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Research article

A novel robust estimator for addressing multicollinearity and outliers in Beta regression: simulation and application

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Abstract: The beta regression model (BRM) is a popular and widely applied modeling approach, especially when dealing with data bounded within the interval (0, 1). It has been used extensively in various fields, including chemistry, environmental science, medicine, and biology. BRM aims to estimate unknown model parameters, typically achieved using the maximum likelihood estimator (MLE). However, MLE is not without limitations. It can be highly sensitive to multicollinearity and outliers, which can distort coefficient estimates, lead to misleading conclusions, and inflate variance, ultimately increasing the mean squared error (MSE). To address these challenges, this study proposed new robust estimators for BRM that incorporated robust modified ridge-type estimators. These estimators were specifically designed to reduce the adverse effects of multicollinearity and outliers. Their performance was theoretically compared to that of the traditional MLE and robust ridge estimators. In addition, an extensive simulation study was carried out in various scenarios to evaluate their effectiveness. Both theoretical comparisons and simulation results demonstrated the clear advantages of the proposed robust estimators in managing multicollinearity and handling outliers. To further validate the findings, the estimators were applied to real-world data from breast cancer patients. The results confirmed that the proposed robust estimators offer greater robustness and reliability compared to MLE and robust ridge methods. These findings highlighted the practical importance of using robust estimation techniques to improve the accuracy and dependability of BRMs, particularly in empirical research involving highly multicollinear and outlier data.

Keywords: Beta regression; multicollinearity; outliers; robust modified ridge-type estimator; robust ridge estimator

Mathematics Subject Classification:62H12, 62J05, 62J07, 65C05

1. Introduction

The beta regression model (BRM) has emerged as a practical and flexible approach for analyzing data bounded between 0 and 1, such as proportions and rates, which we often find in fields such as biology, chemistry, and environmental sciences [1, 2]. What makes this type of data tricky is that traditional methods, like linear or logistic regression, are not designed to handle the constraints of the (0,1) interval, and they often fall short. That's where the beta regression model proves its worth. As part of the broader class of generalized linear models (GLMs), it gives us a powerful framework to model such data appropriately, while also allowing for flexible relationships between variables [3, 4]. The standard method for estimating the parameters in a beta regression model is maximum likelihood estimation (MLE). However, as any researcher who has worked with real-world data knows, things are rarely clean and simple. Two issues in particular can cause serious problems: multicollinearity and outliers.

Multicollinearity arises when two or more predictors are highly correlated. It may sound harmless at first, but it complicates the model by making it difficult to isolate the effect of each variable. This results in unstable parameter estimates, inflated variances, and often misleading conclusions. This challenge has been known for decades, going back to Frisch [5], and has since been explored extensively, including in the context of GLMs by Segerstedt [6]. To mitigate its effects, ridge regression was introduced by Hoerl and Kennard [7], and later Liu proposed an alternative estimator that also introduces a shrinkage parameter to stabilize the estimates [8]. Since then, researchers have continued to build on these foundations, offering several modifications like the modified ridge-type estimator [9], the Dawoud-Kibria estimator [10], the modified two-parameter Liu estimator [11], feasible Stein-type [12], and the Kibria-Lukman estimator [13] to better handle multicollinearity, particularly within the context of GLMs. More recently, work by Qasim et al. [14], Hammad et al. [15], and Akram et al. [16] has shown that these biased estimators can outperform standard MLE, especially under challenging conditions.

On the other hand, outlier data points that significantly deviate from the general trend—pose another threat. Even one or two of these unusual values can throw off your results, particularly if the dataset is small [17]. This issue is well-documented, and several robust estimation techniques have been proposed over the years to counteract it, including maximum likelihood-type estimators (M-estimators), scale estimators (S-estimators), and Mallows M-estimators (MM-estimators) [18, 19], as well as least absolute deviations (LAD) and least trimmed squares (LTS) [20]. In beta regression specifically, where the response is strictly between 0 and 1, even a minor distortion can have a significant impact. Robust methods help reduce the influence of these anomalies, making the model more reliable and the conclusions more trustworthy—even when the data isn't perfect [21, 22].

Now, what happens when both multicollinearity and outliers occur together in a dataset? This, unfortunately, is not a rare situation, and it creates a particularly challenging scenario for statistical modeling. While several estimators handle either issue on their own, their performance tends to suffer when the two coexist. The distortions from multicollinearity can amplify the effects of outliers, and vice versa, leading to severely biased estimates and questionable inference [23,24]. This intersection of problems has motivated researchers to explore more robust and adaptive estimation techniques—ones that can simultaneously withstand the effects of both issues, ensuring more stable and accurate regression results under real-world conditions. This research recognizes this methodological gap and builds upon recent work by Lukman et al. [24], who defined a robust Kibria-Lukman estimator for logistic regression; Altukhaes et al. [25], who proposed a robust Liu estimator for a restricted semiparametric regression model; Olaluwoye et al. [26], who developed robust methods for beta regression; Mohammad et al. [27], who introduced a new robust two-parameter estimator for Poisson regression; Lukman et al. [28] on robust enhanced ridge-type estimation for Poisson regression; Suhail et al. [29] on a quantile-based ridge M-estimator for linear regression; Alqasem et al. [30] in a comprehensive study on a robust Poisson James-Stein estimator; Oyeleke et al. [31] in a comparative study of one and two-parameter estimators; Majid et al. [32] on a robust Kibria-Lukman estimator for linear regression; Norouzirad and Arashi [33] on preliminary test and Stein-type shrinkage ridge estimators in robust regression; and Alghamdi et al. [34] on robust modified Liu estimation in Poisson regression, all of whom address the joint problem of multicollinearity and outliers across various regression models.

Conventional MLE delivers asymptotically efficient inference but proves highly unstable under conditions of multicollinearity or data contamination. While biased estimators mitigate the effect of multicollinearity, this is achieved at the expense of bias, and their performance remains susceptible to outliers. Conversely, robust regression methods enhance stability against anomalous observations but fail to explicitly address multicollinearity. Although recent improved methodologies have sought to integrate shrinkage and robustness, such approaches remain limited and, crucially, are markedly understudied within the context of BRM. This identified gap motivates the present work, which aims to develop an estimator capable of simultaneously curtailing variance, managing bias, and ensuring robustness in finite samples.

To this end, this paper introduces a novel robust estimator specifically designed for the BRM framework that handles the dual challenges of multicollinearity and outliers. The performance of the proposed estimator is evaluated through theoretical comparisons and an extensive Monte Carlo simulation study. Its empirical validity is further demonstrated through an application to real-world breast cancer data. Results confirm the superiority of the proposed estimator over competing methods, including traditional MLE and robust ridge regression estimators. By concurrently addressing these two common data pathologies, the proposed method provides a more reliable and stable analytical tool for applied researchers working with real-world data that violates ideal statistical assumptions.

This paper is structured as follows: Section 2 provides a comprehensive review of the literature, covering the beta regression model along with both non-robust and robust estimators, highlighting their statistical properties. Section 3 introduces the proposed estimator, discusses its theoretical comparison, and compares it analytically with other estimators. Section 4 outlines an extensive simulation study conducted under scenarios involving multicollinearity and outliers, showcasing the results from the simulation. Section 5 applies the proposed methods to real-world data from breast cancer patients, demonstrating how the estimators perform in practice. Finally, Section 6 summarizes the key findings and offers suggestions for future research directions.

2. Methodology

2.1. Beta regression model

Consider a set of observations $y_1, y_2, \ldots, y_n$, drawn from a random variable y that follows a beta distribution, denoted as $\beta_e(a, b)$, where both parameters a and b are positive. The probability density function (PDF) of the beta distribution is given by

$$f(y; a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}, \quad \text{for } y \in (0, 1), \tag{2.1}$$

where $\Gamma(\cdot)$ represents the gamma function, and the parameters a and b control the shape of the distribution. The mean and variance of y are

$$\mathrm{E}(y) = \frac{a}{a+b}, \quad \mathrm{Var}(y) = \frac{ab}{(a+b)^2(a+b+1)}.$$

To adapt the beta distribution for regression modeling, Ferrari and Cribari-Neto [1] introduced a useful reparameterization. Instead of working directly with a and b, by $\mu = \frac{a}{a+b}$ and $\phi = a + b$, using this transformation, we express $a = \mu\phi$ and $b = (1-\mu)\phi$. The PDF becomes

$$f(y; \mu, \phi) = \frac{\Gamma(\phi)}{\Gamma(\mu\phi)\Gamma((1-\mu)\phi)} y^{\mu\phi-1}(1-y)^{(1-\mu)\phi-1}, \quad \text{for } y \in (0, 1). \tag{2.2}$$

In this form, $y \sim \beta_e(\mu, \phi)$, where $\mu \in (0, 1)$ is the mean, and $\phi > 0$ is the precision. The mean and variance under this parameterization are

$$\mathrm{E}(y) = \mu, \quad \mathrm{Var}(y) = \frac{\mu(1-\mu)}{1+\phi}.$$

The BRM is based on this reparameterized beta distribution. It assumes each response y_i follows a beta distribution with mean μ_i and a shared precision ϕ. The mean is linked to predictors through a link function $g(\mu_i) = \eta_i = \mathbf{x}_i^T\boldsymbol{\beta}$, where x_i is the i-th row of the design matrix $\mathbf{X}$, which contains $p + 1$ explanatory variables, and $\boldsymbol{\beta}$ is a vector of unknown regression coefficients. The function $g(\cdot)$ ensures that the estimated means μ_i stay within the (0,1) interval. The link is defined as

$$g(\mu_i) = \log\left(\frac{\mu_i}{1-\mu_i}\right), \quad \text{so that } \mu_i = \frac{e^{\mathbf{x}_i^T\beta}}{1+e^{\mathbf{x}_i^T\beta}}. \tag{2.3}$$

The log-likelihood function for this model, which we use to estimate β, is

$$\ell(\boldsymbol{\beta}) = \sum_{i=1}^{n} \left[\log\Gamma(\phi) - \log\Gamma(\mu_i\phi) - \log\Gamma((1-\mu_i)\phi) + (\mu_i\phi - 1)\log y_i + ((1-\mu_i)\phi - 1)\log(1-y_i)\right]. \tag{2.4}$$

To estimate the regression coefficients β, the MLE is typically used. It's obtained using an iterative reweighted least squares algorithm, and the beta maximum likelihood estimator (BMLE) is given by

$$\hat{\boldsymbol{\beta}}_{\mathrm{BMLE}} = \mathbf{U}^{-1}\mathbf{X}^T\hat{\mathbf{W}}\hat{\mathbf{c}}, \tag{2.5}$$

where $\mathbf{U} = \mathbf{X}^T\hat{\mathbf{W}}\mathbf{X}$, $\hat{\mathbf{c}} = \hat{\boldsymbol{\eta}}+\hat{\mathbf{W}}^{-1}\hat{\mathbf{Q}}(\mathbf{y}^*-\boldsymbol{\mu}^*)$, and $\hat{\mathbf{W}} = \mathrm{diag}(\hat{w}_1,\ldots,\hat{w}_n)$. Here, $\hat{\mathbf{W}}$ and $\hat{\mathbf{Q}}$ are the matrices $\mathbf{W}$ and $\mathbf{Q}$, respectively, evaluated at the BMLE estimator [26, 35].

The variance-covariance, matrix mean squared error (MMSE), and MSE of $\hat{\beta}_{\mathrm{BMLE}}$ are

$$\text{Var-Cov}(\hat{\boldsymbol{\beta}}_{\mathrm{BMLE}}) = \hat{\phi}\,\mathbf{U}^{-1}, \tag{2.6}$$

$$\mathrm{MMSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMLE}}) = \hat{\phi}\,\boldsymbol{\gamma}\boldsymbol{\Lambda}^{-1}\boldsymbol{\gamma}^T, \tag{2.7}$$

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMLE}}) = \hat{\phi}\sum_{j=1}^{p+1}\frac{1}{\lambda_j}. \tag{2.8}$$

To compute the MMSE and the MSE, we start by assuming that $\boldsymbol{\alpha} = \boldsymbol{\gamma}^T\hat{\boldsymbol{\beta}}$. Additionally, we define the diagonal matrix $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_{p+1})$, which corresponds to the expression $\boldsymbol{\gamma}\boldsymbol{\Lambda}\boldsymbol{\gamma}^T$. In this context, $\boldsymbol{\gamma}$ is an orthogonal matrix whose columns are the eigenvectors of the matrix $\mathbf{U}$, denoted as $\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{p+1}$. The values $\lambda_1,\lambda_2,\ldots,\lambda_{p+1}$ are the corresponding eigenvalues of $\mathbf{U}$, arranged in descending order such that $\lambda_1 > \lambda_2 > \ldots > \lambda_{p+1} > 0$.

When explanatory variables in a BRM are highly correlated, the matrix $\mathbf{U}$ becomes ill-conditioned, which means that some of its eigenvalues are very small. As a result, the estimated MSE of the MLE tends to be inflated. Multicollinearity not only increases the variance of the estimates but also leads to wider confidence intervals and unstable, unreliable parameter estimates. To overcome these issues, researchers have turned to regularization techniques, such as ridge regression and a modified ridge-type estimator, which provide more stable solutions [36]. Furthermore, the optimal subset selection criterion that is related to covariance matrices, observation matrices, and response vectors (COR criterion) has been introduced as an alternative strategy for subset selection in regression models, offering a novel means of improving the accuracy of the estimation under multicollinearity [37]. In this study, we focus on the beta modified ridge-type estimator, a variant specifically suited to beta regression models, which will be explored in detail in the following subsection.

