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Research article

Estimation of the integral funnel for solutions of equations of perturbed motion with uncertain parameters

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Abstract: In this article, for scalar nonlinear equations with uncertain parameters, a new approach has been proposed for studying the integral funnel (antifunnel) of a family of their solutions. Efficient conditions for a family of solutions to remain in an integral funnel (antifunnel) and conditions for narrowing of the integral funnel were established. The proposed approach is based on a comparison scheme and a set of regularized differential equations. The obtained results contribute to the development of the qualitative theory of uncertain systems and to the expansions of the appropriate applications.

Keywords: scalar nonlinear equation; uncertain parameters; nonporous fence; funnel; antifunnel
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1. Introduction

The properties of integral funnels for some classes of differential equations have been studied in [4, 6, 20, 23]. Understanding integral funnels is essential in the fundamental theory of differential systems, as well as for their qualitative behavior. However, there is not much research that investigates the integral funnels of uncertain dynamical systems [14, 24]. Therefore, the topic of the application of some efficient approaches in the study of integral funnels of uncertain systems needs further development. This is the first goal of the proposed research.

The fundamental problem in the qualitative analysis of equations of perturbed motion of nonlinear systems is to obtain some information about the behavior of the solutions of the system if there are incomplete or uncertain details in the parameters of the system. In [9], it was proposed to use the comparison principle and elements of the theory of fuzzy equations to solve this problem. A constructive application of the direct Lyapunov method for the analysis of the stability of motion of systems with inaccurate parameter values was proposed in [11]. [16] presented the results of the analysis of the stability of motion of some classes of systems of equations of disturbed motion with inaccurate values of parameters based on vector and matrix-valued Lyapunov functions.

One approach to investigate the behavior of a system is to obtain two-sided estimates of solutions or to estimate the magnitude and narrowing of the integral funnel [15, 20]. Note that two-sided estimates for solutions of ordinary differential equations were considered by S. A. Chaplygin [2] and were used in the study of a specific mechanical system–the movement of a train on a non-linear incline (see [1] and the bibliography therein). Also, some studies of the integral funnel (anti-funnel) for solutions of a scalar differential equation in the absence of uncertain parameters were given in [7].

In [8], various conditions were given for the existence of two-sided estimates of solutions of systems of ordinary differential equations and partial differential equations of various types without taking into the account the inaccuracy of the systems' parameters.

In fact, uncertain parameters often exist in real-world problems and the study of their effects on the qualitative behavior of the models is crucial. In particular, while developing complex engineering systems one should provide for long life of its practical operation under the conditions of uncertainties. One of the directions in the development of the theory of uncertain systems that has been elaborated intensely is based on the assumption that parameters of a system are changed on a certain interval and there are no controls. Important results obtained in this direction are presented in [21]. We refer to [13, 16, 22] for more detailed information on the methods of analysis of dynamic systems, whose parameters are specified uncertainly (uncertain systems, for short).

On the other hand, the integral funnel methods are very helpful in the study of some qualitative properties of uncertain systems [25].

This article considers a scalar nonlinear equation with an uncertain parameter taking a value in some compact set. A new approach is proposed for studying the integral funnel (antifunnel) for a family of solutions of the equation under consideration using a comparison scheme and a set of regularized differential equations in order to obtain two-sided estimates of solutions. This approach is based on some recent results for regularized sets of fuzzy differential equations with respect to uncertain parameters established in [17, 18]. Although the method used is very powerful, it is not applied to the study of the integral funnel (antifunnel) for uncertain systems, which is the main aim of this research.

The main contributions of the paper can be summarized as follows:

(a) A new modeling approach is proposed to study important properties of uncertain systems with structural uncertainty. The method is based on a comparison scheme and a set of regularized differential equations, and is quite different from the applied approaches such as the Lyapunov function method [14, 16, 21] and the method of integral manifolds [22];

(b) The narrowing of the integral funnel of the proposed uncertain problem is investigated;

(c) Conditions under which a family of solutions of the uncertain problem remains in the integral funnel (or antifunnel) for the entire interval of existence are established;

(d) Efficient criteria under which the narrowing of the integral funnel does not exceed some predetermined value or tends to zero as $t \to \infty$ are derived.

