AIMS Mathematics
https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(9): 21595–21616.
DOI: 10.3934/math.2025960
Received: 08 May 2025
Revised: 20 July 2025
Accepted: 25 July 2025
Published: 18 September 2025

Research article

Oscillatory dynamics and recursive monotonicity of positive solutions to second-order neutral delay differential equations: Analytical framework and applications

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Abstract: This paper studies the oscillatory behavior of a class of second-order neutral delay differential equations (NDDEs). Using comparison principles and Riccati transformation techniques, we derived new sufficient conditions for oscillation. A recursive framework was introduced to enhance the monotonic properties of positive solutions, leading to sharper criteria. Auxiliary inequalities, exponential estimates, and integral averaging methods support the analysis. Illustrative examples demonstrate the applicability and improvements over existing results, positioning our findings within a broader context relevant to engineering and biological models.

Keywords: differential equations; neutral delay differential equations; oscillation; positive solutions; monotonic properties; recursive dynamics; numerical validation; applied modeling

1. Introduction

Second-order functional differential equations, especially neutral delay differential equations (NDDEs), arise naturally in the modeling of dynamic systems with memory and after-effects, such as mechanical vibrations, feedback-controlled circuits, and biological processes [1–3]. Unlike retarded equations, NDDEs involve delays in both the function and its derivative, leading to more complex dynamics and analytical challenges [4, 5]. Understanding the oscillatory behavior of their solutions is essential in assessing system stability, particularly in engineering and biological contexts [6–8]. This work focuses on developing oscillation criteria for NDDEs that can predict qualitative behavior without requiring explicit solutions.

In this study, we focus on analyzing the oscillatory nature of solutions to a class of second-order differential equations that feature a superlinear neutral term. The main equation under consideration is given by

$$(r(\mathfrak{s})(z'(\mathfrak{s}))^{\kappa})' + q(\mathfrak{s})x^{\delta}(\sigma(\mathfrak{s})) = 0, \tag{1.1}$$

where $\mathfrak{s} \geq \mathfrak{s}_0$ and the function $z(\mathfrak{s})$ is defined as

$$z(\mathfrak{s}) := x(\mathfrak{s}) + p(\mathfrak{s})x(\tau(\mathfrak{s})).$$

This equation encompasses a wide range of neutral terms by incorporating both the state variable $x(\mathfrak{s})$ and its delayed form $x(\tau(\mathfrak{s}))$ through the function $p(\mathfrak{s})$. In order to establish the theoretical results, the following assumptions are made throughout the paper:

(H_1) The parameters κ and δ are assumed to be ratios of odd positive integers;
(H_2) The coefficient function $r \in C^1([\mathfrak{s}_0, \infty), \mathbb{R}^+)$, $r' \geq 0$, and satisfies the integral condition

$$\int_{\mathfrak{s}_0}^{\infty} r^{-1/\kappa}(\zeta)\,\mathrm{d}\zeta = \infty;$$

(H_3) The functions q and p belong to $C([\mathfrak{s}_0, \infty), [0, \infty))$, with $0 \leq p(\mathfrak{s}) < 1$ and a constant $p_0 \geq p(\mathfrak{s})$ for all $\mathfrak{s} \geq \mathfrak{s}_0$. Moreover, q does not vanish identically on any half-line $[\mathfrak{s}_*, \infty)$, for some $\mathfrak{s}_* \geq \mathfrak{s}_0$;
(H_4) The delay functions $\tau, \sigma \in C([\mathfrak{s}_0, \infty), \mathbb{R})$ satisfy $\tau(\mathfrak{s}) \leq \mathfrak{s}$, $\sigma(\mathfrak{s}) < \mathfrak{s}$, and both limits

$$\lim_{\mathfrak{s}\to\infty} \tau(\mathfrak{s}) = \infty, \quad \lim_{\mathfrak{s}\to\infty} \sigma(\mathfrak{s}) = \infty$$

hold.

A solution x of Eq (1.1) is understood as a real-valued function $x \in C([\mathfrak{s}_*, \infty), \mathbb{R})$, where $\mathfrak{s}_* \geq \mathfrak{s}_0$, satisfying the conditions that $r(z'(\mathfrak{s}))^{\kappa} \in C^1([\mathfrak{s}_*, \infty), \mathbb{R})$ and that Eq (1.1) holds on the interval $[\mathfrak{s}_*, \infty)$. We restrict our attention to nontrivial solutions, that is, those for which

$$\sup\{|x(\mathfrak{s})| : \mathfrak{s} \geq \mathfrak{s}_*\} > 0.$$

Such a solution x is said to be *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*. This classification is critical in understanding the qualitative behavior of solutions, particularly in determining whether oscillatory phenomena dominate the long-term dynamics of the modeled system.

The study of oscillatory behavior in differential equations plays a pivotal role in analyzing the stability and dynamic response of numerous systems in physics, biology, and engineering. Oscillation reflects the system's inherent sensitivity to initial conditions and external disturbances, which is a critical factor in anticipating long-term system behavior. Rather than solving the equations directly, researchers often employ powerful analytical tools—such as comparison theorems and Riccati transformation techniques—to derive sufficient conditions for oscillation. These methods provide valuable insight into the qualitative nature of solutions. The importance of oscillation theory is particularly evident in real-world applications such as population dynamics, automatic control systems, and mechanical vibration analysis [9, 10].

The exploration of oscillatory phenomena in differential equations involving delays, particularly neutral delays, has seen significant advancement over the past decades. Foundational work laid down essential criteria to determine the existence of oscillatory solutions in systems governed by various forms of delayed arguments [11–13]. As the field progressed, attention turned toward neutral differential equations—systems in which delays affect not only the function but also its derivative—thereby introducing additional analytical challenges and complexities [14, 15]. To address these, mathematicians developed a suite of refined analytical tools, including comparison principles, integral averaging methods, and Riccati-type transformations, which allow oscillatory behavior to be investigated without the need for explicit solutions [16–18]. More recent studies have aimed to generalize and sharpen these criteria, establishing more comprehensive and flexible conditions—both sufficient and necessary—for oscillation in increasingly complex and realistic models [19, 20].

In a seminal contribution from 1978, Brands [21] investigated a class of delay differential equations (DDEs) of the form

$$x''(\mathfrak{s}) + q(\mathfrak{s})\, x(\mathfrak{s} - \sigma) = 0,$$

and demonstrated that the oscillatory nature of its solutions is equivalent to the oscillatory behavior of solutions to the corresponding non-delayed second-order equation:

$$x''(\mathfrak{s}) + q(\mathfrak{s})\, x(\mathfrak{s}) = 0.$$

This equivalence provided a foundation for subsequent studies into more general forms of delay differential equations. A notable advancement was introduced by Ladas et al. [22], who extended the oscillation theory from first-order to second-order DDEs of the form

$$x''(\mathfrak{s}) + q(\mathfrak{s})\, x(\sigma(\mathfrak{s})) = 0,$$

offering refined criteria that improved the understanding of when such equations exhibit oscillatory solutions.

As research progressed, Riccati transformation techniques gained prominence due to their effectiveness in converting higher-order delay differential equations (DDEs) into equivalent first-order forms. This simplification facilitates the application of classical oscillation theory to more complex equations. Utilizing this method, Džurina and Stavroulakis [23] developed an oscillation criterion tailored for half-linear DDEs of the form

$$\left(r(\mathfrak{s})\left(x'(\mathfrak{s})\right)^{\kappa}\right)' + q(\mathfrak{s})\, x^{\kappa}(\sigma(\mathfrak{s})) = 0,$$

where the oscillatory nature of solutions was characterized in terms of integral conditions involving the delay function $\sigma(\mathfrak{s})$ and the coefficient function $q(\mathfrak{s})$. Their approach provided a critical link between the structure of the equation and its qualitative behavior. By the mid-2000s, the investigation of neutral delay differential equations (NDDEs) evolved toward more generalized formulations. Xu and Meng [24] broadened earlier oscillation criteria by considering the neutral DDE

$$\left(r(\mathfrak{s})\left(z'(\mathfrak{s})\right)^{\kappa}\right)' + q(\mathfrak{s})\, x^{\kappa}(\sigma(\mathfrak{s})) = 0, \tag{1.2}$$

which allowed both the delay function $\sigma(\mathfrak{s})$ and the neutral coefficient $p(\mathfrak{s})$ to vary with time. This generalization marked a significant step toward capturing more realistic dynamics in complex systems.

Further progress was made by Grace et al. [25], who improved the oscillation criteria associated with Eq (1.2) by introducing new sufficient conditions that account for variable delays and coefficients. Concurrently, Baculikova and Džurina [26] focused on second-order Emden-Fowler-type

NDDEs, employing comparison theorems to derive oscillation results. Their work provided insight into solution behavior, particularly in the case when both parameters κ and δ equal one in Eq (1.1), thus addressing an important subclass of nonlinear delay models.