2.2. Beta ridge regression estimator

To address the instability caused by multicollinearity in BRM. Abonazel and Taha [35], along with Qasim et al. [14], introduced the beta ridge regression estimator (BRRE). The BRRE is formulated as

$$\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}} = (\mathbf{U} + k\mathbf{I}_s)^{-1}\mathbf{U}\hat{\boldsymbol{\beta}}_{\mathrm{BMLE}}, \quad k > 0, \tag{2.9}$$

where $k = \frac{\hat{\phi}}{(p+1)\sum_{j=1}^{p+1}\hat{\alpha}_j^2}$ is the BRRE parameter, controlling the degree of shrinkage applied, and $\mathbf{I}_s$ denotes the identity matrix of dimension $(p+1)\times(p+1)$. Notably, when $k = 0$, the BRRE simplifies to the BMLE.

The MMSE and MSE of BRRE are given by

$$\mathrm{MMSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}}) = \hat{\phi}\boldsymbol{\gamma}\boldsymbol{\Lambda}_k^{-1}\boldsymbol{\Lambda}\boldsymbol{\Lambda}_k^{-1}\boldsymbol{\gamma}^T + k^2\boldsymbol{\gamma}\boldsymbol{\Lambda}_k^{-1}\boldsymbol{\alpha}_{\mathrm{MLE}}\boldsymbol{\alpha}_{\mathrm{MLE}}{}^T\boldsymbol{\Lambda}_k^{-1}\boldsymbol{\gamma}^T, \tag{2.10}$$

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}}) = \hat{\phi}\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j+k)^2} + k^2\sum_{j=1}^{p+1}\frac{\alpha^2_{\mathrm{MLE}\,j}}{(\lambda_j+k)^2}, \tag{2.11}$$

where $\hat{\boldsymbol{\alpha}}^2{}_{\mathrm{MLE}} = \boldsymbol{\gamma}^T\hat{\boldsymbol{\beta}}_{\mathrm{BMLE}}$.

2.3. Beta modified ridge-type estimator

Akram et al. [16] proposed the beta-modified ridge-type estimator (BMRTE), an extension of the beta ridge regression, which incorporates two shrinkage parameters to provide a more flexible approach to regularization. The BMRTE is defined as

$$\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}} = (\mathbf{U} + k(1+d)\mathbf{I}_s)^{-1}\mathbf{U}\hat{\boldsymbol{\beta}}_{\mathrm{BMLE}}, \quad 0 < d < 1,\ k > 0, \tag{2.12}$$

where d and k are the modified ridge-type shrinkage parameters that control the regularization effect in this case, using $\hat{k}$ as in BRRE and $\hat{d} = \max\left(\frac{\hat{\alpha}^2{}_{\mathrm{MLE}\,j} - 1}{\hat{\alpha}^2{}_{\mathrm{MLE}\,j} + \frac{1}{\lambda_j}}\right)$ [26, 38]. When $k = 0$, the BMRTE reduces to the BMLE. Similarly, when $d = 0$, the estimator simplifies to the BRRE.

The bias vector and variance-covariance matrix for BMRTE are given as follows:

$$\mathrm{Bias}\left(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}}\right) = \mathrm{E}\left(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}}\right) - \boldsymbol{\beta} = -k(1+d)\boldsymbol{\Lambda}_{kd}^{-1}\boldsymbol{\beta}, \tag{2.13}$$

$$\text{Var-Cov}\left(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}}\right) = (\mathbf{U}+k(1+d)\mathbf{I}_s)^{-1}\mathbf{U}\text{Var-Cov}(\hat{\boldsymbol{\beta}}_{\mathrm{BMLE}})(\mathbf{U}+k(1+d)\mathbf{I}_s)^{-1}\mathbf{U} = \hat{\phi}\boldsymbol{\Lambda}_{kd}^{-1}\boldsymbol{\Lambda}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{kd}^{-1}\boldsymbol{\Lambda}. \tag{2.14}$$

Then the MMSE and MSE for BMRTE are given by

$$\mathrm{MMSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}}) = \hat{\phi}\boldsymbol{\gamma}\boldsymbol{\Lambda}_{kd}^{-1}\boldsymbol{\Lambda}\boldsymbol{\Lambda}_{kd}^{-1}\boldsymbol{\gamma}^T + k^2(d+1)^2\boldsymbol{\gamma}\boldsymbol{\Lambda}_{kd}^{-1}\hat{\boldsymbol{\alpha}}_{\mathrm{MLE}}\hat{\boldsymbol{\alpha}}_{\mathrm{MLE}}{}^T\boldsymbol{\Lambda}_{kd}^{-1}\boldsymbol{\gamma}^T, \tag{2.15}$$

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}}) = \hat{\phi}\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j + k(d+1))^2} + \sum_{j=1}^{p+1}\frac{k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{MLE}\,j}}{(\lambda_j + k(d+1))^2}. \tag{2.16}$$

2.4. Beta robust estimators

The MLE, BRRE, and BMRTE are known to be sensitive to outliers. This sensitivity highlights the need for robust alternatives in BRM. As a result, several robust estimators have been developed to improve estimation accuracy in the presence of outliers. This section summarizes two well-known robust methods designed for BRM.

Ghosh [21] introduced the minimum density power divergence estimator (MDPDE), a robust approach tailored for BRM. This method introduces a tuning parameter $\delta \geq 0$, which controls the trade-off between efficiency and robustness, and is based on the concept of density power divergence. The estimator solves the following estimating equation:

$$\sum_{i=1}^{n}\left[A(y_i;\theta)f_\theta(y_i;\mu_i,\phi_i)^\delta - E_{i,1-\delta}(\theta)\right] = 0, \tag{2.17}$$

where

- $f_\theta(y_i;\mu_i,\phi_i)$ is the beta density function,
- $A(y_i;\theta)$ is the score function (i.e., the gradient of the log-likelihood),
- $E_{i,1-\delta}(\theta)$ is the expected value of the weighted score function.

In this formulation, the term f_θ^δ acts as a weight, lowering observations that do not align well with the model, especially when $\delta \in (0, 1)$ [26].

Ribeiro and Ferrari [22] proposed a surrogate maximum likelihood estimator (SMLE) based on a modified likelihood function, known as the L_q-likelihood. This method reparametrizes the likelihood using the function $L_q(a)$, leading to the objective function $L_q(\theta) = \sum_{i=1}^n L_q(f_\theta(y_i; \mu_i, \phi_i))$ with $L_q(a) = \frac{a^{1-q}-1}{1-q}, \quad \text{for } q \in (0, 1), \text{ and } q = 1 - \delta.$

The estimation equation becomes

$$\sum_{i=1}^{n} \left[A(y_i; \theta) f_\theta(y_i; \mu_i, \phi_i)^\delta \right] = 0. \tag{2.18}$$

Although this approach is simple and useful in practice, the resulting estimator is not Fisher consistent unless $\delta = 0$. Both the MDPDE and SMLE estimators rely on the assumption that the beta densities are bounded, ensuring the robustness and validity of their asymptotic properties. Ghosh [21] achieved Fisher consistency by adjusting the weighted score function, while Ribeiro and Ferrari [22] obtained a Fisher-consistent estimator by carefully parameterizing the L_q-likelihood.

Although the MDPDE and SMLE methods offer robust estimation, they rely on the assumption that the beta densities are bounded. However, when this assumption doesn't hold (i.e., the beta densities are unbounded), numerical issues may occur, as observed in simulation studies. This means these estimators may not always work reliably in all scenarios. To address this limitation, we propose alternative robust estimators in the following section that perform well regardless of the type of beta density.

The main issue with MDPDE and SMLE is that beta densities are not always closed under power transformations. For any given density v and a constant $\xi > 0$, the power transformation is $v^\xi(y) = \frac{v(y)^\xi}{\int v(y)^\xi dy}$ This works only if the integral is finite. For the beta density, this transformation is valid only when certain conditions hold: $\mu\phi \geq 1$ and $(1 - \mu)\phi \geq 1$. In simpler terms, only a specific subset of beta distributions (bounded ones) can handle this transformation, limiting its use.

To overcome this issue, a logit transformation is applied to the response variable: $y^* = \log\left(\frac{y}{1-y}\right)$ If y follows a beta distribution, then y^* follows what's known as the exponential generalized beta of the second type (EGB) distribution.

Using the transformed variable y^*, we define two robust estimators:

- Logit minimum density power divergence estimator (LMDPDE): This estimator builds on earlier methods by Ribeiro and Ferrari [22] and Ghosh [21]. It minimizes the divergence between observed and expected densities using the transformed data. The objective function is

$$H_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \left[K_{i,1+\delta}(\theta) - \alpha h_\theta(y_i^*; \mu_i, \phi_i)^\delta \right]. \tag{2.19}$$

  The corresponding estimating equation balances the influence of each data point based on how likely it is under the model. The parameter $\delta \in [0, 1)$ controls robustness such that when $\delta = 0$, it reduces to the usual MLE and for $\delta > 0$, the method becomes more robust to outliers.
- Logit surrogate maximum likelihood estimator (LSMLE): Inspired by the work of Ribeiro and Ferrari [22], this method maximizes a modified likelihood function—specifically, the L_q-likelihood—using the EGB-transformed densities. This approach maintains statistical properties

like consistency while being robust. The LSMLE solves the equation:

$$\sum_{i=1}^{n} A^*(y_i^*;\theta) h_\theta^*(y_i^*;\mu_i,\phi_i)^\delta = 0, \tag{2.20}$$

where $A^*(y_i^*;\theta)$ is the modified score vector [26]. By leveraging the stability of the EGB distribution under power transformations, LSMLE offers a reliable alternative when the original methods (MDPDE and SMLE) fall short.

2.5. Robust beta ridge regression estimators

Based on the works of Ribeiro and Ferrari [22] and Ghosh [21], Olaluwoye et al. [26] proposed a robust beta ridge regression estimator, which is derived by combining the ridge estimator with four robust beta estimators: MDPDE, SMLE, LMDPDE, and LSMLE. The expressions for the robust beta estimators of $\boldsymbol{\beta}$ are given by

$$\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{LSMLE}}} = (\mathbf{U} + k\mathbf{I}_s)^{-1}\mathbf{U}\hat{\boldsymbol{\beta}}_{\mathrm{LSMLE}}, \tag{2.21}$$

$$\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{LMDPDE}}} = (\mathbf{U} + k\mathbf{I}_s)^{-1}\mathbf{U}\hat{\boldsymbol{\beta}}_{\mathrm{LMDPDE}}, \tag{2.22}$$

$$\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{SMLE}}} = (\mathbf{U} + k\mathbf{I}_s)^{-1}\mathbf{U}\hat{\boldsymbol{\beta}}_{\mathrm{SMLE}}, \tag{2.23}$$

$$\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{MDPDE}}} = (\mathbf{U} + k\mathbf{I}_s)^{-1}\mathbf{U}\hat{\boldsymbol{\beta}}_{\mathrm{MDPDE}}, \tag{2.24}$$

where $k(k > 0)$ is ridge parameter, and I_s represents the identity matrix of order $(p+1)\times(p+1)$.

The MSE expressions for these estimators are as follows:

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{LSMLE}}}) = \hat{\phi}\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j+k)^2} + k^2\sum_{j=1}^{p+1}\frac{\alpha^2_{\mathrm{LSMLE}\,j}}{(\lambda_j+k)^2}, \tag{2.25}$$

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{LMDPDE}}}) = \hat{\phi}\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j+k)^2} + k^2\sum_{j=1}^{p+1}\frac{\alpha^2_{\mathrm{LMDPDE}\,j}}{(\lambda_j+k)^2}, \tag{2.26}$$

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{SMLE}}}) = \hat{\phi}\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j+k)^2} + k^2\sum_{j=1}^{p+1}\frac{\alpha^2_{\mathrm{SMLE}\,j}}{(\lambda_j+k)^2}, \tag{2.27}$$

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{MDPDE}}}) = \hat{\phi}\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j+k)^2} + k^2\sum_{j=1}^{p+1}\frac{\alpha^2_{\mathrm{MDPDE}\,j}}{(\lambda_j+k)^2}, \tag{2.28}$$

where $\boldsymbol{\alpha}^2_{\mathrm{r}\,j}$ denotes the squared weight associated with the j-th predictor under estimator r, and is computed as the inner product of the vector $\boldsymbol{\gamma}$ and the estimated coefficient vector $\hat{\beta}_r$, for each $r \in$ {LSMLE, LMDPDE, SMLE, MDPDE}.

3. Proposed estimator

Building on the work of Ghosh [21] and Ribeiro and Ferrari [22], and extending the approach proposed by Olaluwoye et al. [26], we propose a novel robust estimator for beta regression. This estimator is constructed by combining a modified ridge-type estimator with four robust beta estimators: the MDPDE, the SMLE, the LMDPDE, and the LSMLE. The resulting estimator effectively addresses both multicollinearity and the presence of outliers. Additionally, it encompasses several existing estimators, including robust beta ridge regression and the classical maximum likelihood estimator, as special cases, thereby offering a more flexible and comprehensive modeling approach.