The article is structured according to the following plan: Section 2 provides some definitions from the qualitative theory of differential equations of perturbed motion and formulates the research goals. In Section 3, new criteria for a set of solutions of the problem under consideration to be in the integral funnel (anti-funnel) are established. Section 4 presents the results of estimating the solution of the integral funnel for the set of solutions. Section 5 specifies the conditions for the narrowing of the integral funnel. Some linear systems with uncertain parameters are discussed in Section 6. The final Section 7 presents the main conclusions from the conducted research.

2. Statement of the problem

We consider an initial value problem (IVP) related to the scalar differential equation that describes the disturbed motion of a certain mechanical system in the form

$$\frac{dx}{dt} = f(t, x, \alpha), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C(\mathbb{R}_\tau \times D \times G,\ \mathbb{R})$, $\mathbb{R}_\tau \subseteq \mathbb{R}_+$, τ is a finite number or the symbol ∞, D is an open interval in $\mathbb{R}$, $\alpha \in G$ is the uncertain parameter in (2.1), $G \subseteq \mathbb{R}^d$, and $d > 1$.

We assume that for $(t_0, x_0) \in \mathbb{R}_\tau \times D$ there exists a family of solutions $X(t) = [x(t, \alpha)]$ of the uncertain differential equation in (2.1) for all $t \geq t_0$ and any value of $\alpha \in G$.

For the right-hand side of the differential equation in (2.1) we determine the boundary functions

$$f^m(t, x) = \min_{\alpha \in G} f(t, x, \alpha), \tag{2.2}$$

$$f^M(t, x) = \max_{\alpha \in G} f(t, x, \alpha), \tag{2.3}$$

regularized with respect to the uncertain parameter $\alpha \in G$ [17, 18].

For the family of solutions $X(t)$ we introduce the norm $\|X(t)\| = \max_i |x_i(t, \alpha)|$ for any value of the uncertain parameter $\alpha \in G$ for $i = 1, 2, \dots, p$ and $1 < p < \infty$.

Together with the IVP (2.1) we will consider the set of IVPs:

$$\frac{dy}{dt} = f^m(t, y, (t)), \ y(t_0) = y_0, \tag{2.4}$$

$$\frac{dz}{dt} = f^M(t, z(t)), \ z(t_0) = z_0, \tag{2.5}$$

with solutions $y(t)$, $z(t)$ providing the upper (lower) bounds of the solutions of the IVP (2.1) for $t \geq t_0$ and $\alpha \in G$, and will introduce the following definitions.

Definition 2.1. The continuous and continuously differentiable function $y(t)$ is a lower fence for the family of solutions of the IVP (2.1) if in the IVP (2.4) we have

$$f^m(t, y(t)) \leq f(t, y(t), \alpha), \tag{2.6}$$

for all $t \in \mathbb{R}_\tau$ and any value of $\alpha \in G$.

Definition 2.2. The continuous and continuously differentiable function $z(t)$ is an upper fence for the family of solutions of the IVP (2.1) if in the IVP (2.5) we have

$$f^M(t, z(t)) \geq f(t, z(t), \alpha), \tag{2.7}$$

for all $t \in \mathbb{R}_\tau$ and any value of $\alpha \in G$.

When satisfying the inequalities

$$\frac{dy}{dt} < f^m(t, y(t)), \tag{2.8}$$

$$\frac{dz}{dt} > f^M(t, z(t)), \tag{2.9}$$

the fences $y(t)$, $z(t)$ will be called strict or weak if for some value $t \in \mathbb{R}_\tau$ the inequalities (2.8), (2.9) become non-strict.

Definition 2.3. For the family of solutions $X(t)$ of the IVP (2.1):

(a) the lower fence $y(t)$ is non-porous if $y(t_0) \leq x(t_0)$ implies $y(t) < \|X(t)\|$ for all $t \in \mathbb{R}_\tau$;

(b) the upper fence $z(t)$ is non-porous if $x(t_0) \leq z(t_0)$ implies $\|X(t)\| < z(t)$ for all $t \in R_\tau$.