More recently, Chatzarakis et al. [27] contributed to the theory of oscillation by developing results for non-canonical DDEs, particularly in scenarios where the reciprocal power $r^{-1/\kappa}(\mathfrak{s})$ is integrable over an infinite interval. These results advanced the understanding of how the integrability of coefficient functions affects solution behavior. Building upon this foundation, Bohner et al. [28] extended the analysis to include both canonical and non-canonical forms of second-order NDDEs. Their work introduced significant refinements to existing oscillation criteria, emphasizing the role of new functional constructs and integral conditions that guarantee oscillatory behavior of solutions.

The most recent advancements in this area were made by Essam et al. [29], who broadened the oscillation framework further by examining equations of the form

$$\left(r\left(\mathfrak{s}\right)\left(x'\left(\mathfrak{s}\right)\right)^{\kappa}\right)' + \sum_{i=1}^{m} q_i\left(\mathfrak{s}\right) y^{\delta}\left(\sigma_i\left(\mathfrak{s}\right)\right) = 0,$$

for $m \in \mathbb{Z}^{+}$. Their study introduced additional delay-dependent terms and conditions, resulting in sharper and more comprehensive oscillation criteria. These refinements enhance the applicability of the theory to more complex dynamical systems with multiple interacting delays.

Beyond the broad classifications of differential equations, many researchers have concentrated their efforts on the oscillatory nature of solutions across various orders and structural types. For instance, the works in [30–32] investigated third-order delay differential equations, emphasizing how time-dependent delays influence the onset of oscillations. Meanwhile, studies such as [33, 35] focused on even-order neutral delay differential equations, delivering essential findings that deepen our understanding of oscillation mechanisms in these more intricate frameworks. The contributions of [36] further extended the theoretical landscape by tackling higher-order equations and proposing novel oscillation criteria applicable to nonlinear dynamical systems. Additionally, the investigations in [37–40] addressed differential equations with advanced arguments, offering a comprehensive theoretical structure to capture the dynamic behaviors unique to such equations. Collectively, these studies underscore the breadth and depth of oscillation theory, with each contribution enriching the analytical tools and criteria used to characterize solution behaviors across diverse classes of differential equations.

In this work, we aim to broaden the existing literature by analyzing the oscillatory behavior of solutions to Eq (1.1) through multiple methodological perspectives. Specifically, we apply comparison techniques with first-order equations alongside the Riccati transformation approach. Following this, we introduce refined analytical conditions that deepen the relationship between the dependent variable, its derivatives, and the associated functional terms. These refinements establish a recursive structure within the oscillation criteria, enhancing their robustness over time and ensuring their effectiveness across broader settings.

The remainder of this paper is structured as follows. Section 2 provides key definitions, auxiliary functions, and lemmas refining solution and derivative relations. In Section 3, we establish a series of oscillation theorems for second-order neutral delay differential equations (NDDEs), employing both comparison principles and Riccati transformation techniques. Building upon these core results, we enhance the oscillation criteria by introducing recursive constructs and refined functional estimates that incorporate the interplay between delay terms and their derivatives. In Section 4, we present a detailed discussion of our theoretical findings alongside a collection of illustrative examples designed to validate and highlight the novelty of the proposed results. The paper concludes with Section 5, which offers a synthesis of our contributions and outlines potential directions for future research in the theory of NDDEs. Compared to previous studies (e.g., [12, 21, 25]), our approach provides greater flexibility through the use of non-constant delays and neutral coefficients. The integration of recursive inequalities and integral averaging techniques reflects a methodological advancement aligned with recent developments in the field of oscillation theory.

2. Preliminaries and foundational lemmas

To facilitate the development of our main results, we introduce several auxiliary functions and constructs that simplify expressions and highlight key qualitative features of the solutions. These tools are essential for deriving foundational estimates and lemmas that support the analytic framework of the NDDE under study. We begin by defining the following quantities:

$$R(\mathfrak{s}) := \int_{\mathfrak{s}_1}^{\mathfrak{s}} r^{-1/\kappa}(\zeta)\, \mathrm{d}\zeta,$$

$$Q(\mathfrak{s}) := (1 - p(\sigma(\mathfrak{s})))^{\delta} q(\mathfrak{s}),$$

$$\tilde{R}_1(\mathfrak{s}) := R(\mathfrak{s}, \mathfrak{s}_1) + \frac{\epsilon^{\delta}}{\kappa} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \left(\frac{\sigma(\zeta)}{\zeta}\right)^{\delta} R^{1+\kappa}(\zeta, \mathfrak{s}_1)\, \pi(\zeta) Q(\zeta)\, \mathrm{d}\zeta,$$

$$\tilde{R}_2(\mathfrak{s}) := R(\mathfrak{s}, \mathfrak{s}_1) + \frac{K_4^{\delta-\kappa}}{\kappa} \int_{\mathfrak{s}_1}^{\mathfrak{s}} R(\zeta, \mathfrak{s}_1)\, R^{\kappa}(\sigma(\zeta), \mathfrak{s}_1)\, Q(\zeta)\, \mathrm{d}\zeta,$$

$$\hat{R}(\mathfrak{s}) := Q(\mathfrak{s})\pi(\mathfrak{s}) \exp\left(-\kappa \int_{\sigma(\mathfrak{s})}^{\mathfrak{s}} \frac{r^{-1/\kappa}(\zeta)}{\tilde{R}_1(\zeta)}\, \mathrm{d}\zeta\right),$$

for all $\mathfrak{s} \geq \mathfrak{s}_1 \geq \mathfrak{s}_0$ and for any $\epsilon \in (0, 1)$.

Lemma 1. [Lemma 2.3, [44]] *Let $g(u) = Au - Bu^{(\kappa+1)/\kappa}$, where A and $B > 0$ are constants, and κ is the quotient of two odd natural numbers. Then g attains its maximum value on $\mathbb{R}$ at*

$$u^* = \left(\frac{\kappa A}{(\kappa + 1)B}\right)^{\kappa},$$

and the maximum value is given by

$$\max_{u\in\mathbb{R}} g(u) = g(u^*) = \frac{\kappa^{\kappa}}{(\kappa + 1)^{\kappa+1}} \cdot \frac{A^{\kappa+1}}{B^{\kappa}}. \tag{2.1}$$

Lemma 2. [Lemma 3, [45]] *Suppose that x is a positive solution of* (1.1) *on* $[\mathfrak{s}_0, \infty)$ *such that the associated function z is positive and increasing, and z' is positive and non-increasing. Then, there exist $\epsilon \in (0, 1)$ and $\mathfrak{s}_{\epsilon} \geq \mathfrak{s}_0$ such that*

$$\mathfrak{s} \cdot z(\sigma(\mathfrak{s})) \geq \epsilon \cdot \sigma(\mathfrak{s}) \cdot z(\mathfrak{s}), \tag{2.2}$$

for all sufficiently large $\mathfrak{s}$.

Lemma 3. [Theorem 2.1, [46]] *Assume that x is an increasing positive solution of* (1.1) *on* $[\mathfrak{s}_0, \infty)$. *Then it holds that*

$$z^{\delta-\kappa}(\mathfrak{s}) \geq \pi(\mathfrak{s}),$$

where the function $\pi(\mathfrak{s})$ is defined as

$$\pi(\mathfrak{s}) := \begin{cases} m_1, & \text{if } \kappa < \delta, \\ 1, & \text{if } \kappa = \delta, \\ m_2 R^{\delta-\kappa}(\mathfrak{s}, \mathfrak{s}_0), & \text{if } \kappa > \delta, \end{cases}$$

with m_1 and m_2 being positive real constants.

Now, we proceed to classify the positive solutions x of Eq (1.1).

Lemma 4. *Assume that x is a positive solution of Eq* (1.1) *on the interval* $[\mathfrak{s}_0, \infty)$ *and that hypothesis* (H_2) *holds eventually. Then, there exists a constant $\mathfrak{s}_1 \geq \mathfrak{s}_0$ such that, for all $\mathfrak{s} \geq \mathfrak{s}_1$, the following inequalities are satisfied:*

$$z(\mathfrak{s}) > 0, \quad z'(\mathfrak{s}) > 0, \quad \text{and} \quad \left(r(\mathfrak{s})\left(z'(\mathfrak{s})\right)^{\kappa}\right)' \leq 0. \tag{2.3}$$

Proof. Let x be a positive solution of Eq (1.1) on $[\mathfrak{s}_0, \infty)$. Then, from (1.1), we have

$$\left(r(\mathfrak{s})\left(z'(\mathfrak{s})\right)^{\kappa}\right)' = -q(\mathfrak{s})x^{\delta}(\sigma(\mathfrak{s})) \leq 0.$$

This shows that the function $r(\mathfrak{s})\left(z'(\mathfrak{s})\right)^{\kappa}$ is non-increasing. As a consequence, $z'(\mathfrak{s})$ must eventually be of constant sign—either strictly positive or strictly negative.