The expressions for the robust beta modified ridge-type estimators for $\boldsymbol{\beta}$ are given by the following equations:

$$\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}} = (\mathbf{U} + k(1+d)\mathbf{I}_s)^{-1}\mathbf{U}\hat{\boldsymbol{\beta}}_{\text{LSMLE}}, \tag{3.1}$$

$$\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LMDPDE}}} = (\mathbf{U} + k(1+d)\mathbf{I}_s)^{-1}\mathbf{U}\hat{\boldsymbol{\beta}}_{\text{LMDPDE}}, \tag{3.2}$$

$$\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{SMLE}}} = (\mathbf{U} + k(1+d)\mathbf{I}_s)^{-1}\mathbf{U}\hat{\boldsymbol{\beta}}_{\text{SMLE}}, \tag{3.3}$$

$$\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{MDPDE}}} = (\mathbf{U} + k(1+d)\mathbf{I}_s)^{-1}\mathbf{U}\hat{\boldsymbol{\beta}}_{\text{MDPDE}}. \tag{3.4}$$

The proposed robust beta modified ridge-type estimators are flexible and encompass several well-known estimators as special cases under specific choices of the shrinkage parameters k and d. In particular:

- When $k = 0$: The robust beta modified ridge-type estimators reduce to their corresponding base robust estimators. For instance:

$$\begin{aligned}
&\text{i-}\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}} = \hat{\boldsymbol{\beta}}_{\text{LSMLE}},\\
&\text{ii-}\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LMDPDE}}} = \hat{\boldsymbol{\beta}}_{\text{LMDPDE}},\\
&\text{iii-}\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{SMLE}}} = \hat{\boldsymbol{\beta}}_{\text{SMLE}},\\
&\text{iv-}\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{MDPDE}}} = \hat{\boldsymbol{\beta}}_{\text{MDPDE}}.
\end{aligned}$$

- When $d = 0$: The estimators reduce to the robust beta ridge estimator forms based on the corresponding robust estimators:

$$\hat{\boldsymbol{\beta}}_{\text{BMRTE}_r} = \hat{\boldsymbol{\beta}}_{\text{BRRE}_r}.$$

- When $k = 0$ and $d = 0$: The estimators reduce to the MLE if the base estimator used is the MLE:

$$\hat{\boldsymbol{\beta}}_{\text{BMRTE}_r} = \hat{\boldsymbol{\beta}}_{\text{MLE}},$$

where $r \in \{\text{LSMLE}, \text{LMDPDE}, \text{SMLE}, \text{MDPDE}\}$. These special cases demonstrate that the BMRTE framework generalizes several existing estimators, offering robustness against multicollinearity and outliers, while maintaining interpretability and flexibility.

The SMSE expressions for these estimators are as follows:

$$\text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}}) = \hat{\phi}\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j + k(d+1))^2} + \sum_{j=1}^{p+1}\frac{k^2(d+1)^2\hat{\alpha}^2{}_{\text{LSMLE}j}}{(\lambda_j + k(d+1))^2}, \tag{3.5}$$

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LMDPDE}}}) = \hat{\phi}\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j + k(d+1))^2} + \sum_{j=1}^{p+1}\frac{k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LMDPDE}\,j}}{(\lambda_j + k(d+1))^2}, \tag{3.6}$$

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{SMLE}}}) = \hat{\phi}\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j + k(d+1))^2} + \sum_{j=1}^{p+1}\frac{k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{SMLE}\,j}}{(\lambda_j + k(d+1))^2}, \tag{3.7}$$

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{MDPDE}}}) = \hat{\phi}\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j + k(d+1))^2} + \sum_{j=1}^{p+1}\frac{k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{MDPDE}\,j}}{(\lambda_j + k(d+1))^2}. \tag{3.8}$$

3.1. Theoretical comparisons between estimators

In this section, we present the theoretical comparisons between the proposed estimators and the existing ones, using the MSE as the primary criterion for evaluation. Since we introduced four types of robust estimators, we focus our theoretical comparison on one of them, for instance, the estimator based on the LSMLE, to illustrate the methodology. The same approach can be applied analogously to the other robust estimators (SMLE, MDPDE, and LMDPDE), as they share a similar mathematical structure. This allows us to assess the efficiency of the proposed estimators in terms of reducing MSE while addressing issues such as multicollinearity and potential outliers in the data.

Theorem 1. *The proposed estimator $\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}$ is superior to $\hat{\boldsymbol{\beta}}_{BMLE}$ in the MSE sense. Formally,*

$$MSE(\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}) - MSE(\hat{\boldsymbol{\beta}}_{BMLE}) < 0,$$

for $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$, if, and only if,

$$\hat{\phi}\lambda_j^2 + \lambda_j k^2(d+1)^2\hat{\alpha}^2{}_{LSMLE\,j} - \hat{\phi}(\lambda_j + k(d+1))^2 < 0, \quad \textit{for all } j = 1, ..., p+1.$$

Proof. The difference between $\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}})$ and $\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMLE}})$, as computed using Eq (3.5) and Eq (2.8), is given by

$$\begin{aligned}\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}) - \mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMLE}}) =& \hat{\phi}\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j + k(d+1))^2} + \sum_{j=1}^{p+1}\frac{k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j}}{(\lambda_j + k(d+1))^2} - \hat{\phi}\sum_{j=1}^{p+1}\frac{1}{\lambda_j} \\ =& \sum_{j=1}^{p+1}\left(\frac{\hat{\phi}\lambda_j^2 + \lambda_j k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j} - \hat{\phi}(\lambda_j + k(d+1))^2}{\lambda_j(\lambda_j + k(d+1))^2}\right).\end{aligned} \tag{3.9}$$

Hence, the difference is negative, i.e., $\mathrm{MSE}\left(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}\right) - \mathrm{MSE}\left(\hat{\boldsymbol{\beta}}_{\mathrm{BMLE}}\right) < 0$, whenever $\hat{\phi}\lambda_j^2 + \lambda_j k^2(d+1)^2\hat{\alpha}^2_{\mathrm{LSMLE},j} - \hat{\phi}(\lambda_j + k(d+1))^2 < 0$, under the regularity conditions $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$, for all $j = 1, \ldots, p+1$. The proof is completed. □

Theorem 2. *The proposed estimator $\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}$ is superior to $\hat{\boldsymbol{\beta}}_{BRRE}$ in the MSE sense. Formally,*

$$MSE(\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}) - MSE(\hat{\boldsymbol{\beta}}_{BRRE}) < 0,$$

for $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$ if, and only if,

$$(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{LSMLE\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{MLE\,j}) < 0, \quad \textit{for all } j = 1, ..., p+1.$$

Proof. The difference between $\text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}})$ and $\text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BRRE}})$, as computed using Eq (3.5) and Eq (2.11), is given by

$$\begin{aligned}\text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}}) - \text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BRRE}}) &= \sum_{j=1}^{p+1} \frac{\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\text{LSMLE}\,j}}{(\lambda_j + k(d+1))^2} - \sum_{j=1}^{p+1} \frac{\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\text{MLE}\,j}}{(\lambda_j + k)^2} \\ = \sum_{j=1}^{p+1} \left(\frac{(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\text{LSMLE}\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\text{MLE}\,j})}{(\lambda_j + k)^2(\lambda_j + k(d+1))^2} \right)&.\end{aligned} \tag{3.10}$$

Hence, the difference is negative, i.e., $\text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}}) - \text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BRRE}}) < 0$, whenever $(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\text{LSMLE}\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\text{MLE}\,j}) < 0$, under the regularity conditions $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$, for all $j = 1, ..., p+1$. The proof is completed. □

Theorem 3. *The proposed estimator $\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}$ is superior to $\hat{\boldsymbol{\beta}}_{BMRTE}$ in the MSE sense. Formally,*

$$MSE(\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}) - MSE(\hat{\boldsymbol{\beta}}_{BMRTE}) < 0,$$

for $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$, if, and only if,

$$k^2(d+1)^2\hat{\alpha}^2{}_{LSMLE\,j} - k^2(d+1)^2\hat{\alpha}^2{}_{MLE\,j} < 0, \quad \textit{for all } j = 1, ..., p+1.$$

Proof. The difference between $\text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}})$ and $\text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BMRTE}})$, as computed using Eq (3.5) and Eq (2.16), is given by

$$\begin{aligned}\text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}}) - \text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BMRTE}}) &= \sum_{j=1}^{p+1} \frac{\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\text{LSMLE}\,j}}{(\lambda_j + k(d+1))^2} - \sum_{j=1}^{p+1} \frac{\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\text{MLE}\,j}}{(\lambda_j + k(d+1))^2} \\ &= \sum_{j=1}^{p+1} \left(\frac{k^2(d+1)^2\hat{\alpha}^2{}_{\text{LSMLE}\,j} - k^2(d+1)^2\hat{\alpha}^2{}_{\text{MLE}\,j}}{(\lambda_j + k(d+1))^2} \right).\end{aligned} \tag{3.11}$$

Hence, the difference is negative, i.e., $\text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}}) - \text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BMRTE}}) < 0$, whenever $k^2(d+1)^2\hat{\alpha}^2{}_{\text{LSMLE}\,j} - k^2(d+1)^2\hat{\alpha}^2{}_{\text{MLE}\,j} < 0$, under the regularity conditions $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$, for all $j = 1, ..., p+1$. The proof is completed. □

Theorem 4. *The proposed estimator $\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}$ is superior to $\hat{\boldsymbol{\beta}}_{BRRE_{LSMLE}}$ in the MSE sense. Formally,*

$$MSE(\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}) - MSE(\hat{\boldsymbol{\beta}}_{BRRE_{LSMLE}}) < 0,$$

for $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$, if, and only if,

$$d\lambda_j \left[(k^2\hat{\alpha}^2_{LSMLE,j}(d+2) - 2\hat{\phi}k)\lambda_j + k^2(2k\hat{\alpha}^2_{LSMLE,j}(d+1) - \hat{\phi}(d+2)) \right] < 0, \quad \textit{for all } j = 1, ..., p+1.$$

Proof. The difference between $\text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}})$ and $\text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BRRE}_{\text{LSMLE}}})$, as computed using Eq (3.5) and Eq (2.25), is given by

$$\begin{aligned}\text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}}) - \text{MSE}(\hat{\boldsymbol{\beta}}_{\text{BRRE}_{\text{LSMLE}}}) &= \sum_{j=1}^{p+1} \frac{\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\text{LSMLE}\,j}}{(\lambda_j + k(d+1))^2} - \sum_{j=1}^{p+1} \frac{\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\text{LSMLE}\,j}}{(\lambda_j + k)^2} \\ = \sum_{j=1}^{p+1} \left(\frac{d\lambda_j \left[(k^2\hat{\alpha}^2_{\text{LSMLE},j}(d+2) - 2\hat{\phi}k)\lambda_j + k^2(2k\hat{\alpha}^2_{\text{LSMLE},j}(d+1) - \hat{\phi}(d+2)) \right]}{(\lambda_j + k)^2(\lambda_j + k(d+1))^2} \right)&.\end{aligned} \tag{3.12}$$

Hence, the difference is negative, i.e., $\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}) - \mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{LSMLE}}}) < 0$, whenever $d\lambda_j\left[(k^2\hat{\alpha}^2_{\mathrm{LSMLE},j}(d+2) - 2\hat{\phi}k)\lambda_j + k^2(2k\hat{\alpha}^2_{\mathrm{LSMLE},j}(d+1) - \hat{\phi}(d+2))\right] < 0$, under the regularity conditions $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$, for all $j = 1, ..., p+1$. The proof is completed. □

Theorem 5. *The proposed estimator $\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}$ is superior to $\hat{\boldsymbol{\beta}}_{BRRE_{LMDPDE}}$ in the MSE sense. Formally,*

$$MSE(\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}) - MSE(\hat{\boldsymbol{\beta}}_{BRRE_{LMDPDE}}) < 0,$$

for $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$, if, and only if,

$$(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{LSMLE\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{LMDPDE\,j}) < 0, \textit{ for all } j = 1, ..., p+1.$$

Proof. The difference between $\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}})$ and $\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{LMDPDE}}})$, as computed using Eq (3.5) and Eq (2.26), is given by

$$\begin{aligned} \mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}) - \mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{LMDPDE}}}) &= \sum_{j=1}^{p+1} \frac{\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j}}{(\lambda_j + k(d+1))^2} - \sum_{j=1}^{p+1} \frac{\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\mathrm{LMDPDE}\,j}}{(\lambda_j + k)^2} \\ = \sum_{j=1}^{p+1}\left(\frac{(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\mathrm{LMDPDE}\,j})}{(\lambda_j + k)^2(\lambda_j + k(d+1))^2}\right). \end{aligned} \tag{3.13}$$

Hence, the difference is negative, i.e., $\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}) - \mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{LMDPDE}}}) < 0$, whenever $(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\mathrm{LMDPDE}\,j}) < 0$, under the regularity conditions $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$, for all $j = 1, ..., p+1$. The proof is completed. □