Note that the meaning of the boundary porosity for the uncertain differential equation in (2.1) is similar in content to the concept of membrane semipermeability in biology or chemistry [7].

Definition 2.4. If for a family of solutions $X(t)$ of the IVP (2.1) on some interval $\mathbb{R}_\tau$ the solution $y(t)$ of the IVP (2.4) is a non-porous lower fence and the solution $z(t)$ of the IVP (2.5) is a non-porous upper fence, and the inequality $y(t) < z(t)$, $t \in \mathbb{R}_\tau$, is satisfied, then the set of points $(t, x) \in \mathbb{R}_\tau \times D$, for which for any value of $\alpha \in G$ the two-sided estimate

$$y(t) \leq \|X(t)\| \leq z(t)$$

is satisfied, is called an integral funnel for the IVP (2.1). The value $d[z(t), y(t)]$, $t \in \mathbb{R}_\tau$, where $d[\cdot, \cdot]$ is the distance between functions, which determines the *magnitude* of the integral funnel.

Hence, an integral funnel refers to a pair of fences that confine the behavior of solutions to the equation [4, 20].

Definition 2.5. If for a family of solutions $X(t)$ of the IVP (2.1) on some interval $\mathbb{R}_\tau$ the solution $y(t)$ of the IVP (2.4) is a non-porous lower fence and the solution $z(t)$ of the IVP (2.5) is a non-porous upper fence, and the inequality $y(t) > z(t)$, $t \in \mathbb{R}_\tau$, is satisfied, then the set of points $(t, x) \in \mathbb{R}_\tau \times D$, for which for any value of $\alpha \in G$ the two-sided estimate

$$y(t) \geq \|X(t)\| \geq z(t)$$

is satisfied, is called an antifunnel for the IVP (2.1).

For the qualitative behavior of the solutions of the IVP (2.1) it is of interest to obtain conditions under which:

- the family of solutions $X(t)$ of the IVP (2.1) remains in the integral funnel (or antifunnel) for $t \in \mathbb{R}_\tau$;
- the narrowing of the integral funnel does not exceed some predetermined value or tends to zero as $t \to \infty$.

Such criteria are also of essential importance for the numerous applications of uncertain differential equations of type (2.1) [5, 22].

3. On some properties of the integral funnel and antifunnel

We will show that the following statements on the properties of the integral funnel and antifunnel hold.

Theorem 3.1. *Assume that:*

(1) The pair of fences $y(t)$ and $z(t)$, $y(t) \leq z(t)$, that are solutions of the IVPs (2.4), (2.5) *for $t \in \mathbb{R}_\tau$ form an integral funnel with non-porous fences for the family of solutions $X(t)$ of the IVP* (2.1).

(2) The function $f(t, x, \alpha)$ is Lipschitz continuous for any $\alpha \in G$ and any values of (t, x) located in the integral funnel.

Then, the family of solutions $X(t)$ corresponding to the initial value x_0, located in the funnel, remains there for all $t \in \mathbb{R}_\tau$.

Proof. According to Definition 2.4, the family of solutions $X(t)$ of the IVP (2.1) will remain in the funnel if the fences (upper fence $z(t)$ and lower fence $y(t)$) are non-porous.

Consider the case of the lower fence. From the assumption $y(t_0) \leq x(t_0)$ at time t_0 it follows that the family of solutions $X(t)$ is above or on the fence $y(t)$ for $t \in \mathbb{R}_\tau$. Since the funnel is formed by non-porous fences, the strictness of the lower fence means that

$$\frac{dy}{dt} - f^m(t, y(t)) < 0.$$

Let $y(t_0) < x(t_0)$ and for some $\tilde{t} > t_0$ the estimate $y(\tilde{t}) \geq \|X(\tilde{t})\|$ holds.

Let t_1 be the first value of $t > t_0$ for which $y(t_1) = \|X(t_1)\|$. Then, at the moment t_1, we have

$$\left(\frac{dx}{dt}\right)_{t=t_1} = f(t_1, x(t_1), \alpha) \geq f^m(t_1, y(t_1)) > \left(\frac{dy}{dt}\right)_{t=t_1}. \tag{3.1}$$

From inequality (3.1) it follows that the difference $\|X(t)\| - y(t)$ increases at the moment t_1. However, this contradicts the fact that this difference is positive to the left of t_1, since $\|X(t_1)\| - y(t_1) = 0$.