Suppose, for the sake of contradiction, that $z'(\mathfrak{s}) < 0$ eventually. Then there exists a constant $K_1 > 0$ such that

$$r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s}) \leq -K_1^{1/\kappa}.$$

Integrating this inequality over the interval $[\mathfrak{s}_1, \mathfrak{s}]$ yields

$$z(\mathfrak{s}) - z(\mathfrak{s}_1) \leq -K_1^{1/\kappa} \int_{\mathfrak{s}_1}^{\mathfrak{s}} r^{-1/\kappa}(\zeta)\, \mathrm{d}\zeta.$$

Taking the limit as $\mathfrak{s} \to \infty$ and applying condition (H_2), we observe that the integral diverges, leading to $z(\mathfrak{s}) \to -\infty$, which contradicts the positivity of z on $[\mathfrak{s}_0, \infty)$. Therefore, $z'(\mathfrak{s})$ must be eventually positive, completing the proof. □

Lemma 5. *Assume that x is a positive solution of Eq* (1.1) *on the interval* $[\mathfrak{s}_0, \infty)$. *Then, the following inequalities hold eventually:*

- (P_1) $x(\mathfrak{s}) \geq (1 - p(\mathfrak{s}))\, z(\mathfrak{s})$;
- (P_2) $\left(r(\mathfrak{s})\left(z'(\mathfrak{s})\right)^{\kappa}\right)' + Q(\mathfrak{s})z^{\delta}(\sigma(\mathfrak{s})) \leq 0$;
- (P_3) $z'(\mathfrak{s}) - r^{-1/\kappa}(\mathfrak{s})R^{-1}(\mathfrak{s}, \mathfrak{s}_1)z(\mathfrak{s}) \leq 0$;
- (P_4) *The function* $\dfrac{z(\mathfrak{s})}{R(\mathfrak{s}, \mathfrak{s}_1)}$ *is non-increasing.*

Proof. Let x be a positive solution of Eq (1.1) on $[\mathfrak{s}_0, \infty)$. For $\mathfrak{s} \geq \mathfrak{s}_1$, using the definition of z, we have:

$$x(\mathfrak{s}) = z(\mathfrak{s}) - p(\mathfrak{s})x(\tau(\mathfrak{s})) \geq z(\mathfrak{s}) - p(\mathfrak{s})z(\tau(\mathfrak{s})) \geq (1 - p(\mathfrak{s}))z(\mathfrak{s}),$$

completing the proof of (P_1).

Substituting into (1.1) and using the result from (P_1), we obtain:

$$\begin{aligned}\left(r(\mathfrak{s})\left(z'(\mathfrak{s})\right)^{\kappa}\right)' &= -q(\mathfrak{s})x^{\delta}(\sigma(\mathfrak{s}))\\ &\leq -q(\mathfrak{s})\left(1 - p(\sigma(\mathfrak{s}))\right)^{\delta} z^{\delta}(\sigma(\mathfrak{s}))\\ &= -Q(\mathfrak{s})z^{\delta}(\sigma(\mathfrak{s})).\end{aligned}$$

To prove (P_3), observe that the monotonicity of $r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})$ implies:

$$z(\mathfrak{s}) = \int_{\mathfrak{s}_1}^{\mathfrak{s}} z'(\zeta)\, d\zeta \geq \int_{\mathfrak{s}_1}^{\mathfrak{s}} r^{-1/\kappa}(\zeta)r^{1/\kappa}(\zeta)z'(\zeta)\, d\zeta \geq r^{1/\kappa}(\mathfrak{s})R(\mathfrak{s}, \mathfrak{s}_1)z'(\mathfrak{s}),$$

which gives

$$z'(\mathfrak{s}) - r^{-1/\kappa}(\mathfrak{s})R^{-1}(\mathfrak{s}, \mathfrak{s}_1)z(\mathfrak{s}) \leq 0.$$

For (P_4), consider the derivative:

$$\begin{aligned}\left(\frac{z(\mathfrak{s})}{R(\mathfrak{s}, \mathfrak{s}_1)}\right)' &= \frac{R(\mathfrak{s}, \mathfrak{s}_1)z'(\mathfrak{s}) - z(\mathfrak{s})r^{-1/\kappa}(\mathfrak{s})}{R^2(\mathfrak{s}, \mathfrak{s}_1)}\\ &= \frac{r^{-1/\kappa}(\mathfrak{s})}{R^2(\mathfrak{s}, \mathfrak{s}_1)}\left[r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})R(\mathfrak{s}, \mathfrak{s}_1) - z(\mathfrak{s})\right] \leq 0,\end{aligned}$$

which confirms the non-increasing nature of $\dfrac{z(\mathfrak{s})}{R(\mathfrak{s}, \mathfrak{s}_1)}$ and thus proves (P_4). □

Lemma 6. *Assume that x is a positive solution of Eq* (1.1) *on the interval* $[\mathfrak{s}_0, \infty)$*, and that hypothesis* (H_2) *is satisfied. Suppose further that σ is a strictly increasing function and $\mu > 0$. If the inequality*

$$\int_{\sigma(\mathfrak{s})}^{\mathfrak{s}} Q(\zeta)\tilde{R}_1^{\delta}\left(\sigma(\zeta)\right)\, \mathrm{d}\zeta \geq \mu \tag{2.4}$$

holds, and the sequence $\{f_i(\mu)\}_{i=0}^{\infty}$ is defined recursively by

$$f_0(\mu) := 1, \quad f_{i+1}(\mu) := \exp\left(\mu f_i(\mu)\right), \quad i = 0, 1, 2, \ldots, \tag{2.5}$$

then the following inequality is satisfied eventually:

$$\frac{r\left(\sigma(\mathfrak{s})\right)\left(z'\left(\sigma(\mathfrak{s})\right)\right)^{\kappa}}{r(\mathfrak{s})\left(z'(\mathfrak{s})\right)^{\kappa}} \geq f_i(\mu). \tag{2.6}$$

Proof. Assume that x is a positive solution of (1.1) on $[\mathfrak{s}_0, \infty)$. Consequently, $x(\tau(\mathfrak{s}))$ and $x(\sigma(\mathfrak{s}))$ are also positive for all $\mathfrak{s} \geq \mathfrak{s}_1$. For simplicity, define

$$W_1 := z(\mathfrak{s}) - R(\mathfrak{s}, \mathfrak{s}_1)r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s}).$$

It is straightforward to compute its derivative:

$$W_1' = -\left(r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})\right)' R(\mathfrak{s},\mathfrak{s}_1). \tag{2.7}$$

Applying the chain rule and using part (P_2) of Lemma 5, we obtain:

$$\begin{aligned} -Q(\mathfrak{s})R(\mathfrak{s},\mathfrak{s}_1)z^{\delta}(\sigma(\mathfrak{s})) &\geq (r(\mathfrak{s})z'(\mathfrak{s})^{\kappa})' R(\mathfrak{s},\mathfrak{s}_1) \\ &= \kappa\left(r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})\right)' R(\mathfrak{s},\mathfrak{s}_1)r^{1-1/\kappa}(\mathfrak{s})\left(z'(\mathfrak{s})\right)^{\kappa-1}, \end{aligned}$$

which implies that

$$-\left(r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})\right)' \geq \frac{1}{\kappa}Q(\mathfrak{s})\left(r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})\right)^{1-\kappa} z^{\delta}(\sigma(\mathfrak{s})).$$

Substituting this inequality into (2.7), we get:

$$W_1' \geq \frac{1}{\kappa}Q(\mathfrak{s})R(\mathfrak{s},\mathfrak{s}_1)\left(r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})\right)^{1-\kappa} z^{\delta}(\sigma(\mathfrak{s})). \tag{2.8}$$

Now, by invoking hypothesis (H_2) and Lemma 2, we deduce

$$W_1' \geq \frac{\epsilon^{\delta}}{\kappa}\left(\frac{\sigma(\mathfrak{s})}{\mathfrak{s}}\right)^{\delta} Q(\mathfrak{s})R(\mathfrak{s},\mathfrak{s}_1)\left(r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})\right)^{1-\kappa} z^{\delta}(\mathfrak{s}),$$

and by applying part (P_3) of Lemma 5, we find:

$$W_1' \geq \frac{\epsilon^{\delta}}{\kappa}\left(\frac{\sigma(\mathfrak{s})}{\mathfrak{s}}\right)^{\delta} Q(\mathfrak{s})R^{1+\kappa}(\mathfrak{s},\mathfrak{s}_1)r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})z^{\delta-\kappa}(\mathfrak{s}).$$