Theorem 6. *The proposed estimator $\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}$ is superior to $\hat{\boldsymbol{\beta}}_{BRRE_{SMLE}}$ in the MSE sense. Formally,*

$$MSE(\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}) - MSE(\hat{\boldsymbol{\beta}}_{BRRE_{SMLE}}) < 0,$$

for $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$, if, and only if,

$$(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{LSMLE\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{SMLE\,j}) < 0, \textit{ for all } j = 1, ..., p+1.$$

Proof. The difference between $\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}})$ and $\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{SMLE}}})$, as computed using Eq (3.5) and Eq (2.27), is given by

$$\begin{aligned} \mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}) - \mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{SMLE}}}) &= \sum_{j=1}^{p+1} \frac{\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j}}{(\lambda_j + k(d+1))^2} - \sum_{j=1}^{p+1} \frac{\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\mathrm{SMLE}\,j}}{(\lambda_j + k)^2} \\ = \sum_{j=1}^{p+1}\left(\frac{(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\mathrm{SMLE}\,j})}{(\lambda_j + k)^2(\lambda_j + k(d+1))^2}\right). \end{aligned} \tag{3.14}$$

Hence, the difference is negative, i.e., $\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}) - \mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{SMLE}}}) < 0$, whenever $(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\mathrm{SMLE}\,j}) < 0$, under the regularity conditions $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$, for all $j = 1, ..., p+1$. The proof is completed. □

Theorem 7. *The proposed estimator* $\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}$ *is superior to* $\hat{\boldsymbol{\beta}}_{BRRE_{MDPDE}}$ *in the MSE sense. Formally,*

$$MSE(\hat{\boldsymbol{\beta}}_{BMRTE_{LSMLE}}) - MSE(\hat{\boldsymbol{\beta}}_{BRRE_{MDPDE}}) < 0,$$

$\hat{\phi} > 0$, $k > 0$, *and* $0 < d < 1$, *if, and only if,*

$$(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{LSMLE\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{MDPDE\,j}) < 0, \textit{ for all } j = 1, ..., p+1.$$

Proof. The difference between $\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}})$ and $\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{MDPDE}}})$, as computed using Eq (3.5) and Eq (2.28), is given by

$$\begin{aligned}\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}) - \mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{MDPDE}}}) &= \sum_{j=1}^{p+1}\frac{\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j}}{(\lambda_j + k(d+1))^2} - \sum_{j=1}^{p+1}\frac{\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\mathrm{MDPDE}\,j}}{(\lambda_j + k)^2} \\ = \sum_{j=1}^{p+1}\left(\frac{(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\mathrm{MDPDE}\,j})}{(\lambda_j + k)^2(\lambda_j + k(d+1))^2}\right).\end{aligned} \tag{3.15}$$

The difference $\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}) - \mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{MDPDE}}}) < 0$, whenever $(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\mathrm{MDPDE}\,j}) < 0$, under the regularity conditions $\hat{\phi} > 0$, $k > 0$, and $0 < d < 1$, for all $j = 1, ..., p+1$. The proof is completed. □

These methodologies are applied to the breast cancer patient dataset in Section 5.

4. Monte Carlo simulation study

We conducted a comprehensive Monte Carlo simulation study to assess the performance of different robust and non-robust estimators in BRM. Similar simulation frameworks have been employed in previous research [26,39,40]. Following the approach used in Lukman et al. [28], Alghamdi et al. [34], and Mohammad et al. [27], the explanatory variables were generated using the following model:

$$x_{ij} = \sqrt{1-\rho^2}\, z_{ij} + \rho z_{i(p+1)}, \qquad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, (p+1), \tag{4.1}$$

where n is the sample size, p is the number of predictors, and ρ controls the degree of multicollinearity among the predictors. The variables z_{ij} are drawn from a standard normal distribution with mean 0 and standard deviation 1. This structure ensures that the correlation between predictors is approximately ρ^2. The MSE is calculated as a function of $\boldsymbol{\beta}$, with the constraint $\boldsymbol{\beta}'\boldsymbol{\beta} = 1$ as discussed in Alghamdi et al. [34] and Olaluwoye et al. [26].

To evaluate the robustness of the estimators, we introduced outliers into the dataset through a controlled process. First, a set number of outliers, either 10% or 20% of the total observations, was determined. These outliers were then randomly assigned to specific observation indices. For each selected index, outliers were added to the first and fourth predictors (x_1 and x_4) by replacing their original values with samples drawn from a normal distribution with a mean of 5 and a standard deviation of 2. This intentional distortion enables us to rigorously examine how well the estimators perform in the presence of anomalous data.

Following the works of Alghamdi et al. [34], Mohammad et al. [27], and Alqasem et al. [30] in robust regression, this study conducts an extensive Monte Carlo simulation to evaluate the performance of competing estimators under a wide range of data conditions. The primary objective of the simulation is to generate data that accurately emulates challenging real-world scenarios, including multicollinearity and outliers, thereby providing a rigorous test of estimator robustness and efficiency. The simulation design incorporates a comprehensive factorial array of parameters known to influence estimator performance. This includes varying sample sizes ($n = 35, 50, 75, 100, 200, 300$), numbers of predictors ($p = 4, 8$), and degrees of multicollinearity ($\rho = 0.80, 0.90, 0.95, 0.99, 0.999$). Furthermore, the study investigates the impact of overdispersion ($\phi = 5, 10$) and the presence of outliers at contamination levels of 10% and 20%. Estimator performance was evaluated using both the MSE and the median squared error, calculated as the average squared deviation between the estimated and true parameter values across 1000 replications. All simulations were implemented in R, using the `betareg()` function for standard estimation.

The MSE was computed using the following formula:

$$\mathrm{MSE}(\widehat{\boldsymbol{\beta}}) = \frac{1}{1000}\sum_{j=1}^{1000}(\widehat{\boldsymbol{\beta}}_{ij} - \boldsymbol{\beta}_i)^T(\widehat{\boldsymbol{\beta}}_{ij} - \boldsymbol{\beta}_i), \tag{4.2}$$

where $\widehat{\boldsymbol{\beta}}_{ij}$ is the estimate of the i^{th} parameter in the j^{th} replication, and $\boldsymbol{\beta}_i$ is the true parameter value. In addition to classical estimators, the simulation also incorporated four robust estimators available through the `robustbetareg` package in R.

In both the simulation study and the empirical application, the shrinkage parameter k for the robust and non-robust BRREs was determined following the approaches of Abonazel and Taha [35], Akram et al. [16], and Qasim et al. [14], defined as $k = \frac{\hat{\phi}}{(p+1)\sum_{j=1}^{p+1}\hat{\alpha}_j^2}$. For the robust and non-robust BMRTEs, the same value of k was employed as in the BRRE, while the additional shrinkage parameter d was estimated using $\hat{d} = \max\left(\frac{\hat{\alpha}^2_{\mathrm{MLE},j}-1}{\hat{\alpha}^2_{\mathrm{MLE},j}+\frac{1}{\lambda_j}}\right)$.

The simulation results are summarized in Tables 1–8 and Figures 1–8. These results comprehensively assess the performance of MLE, BRRE, BMRTE, $\mathrm{BRRE_{LSMLE}}$, $\mathrm{BRRE_{LMDPDE}}$, $\mathrm{BRRE_{SMLE}}$, $\mathrm{BRRE_{MDPDE}}$, $\mathrm{BMRTE_{LSMLE}}$, $\mathrm{BMRTE_{LMDPDE}}$, $\mathrm{BMRTE_{SMLE}}$, and $\mathrm{BMRTE_{MDPDE}}$, under varying conditions, including different sample sizes (n), multicollinearity levels (ρ), numbers of predictors (p), levels of contamination by outliers, and standard deviations (ϕ). The key findings from these simulations are as follows:

- **Sample Size (n):** As expected, MSE decreased for all estimators as sample size increased from $n = 30$ to $n = 300$. Reflecting improved estimation accuracy with more data. This pattern holds across both clean and contaminated datasets. However, the performance advantage of the robust BMRTE estimators was most pronounced in smaller samples ($n = 30, 50, 75$), where their stability is most critical.
- **Multicollinearity (ρ):** Increased multicollinearity increases MSE for all robust and non-robust estimators, especially for non-robust methods such as MLE, BRRE, and BMRTE. This suggests that multicollinearity adversely affects the precision of parameter estimates.
- **Number of explanatory variables(p):** Increasing model complexity from $p = 4$ to $p = 8$ led to higher MSE across all scenarios, particularly under small sample sizes and strong multicollinearity.

- **Dispersion (ϕ):** A higher dispersion parameter ($\phi = 10$ vs. $\phi = 5$) increased MSE for all estimators, as expected. The robust estimators maintained their relative superiority, proving effective in high-noise environments.
- **Outliers:** Increasing the proportion of outliers from 10% to 20% led to higher MSE values across all estimators, with the effect being particularly severe for non-robust estimators, thereby reducing their reliability.
- **BMLE:** Exhibited pronounced sensitivity to both multicollinearity and outliers, which consistently resulted in higher MSE values and substantially reduced its reliability in practical applications.

  Non-robust biased estimators such: Include BRRE and BMRTE. While these estimators handled multicollinearity well, they offered no protection against outliers. Their MSE values frequently doubled or tripled with 20% outliers, demonstrating that shrinkage alone does not confer robustness.
- **Robust biased estimators:** These include the robust versions of BRRE and BMRTE implemented with LSMLE, MDPDE, LSMLE, and SMLE. In clear contrast to their non-robust counterparts, the robust estimators maintained notable stability under both multicollinearity and contamination. This highlights their primary role in ensuring reliable inference when data are simultaneously affected by strong multicollinearity and outliers.
- The biased estimators, BRRE and BMRTE, clearly outperform BMLE in the presence of multicollinearity, primarily due to their built-in shrinkage mechanisms that stabilize parameter estimates and reduce variance. Furthermore, BMRTE outperforms BRRE in reducing MSE. However, a critical limitation of these non-robust estimators is their pronounced susceptibility to contamination by outliers, which significantly inflates their MSE and significantly reduces their reliability. In contrast, the proposed robust estimators demonstrate superior performance in the presence of data contamination, effectively mitigating the impact of outlier observations. It is worth noting that the robust $\text{BMRTE}_{\text{LSMLE}}$, $\text{BMRTE}_{\text{SMLE}}$, and $\text{BMRTE}_{\text{MDPDE}}$ offer exceptional estimation accuracy and stability. While the robust $\text{BRRE}_{\text{LMDPDE}}$ and $\text{BRRE}_{\text{LSMLE}}$, show reasonably good and robust performance, the BMRTE-based estimators consistently dominate, achieving the lowest mean estimation error across a wide range of conditions, proving to be the most superior and reliable class of estimators for practical application.
- Figures 1–8 illustrate that the BMLE does the worst (in figures RBRR1 is shorthand for $\text{BRRE}_{\text{LSMLE}}$, to RBRR4 for $\text{BRRE}_{\text{MDPDE}}$. The same applies for the BMRTE estimators, with RBMRT1 representing $\text{BMRTE}_{\text{LSMLE}}$, and so on.), especially when there are problems like multicollinearity and outliers. The BMLE is very sensitive to these issues, leading to poor performance. On the other hand, non-robust estimators BRRE and BMRTE do better than BMLE, but they still struggle when the data has a lot of outliers. Among these, the robust versions of the BRRE estimators (like $\text{BRRE}_{\text{LSMLE}}$, $\text{BRRE}_{\text{MDPDE}}$, and $\text{BRRE}_{\text{SMLE}}$) show clear improvements, especially when there are 10% or 20% outliers. This shows that adding robustness helps the estimator handle bad data better. However, the best-performing estimators are from the BMRTE family (like $\text{BMRTE}_{\text{LSMLE}}$, $\text{BMRTE}_{\text{MDPDE}}$, and $\text{BMRTE}_{\text{SMLE}}$). These estimators consistently perform the best in all situations, with the lowest MSE, meaning they are the most reliable even when the data is messy with outliers or correlations between variables.

Table 1. Estimated MSE for non-robust and robust estimation methods when $p = 4$, $\phi = 5$, and 10% outliers.