If $y(t_0) = x(t_0)$ and

$$\left(\frac{dx}{dt}\right)_{t=t_0} = f(t_0, x(t_0), \alpha) \geq f^m(t_0, y(t_0)) > \left(\frac{dy}{dt}\right)_{t=t_0}, \tag{3.2}$$

then the difference $\|X(t)\| - y(t)$ increases at t_0. This means that the family of solutions $X(t)$ is above the fence $y(t)$ for all $t \geq t_0$ and all values of $\alpha \in G$.

The case of non-porosity of the upper fence $z(t)$ is considered similarly.

Therefore, the family of solutions $X(t)$ of the IVP (2.1) remains in the integral funnel for all $t > t_1 \in \mathbb{R}_\tau$. This proves Theorem 3.1 □

Theorem 3.1 has the following corollary.

Corollary 3.2. *Let for $x(t_0) = y(t_0) = z(t_0) = x_0$ the solutions $x(t)$, $y(t)$, $z(t)$ of the IVPs* (2.1), (2.4), (2.5) *exist for all $t \in \mathbb{R}_\tau$. If*

$$\frac{dy}{dt} - f^m(t, y(t)) < 0, \tag{3.3}$$

$$\frac{dz}{dt} - f^M(t, z(t)) > 0, \tag{3.4}$$

for all $t \in \mathbb{R}_\tau$, then

$$y(t, t_0, x_0) < \|X(t)\|$$

and

$$\|X(t)\| < z(t, t_0, x_0)$$

for all $t \in \mathbb{R}_\tau$ and any values of $\alpha \in G$.

Note that in this case the functions $y(t)$ and $z(t)$ form a Chaplygin's fork [2] and the family of solutions $X(t)$ of the IVP (2.1) is located in this fork.

Theorem 3.3. *Assume that:*

(1) The solutions $y(t)$, $z(t)$ of the IVPs (2.4), (2.5) *form an antifunnel for the family of solutions $X(t)$ of the IVP* (2.1).

(2) The function $f(t, x, \alpha)$ satisfies the Lipschitz condition with respect to x for any value of $\alpha \in G$ and any values of (t, x) located in the antifunnel.

Then, for $(t, x, \alpha) \in \mathbb{R}_\tau \times D \times G$ there exists a family of solutions $X(t)$ of the IVP (2.1) *such that*

$$z(t) \leq \|X(t)\| \leq y(t) \tag{3.5}$$

for all $t \in \mathbb{R}_\tau$.

Proof. The proof is analogous to the proof of Theorem 3.1 [7]. □

Theorem 3.4. *Assume that:*

(1) The solutions $y(t)$, $z(t)$ of the IVPs (2.4), (2.5) *form a narrowing anti-funnel for the family of solutions $X(t)$ of the IVP* (2.1), *i.e., for some $t_1 \in \mathbb{R}_\tau$ we have*

$$\lim_{t \to t_1} |y(t) - z(t)| = 0. \tag{3.6}$$

(2) The function $f(t, x, \alpha)$ satisfies the Lipschitz condition with respect to x for any value of $\alpha \in G$ and any values of (t, x) located in the antifunnel, and the following inequality

$$\frac{\partial f}{\partial x}(t, x, \alpha) \geq 0 \tag{3.7}$$

holds for any $\alpha \in G$.