Furthermore, using Lemma 3, we obtain:

$$W_1' \geq \frac{\epsilon^{\delta}}{\kappa}\left(\frac{\sigma(\mathfrak{s})}{\mathfrak{s}}\right)^{\delta} Q(\mathfrak{s})R^{1+\kappa}(\mathfrak{s},\mathfrak{s}_1)r^{1/\kappa}(\mathfrak{s})\pi(\mathfrak{s})z'(\mathfrak{s}),$$

eventually. Integrating both sides of the above inequality over $[\mathfrak{s}_1,\mathfrak{s}]$, we get:

$$\begin{aligned} z(\mathfrak{s}) &\geq R(\mathfrak{s},\mathfrak{s}_1)r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s}) \\ &\quad + \int_{\mathfrak{s}_1}^{\mathfrak{s}} \frac{\epsilon^{\delta}}{\kappa}\left(\frac{\sigma(\zeta)}{\zeta}\right)^{\delta} Q(\zeta)R^{1+\kappa}(\zeta,\mathfrak{s}_1)r^{1/\kappa}(\zeta)\pi(\zeta)z'(\zeta)\,\mathrm{d}\zeta \\ &\geq r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})\left[R(\mathfrak{s},\mathfrak{s}_1) + \frac{\epsilon^{\delta}}{\kappa}\int_{\mathfrak{s}_1}^{\mathfrak{s}}\left(\frac{\sigma(\zeta)}{\zeta}\right)^{\delta} R^{1+\kappa}(\zeta,\mathfrak{s}_1)\pi(\zeta)Q(\zeta)\,\mathrm{d}\zeta\right]. \end{aligned}$$

Thus,

$$z^{\delta}(\mathfrak{s}) \geq \left(\tilde{R}_1(\mathfrak{s})r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})\right)^{\delta}. \tag{2.9}$$

Substituting this result into part (P_2) of Lemma 5, we arrive at:

$$(r(\mathfrak{s})(z'(\mathfrak{s}))^{\kappa})' + Q(\mathfrak{s})\left(\tilde{R}_1(\sigma(\mathfrak{s}))r^{1/\kappa}(\sigma(\mathfrak{s}))z'(\sigma(\mathfrak{s}))\right)^{\delta} \leq 0. \tag{2.10}$$

Finally, Lemma 4 guarantees that $r(\mathfrak{s})(z'(\mathfrak{s}))^{\kappa}$ is a positive solution of (2.10). The rest of the proof follows identically from Lemma 1 in [47], which completes the argument. □

3. Main oscillation results

In this section, we present our main oscillation theorems, which are derived using two principal methods: the comparison principle applied to associated first-order DDEs, and the Riccati transformation technique. These foundational results establish baseline oscillation criteria.

The following results refine and generalize earlier criteria by incorporating variable delays and constructing sharper bounds based on novel functional transformations.

Theorem 1. *Suppose that* $r'(\mathfrak{s}) > 0$*. If the first-order delay differential equation*

$$U'(\mathfrak{s}) + Q(\mathfrak{s})\tilde{R}_1^{\delta}(\sigma(\mathfrak{s}))U^{\delta/\kappa}(\sigma(\mathfrak{s})) = 0 \tag{3.1}$$

is oscillatory, then every solution of the second-order Eq (1.1) *is also oscillatory.*

Proof. Assume, for contradiction, that x is a positive solution of (1.1) on $[\mathfrak{s}_0, \infty)$. Define $U(\mathfrak{s}) := r(\mathfrak{s})(z'(\mathfrak{s}))^{\kappa}$. Substituting into inequality (2.10) from Lemma 6 yields

$$U'(\mathfrak{s}) + Q(\mathfrak{s})\tilde{R}_1^{\delta}(\sigma(\mathfrak{s}))U^{\delta/\kappa}(\sigma(\mathfrak{s})) \leq 0.$$

However, according to Theorem 1 in [11], this inequality admits a positive solution, which contradicts the assumed oscillatory nature of Eq (3.1). Hence, the assumption must be false, and all solutions of (1.1) must be oscillatory. This completes the proof. □

To capture a wider range of oscillatory behaviors, we consider different sets of hypotheses for the Riccati equation. Each framework allows for the derivation of distinct criteria suited to various structural forms of the NDDEs, thereby enhancing the generality and applicability of the results.

Theorem 2. *Let* $K_4 > 0$*. If the first-order delay differential equation*

$$U'(\mathfrak{s}) + Q(\mathfrak{s})\tilde{R}_2^{\delta}(\sigma(\mathfrak{s}))U^{\delta/\kappa}(\sigma(\mathfrak{s})) = 0$$

is oscillatory, then every solution of Eq (1.1) *also oscillates.*

Proof. Suppose, on the contrary, that x is a positive solution of (1.1) on $[\mathfrak{s}_0, \infty)$. Then $x(\tau(\mathfrak{s}))$ and $x(\sigma(\mathfrak{s}))$ are also positive for all $\mathfrak{s} \geq \mathfrak{s}_1$. Using inequality (2.8) and part (P_3) of Lemma 5, we obtain:

$$\begin{aligned} W_1' &\geq \frac{1}{\kappa}Q(\mathfrak{s})R(\mathfrak{s},\mathfrak{s}_1)R^{\delta}(\sigma(\mathfrak{s}),\mathfrak{s}_1)\left(r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})\right)^{1-\kappa}\left(r^{1/\kappa}(\sigma(\mathfrak{s}))z'(\sigma(\mathfrak{s}))\right)^{\delta} \\ &\geq \frac{1}{\kappa}Q(\mathfrak{s})R(\mathfrak{s},\mathfrak{s}_1)R^{\delta}(\sigma(\mathfrak{s}),\mathfrak{s}_1)\left(r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})\right)^{1+\delta-\kappa}. \end{aligned} \tag{3.2}$$

Since $r^{1/\kappa}z'$ is a positive, non-increasing function, there exists a constant $K_2 > 0$ such that $r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s}) \leq K_2$. If $\kappa > \delta$, then

$$\left(r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s})\right)^{\delta-\kappa} \geq K_2^{\delta-\kappa},$$

and inequality (3.2) becomes:

$$W_1' \geq \frac{K_2^{\delta-\kappa}}{\kappa}Q(\mathfrak{s})R(\mathfrak{s},\mathfrak{s}_1)R^{\delta}(\sigma(\mathfrak{s}),\mathfrak{s}_1)r^{1/\kappa}(\mathfrak{s})z'(\mathfrak{s}). \tag{3.3}$$

On the other hand, if $\kappa \leq \delta$, the monotonicity of z implies that there exists $K_3 > 0$ such that $z(\mathfrak{s}) \geq K_3$, and we obtain:

$$\begin{aligned} W_1' &\geq \frac{K_3^{\delta-\kappa}}{\kappa} Q(\mathfrak{s}) R(\mathfrak{s}, \mathfrak{s}_1) \left(r^{1/\kappa}(\mathfrak{s}) z'(\mathfrak{s}) \right)^{1-\kappa} z^{\kappa}(\sigma(\mathfrak{s})) \\ &\geq \frac{K_3^{\delta-\kappa}}{\kappa} Q(\mathfrak{s}) R(\mathfrak{s}, \mathfrak{s}_1) R^{\kappa}(\sigma(\mathfrak{s}), \mathfrak{s}_1) r^{1/\kappa}(\mathfrak{s}) z'(\mathfrak{s}). \end{aligned} \tag{3.4}$$

Define $K_4 := \min\{K_2, K_3\}$. Then, combining both cases, we deduce:

$$W_1' \geq \frac{K_4^{\delta-\kappa}}{\kappa} Q(\mathfrak{s}) R(\mathfrak{s}, \mathfrak{s}_1) R^{\kappa}(\sigma(\mathfrak{s}), \mathfrak{s}_1) r^{1/\kappa}(\mathfrak{s}) z'(\mathfrak{s}).$$

Proceeding analogously to the proof of Lemma 6 and Theorem 1, we conclude that the original Eq (1.1) must oscillate. This contradiction completes the proof. □

Theorem 3. *Assume that* $\sigma' > 0$, *condition* (2.4) *holds for all* $\mu > 0$, *and let* θ *be a differentiable and continuous real-valued function on* $[\mathfrak{s}_0, \infty)$. *If*

$$\limsup_{\mathfrak{s}\to\infty} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \left(\theta(\zeta) Q(\zeta) - \left(\frac{\theta_+'(\zeta)}{\kappa+1} \right)^{\kappa+1} \left(\frac{\kappa}{\delta\theta(\zeta)\sigma'(\zeta)} \right)^{\kappa} \frac{r(\sigma(\zeta))}{f_i(\mu)\pi(\sigma(\zeta))} \right) d\zeta = \infty, \tag{3.5}$$

where $\theta_+'(\mathfrak{s}) := \max\{0, \theta'(\mathfrak{s})\}$, $f_i(\mu)$ *is defined by* (2.5), *and* $i = 0, 1, 2, \ldots$, *then all solutions of Eq* (1.1) *are oscillatory.*

