of fit for iron material datasets.

	$-\ln L$	AIC	AICc	BIC	HQIC	CAIC
FGM-QEXD	-91.362	-172.72	-171.36	-163.164	-169.084	-158.164
FGM-LD	-77.994	-145.987	-144.624	-136.427	-142.347	-131.427
FGM-IWD	-71.9757	-133.951	-132.588	-124.391	-130.311	-119.391
FGM-RD	-68.8497	-131.699	-131.178	-125.963	-129.515	-122.963
FGM-CD	-78.7199	-147.44	-146.076	-137.88	-143.799	-132.88

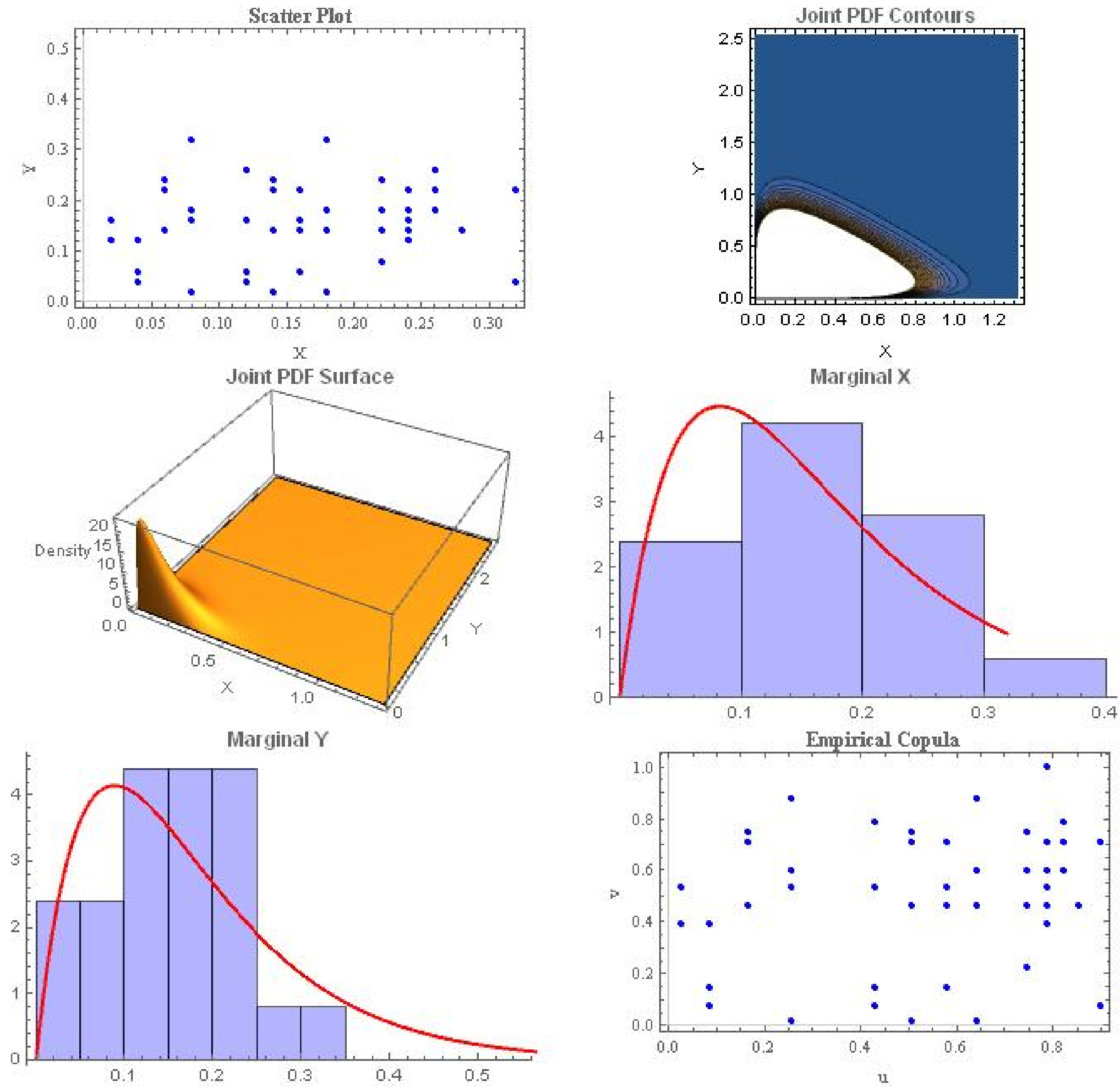

Figure 10. Statistical representation of iron material data.

Remark

The proposed FGM-QEXD copula model was applied to two real data sets and compared its performance with that of several alternative copula models, including Frank, Gumbel, and Clayton copulas (see [7]), as well as the FGM copula combined with a different marginal distribution (see [37]). The comparison was conducted using ML values and the AIC. Across both datasets, the proposed model consistently achieved higher log-likelihood values and notably smaller AIC scores, confirming its superior goodness of fit. The reduction in AIC indicates that the FGM copula with QED marginals provides a more efficient balance between model complexity and explanatory power than the competing models. These results demonstrate that the proposed construction offers a more accurate and parsimonious representation of the joint behavior of the observed data.

8. Conclusions

- By merging the FGM copula with the quadratic exponential marginal distributions, a new bivariate distribution (FGM-QEXD) has been suggested. This model captures modest interdependence between variables and provides an analytically tractable form.
- Product moments, MGF, HR, conditional distributions, expectancies, vital reliability tools (MRL, vitality function, etc.), and internal correlation coefficient structure are all derived from comprehensive statistical features of the FGM-QEXD model.
- This approach is ideal for reliability studies of systems and lifetime analyses since the dependability and reliability functions are both practical and interpretable.
- The FGM-QEXD model outperformed both conventional and modern FGM-based bivariate models in terms of fitting performance when applied to two real datasets, one pertaining to computer science and the other to the strength of iron materials.
- The suggested FGM-QEXD bivariate family possesses numerous practical applications beyond its theoretical attributes. The model effectively incorporates both positive and negative dependence via the parameter θ, making it ideal for the analysis of reliability data where system components may fail in a coupled or compensating fashion. In materials research, such as the examined iron-material dataset, the model can characterize the combined behavior of mechanical characteristics under stress or deterioration. In engineering systems, it can evaluate the reliability of components functioning under common environmental circumstances. In survival and biological research, the model serves as a versatile instrument for concurrently simulating interrelated lifetimes, such as paired organ failure durations or therapy results. The practical capabilities indicate that the FGM-QEXD family is both mathematically manageable and widely applicable to real-world issues concerning dependent lifespan data.

Author contributions

I. A. Husseiny: Conceptualization, writing original draft, formal analysis, software, investigation, methodology, supervision; Abdulrahman M. A. Aldawsari: Validation, resources, writing–review & editing, data curation, methodology; Asamh Saleh M. Al Luhayb: Writing–review & editing, investigation, software, formal analysis; Reid Alotaibi: Conceptualization, formal analysis, writing original draft, software, investigation, methodology, supervision; All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

Data availability

The corresponding author can provide the datasets created and/or studied during the current work upon reasonable request.

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