Then, there exists a family of solutions $X(t)$ of the IVP (2.1) *that remains in the anti-funnel for $t \in \mathbb{R}_\tau$.*

Proof. From condition (3.7) and any values of $\alpha \in G$ it follows that for solutions $y(t)$, $z(t)$ for which $z(t) > y(t)$, $t \in R_\tau$, we have the relation

$$\frac{d}{dt}(z(t) - y(t)) = f(t, z(t), \alpha) - f(t, y(t), \alpha) \equiv \int_{y(t)}^{z(t)} \frac{\partial}{\partial x} f(t, x, \alpha) dx \geq 0. \tag{3.8}$$

Relation (3.8) indicates that the distance between solutions $y(t)$ and $z(t)$ cannot decrease for $t \in \mathbb{R}_\tau$. Therefore, between these solutions there may be a family of solutions $X(t)$ of the IVP (2.1) for all $t \in \mathbb{R}_\tau$. This proves Theorem 3.4. □

4. Estimates of the integral funnel magnitude

When the conditions of Theorem 3.1 are met, the solutions $y(t)$ and $z(t)$ form an integral funnel, i.e., Definition 2.4 is satisfied. The magnitude of the integral funnel is determined by the distance between the solutions $z(t)$, $y(t)$ for all $t \in \mathbb{R}_\tau$.

The following statements hold.

Theorem 4.1. *Assume that:*

(1) The solutions $y(t)$ and $z(t)$ of the IVPs (2.4), (2.5) *are defined for $t \in \mathbb{R}_\tau$.*

(2) There exists a continuous function $g(t,u) : \mathbb{R}_\tau \times \mathbb{R}$, non-decreasing with respect to the second argument such that

$$d[f^M(t,z(t)),\ f^m(t,y(t))] \le g(t, d[z(t),y(t)]),\ t \in \mathbb{R}_\tau.$$

(3) There exists a maximal solution $u(t,t_0,u_0)$ of the scalar IVP

$$\frac{du}{dt} = g(t,u),\ u(t_0) = u_0 \ge 0.$$

(4) The fences forming a funnel $y(t)$ and $z(t)$ have initial values (y_0, z_0) such that $d[z_0, y_0] \le u_0$. Then, the magnitude of the integral funnel is estimated by the inequality

$$d[z(t),y(t)] \le u(t,t_0,u_0) \tag{4.1}$$

for all $t \in \mathbb{R}_\tau$.

Proof. From the differential equations in (2.4), (2.5) it follows that

$$\begin{aligned} d[z(t),y(t)] &= d\left[z_0 + \int_{t_0}^{t} f^M(\tau,z(\tau))d\tau,\ \ y_0 + \int_{t_0}^{t} f^m(\tau,y(\tau))d\tau\right] \\ &\le d\left[\int_{t_0}^{t} f^M(\tau,z(\tau))d\tau,\ \ \int_{t_0}^{t} f^m(\tau,y(\tau))d\tau\right] + d[z_0,y_0]. \end{aligned} \tag{4.2}$$

Taking into account condition (2) of Theorem 4.1, from estimate (4.2), the inequalities follow:

$$\begin{aligned} d[z(t),y(t)] &\le d[z_0,y_0] + \int_{t_0}^{t} d[f^M(\tau,z(\tau)),\ f^m(\tau,y(\tau))]d\tau \\ &\le d[z_0,y_0] + \int_{t_0}^{t} g(\tau, d[z(\tau),y(\tau)]d\tau \\ &= \rho(t_0) + \int_{t_0}^{t} g(\tau,\rho(\tau))d\tau, \end{aligned} \tag{4.3}$$

where $\rho(t) = d[z(t), y(t)], \rho(t_0) = d[z_0, y_0]$.

Taking into account conditions (3), (4) of Theorem 4.1, and applying Theorem 1.6.1 from [10], we obtain an estimate of the magnitude of the integral funnel for the family of solutions $X(t)$ of the IVP (2.1) in the form (4.1). This proves Theorem 4.1. □

Corollary 4.2. *If under the conditions of Theorem 4.1 the function* $g(t, u) = \lambda(t)d[z(t), y(t)]$*,* $\lambda(t) > 0$ *for all* $t \in \mathbb{R}_\tau$*, then*

$$d[z(t), y(t)] \le d[z_0, y_0] \exp \int_{t_0}^{t} \lambda(\tau)d\tau, \quad t \in \mathbb{R}_\tau.$$

Corollary 4.3. *If under the conditions of Theorem 4.1 the function* $g(t, u) = \pi(t)d[z(t), y(t)]$*,* $\pi(t) < 0$ *for all* $t \in \mathbb{R}_\tau$*, then*

$$d[z(t), y(t)] \le d[z_0, y_0] \exp\left(-\int_{t_0}^{t} \pi(s)ds\right), \quad t \in \mathbb{R}_\tau.$$