Proof. Suppose, on the contrary, that x is a positive solution of (1.1) on $[\mathfrak{s}_0, \infty)$. Then $x(\tau(\mathfrak{s}))$ and $x(\sigma(\mathfrak{s}))$ remain positive for $\mathfrak{s} \geq \mathfrak{s}_1$. Define the auxiliary function:

$$W_2(\mathfrak{s}) := \theta(\mathfrak{s}) r(\mathfrak{s}) \frac{(z'(\mathfrak{s}))^{\kappa}}{z^{\delta}(\sigma(\mathfrak{s}))}.$$

Differentiating W_2 and applying part (P_2) of Lemma 5, we get:

$$W_2'(\mathfrak{s}) = \frac{\theta'(\mathfrak{s})}{\theta(\mathfrak{s})} W_2(\mathfrak{s}) + \theta(\mathfrak{s}) \frac{(r(\mathfrak{s})(z'(\mathfrak{s}))^{\kappa})'}{z^{\delta}(\sigma(\mathfrak{s}))} - \delta\theta(\mathfrak{s})\sigma'(\mathfrak{s}) r(\mathfrak{s}) \frac{(z'(\mathfrak{s}))^{\kappa+1}}{z^{\delta+1}(\sigma(\mathfrak{s}))} \frac{z'(\sigma(\mathfrak{s}))}{z'(\mathfrak{s})}.$$

Applying Lemma 3, we obtain:

$$z^{\kappa-\delta}(\mathfrak{s}) \leq \pi^{-1}(\mathfrak{s}) \quad \Rightarrow \quad W_2(\mathfrak{s}) \leq \theta(\mathfrak{s}) r(\mathfrak{s}) \left(\frac{z'(\mathfrak{s})}{z(\sigma(\mathfrak{s}))} \right)^{\kappa} \pi^{-1}(\mathfrak{s}).$$

Also, it follows that:

$$\theta^{-1/\kappa}(\mathfrak{s}) r^{-1/\kappa}(\mathfrak{s}) \pi^{1/\kappa}(\sigma(\mathfrak{s})) W_2^{1+1/\kappa}(\mathfrak{s}) \leq \frac{z'(\mathfrak{s})}{z(\sigma(\mathfrak{s}))} W_2(\mathfrak{s}) = \theta(\mathfrak{s}) r(\mathfrak{s}) \frac{(z'(\mathfrak{s}))^{\kappa+1}}{z^{\delta+1}(\sigma(\mathfrak{s}))}.$$

Thus, the inequality becomes:

$$W_2'(\mathfrak{s}) \leq \frac{\theta'(\mathfrak{s})}{\theta(\mathfrak{s})} W_2(\mathfrak{s}) - \theta(\mathfrak{s}) Q(\mathfrak{s}) - \delta\sigma'(\mathfrak{s}) \theta^{-1/\kappa}(\mathfrak{s}) r^{-1/\kappa}(\mathfrak{s}) \pi^{1/\kappa}(\sigma(\mathfrak{s})) W_2^{1+1/\kappa}(\mathfrak{s}) \frac{z'(\sigma(\mathfrak{s}))}{z'(\mathfrak{s})}.$$

From Lemma 6, we know:

$$\frac{z'(\sigma(\mathfrak{s}))}{z'(\mathfrak{s})} \geq f_i^{1/\kappa}(\mu)\left(\frac{r(\mathfrak{s})}{r(\sigma(\mathfrak{s}))}\right)^{1/\kappa}.$$

Substituting into the previous inequality, we obtain:

$$W_2'(\mathfrak{s}) \leq \frac{\theta'(\mathfrak{s})}{\theta(\mathfrak{s})}W_2(\mathfrak{s}) - \theta(\mathfrak{s})Q(\mathfrak{s}) - \delta\sigma'(\mathfrak{s})\left(\frac{f_i(\mu)\pi(\sigma(\mathfrak{s}))}{\theta(\mathfrak{s})r(\sigma(\mathfrak{s}))}\right)^{1/\kappa} W_2^{(1+1/\kappa)}(\mathfrak{s}).$$

Now, by applying Lemma 1 with:

$$A := \frac{\theta'_+(\mathfrak{s})}{\theta(\mathfrak{s})}, \quad B := \delta\sigma'(\mathfrak{s})\left(\frac{f_i(\mu)\pi(\sigma(\mathfrak{s}))}{\theta(\mathfrak{s})r(\sigma(\mathfrak{s}))}\right)^{1/\kappa},$$

we get:

$$W_2'(\mathfrak{s}) \leq -\theta(\mathfrak{s})Q(\mathfrak{s}) + \left(\frac{\theta'_+(\mathfrak{s})}{\kappa+1}\right)^{\kappa+1}\left(\frac{\kappa}{\delta\theta(\mathfrak{s})\sigma'(\mathfrak{s})}\right)^{\kappa}\frac{r(\sigma(\mathfrak{s}))}{f_i(\mu)\pi(\sigma(\mathfrak{s}))}.$$

Integrating from $\mathfrak{s}_1$ to $\mathfrak{s}$, we conclude that

$$W_2(\mathfrak{s}_1) \geq \int_{\mathfrak{s}_1}^{\mathfrak{s}}\left(\theta(\zeta)Q(\zeta) - \left(\frac{\theta'_+(\zeta)}{\kappa+1}\right)^{\kappa+1}\left(\frac{\kappa}{\delta\theta(\zeta)\sigma'(\zeta)}\right)^{\kappa}\frac{r(\sigma(\zeta))}{f_i(\mu)\pi(\sigma(\zeta))}\right)d\zeta.$$

This contradicts the assumption in (3.5), completing the proof. □

Inspired by the Riccati-based techniques of Bohner et al. [12], we present a new oscillation criterion that emphasizes the role of exponential damping and functional bounds.

Theorem 4. *Assume that there exists a differentiable and continuous real function θ on $[\mathfrak{s}_0, \infty)$. If*

$$\limsup_{\mathfrak{s}\to\infty}\int_{\mathfrak{s}_1}^{\mathfrak{s}}\left(\theta(\zeta)\hat{R}(\zeta) - \frac{1}{(\kappa+1)^{\kappa+1}}\frac{(\theta'(\zeta))^{\kappa+1}}{\theta^{\kappa}(\zeta)}r(\zeta)\right)\mathrm{d}\zeta = \infty, \tag{3.6}$$

where $\theta'(\mathfrak{s}) = \max\{0, \theta'(\mathfrak{s})\}$, then all solutions of Eq (1.1) *are oscillatory.*

Proof. Assume, for contradiction, that x is a positive solution of (1.1) on $[\mathfrak{s}_0, \infty)$. Define the Riccati-type function

$$W_3(\mathfrak{s}) := \theta(\mathfrak{s})r(\mathfrak{s})\left(\frac{z'(\mathfrak{s})}{z(\mathfrak{s})}\right)^{\kappa}. \tag{3.7}$$

Differentiating W_3, we obtain:

$$W_3'(\mathfrak{s}) = \theta'(\mathfrak{s})r(\mathfrak{s})\left(\frac{z'}{z}\right)^{\kappa} + \theta(\mathfrak{s})\left[\frac{(r(\mathfrak{s})(z'(\mathfrak{s}))^{\kappa})'}{z^{\kappa}(\mathfrak{s})} - \kappa r(\mathfrak{s})\left(\frac{z'}{z}\right)^{\kappa+1}\right]. \tag{3.8}$$

Using the inequality:

$$\ln\left(\frac{z(\mathfrak{s})}{z(\sigma(\mathfrak{s}))}\right) = \int_{\sigma(\mathfrak{s})}^{\mathfrak{s}}\frac{z'(\zeta)}{z(\zeta)}\mathrm{d}\zeta \leq \int_{\sigma(\mathfrak{s})}^{\mathfrak{s}}\frac{r^{-1/\kappa}(\zeta)}{\tilde{R}_1(\zeta)}\mathrm{d}\zeta,$$

we obtain:

$$\frac{z(\mathfrak{s})}{z(\sigma(\mathfrak{s}))} \leq \exp\left(\int_{\sigma(\mathfrak{s})}^{\mathfrak{s}} \frac{r^{-1/\kappa}(\zeta)}{\tilde{R}_1(\zeta)} \mathrm{d}\zeta\right).$$

Now, using part (P_2) of Lemma 5, we have:

$$\begin{aligned}\frac{(r(\mathfrak{s})(z'(\mathfrak{s}))^{\kappa})'}{z^{\kappa}(\mathfrak{s})} &\leq -Q(\mathfrak{s})\left(\frac{z(\sigma(\mathfrak{s}))}{z(\mathfrak{s})}\right)^{\kappa} z^{\delta-\kappa}(\sigma(\mathfrak{s})) \\ &\leq -Q(\mathfrak{s}) \exp\left(-\kappa \int_{\sigma(\mathfrak{s})}^{\mathfrak{s}} \frac{r^{-1/\kappa}(\zeta)}{\tilde{R}_1(\zeta)} \mathrm{d}\zeta\right) z^{\delta-\kappa}(\sigma(\mathfrak{s})) \\ &\leq -Q(\mathfrak{s})\pi(\mathfrak{s}) \exp\left(-\kappa \int_{\sigma(\mathfrak{s})}^{\mathfrak{s}} \frac{r^{-1/\kappa}(\zeta)}{\tilde{R}_1(\zeta)} \mathrm{d}\zeta\right) = -\hat{R}(\mathfrak{s}).\end{aligned}$$