ρ	n	Traditional	Non-robust	Robust estimator				Non-robust	Robust estimator			
		BMLE	BRRE	BRRE (LSMLE)	BRRE (LMDPDE)	BRRE (SMLE)	BRRE (MDPDE)	BMRTE	BMRTE (LSMLE)	BMRTE (LMDPDE)	BMRTE (SMLE)	BMRTE (MDPDE)
0.80	30	2.23818	1.65113	0.2469	0.1849	0.1961	0.25378	1.43344	0.17892	0.15282	0.17672	0.24456
	50	1.7627	1.21559	0.06198	0.03763	0.04043	0.06228	1.32083	0.0614	0.03273	0.03621	0.06191
	75	1.61344	1.41339	0.04887	0.032	0.03468	0.04919	0.98612	0.0404	0.02607	0.03075	0.0488
	100	1.28577	0.98369	0.04878	0.03621	0.03837	0.04901	0.94346	0.0391	0.03015	0.03494	0.04873
	200	1.232	1.03634	0.02694	0.01988	0.02096	0.02703	0.83722	0.02696	0.01717	0.01927	0.02693
	300	1.07887	0.89437	0.02278	0.01999	0.02053	0.02283	0.79213	0.02343	0.01707	0.0191	0.02277
0.85	30	3.22865	1.96417	0.25666	0.20452	0.21449	0.26139	1.65916	0.18708	0.17228	0.19646	0.25512
	50	2.10159	1.31215	0.08194	0.06056	0.06394	0.08234	1.3489	0.06645	0.05046	0.05832	0.08185
	75	1.62731	1.42954	0.06644	0.06067	0.06192	0.06681	1.0563	0.05772	0.05593	0.05965	0.06634
	100	1.44806	0.85339	0.04562	0.03897	0.04023	0.04576	0.88682	0.04098	0.03465	0.03811	0.04559
	200	1.21578	1.11681	0.03867	0.03109	0.03254	0.03882	0.81613	0.03059	0.02629	0.03009	0.03863
	300	1.18108	0.98471	0.02368	0.01839	0.01935	0.02374	0.50682	0.02364	0.01521	0.01775	0.02367
0.90	30	4.62448	1.36292	0.53312	0.33239	0.36249	0.54981	2.05495	0.30226	0.25164	0.31227	0.52846
	50	3.42235	2.32639	0.0934	0.06236	0.06661	0.09406	1.58964	0.06187	0.05136	0.05987	0.09325
	75	1.83988	1.66437	0.0844	0.06058	0.06459	0.08498	1.19259	0.06099	0.04922	0.05822	0.08426
	100	1.7386	1.34889	0.06919	0.05849	0.06049	0.06949	0.97203	0.05996	0.05151	0.05698	0.06911
	200	1.57342	1.14832	0.04923	0.03934	0.04112	0.04944	0.74296	0.03694	0.03355	0.03815	0.04918
	300	1.387	0.80407	0.03064	0.02333	0.02466	0.03071	0.58489	0.02602	0.01923	0.02254	0.03062
0.95	30	5.33023	3.19626	0.66259	0.34806	0.38753	0.68406	2.8168	0.35071	0.24318	0.31917	0.65586
	50	5.22417	1.98249	0.19377	0.12032	0.1307	0.19628	1.76749	0.10448	0.0916	0.11387	0.19321
	75	2.77837	1.91867	0.08258	0.05171	0.05628	0.08329	1.73365	0.04796	0.03972	0.04918	0.08242
	100	2.49888	1.71151	0.06697	0.04775	0.05093	0.0674	1.61904	0.04679	0.03822	0.04576	0.06686
	200	1.95866	1.22028	0.04968	0.03664	0.03884	0.04993	1.065	0.03954	0.02991	0.03524	0.04962
	300	1.63745	1.1526	0.0432	0.03283	0.03465	0.04336	0.55228	0.03009	0.02727	0.03174	0.04317
0.99	30	26.60737	3.67301	1.35745	0.35143	0.43758	1.59225	3.09081	0.35566	0.18459	0.3016	1.30339
	50	12.2836	3.62686	1.06275	0.45277	0.53781	1.18304	2.57773	0.44368	0.2629	0.39931	1.03062
	75	10.55861	1.79743	0.56299	0.23159	0.26936	0.58793	1.37667	0.22737	0.14151	0.20805	0.55727
	100	5.35789	1.6595	0.498	0.22175	0.25642	0.52481	1.02274	0.21537	0.13764	0.1997	0.49141
	200	3.14077	1.34161	0.18761	0.10514	0.11691	0.19243	1.00173	0.08777	0.07471	0.09821	0.18651
	300	2.60641	1.09939	0.12658	0.08506	0.09176	0.1286	0.60674	0.07398	0.06515	0.08051	0.12608

Table 2. Estimated MSE for non-robust and robust estimation methods when $p = 4$, $\phi = 5$, and 20% outliers.

ρ	n	Traditional	Non-robust	Robust estimator				Non-robust	Robust estimator			
		BMLE	BRRE	BRRE (LSMLE)	BRRE (LMDPDE)	BRRE (SMLE)	BRRE (MDPDE)	BMRTE	BMRTE (LSMLE)	BMRTE (LMDPDE)	BMRTE (SMLE)	BMRTE (MDPDE)
0.80	30	3.8749	3.17921	0.47107	0.32887	0.35115	0.47975	2.99636	0.31343	0.26813	0.31433	0.4682
	50	3.325	3.06512	0.14821	0.08837	0.09781	0.14862	2.97475	0.09708	0.06551	0.08403	0.14812
	75	2.87535	2.41637	0.1294	0.09476	0.1012	0.12986	2.30602	0.09006	0.07654	0.0915	0.12929
	100	2.67993	2.26012	0.13879	0.10838	0.114	0.13912	2.18952	0.09771	0.09188	0.10543	0.13872
	200	2.531	2.11006	0.09313	0.07192	0.07569	0.09327	1.9834	0.06472	0.05935	0.06947	0.09311
	300	2.32275	2.05674	0.08887	0.08229	0.08358	0.08896	1.96814	0.06976	0.07736	0.08137	0.08886
0.85	30	5.32214	3.90514	0.63928	0.50024	0.52512	0.64674	3.65478	0.45125	0.41869	0.48111	0.63711
	50	3.36755	3.11516	0.19667	0.14936	0.1572	0.19727	3.05415	0.12887	0.12453	0.14435	0.19654
	75	3.28625	2.35553	0.17729	0.15529	0.15963	0.17787	2.31443	0.14127	0.1407	0.15246	0.17714
	100	3.13969	2.27581	0.14238	0.12167	0.12552	0.1426	1.96636	0.10667	0.1082	0.11909	0.14233
	200	2.48196	2.1461	0.11743	0.10133	0.1045	0.11764	1.99094	0.09482	0.09065	0.09932	0.11738
	300	2.3857	2.10208	0.084	0.06896	0.07179	0.08409	1.85218	0.06058	0.05949	0.06722	0.08399
0.90	30	5.74657	4.2361	1.3659	0.85108	0.9255	1.39447	4.03287	0.74767	0.64506	0.80126	1.35832
	50	5.35932	3.41922	0.19856	0.14214	0.15079	0.19937	3.37099	0.1322	0.11677	0.13692	0.19838
	75	3.61779	2.76638	0.20698	0.15808	0.16669	0.20777	2.65483	0.14584	0.13243	0.15321	0.2068
	100	3.23058	2.47441	0.19277	0.16617	0.17114	0.19332	2.31909	0.13275	0.14869	0.16274	0.19265
	200	3.05713	2.01524	0.13332	0.11002	0.11423	0.13362	1.50588	0.0932	0.09614	0.10735	0.13325
	300	2.82472	1.76739	0.09828	0.08239	0.08545	0.09838	1.37372	0.08151	0.07288	0.08075	0.09826
0.95	30	7.10833	5.25115	2.25894	1.13513	1.26968	2.30967	4.30734	1.1106	0.7868	1.04717	2.24639
	50	7.02289	3.95326	0.42148	0.27457	0.29506	0.42497	3.72787	0.24127	0.21716	0.26198	0.42071
	75	4.8097	3.88319	0.18895	0.13594	0.14437	0.18989	3.18977	0.11416	0.11144	0.13107	0.18875
	100	4.51296	3.5039	0.16369	0.12935	0.13516	0.16425	2.59601	0.11522	0.111	0.1257	0.16357
	200	3.78428	2.805	0.13883	0.11048	0.1154	0.13925	2.31029	0.09687	0.0946	0.10732	0.13874
	300	3.37906	2.44077	0.11619	0.09729	0.10068	0.11639	1.51024	0.08838	0.08645	0.09525	0.11614
0.99	30	34.41163	5.33942	3.84577	1.07395	1.30803	3.98942	5.03074	1.02948	0.60297	0.93931	3.74442
	50	14.01101	4.49891	2.0611	0.85588	1.0012	2.24079	3.80164	0.74526	0.51913	0.75559	2.00624
	75	11.71253	3.44133	1.37424	0.56016	0.64667	1.41905	3.75141	0.53074	0.3558	0.50771	1.36438
	100	7.96278	3.33563	1.11703	0.52045	0.59258	1.16129	3.43875	0.50676	0.3422	0.4744	1.10642
	200	5.14339	2.96119	0.352	0.21534	0.23432	0.35834	2.65998	0.18114	0.16504	0.20386	0.35058
	300	4.8189	2.1478	0.27411	0.19432	0.20691	0.2772	1.80348	0.16498	0.15663	0.18585	0.27339

Table 3. Estimated MSE for non-robust and robust estimation methods when $p = 4$, $\phi = 10$, and 10% outliers.

ρ	n	Traditional	Non-robust	Robust estimator				Non-robust	Robust estimator			
		BMLE	BRRE	BRRE (LSMLE)	BRRE (LMDPDE)	BRRE (SMLE)	BRRE (MDPDE)	BMRTE	BMRTE (LSMLE)	BMRTE (LMDPDE)	BMRTE (SMLE)	BMRTE (MDPDE)
0.80	30	3.98398	3.54757	1.25671	0.91549	1.00968	1.39312	3.26314	0.85787	0.73889	0.85643	1.1741
	50	3.44878	2.98579	0.41431	0.40422	0.40336	0.41742	2.56463	0.46234	0.41064	0.40285	0.41325
	75	3.10808	2.7527	0.3258	0.33151	0.32621	0.32765	2.89138	0.37905	0.3468	0.33113	0.32502
	100	2.66266	2.51306	0.33207	0.31713	0.31891	0.33474	2.43744	0.34533	0.31396	0.3145	0.33076
	200	2.48042	2.34252	0.19163	0.20892	0.20306	0.19145	2.24961	0.2706	0.2257	0.21035	0.19167
	300	2.26274	2.1264	0.17207	0.18498	0.18135	0.17169	2.04517	0.22433	0.19674	0.18671	0.1722
0.85	30	5.53882	4.63301	1.61686	1.1783	1.28277	1.7551	4.09746	1.15438	0.9333	1.09126	1.53435
	50	3.6307	3.03408	0.41728	0.3744	0.38075	0.42256	2.69429	0.39999	0.36123	0.36969	0.41566
	75	3.16613	2.94942	0.35143	0.35661	0.35167	0.35364	2.92726	0.41256	0.36981	0.35576	0.35039
	100	3.03526	2.60422	0.24874	0.24471	0.24403	0.24963	2.38361	0.27134	0.25104	0.24492	0.24839
	200	2.42572	2.3324	0.28482	0.28455	0.28314	0.28563	2.26363	0.31504	0.28974	0.28411	0.28435
	300	2.27336	2.12996	0.1839	0.19011	0.18813	0.18378	2.03023	0.22154	0.19697	0.19095	0.18392
0.90	30	8.33461	4.87062	2.24868	1.33759	1.53561	2.57536	4.35058	1.2906	0.90577	1.18822	2.07747
	50	5.83611	4.86237	0.59784	0.48951	0.51036	0.61414	4.1203	0.50049	0.43585	0.47413	0.59174
	75	3.49394	3.33958	0.53035	0.45828	0.47331	0.54293	3.19863	0.47892	0.41908	0.44569	0.52429
	100	3.30985	2.95091	0.36493	0.32028	0.32699	0.37061	2.71265	0.35019	0.30247	0.31335	0.36247
	200	3.00538	2.68443	0.32727	0.29382	0.30006	0.33122	2.47628	0.30295	0.27599	0.28786	0.32533
	300	2.55248	2.16614	0.21843	0.21631	0.21618	0.21889	1.9836	0.23654	0.21703	0.21583	0.21825
0.95	30	9.07345	6.44803	1.92464	0.89093	1.03623	2.21386	5.72455	0.87063	0.61113	0.79438	1.81085
	50	8.58366	4.96358	1.07715	0.70815	0.77506	1.14388	4.28323	0.67574	0.54232	0.66056	1.05309
	75	5.02237	4.21155	0.49106	0.38985	0.40986	0.50846	3.79318	0.40019	0.34138	0.37638	0.48507
	100	4.58647	3.99663	0.44273	0.36976	0.3852	0.45399	3.61819	0.36391	0.32799	0.35828	0.43834
	200	3.72993	3.464	0.31665	0.27235	0.28079	0.32207	3.25687	0.28429	0.251	0.26641	0.31471
	300	3.30791	2.922	0.29893	0.2716	0.2775	0.30216	2.71478	0.27657	0.25498	0.26744	0.29774
0.99	30	47.12558	12.31155	3.27466	1.55295	1.93704	4.39102	10.80985	1.52496	0.84349	1.25138	2.63319
	50	20.64951	10.99776	3.12991	1.80045	2.21345	4.00063	8.81813	1.55186	0.94773	1.44494	2.56159
	75	15.76569	6.98031	1.82486	0.76155	0.91473	2.15354	5.5116	0.85901	0.44326	0.64885	1.69934
	100	9.11107	4.43714	1.59133	0.77378	0.93627	1.92081	3.74876	0.59998	0.45476	0.66593	1.4495
	200	6.36343	4.23202	0.8896	0.51662	0.59113	0.9928	3.56591	0.54827	0.35391	0.46586	0.84745
	300	4.74903	3.36754	0.63524	0.40483	0.4494	0.68401	2.98224	0.40519	0.29488	0.37154	0.61515

Table 4. Estimated MSE for non-robust and robust estimation methods when $p = 4$, $\phi = 10$, and 20% outliers.