Corollary 4.4. *If all the conditions of Theorem 4.1 are satisfied under the initial conditions* $x(t_0) = y(t_0) = z(t_0)$ *and* $u_0 = 0$*, then estimate* (4.1) *of the magnitude of the integral funnel is preserved.*

Theorem 4.5. *Assume that:*

(1) Condition (1) of Theorem 4.1 holds.
(2) There exist nonnegative functions $\varphi_1(t)$*,* $\varphi_2(t)$ *such that*

$$d[f^M(t, z(t)),\ f^m(t, y(t))] \le \varphi_1(t)d[z(t), y(t)] + \varphi_2(t)d^q[z(t), y(t)], \tag{4.4}$$

for all $t \in \mathbb{R}_\tau$ *and* $q > 1$*.*
(3) There exists a maximal solution $w(t, t_0, w_0)$ *of the IVP*

$$\frac{dw}{dt} = \varphi_1(t)w(t) + \varphi_2(t)w^q(t),\ w(t_0) \le w_0,\ w_0 = d[z_0, y_0]. \tag{4.5}$$

(4) The inequality

$$\psi(t_0, t) = (q-1)w_0^{q-1} \int_{t_0}^{t} \varphi_2(s) \exp\left((q-1)\int_{t_0}^{s} \varphi_1(\tau)d\tau\right) ds < 1,$$

holds for $(t, s) \in J \subset \mathbb{R}_\tau$*.*

Then, the magnitude of the integral funnel is estimated by the inequality

$$d[z(t), y(t)] \le w(t, t_0, w_0), \tag{4.6}$$

where $w(t, t_0, w_0) = d[z_0, y_0] \exp\left(\int_{t_0}^{t} \varphi_1(s)ds\right)(1 - \psi(t_0, t))^{-\frac{1}{q-1}}$ *for all* $t \in J \subset \mathbb{R}_\tau$*.*

Proof. From the fact that the solutions of the IVPs (2.4), (2.5) are represented in the forms

$$z(t) = z_0 + \int_{t_0}^{t} f^M(s, z(s))ds, \tag{4.7}$$

$$y(t) = y_0 + \int_{t_0}^{t} f^m(s, y(s))ds, \tag{4.8}$$

it follows that the estimate (4.2) is satisfied. From the estimate (4.4) and the IVP (4.5) we find that

$$w(t) \le w_0 + \int_{t_0}^{t} (\varphi_1(s)w(s) + \varphi_2(s)w^q(s))ds, \tag{4.9}$$

for all $t \in \mathbb{R}_\tau$, $w_0 \ge 0$. Applying the estimation technique from [15, 19] to inequality (4.9), we obtain an estimate of $w(t, t_0, w_0)$ in the form

$$w(t, t_0, w_0) \le w_0 \exp\left(\int_{t_0}^{t} \varphi_1(s)ds\right)(1 - \psi(t_0, t))^{-\frac{1}{q-1}}, \tag{4.10}$$

for all $t \in J$ for which condition (4) of Theorem 4.2 is satisfied. This proves Theorem 4.2. □

Corollary 4.6. *If condition (2) of Theorem 4.2 is satisfied for* $0 < q < 1$*, then estimate* (4.10) *takes the form*

$$w(t, t_0, w_0) \le w_0 \exp(1 - \psi(t_0, t))^{-\frac{1}{q-1}}, \tag{4.11}$$

for all $t \in \mathbb{R}_\tau$.

5. Conditions for the narrowing of the integral funnel

Theorems 4.1 and 4.2 allow us to specify sufficient conditions for the narrowing of the integral funnel for the family of solutions $X(t)$ of the IVP (1). Let us consider the scalar equation

$$\frac{du}{dt} = g(t, u), \quad u(t_0) = u_0, \tag{5.1}$$

under the condition $g(t, 0) = 0$, $t \in \mathbb{R}_\tau$, and introduce the following definitions.