Substituting this into (3.8) and using (3.7), we find:

$$W_3'(\mathfrak{s}) \leq \theta(\mathfrak{s}) \left[\frac{\theta'(\mathfrak{s})}{\theta^2(\mathfrak{s})} W_3(\mathfrak{s}) - \hat{R}(\mathfrak{s}) - \kappa r^{-1/\kappa}(\mathfrak{s})\theta^{-1/\kappa-1}(\mathfrak{s}) W_3^{1+1/\kappa}(\mathfrak{s})\right].$$

Now, applying Lemma 1 with

$$A := \frac{\theta'(\mathfrak{s})}{\theta(\mathfrak{s})}, \quad B := \frac{\kappa}{(\theta(\mathfrak{s})r(\mathfrak{s}))^{1/\kappa}},$$

we get:

$$W_3'(\mathfrak{s}) \leq \frac{1}{(\kappa+1)^{\kappa+1}} \frac{(\theta'(\mathfrak{s}))^{\kappa+1}}{\theta^{\kappa}(\mathfrak{s})} r(\mathfrak{s}) - \theta(\mathfrak{s})\hat{R}(\mathfrak{s}).$$

Integrating both sides over $[\mathfrak{s}_1, \mathfrak{s}]$, we obtain:

$$W_3(\mathfrak{s}_1) \geq \int_{\mathfrak{s}_1}^{\mathfrak{s}} \left(\theta(\zeta)\hat{R}(\zeta) - \frac{1}{(\kappa+1)^{\kappa+1}} \frac{(\theta'(\zeta))^{\kappa+1}}{\theta^{\kappa}(\zeta)} r(\zeta)\right) \mathrm{d}\zeta.$$

This contradicts the assumption in (3.6). Hence, all solutions of (1.1) must be oscillatory. □

4. Illustrative examples and discussion

Through this paper, we have investigated the oscillatory behavior of the second-order neutral delay differential equation (NDDE) (1.1), and, moreover, we have established several monotonic properties for its positive solutions. In Section 3, we introduced various oscillation criteria derived via multiple approaches, including the comparison method and Riccati transformation techniques, each relying on different auxiliary functions. These criteria were subsequently refined by incorporating recurrence properties that enhance their effectiveness.

In the subsequent section, we offered an analytical perspective on these results, presenting comparative remarks that highlight the novelty and improvements over previous works. Furthermore, we provided additional oscillation theorems that strengthen the theoretical framework. Finally, we applied our results to special cases of second-order differential equations, demonstrating the applicability and originality of our criteria through direct comparison.

In alignment with methodologies presented in [41, 47], the following corollaries summarize the oscillation conditions derived from the main theorems and serve as the theoretical backbone for the subsequent examples.

Corollary 1. *Equation* (1.1) *is oscillatory if either of the following conditions holds:*

$$\int_{\mathfrak{s}_1}^{\infty} Q(\zeta)\,\tilde{R}_1^{\delta}(\sigma(\zeta))\,\mathrm{d}\zeta = \infty, \tag{4.1}$$

or

$$\int_{\mathfrak{s}_1}^{\infty} Q(\zeta)\,\tilde{R}_2^{\delta}(\sigma(\zeta))\,\mathrm{d}\zeta = \infty, \tag{4.2}$$

provided that $\kappa > \delta$.

Corollary 2. *Equation* (1.1) *is oscillatory if either of the following inequalities is satisfied:*

$$\liminf_{\mathfrak{s}\to\infty} \int_{\sigma(\mathfrak{s})}^{\mathfrak{s}} Q(\zeta)\,\tilde{R}_1^{\delta}(\sigma(\zeta))\,\mathrm{d}\zeta > \frac{1}{\mathrm{e}}, \tag{4.3}$$

or

$$\liminf_{\mathfrak{s}\to\infty} \int_{\sigma(\mathfrak{s})}^{\mathfrak{s}} Q(\zeta)\,\tilde{R}_2^{\delta}(\sigma(\zeta))\,\mathrm{d}\zeta > \frac{1}{\mathrm{e}}, \tag{4.4}$$

for $\kappa \geq \delta$.

Corollary 3. *Let $\kappa < \delta$ and suppose there exists a differentiable, continuous real function ϑ on* $[\mathfrak{s}_0, \infty)$ *such that $\vartheta' > 0$ and $\lim_{\mathfrak{s}\to\infty} \vartheta(\mathfrak{s}) = \infty$. If*

$$\limsup_{\mathfrak{s}\to\infty} \frac{\delta\,\vartheta'(\sigma(\mathfrak{s}))\,\sigma'(\mathfrak{s})}{\kappa\,\vartheta'(\mathfrak{s})} < 1, \tag{4.5}$$

and

$$\liminf_{\mathfrak{s}\to\infty} \int_{\sigma(\mathfrak{s})}^{\mathfrak{s}} \frac{1}{\vartheta'(\zeta)} e^{-\vartheta(\zeta)} Q(\zeta)\,\tilde{R}_1^{\delta}(\sigma(\zeta))\,\mathrm{d}\zeta > 0, \tag{4.6}$$

then Eq (1.1) *is oscillatory.*

Corollary 4. *Assume that $\sigma' > 0$ and that condition* (2.4) *holds for any $\mu > 0$. If*

$$\limsup_{\mathfrak{s}\to\infty} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \left(\tilde{R}_2^{\kappa}(\sigma(\zeta))\,Q(\zeta) - \left(\frac{\kappa}{\kappa+1}\right)^{\kappa+1} \left(\frac{\kappa}{\delta}\right)^{\kappa} \frac{\sigma'(\zeta)}{f_i(\mu)\pi(\sigma(\zeta))\,r^{1/\kappa}(\sigma(\zeta))\,\tilde{R}_2(\sigma(\zeta))} \right) \mathrm{d}\zeta = \infty, \tag{4.7}$$

where $f_i(\mu)$ is defined by (2.5) *and $i = 0, 1, 2, \ldots,$ then Eq* (1.1) *is oscillatory.*

In the following, we present illustrative examples to demonstrate the applicability of our main results and oscillation theorems established in Section 3. These examples serve to validate the sharpness and usability of the criteria developed for the second-order neutral delay differential equation (1.1).

Example 1. *Consider the second-order delay differential equation:*

$$\left(\mathfrak{s}^{\alpha}\,(x'(\mathfrak{s}))^{\kappa}\right)' + \frac{1}{\mathfrak{s}^{\beta}} x^{\kappa}(\sigma(\mathfrak{s})) = 0, \quad \mathfrak{s} \geq 1, \tag{4.8}$$

where $\alpha, \beta > 0$, κ is a ratio of two positive odd integers, $\delta = \kappa$, and $\sigma(\mathfrak{s}) = \mathfrak{s} - h$ for some constant $h \in (0, 1)$.

Then the auxiliary functions become:

$$r(\mathfrak{s}) = \mathfrak{s}^{\alpha}, \quad q(\mathfrak{s}) = \frac{1}{\mathfrak{s}^{\beta}},$$
$$Q(\mathfrak{s}) = (1-p)^{\delta} q(\mathfrak{s}) = \frac{(1-p)^{\kappa}}{\mathfrak{s}^{\beta}},$$
$$R(\mathfrak{s}) = \int_1^{\mathfrak{s}} \zeta^{-\alpha/\kappa} \mathrm{d}\zeta.$$

If $\alpha/\kappa < 1$*, then* $R(\mathfrak{s}) \to \infty$ *as* $\mathfrak{s} \to \infty$*. Consequently, by Corollary 1, if*

$$\int_1^{\infty} Q(\zeta) R^{\delta}(\sigma(\zeta))\, \mathrm{d}\zeta = \infty,$$

then all solutions of (4.8) *are oscillatory.*

Let $\alpha = 1$*,* $\beta = 0.5$*,* $\kappa = 1$*, and* $p = 0.5$*. Then* $R(\mathfrak{s}) = \int_1^{\mathfrak{s}} \zeta^{-1} \mathrm{d}\zeta = \log(\mathfrak{s})$*, and*

$$\int_1^{\infty} \frac{(1-0.5)}{(\zeta-h)^{0.5}} (\log(\zeta-h))\, \mathrm{d}\zeta = \infty.$$