ρ	n	Traditional	Non-robust	Robust estimator				Non-robust	Robust estimator			
		BMLE	BRRE	BRRE (LSMLE)	BRRE (LMDPDE)	BRRE (SMLE)	BRRE (MDPDE)	BMRTE	BMRTE (LSMLE)	BMRTE (LMDPDE)	BMRTE (SMLE)	BMRTE (MDPDE)
0.80	30	6.15759	5.66758	2.46623	1.85247	1.98972	2.64361	5.47157	1.65757	1.56496	1.77072	2.38174
	50	5.48342	5.35034	0.98921	0.93796	0.94497	0.99334	5.29175	0.95757	0.92152	0.93269	0.98804
	75	4.97408	4.75551	0.82176	0.8117	0.81293	0.82395	4.64926	0.83298	0.80878	0.80967	0.82088
	100	4.48439	4.11387	0.89272	0.8655	0.87101	0.89557	4.00313	0.86504	0.84995	0.86134	0.89162
	200	4.28987	3.82816	0.63012	0.64436	0.63929	0.63016	3.65964	0.68826	0.65696	0.6449	0.6301
	300	4.05184	3.8001	0.63611	0.64348	0.64155	0.63598	3.54683	0.66487	0.65022	0.64451	0.63615
0.85	30	8.14544	7.00649	2.80673	2.19393	2.3361	2.93415	6.9246	2.06453	1.82003	2.08779	2.74822
	50	5.62173	5.48519	0.95973	0.87159	0.88602	0.96602	5.425	0.86657	0.83387	0.86228	0.95796
	75	5.61152	4.95521	0.87421	0.86044	0.86366	0.87703	4.53182	0.86973	0.85058	0.85702	0.87275
	100	5.1109	4.39858	0.67913	0.66122	0.66419	0.68046	3.98297	0.67428	0.65326	0.65889	0.67868
	200	4.18637	4.06702	0.82954	0.82154	0.82299	0.83073	3.9609	0.83054	0.81662	0.81964	0.82897
	300	3.99128	3.75828	0.63221	0.63447	0.63342	0.63231	3.52669	0.65448	0.63776	0.63456	0.63217
0.90	30	8.51678	7.26985	4.96261	3.05446	3.41515	5.41446	6.9604	2.90775	2.17232	2.78722	4.77798
	50	8.10939	5.67505	1.19165	1.02159	1.05175	1.20671	5.62142	0.99763	0.94083	1.00174	1.18697
	75	5.81092	5.45683	1.19495	1.08778	1.11077	1.20768	4.72944	1.06428	1.02385	1.07191	1.19
	100	5.30037	4.93497	0.88872	0.82382	0.83805	0.89605	4.61259	0.7857	0.77834	0.81302	0.88633
	200	5.0931	4.7801	0.84004	0.7938	0.80351	0.84446	4.55579	0.77982	0.76324	0.78623	0.8384
	300	4.61631	4.01613	0.69701	0.68872	0.6901	0.69761	3.58719	0.69864	0.68473	0.68747	0.69679
0.95	30	10.36219	8.65022	5.65653	2.8714	3.27023	6.16769	8.17144	2.45896	1.99059	2.62312	5.50729
	50	10.24682	7.12794	2.57568	1.74508	1.87653	2.67018	6.56105	1.60519	1.40514	1.65299	2.54389
	75	7.5605	6.83812	1.10812	0.94556	0.97549	1.12684	6.37414	0.9246	0.86819	0.92617	1.10207
	100	7.08362	6.59366	0.9808	0.88026	0.90006	0.99118	6.25846	0.85507	0.82397	0.86647	0.97728
	200	6.16248	5.94612	0.78963	0.73093	0.74282	0.7957	5.84692	0.71579	0.69637	0.72252	0.78758
	300	5.73657	5.34922	0.79873	0.75733	0.76577	0.80206	5.12244	0.74979	0.73226	0.75147	0.79757
0.99	30	53.54163	13.51391	11.64353	5.42388	6.7005	15.22485	11.40885	4.54351	2.98715	4.55771	10.24972
	50	37.9455	9.89207	6.32926	3.65982	4.38473	7.81185	8.45659	2.67948	2.19213	3.15532	5.62359
	75	25.64605	8.91053	4.05866	2.03371	2.35288	4.56183	7.53901	2.12392	1.31431	1.80952	3.88973
	100	10.99714	6.50208	3.45403	1.77697	2.06563	3.96749	5.97096	1.9646	1.1495	1.56654	3.25095
	200	8.41055	6.00051	1.79108	1.25081	1.36121	1.91694	5.26811	1.21951	0.98716	1.17713	1.74611
	300	7.2251	5.76921	1.3667	1.0409	1.11009	1.42933	5.10499	1.00659	0.85437	0.98944	1.34297

Table 5. Estimated MSE for non-robust and robust estimation methods when $p = 8$, $\phi = 5$, and 10% outliers.

ρ	n	Traditional	Non-robust	Robust estimator				Non-robust	Robust estimator			
		BMLE	BRRE	BRRE (LSMLE)	BRRE (LMDPDE)	BRRE (SMLE)	BRRE (MDPDE)	BMRTE	BMRTE (LSMLE)	BMRTE (LMDPDE)	BMRTE (SMLE)	BMRTE (MDPDE)
0.80	30	5.32426	2.92399	1.38863	0.6332	0.7115	1.52667	2.24503	0.57076	0.4973	0.60153	1.35523
	50	2.78709	2.1314	0.13357	0.11925	0.1203	0.13416	2.00785	0.16759	0.12055	0.11882	0.13344
	75	2.55563	1.86764	0.14575	0.12144	0.12544	0.14646	1.71276	0.13248	0.11079	0.11919	0.14559
	100	2.29911	1.86412	0.05254	0.0463	0.04679	0.05267	1.61635	0.08616	0.0469	0.04609	0.05252
	200	1.88056	1.66862	0.03831	0.03191	0.03265	0.03838	1.41193	0.04543	0.03114	0.03153	0.0383
	300	1.79806	1.46953	0.03903	0.03578	0.03595	0.03907	1.24409	0.07702	0.03681	0.03578	0.03902
0.85	30	6.43345	2.97315	1.5563	0.87829	0.95588	1.59887	2.74044	0.7381	0.70211	0.83733	1.54707
	50	4.12232	2.6096	0.25957	0.20563	0.2067	0.26119	2.34658	0.25993	0.21326	0.2054	0.25924
	75	2.80087	2.4222	0.26012	0.19565	0.20362	0.26198	1.88643	0.22419	0.17856	0.19129	0.25973
	100	2.46475	1.66534	0.08938	0.07467	0.07688	0.08965	1.6037	0.09095	0.0689	0.07333	0.08933
	200	1.97699	1.41782	0.05166	0.04812	0.04695	0.05175	1.13835	0.09407	0.056	0.04888	0.05164
	300	1.90991	1.35075	0.0396	0.03557	0.03585	0.03966	1.05773	0.07181	0.03631	0.03549	0.03958
0.90	30	8.76589	3.2144	1.45174	0.72808	0.80425	1.49344	2.92184	0.5515	0.55464	0.68704	1.44311
	50	3.76869	2.16734	0.3245	0.2236	0.23603	0.3276	1.88151	0.22663	0.19525	0.21679	0.32385
	75	3.07688	2.01468	0.31936	0.2517	0.25938	0.32256	1.57771	0.27778	0.23706	0.24766	0.31867
	100	2.55572	1.79546	0.10235	0.09182	0.09175	0.10274	1.45695	0.12619	0.09686	0.09213	0.10227
	200	2.17002	1.64308	0.09342	0.08396	0.08456	0.09367	1.40509	0.09645	0.08526	0.08377	0.09336
	300	1.94456	1.4171	0.037	0.03317	0.03342	0.03705	1.16475	0.05767	0.03398	0.03311	0.03699
0.95	30	16.72502	4.92647	2.73733	1.22329	1.36596	2.91341	3.27857	1.12525	0.90844	1.14431	2.69959
	50	6.4583	3.25201	0.46961	0.2713	0.28964	0.47818	2.60337	0.26854	0.23724	0.26196	0.46785
	75	4.89723	2.80633	0.4165	0.29689	0.3085	0.42162	2.28323	0.3065	0.27477	0.29087	0.41543
	100	3.81666	2.58717	0.19243	0.14963	0.15384	0.19396	2.26716	0.1697	0.14409	0.14752	0.19211
	200	2.74401	2.33211	0.09503	0.07534	0.07555	0.09549	2.23159	0.12167	0.08153	0.07561	0.09494
	300	2.64417	1.55027	0.0805	0.07295	0.07257	0.08071	1.30024	0.10429	0.07772	0.07321	0.08046
0.99	30	66.38526	21.55779	7.54753	3.30369	4.19356	10.11851	16.15912	3.15141	1.85076	2.92258	6.70302
	50	26.89449	6.21287	2.29039	0.87953	0.9998	2.46892	3.37033	0.86212	0.63621	0.81574	2.25468
	75	17.24697	6.00886	2.29095	1.0865	1.25205	2.53302	2.79798	1.0386	0.74234	0.99791	2.23525
	100	11.21879	3.32262	0.73281	0.35515	0.38222	0.75892	2.26361	0.36605	0.31235	0.34194	0.72757
	200	7.96539	1.9305	0.35063	0.2005	0.2101	0.35764	1.33744	0.2344	0.19265	0.19623	0.34923
	300	6.47835	1.48865	0.28659	0.17747	0.1856	0.28969	0.80457	0.19367	0.1676	0.17351	0.28595

Table 6. Estimated MSE for non-robust and robust estimation methods when $p = 8$, $\phi = 5$, and 20% outliers.

ρ	n	Traditional	Non-robust	Robust estimator				Non-robust	Robust estimator			
		BMLE	BRRE	BRRE (LSMLE)	BRRE (LMDPDE)	BRRE (SMLE)	BRRE (MDPDE)	BMRTE	BMRTE (LSMLE)	BMRTE (LMDPDE)	BMRTE (SMLE)	BMRTE (MDPDE)
0.80	30	6.03183	5.34731	4.39709	1.6374	1.86149	4.64766	5.29159	1.33623	1.21384	1.54012	4.34266
	50	5.6615	4.72255	0.40636	0.34221	0.35059	0.40725	4.63312	0.34434	0.32071	0.33714	0.40618
	75	5.9164	4.35949	0.34714	0.26185	0.27417	0.3481	4.21619	0.26532	0.23144	0.25555	0.34694
	100	5.35227	4.24839	0.14538	0.12581	0.12861	0.14552	4.13829	0.14253	0.11871	0.12412	0.14535
	200	4.58084	3.69734	0.11971	0.10502	0.10721	0.11978	3.33053	0.11104	0.09905	0.10366	0.1197
	300	3.93839	3.40422	0.11732	0.10485	0.10535	0.11738	3.2317	0.14422	0.1075	0.10477	0.11731
0.85	30	8.32019	5.56396	6.31385	3.04238	3.37117	6.41421	5.52672	2.38397	2.25953	2.86054	6.29328
	50	6.60477	5.2986	1.21916	0.7775	0.82952	1.2236	5.16504	0.70237	0.65219	0.75013	1.21825
	75	6.05807	4.45751	0.74262	0.5265	0.55564	0.74542	4.42025	0.46551	0.4488	0.50951	0.74205
	100	5.80524	4.00621	0.23237	0.18787	0.19428	0.23269	3.68514	0.19966	0.17049	0.18402	0.23231
	200	4.80435	3.82202	0.14748	0.13171	0.13276	0.14757	3.3143	0.16	0.13222	0.13125	0.14746
	300	4.78643	2.8635	0.11367	0.10224	0.10351	0.11373	2.58	0.12522	0.10021	0.10158	0.11366
0.90	30	10.71589	5.13428	9.60152	3.83866	4.34121	9.74763	4.81408	2.7806	2.69664	3.57012	5.57216
	50	6.52946	4.97365	0.78749	0.51035	0.54565	0.79202	4.52799	0.4482	0.41893	0.49	0.78656
	75	6.10125	4.57888	0.74464	0.54085	0.56604	0.74906	4.41446	0.50367	0.47964	0.52726	0.74372
	100	5.07246	4.1395	0.26374	0.21969	0.22427	0.26422	3.75331	0.23556	0.21127	0.2173	0.26364
	200	5.05438	3.99594	0.25987	0.21934	0.22407	0.26022	3.74455	0.23225	0.20894	0.21666	0.25979
	300	4.98219	3.6552	0.11341	0.10381	0.10473	0.11346	3.47193	0.1282	0.10311	0.10337	0.1134
0.95	30	18.70803	6.56193	12.87734	7.04438	8.18353	13.78167	5.45291	5.59887	4.59079	6.43105	6.69873
	50	9.48829	5.94035	1.81463	0.97688	1.0685	1.83298	5.2677	0.82661	0.75588	0.92513	1.81091
	75	7.84901	5.59758	0.96788	0.61406	0.65464	0.97438	5.20485	0.55597	0.51545	0.59137	0.96655
	100	6.7178	5.27849	0.39197	0.30118	0.31158	0.39355	5.13766	0.30621	0.27816	0.29539	0.39165
	200	6.14107	5.2455	0.22082	0.1646	0.17015	0.22138	5.05607	0.17714	0.15488	0.16163	0.2207
	300	5.6568	4.52196	0.19699	0.17412	0.17596	0.19724	4.30407	0.18907	0.17262	0.17323	0.19694
0.99	30	98.82904	44.74133	27.27003	7.3956	9.49943	36.18372	40.57217	6.76694	3.89491	6.45721	15.2833
	50	30.2868	8.64646	6.93006	2.16406	2.53639	7.27102	5.59133	1.93273	1.38567	1.96262	6.86274
	75	26.75165	7.9603	5.31479	2.28945	2.64932	5.69947	5.5411	2.08955	1.49833	2.08991	5.23206
	100	15.48847	6.51094	1.82086	0.79244	0.88214	1.85891	4.34398	0.74952	0.60486	0.7438	1.8132
	200	13.38036	5.02655	0.91108	0.46222	0.50309	0.92239	4.12049	0.444	0.37842	0.43999	0.9088
	300	11.14839	3.67117	0.70595	0.39112	0.42204	0.71108	2.39053	0.37476	0.32344	0.37352	0.70491

Table 7. Estimated MSE for non-robust and robust estimation methods when $p = 8$, $\phi = 10$, and 10% outliers.