Definition 5.1. The zero solution $u = 0$ of (5.1) is said to be:

(a) equi-stable, if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_\tau$ there exists a positive function $\delta = \delta(t_0, \varepsilon)$ which is continuous on t_0 for any $\varepsilon > 0$ such that

$$d[u(t), 0] < \varepsilon \text{ for all } t \ge t_0,$$

whenever $d[u_0, 0] < \delta$, where $u(t) = u(t, t_0, u_0)$ is the solution of (5.1);

(b) uniformly stable, if the function δ in Definition 5.1(a) does not depend on $t_0 \in \mathbb{R}_\tau$;

(c) quasi-equi-asymptotically stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_\tau$ there exist $\delta_0 = \delta(t_0)$ and $\tau = \tau(t_0, \varepsilon)$ such that

$$d[u(t), 0] < \varepsilon \ \text{ for all } \ t \geq t_0 + \tau,$$

whenever $d[u_0, 0] < \delta_0$;

(d) quasi-uniformly asymptotically stable if δ_0 and τ in Definition 5.1(c) do not depend on $t_0 \in \mathbb{R}_\tau$;

(e) equi-asymptotically stable if the conditions of Definitions 5.1(a) and 5.1(c) are satisfied simultaneously;

(f) uniformly asymptotically stable if the conditions of Definitions 5.1(b) and 5.1(d) are satisfied simultaneously;

(g) exponentially stable if for any $t_0 \in \mathbb{R}_\tau$ and $d[u_0, 0] < \delta$ the estimate

$$d[u(t), 0] \leq d[u_0, 0] \exp(-\mu(t - t_0))$$

holds for all $t \geq t_0$ and $\mu > 0$.

Let us show that the following statement holds.

Theorem 5.2. *If all the conditions of Theorem 4.1 are satisfied and, in addition, the zero solution of Eq* (5.1) *is stable of a certain type, then the narrowing of the integral funnel is determined by the stability type of the zero solution of Eq* (5.1).

Proof. Under the conditions of Theorem 4.1, the magnitude of the integral funnel is estimated by the inequality (4.1). If the solution $u = 0$ of (5.1) is equi-stable, then according to Definition 6.1(a), for any $\varepsilon > 0$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $d[u(t), 0] < \varepsilon$ for all $t \geq t_0$ whenever $d[u_0, 0] < \delta$. We will show that in this case the funnel solution is equi-narrowing. Assume that it is not true, and for $d[u_0, 0] < \delta$, there exists a $t_1 > t_0$ such that

$$d[u(t_1), 0] = \varepsilon \ \text{ and } \ d[u(t), 0] < \varepsilon,$$

for all $t_0 \leq t < t_1$. On the other hand for all $t \in [t_0, t_1)$ we have

$$d[z(t), y(t)] \leq u(t, t_0, u_0) < d[u(t), 0] < \varepsilon.$$

The above contradiction proves the equi-narrowing of the integral funnel.

If the zero solution of Eq (5.1) is exponentially stable, then

$$d[z(t), y(t)] \leq d[z_0, y_0] \exp(-\mu(t - t_0)) \leq \delta \exp(-\mu(t - t_0)),$$

for all $t \geq t_0$. In this case, the narrowing of the integral funnel is exponential.

Other forms of compression of the integral funnel, corresponding to Definitions 6.1(b)–6.1(f), are considered similarly. □

Theorem 5.3. *If all the conditions of Theorem 4.2 are satisfied and, in addition, the inequality*

$$\exp\left(\int_{t_0}^{t} \varphi_1(s) ds\right) (1 - \psi(t_0, t))^{-\frac{1}{q-1}} < \frac{\varepsilon}{\delta} \tag{5.2}$$

holds for $t \geq t_0$ *and for* $d[z_0, y_0] < \delta$*, then the narrowing of the integral funnel satisfies the inequality*

$$d[z(t), y(t)] < \varepsilon, \tag{5.3}$$

for all $t \geq t_0$*.*

Proof. Under the conditions of Theorem 4.2, the magnitude of the integral funnel is estimated by the inequality (4.6). From (4.6) provided that $d[z_0, y_0] < \delta$ it follows that

$$\delta \exp\left(\int_{t_0}^{t} \varphi_1(s)ds\right)(1 - \psi(t_0, t))^{-\frac{1}{q-1}} < \varepsilon$$

and when inequality (5.2) is satisfied, we obtain an estimate of the narrowing of the integral funnel in the form of (5.3). This proves Theorem 5.2. □

Remark 5.4. Although some results on integral funnels are given in [4, 6, 20, 23], they treated only the nominal systems. The uncertain case is considered very rarely [14, 24] using the funnel control technique and barrier Lyapunov functions. Hence, the proposed research contributes to the study of the theory of integral funnels of uncertain dynamical systems.