Then, (4.8) *is oscillatory. To better support the theoretical findings of Example 1, we provide a graphical representation of two critical functions involved in the oscillation criterion. The first is the auxiliary function* $R(\mathfrak{s}) = \log(\mathfrak{s})$*, which appears in the integral condition. The second is the integrand* $Q(\zeta)R(\zeta-h)$*, where* $Q(\zeta) = \frac{(1-p)}{(\zeta-h)^{\beta}}$ *and* $R(\zeta-h) = \log(\zeta-h)$*. The plot shown in Figure 1 demonstrates their behavior over a selected interval.*

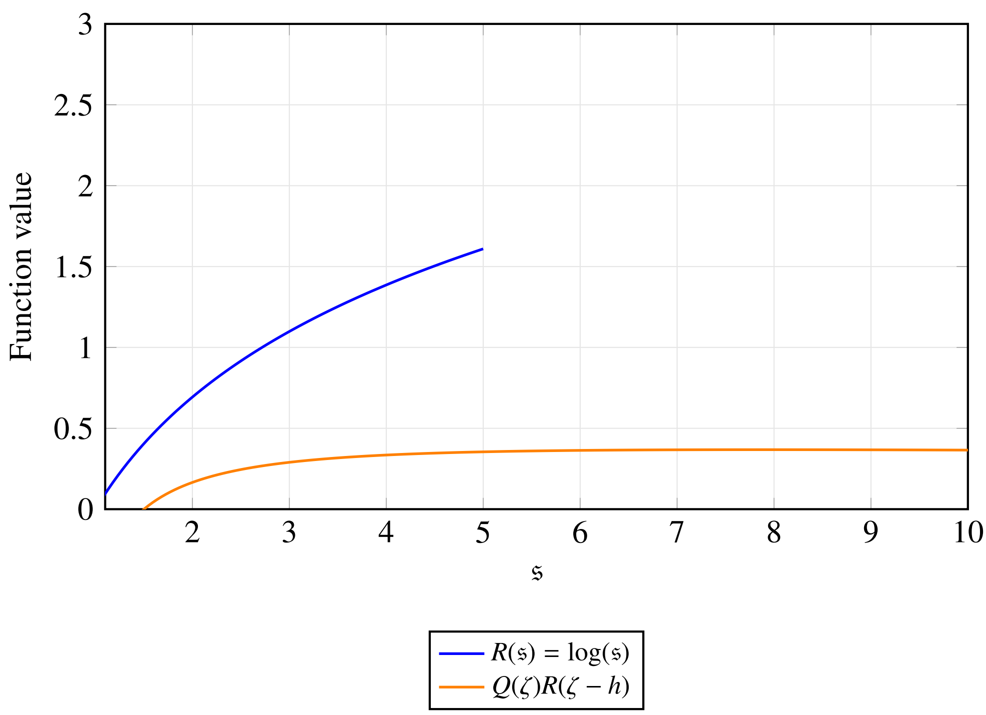

Figure 1. Behavior of the auxiliary function $R(\mathfrak{s})$ and the integrand $Q(\zeta)R(\zeta-h)$ used in the oscillation criterion of Example 1. Parameters: $\alpha = 1, \beta = 0.5, \kappa = 1, p = 0.5, h = 0.5$.

As seen in Figure 1, the function $R(\mathfrak{s}) = \log(\mathfrak{s})$ *increases monotonically and diverges as* $\mathfrak{s} \to \infty$. *The integrand* $Q(\zeta)R(\zeta - h)$ *also remains positive and exhibits growth, confirming that the integral*

$$\int_1^\infty Q(\zeta)R(\zeta - h)\, d\zeta = \infty$$

diverges. This satisfies the condition of Corollary 1 and validates the oscillatory nature of all solutions to the given equation.

Example 2. *Consider the following delay differential equation:*

$$\left(\mathfrak{s}^2 \left(x'(\mathfrak{s})\right)^3\right)' + \frac{\log(\mathfrak{s})}{\mathfrak{s}} x^3(\mathfrak{s} - 1) = 0, \quad \mathfrak{s} \geq 2, \tag{4.9}$$

with $\kappa = \delta = 3$, $r(\mathfrak{s}) = \mathfrak{s}^2$, $q(\mathfrak{s}) = \log(\mathfrak{s})/\mathfrak{s}$, *and* $\sigma(\mathfrak{s}) = \mathfrak{s} - 1$. *We choose the auxiliary function:*

$$\theta(\mathfrak{s}) = \log(\mathfrak{s}).$$

Then,

$$\theta'(\mathfrak{s}) = \frac{1}{\mathfrak{s}}, \quad \frac{(\theta'(\mathfrak{s}))^{\kappa+1}}{\theta^\kappa(\mathfrak{s})} r(\mathfrak{s}) = \frac{1}{\mathfrak{s}^4 \log^3(\mathfrak{s})} \cdot \mathfrak{s}^2 = \frac{1}{\mathfrak{s}^2 \log^2(\mathfrak{s})}.$$

Now, the integral condition from Theorem 4 becomes:

$$\int_2^\infty \left(\log(\zeta) \cdot \hat{R}(\zeta) - \frac{1}{27} \cdot \frac{1}{\zeta^2 \log^2(\zeta)}\right) \mathrm{d}\zeta.$$

Assuming from lemma definitions and earlier calculations that:

$$\hat{R}(\zeta) \sim \frac{\log(\zeta)}{\zeta}, \quad \text{as } \zeta \to \infty,$$

we get:

$$\log(\zeta) \cdot \hat{R}(\zeta) \sim \frac{\log^2(\zeta)}{\zeta}.$$

Thus,

$$\int_2^\infty \left(\frac{\log^2(\zeta)}{\zeta} - \frac{1}{27} \cdot \frac{1}{\zeta^2 \log^2(\zeta)}\right) \mathrm{d}\zeta = \infty,$$

because the first term dominates and diverges. Therefore, the condition of Theorem 4 is satisfied, and Eq (4.9) *is oscillatory. Example 2 involves a second-order neutral delay differential equation with logarithmic growth in the delay coefficient. To better understand the oscillation criterion applied, we graph two essential expressions: the auxiliary function*

$$\frac{(\theta'(\mathfrak{s}))^{\kappa+1}}{\theta^\kappa(\mathfrak{s})} r(\mathfrak{s}) = \frac{1}{\mathfrak{s} \log^2(\mathfrak{s})},$$

and the asymptotic form

$$\log(\mathfrak{s}) \cdot \hat{R}(\mathfrak{s}) \sim \frac{\log^2(\mathfrak{s})}{\mathfrak{s}}.$$

Their comparison is shown in Figure 2 to illustrate the divergence of the corresponding integral condition.

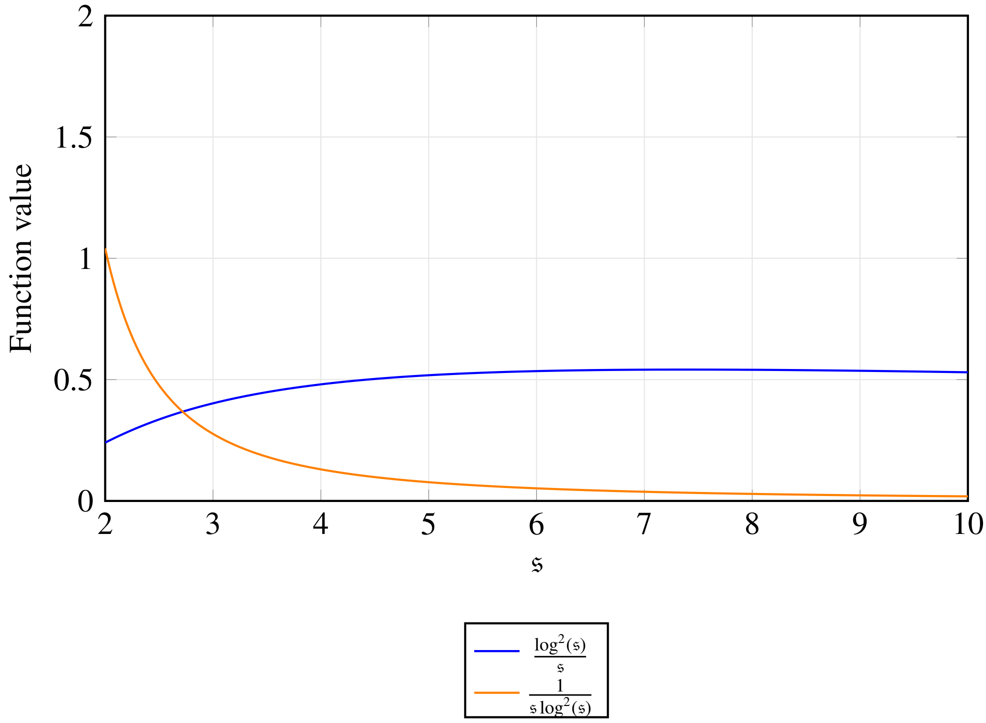

Figure 2. Comparison of the dominant term $\frac{\log^2(\mathfrak{s})}{\mathfrak{s}}$ and the auxiliary term $\frac{1}{\mathfrak{s}\log^2(\mathfrak{s})}$ in the oscillation condition of Example 2.