ρ	n	Traditional	Non-robust	Robust estimator				Non-robust	Robust estimator			
		BMLE	BRRE	BRRE (LSMLE)	BRRE (LMDPDE)	BRRE (SMLE)	BRRE (MDPDE)	BMRTE	BMRTE (LSMLE)	BMRTE (LMDPDE)	BMRTE (SMLE)	BMRTE (MDPDE)
0.80	30	7.08557	6.93734	5.83648	4.11309	4.26103	5.61904	6.28964	2.3878	2.60639	2.01585	3.73374
	50	2.91879	2.59173	0.69559	0.60555	0.61579	0.71067	2.35651	0.68807	0.59961	0.60093	0.69167
	75	2.58506	2.4788	0.71874	0.63035	0.64958	0.73728	2.33675	0.66586	0.60002	0.6228	0.71235
	100	2.33199	2.22761	0.32244	0.33726	0.33039	0.32276	2.11515	0.40716	0.36116	0.33933	0.32235
	200	1.97365	1.88386	0.26458	0.27137	0.26865	0.26464	1.84468	0.32406	0.28365	0.27292	0.26455
	300	1.92564	1.83129	0.26761	0.28211	0.27705	0.26745	1.71793	0.34667	0.29915	0.28394	0.26763
0.85	30	8.11991	5.64452	4.38538	2.59312	3.00239	5.0665	4.89559	2.43279	1.90703	2.43185	4.11524
	50	4.35384	3.7353	1.41192	1.03621	1.11047	1.48003	3.37778	1.00365	0.90295	1.00881	1.39242
	75	2.83664	2.71001	1.07756	0.77908	0.8404	1.13658	2.56065	0.75185	0.66188	0.75273	1.05916
	100	2.73869	2.18233	0.48388	0.43081	0.43761	0.49064	1.87462	0.47849	0.42269	0.42686	0.48198
	200	2.10554	1.92015	0.34782	0.34559	0.34327	0.34904	1.85593	0.41538	0.35653	0.34584	0.3475
	300	1.97612	1.90477	0.26245	0.26516	0.26304	0.26295	1.79201	0.32107	0.27484	0.26589	0.26229
0.90	30	9.94832	6.64374	4.62421	2.60223	3.00552	5.35049	5.54504	2.44701	1.88446	2.41953	4.35285
	50	4.03811	3.13152	1.27656	0.90598	0.98038	1.35725	2.61038	0.87747	0.76908	0.87514	1.2525
	75	3.20767	2.76859	1.19957	0.94683	1.01404	1.26977	2.42977	0.90808	0.82174	0.91961	1.17254
	100	2.9214	2.49382	0.62772	0.52463	0.54294	0.64297	2.1888	0.53943	0.49061	0.51577	0.62332
	200	2.28028	1.97051	0.50675	0.44528	0.45425	0.5148	1.85879	0.48954	0.42966	0.43985	0.50441
	300	1.96218	1.90025	0.25143	0.26857	0.26156	0.25159	1.7727	0.32096	0.29046	0.27024	0.25138
0.95	30	18.67138	11.23936	6.64	3.85106	4.58746	8.08275	9.09761	3.68704	2.60662	3.54148	6.05953
	50	7.48446	5.41869	2.31549	1.41244	1.5988	2.56972	4.53002	1.36062	1.07439	1.33357	2.23409
	75	5.40093	4.15875	1.63528	1.07722	1.1928	1.77176	3.56361	1.02514	0.86816	1.03116	1.59343
	100	4.0764	3.37542	0.96167	0.73057	0.77905	1.00704	2.86618	0.7158	0.63573	0.70893	0.94787
	200	2.91413	2.58948	0.55016	0.40211	0.42298	0.56637	2.41983	0.43931	0.36862	0.39336	0.54633
	300	2.78788	2.31533	0.51131	0.44591	0.4589	0.51986	2.04736	0.47498	0.42066	0.43996	0.50879
0.99	30	72.7248	46.04896	22.44061	12.96224	15.48505	20.85603	41.52265	10.73165	9.70638	11.01371	12.1437
	50	30.73471	18.11691	4.41237	2.52812	3.07086	5.53265	13.61907	2.53113	1.5992	2.27876	3.93261
	75	23.86083	18.08586	3.90374	3.41181	4.21972	4.69542	12.97199	2.4031	1.07141	2.06348	3.05693
	100	12.73777	7.33884	2.65861	1.28559	1.52697	3.14474	5.33166	1.40014	0.89061	1.1873	2.51642
	200	8.54659	4.25763	1.60539	0.79406	0.91813	1.79775	3.07982	0.7515	0.5841	0.74404	1.55666
	300	7.51719	4.21078	1.31344	0.71007	0.79256	1.40952	2.88795	0.66751	0.54343	0.66749	1.28986

Table 8. Estimated MSE for non-robust and robust estimation methods when $p = 8$, $\phi = 10$, and 20% outliers.

ρ	n	Traditional	Non-robust	Robust estimator				Non-robust	Robust estimator			
		BMLE	BRRE	BRRE (LSMLE)	BRRE (LMDPDE)	BRRE (SMLE)	BRRE (MDPDE)	BMRTE	BMRTE (LSMLE)	BMRTE (LMDPDE)	BMRTE (SMLE)	BMRTE (MDPDE)
0.80	30	8.82739	6.22272	5.66219	5.48573	6.0809	6.0253	5.76935	4.84538	3.9829	4.26733	6.22949
	50	5.93879	5.75381	1.49862	1.29035	1.3246	1.51561	5.69714	1.27462	1.22177	1.27619	1.49453
	75	5.67876	5.17575	1.43165	1.28211	1.31238	1.44792	5.05159	1.2639	1.22061	1.27064	1.42723
	100	5.38694	5.15211	0.8571	0.84454	0.84459	0.85815	4.86833	0.84044	0.84905	0.84413	0.85685
	200	4.59554	4.46663	0.75839	0.7567	0.75504	0.75876	4.40105	0.73903	0.73518	0.75718	0.75829
	300	4.02426	3.75397	0.73422	0.73958	0.73697	0.73436	3.51204	0.72048	0.7292	0.72038	0.72418
0.85	30	8.94377	6.34229	6.05103	5.66231	6.00216	6.10451	5.78164	5.04478	4.11317	5.34328	5.14869
	50	6.68658	6.08499	3.32338	2.40621	2.56229	3.40117	5.77901	2.16231	2.08294	2.34621	3.30445
	75	6.0248	5.83793	2.04832	1.55653	1.6467	2.10358	5.45669	1.39753	1.34713	1.5121	2.03471
	100	4.88141	4.71763	1.1199	1.00928	1.02751	1.12691	4.65986	1.01073	0.96838	1.00032	1.11821
	200	4.87559	4.493	0.91085	0.8833	0.88625	0.91226	4.20237	0.9188	0.88061	0.88181	0.91052
	300	4.83208	3.98636	0.75021	0.74541	0.74467	0.75077	3.37286	0.784	0.75045	0.74536	0.75007
0.90	30	11.03616	7.92249	6.29488	5.65315	6.45221	6.30139	6.79138	4.92324	4.13895	5.31808	6.04353
	50	6.70318	5.67534	2.48053	1.83418	1.95432	2.56759	5.25412	1.67786	1.5642	1.77656	2.45831
	75	6.21378	5.55504	2.17497	1.82412	1.90924	2.24172	4.74491	1.70039	1.61787	1.7794	2.15267
	100	5.2906	4.85344	1.39363	1.2125	1.24608	1.40878	4.69528	1.17069	1.12995	1.19511	1.38978
	200	5.06258	4.66659	1.13739	1.00928	1.03167	1.14647	4.05568	0.99728	0.95312	0.99683	1.1351
	300	5.02868	4.43681	0.7204	0.71317	0.71224	0.72117	3.95455	0.75514	0.7193	0.713	0.72022
0.95	30	20.6934	12.12534	7.78511	6.34487	7.27944	8.80189	10.00104	6.80838	5.83209	6.52586	7.71848
	50	9.85152	7.73651	5.59356	3.25304	3.63857	5.96448	6.72439	2.92211	2.47103	3.08117	5.49796
	75	8.11659	6.80471	3.45616	2.28989	2.49783	3.62386	6.03993	2.04761	1.85143	2.19713	3.41387
	100	6.9252	6.10335	1.88736	1.52206	1.59481	1.93332	5.43962	1.44746	1.34934	1.48507	1.87494
	200	6.24698	5.52764	1.15843	0.92608	0.96295	1.17485	5.39129	0.90797	0.84212	0.90757	1.15464
	300	5.77177	5.46502	1.16673	1.05654	1.07719	1.17494	4.9971	1.04146	1.01091	1.04776	1.16469
0.99	30	106.90229	59.40106	14.56561	11.82064	16.19343	18.17106	51.05839	11.01016	6.41234	10.43825	10.07316
	50	39.44154	18.49573	10.44411	5.45969	6.5587	12.8078	15.4478	5.20708	3.40035	4.91328	9.7275
	75	33.38302	21.96648	7.69568	5.82789	6.96489	9.3508	13.20366	5.14659	3.61072	5.19246	6.56767
	100	19.25723	10.06645	5.67036	3.00421	3.46103	6.35904	7.73574	2.77646	2.10588	2.78899	5.5008
	200	17.41936	7.49841	3.55074	1.87745	2.1252	3.83585	5.89416	1.75526	1.37978	1.75901	3.48457
	300	15.5475	7.20389	2.67246	1.41026	1.57168	2.8057	5.28717	1.36285	1.05212	1.32127	2.64304

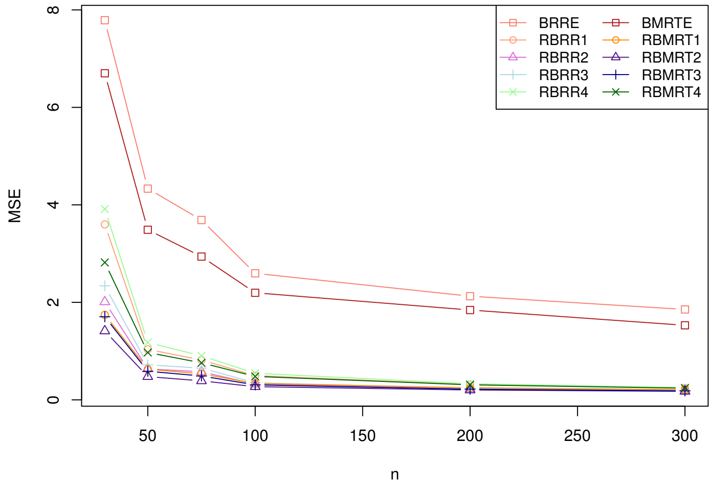

Figure 1. Estimated MSE for non-robust and robust estimators across sample sizes with 10% outliers.

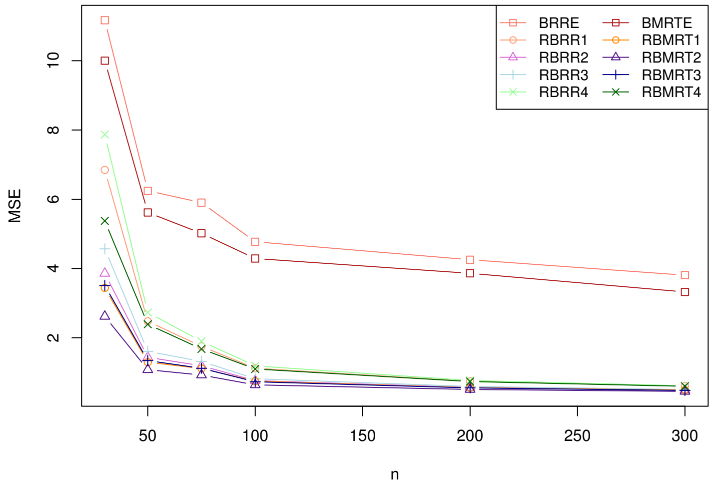

Figure 2. Estimated MSE for non-robust and robust estimators across sample sizes with 20% outliers.