Remark 5.5. Many authors applied the ideas of the method of Lyapunov functions to analyze interval systems (continuous and discrete ones) [14,16]. In addition, the method of integral manifolds is applied for investigating various problems in the theory of uncertain systems [22]. Different from the existing results, here we apply a comparison principle and a set of regularized differential equations in order to obtain two-sided estimates of solutions and study important properties of the integral funnel and antifunnel.

Remark 5.6. The method applied and results obtained can be extended to the case of high-dimensional differential equations with uncertain parameters. Also, delay effects and impulsive perturbations can be considered in future research.

6. Examples of linear systems with uncertain parameters

The study of scalar equations of perturbed motion is important not only for the general theory of these equations, but also because in numerous cases it allows one to model the behavior of solutions of complex systems of higher order [3]. Here, we will consider examples for scalar equations with uncertain parameters.

Example 6.1. Consider the vertical motion of a rocket. Let $m(t)$ denote the mass of the rocket depending on the amount of fuel at time $t \in \mathbb{R}_\tau$, $x(t)$ is the speed of the rocket, and $u(t)$ is the velocity of the exhaust gases. Of all the external forces, we will take into account only the force of gravity g. In this case, the equation of motion of the rocket has the form

$$\frac{dx}{dt} = \frac{1}{m(t)}\frac{dm}{dt}\alpha(x(t) - u(t)) - g, \tag{6.1}$$

where $\alpha \in G$ is the difference $(x(t) - u(t))$ measurement uncertainty parameter.

Example 6.2. Consider a linear non-stationary electric circuit, which is described by a first-order equation. Let the inductance remain constant, $v(t)$ is the voltage at the input, and $x(t)$ is the current in the circuit. The equation describing the oscillatory process in such a circuit has the form

$$\frac{dx}{dt} = -\frac{R_0 + \alpha t}{L} x(t) + \frac{1}{L} v(t), \tag{6.2}$$

where $\alpha \in G$ is an uncertain parameter due to the increasing over time resistance $R(t) = R_0 + \alpha t$.

Since (6.1) and (6.2) are linear equations, it is not difficult to write out their solutions in explicit form and the question of boundaries for their solutions is solved trivially, without using Theorems 4.1 or 5.1.

Remark 6.3. Models (6.1) and (6.2), without uncertain parameters, are given in [3] to illustrate the importance of first-order equations in modeling real-world processes. We present some extensions of these models that take into account uncertain parameters.

7. Conclusions

In this paper, the existence of an integral funnel (antifunnel) for a scalar differential equation with uncertain parameter values is investigated for the first time. To estimate the solution of the integral funnel, the comparison principle in combination with nonlinear integral inequalities is adapted. The main properties of the integral funnel (antifunnel) are established and sufficient conditions for its narrowing are indicated. The results obtained are new and contribute to the development of a qualitative theory of scalar differential equations with uncertain parameter values. The results obtained may be of interest in the study of controlled mechanical, electrical, and neural network systems with uncertain parameter values (see [3,5,12]). It will be important to extend the research on this concept and consider the effects of time delays and data losses. The study of high-dimensional differential equations with uncertain parameters is also a subject of future research.

Author contributions

Anatoliy Martynyuk: Conceptualization, formal analysis, methodology, investigation, writing–original draft, writing–review and editing; Ivanka Stamova: Conceptualization, formal analysis, methodology, investigation, writing–review and editing; Yulya Martynyuk–Chernienko: Conceptualization, formal analysis, methodology, investigation, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare that there are no competing interests.

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