As illustrated in Figure 2, the function $\frac{\log^2(\mathfrak{s})}{\mathfrak{s}}$ *dominates the auxiliary term* $\frac{1}{\mathfrak{s}\log^2(\mathfrak{s})}$ *for large* $\mathfrak{s}$*, ensuring that the integrand remains positive and divergent. This confirms that the integral*

$$\int_2^\infty \left(\log(\zeta)\cdot\hat{R}(\zeta) - \frac{1}{27}\cdot\frac{1}{\zeta\log^2(\zeta)}\right)\mathrm{d}\zeta = \infty$$

satisfies the oscillation condition from Theorem 4, and therefore the solution to the equation is oscillatory.

Example 3. *Consider the delay differential equation:*

$$\left(\mathfrak{s}\left(x'(\mathfrak{s})\right)^3\right)' + \frac{1}{\mathfrak{s}^{1/2}}x^3(\sqrt{\mathfrak{s}}) = 0, \quad \mathfrak{s}\geq 1, \tag{4.10}$$

with parameters: $\kappa = \delta = 3$*,* $r(\mathfrak{s}) = \mathfrak{s}$*,* $q(\mathfrak{s}) = \mathfrak{s}^{-1/2}$*, and delay function* $\sigma(\mathfrak{s}) = \sqrt{\mathfrak{s}}$*. Simple calculations imply that*

$$Q(\mathfrak{s}) = (1 - p(\sigma(\mathfrak{s})))^3 q(\mathfrak{s}) = (1-0.3)^3\cdot\mathfrak{s}^{-1/2} = 0.343\cdot\mathfrak{s}^{-1/2}.$$

We assume from previous results that

$$\tilde{R}_2(\sigma(\zeta)) \sim \zeta^{1/2}, \quad \textit{as } \zeta\to\infty,$$

and

$$r^{1/\kappa}(\sigma(\zeta)) = \sigma(\zeta)^{1/3} = \zeta^{1/6}, \quad \pi(\sigma(\zeta)) \sim 1.$$

Now, we examine the integral condition in Corollary 4:

$$\int_1^\infty \left(\tilde{R}_2^2(\sigma(\zeta))Q(\zeta) - C \cdot \frac{\sigma'(\zeta)}{f_i(\mu)\pi(\sigma(\zeta))r^{1/\kappa}(\sigma(\zeta))\tilde{R}_2(\sigma(\zeta))} \right) \mathrm{d}\zeta,$$

where $C = \left(\frac{\kappa}{\kappa+1}\right)^{\kappa+1} \left(\frac{\kappa}{\delta}\right)^{\kappa}$.

Using: $\tilde{R}_2(\sigma(\zeta)) \sim \zeta^{1/2}$, $Q(\zeta) \sim \zeta^{-1/2}$, *we find:*

$$\tilde{R}_2^2(\sigma(\zeta))Q(\zeta) \sim \zeta \cdot \zeta^{-1/2} = \zeta^{1/2}.$$

Also:

$$\frac{\sigma'(\zeta)}{r^{1/\kappa}(\sigma(\zeta))\tilde{R}_2(\sigma(\zeta))} \sim \frac{1}{2\sqrt{\zeta}} \cdot \frac{1}{\zeta^{1/3} \cdot \zeta^{1/2}} = \frac{1}{2\zeta^{1.16}}.$$

Thus, the integrand behaves like:

$$\zeta^{1/2} - \frac{C}{2f_i(\mu)}\zeta^{-1.16} \to \infty,$$

and the integral diverges:

$$\int_1^\infty \left(\zeta^{1/2} - o(1)\right) \mathrm{d}\zeta = \infty.$$

Hence, by Corollary 4, Eq (4.10) *is oscillatory.*

This example illustrates the impact of nonlinear damping and nontrivial delay structure, revealing richer oscillatory dynamics than previously captured by linear models as discussed in [37, 46]. In Example 3, we examine a nonlinear second-order neutral delay differential equation with a square root delay. To support the analytical verification of the oscillation criterion, we graph the two main terms in the integrand from Corollary 6:

$$\tilde{R}_2^2(\sigma(\zeta))Q(\zeta) \sim \zeta^{1/2} \quad \textit{and} \quad \frac{\sigma'(\zeta)}{r^{1/\kappa}(\sigma(\zeta))\tilde{R}_2(\sigma(\zeta))} \sim \frac{1}{2\zeta^{1.75}}.$$

Their comparison in Figure 3 visually confirms the divergence of the integral required by the oscillation condition.

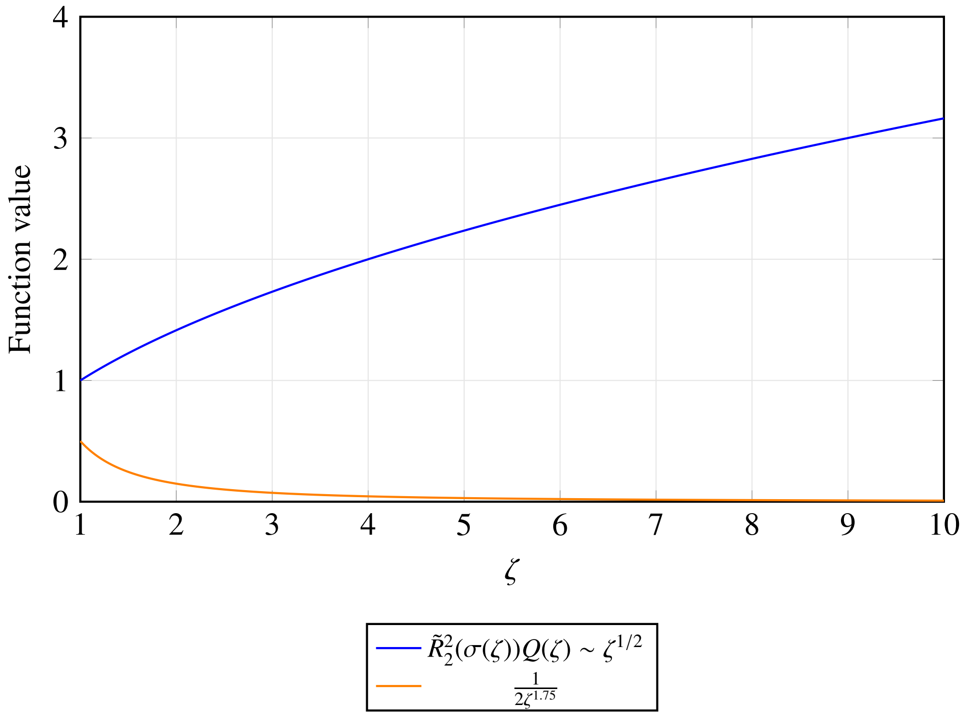

Figure 3. Comparison of the dominant integrand term $\zeta^{1/2}$ and the subdominant correction term $\frac{1}{2\zeta^{1.75}}$ in the oscillation condition of Example 3.

As shown in Figure 3, the leading term $\zeta^{1/2}$ grows while the correction term $\zeta^{-1.75}$ rapidly decays. This visual evidence supports the divergence of the integral

$$\int_1^\infty \left(\tilde{R}_2^2(\sigma(\zeta))Q(\zeta) - C \cdot \frac{\sigma'(\zeta)}{r^{1/\kappa}(\sigma(\zeta))\tilde{R}_2(\sigma(\zeta))} \right) \mathrm{d}\zeta,$$

thereby satisfying the oscillation condition from Corollary 6. Consequently, the equation is oscillatory.

5. Conclusion and future directions

In this work, we studied the oscillatory behavior of a class of second-order neutral delay differential equations. By combining Riccati transformations with a recursive approach, we established new criteria that extend and generalize several known results in the literature.

The analysis relied on auxiliary lemmas that describe the qualitative behavior of positive solutions, and the recursive formulation allowed us to better capture long-term dynamics influenced by delays. The examples provided illustrated how the conditions can be applied effectively, even in cases involving nonlinear or nonuniform delays.

While the results presented here offer a solid theoretical foundation, there are several directions for future work. Extending the approach to higher-order equations, incorporating more general delay types, or using numerical simulations to explore borderline cases could all provide valuable insights.

Overall, the findings contribute to the ongoing development of oscillation theory for delay differential equations and may support further research in both theory and applications.

Author contributions

Maged Alkilayh: Conceptualization, Investigation, Writing original draft preparation, Writing review and editing; Nedhal Almohammed: Conceptualization, Methodology, Investigation, Writing review and editing. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

The authors gratefully acknowledge Qassim University, represented by the Deanship of Graduate Studies and Scientific Research, on the financial support for this research under the number (QU-J-UG-2-2025-55688) during the academic year 1446AH / 2024 AD.

Use of Generative-AI tools declaration

The authors declares that the Artificial Intelligence (AI) tools were not used in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

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