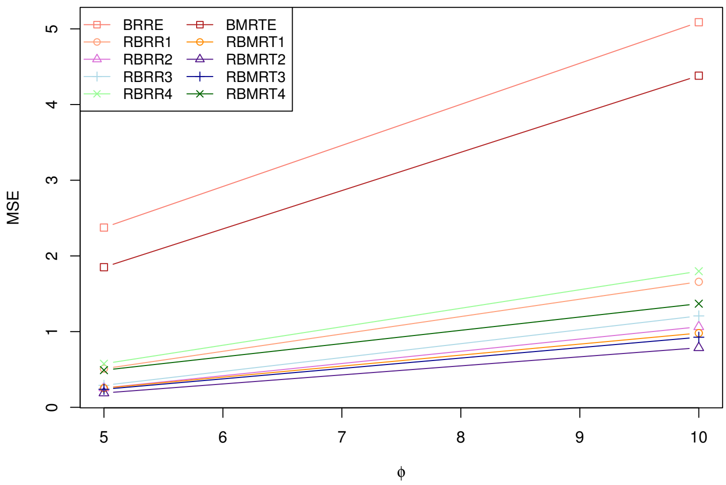

Figure 3. Estimated MSE for non-robust and robust estimators across dispersion parameter with 10% outliers.

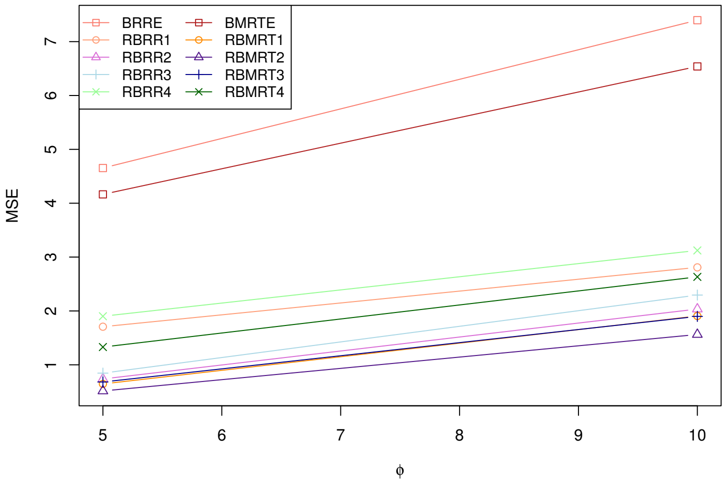

Figure 4. Estimated MSE for non-robust and robust estimators across dispersion parameter with 20% outliers.

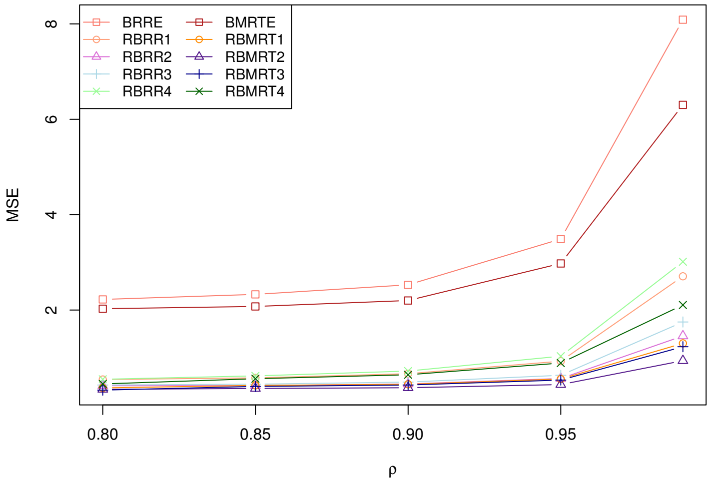

Figure 5. Estimated MSE for non-robust and robust estimators across multicollinearity level with 10% outliers.

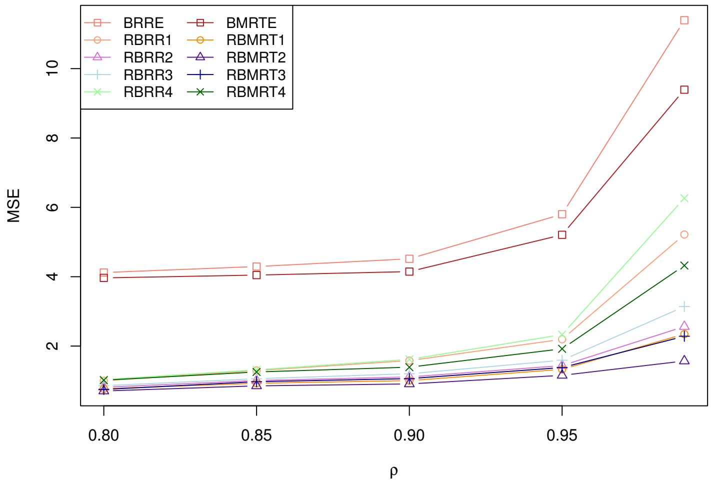

Figure 6. Estimated MSE for non-robust and robust estimators across multicollinearity level with 20% outliers.

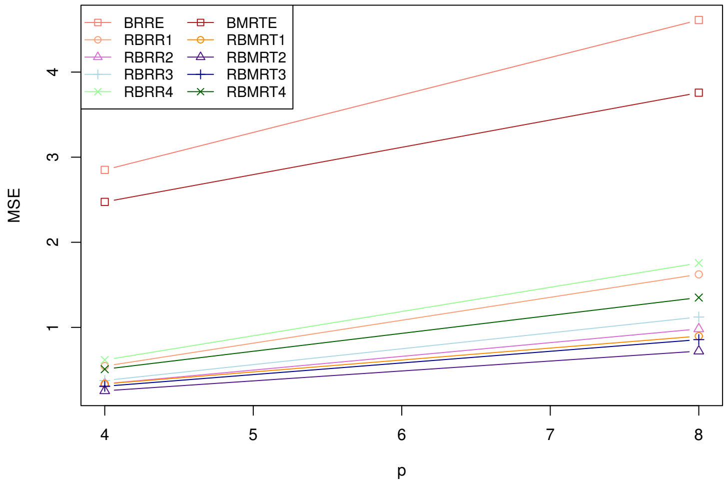

Figure 7. Estimated MSE for non-robust and robust estimators across the number of explanatory variables with 10% outliers.

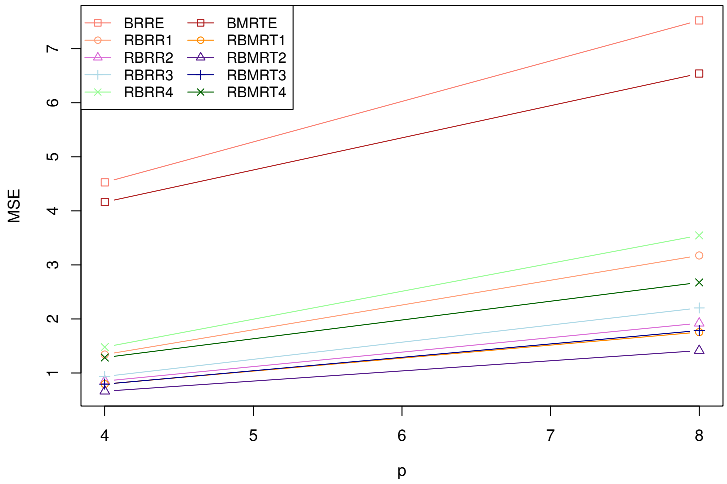

Figure 8. Estimated MSE for non-robust and robust estimators across the number of explanatory variables with 20% outliers.

Across all eight simulation scenarios, the results show that: Robust BMRTE estimators, particularly $\text{BMRTE}_{\text{LSMLE}}$ and $\text{BMRTE}_{\text{MDPDE}}$, consistently provide the lowest MSE values, regardless of sample size, correlation level, or degree of outliers. BMLE and non-robust estimators are unreliable in the presence of outliers and multicollinearity. Therefore, $\text{BMRTE}_{\text{LSMLE}}$ is the most reliable and recommended estimator, offering a practical and powerful solution for real-world count data with outliers and multicollinearity issues.

5. Numerical analysis

This study draws on data sourced from the breast cancer Wisconsin (Diagnostic) dataset, originally compiled from clinical cases at the University of Wisconsin Hospitals between January 1989 and November 1991. The dataset includes diagnostic measurements from 569 patients. Previous analyses of this dataset have been conducted by Abo El-Nasr et al. [41] and Rahmashari et al. [42].

Breast cancer remains a major global public health challenge, representing the most frequently diagnosed malignancy in women and the second most common cause of cancer-related deaths. The disease is marked by uncontrolled proliferation of breast tissue cells, leading to considerable morbidity. The current analysis employs radius mean—defined as the average distance from the center to the periphery of sampled cells—as the response variable. This selection is motivated by the variable's utility in reflecting tumor size and spatial expansion, which serve as indicators of disease progression. Unlike prior studies that emphasized classification of diagnostic outcomes, this work adopts a regression-based approach to model tumor morphology as a function of diagnostic and cellular characteristics. The use of a continuous measure, such as radius mean, permits the application of a broader suite of statistical techniques and supports a more nuanced interpretation of tumor development. This approach aids in identifying relationships between tumor characteristics and disease severity. Alongside radius mean, the study incorporates ten predictor variables: diagnosis, texture, perimeter, area, smoothness, compactness, concavity, concave points, symmetry, and fractal dimension. These covariates provide complementary information on histological attributes that are clinically relevant to understanding tumor behavior and prognosis.

In this study, we investigate the impact of ten independent variables on the dependent variable, mean radius (y), which represents the average distance from the tumor center to points on its perimeter. The independent variables include: Diagnosis (x_1), indicating whether the tumor is malignant (cancerous and likely to spread) or benign (noncancerous and typically requiring less aggressive treatment); texture (x_2), measuring the standard deviation of gray-scale values in the image; perimeter (x_3), reflecting the average perimeter of the core tumor; area (x_4), representing the size of the core tumor; smoothness (x_5), indicating local variation in radius lengths; compactness (x_6), calculated as (perimeter^2 / area); concavity (x_7), capturing the severity of concave portions of the tumor contour; concave points (x_8), which count the number of concave segments along the contour; symmetry (x_9), related to differences in breast density; and fractal dimension (x_{10}), a measure used to approximate the complexity of the tumor's boundary.

Table 9 presents descriptive statistics for variables in the breast cancer dataset, including the number of observations (n), minimum, maximum, mean, and standard deviation for each feature. The mean texture value is 19.29 (range: 9.71–39.28), the mean perimeter is 91.97 pixels (range: 43.79–188.50), and the mean area is 654.89 square pixels (range: 143.50–2501.00). Elevated values in these metrics may reflect increased cellular irregularity, larger tumor dimensions, and more advanced disease progression.

Table 9. Descriptive statistics of the breast cancer dataset variables.

Variable	n	Minimum	Maximum	Mean	Std. Deviation
Diagnosis (x_1)	569	0.0000	1.0000	0.3726	0.4839
Texture (x_2)	569	9.7100	39.2800	19.2896	4.3010
Perimeter (x_3)	569	43.7900	188.5000	91.9690	24.2980
Area (x_4)	569	143.0000	2501.0000	654.8891	351.9141
Smoothness (x_5)	569	0.0526	0.1634	0.0964	0.0141
Compactness (x_6)	569	0.0194	0.3454	0.1043	0.0528
Concavity (x_7)	569	0.0000	0.4268	0.0890	0.0797
Concave points (x_8)	569	0.0000	0.2012	0.0489	0.0388
Symmetry (x_9)	569	0.1060	0.3040	0.1812	0.0274
Fractal dimension (x_{10})	569	0.0490	0.0974	0.0628	0.0071

Smoothness averages 0.096 (range: 0.05–0.16), while compactness averages 0.104 (range: 0.02–0.35). Reduced smoothness and elevated compactness are associated with irregular morphology and higher tissue density, potentially indicating more aggressive tumor growth.

Concavity shows a mean of 0.089 (range: 0.00–0.43), and concave points average 0.049 (range: 0.00–0.20). Higher values suggest more pronounced contour concavities, which may correlate with invasive tumor characteristics.

Symmetry averages 0.181 (range: 0.11–0.30), and fractal dimension averages 0.063 (range: 0.05–0.10). Asymmetry and complex fractal patterns may indicate structural abnormality and aggressive phenotypic behavior.

To begin with, we assess the presence of multicollinearity and outliers in the dataset. In this study, multicollinearity is examined using three different approaches. The first approach involves inspecting the correlation matrix of the explanatory variables. As shown in Figure 9, there are strong correlations among several variables, particularly between x_3 and x_4, x_7 and x_8, and x_6 and x_7, which indicates potential multicollinearity. In addition, we calculate the condition number, which is defined as the square root of the ratio of the matrix U's largest to smallest eigenvalue. The eigenvalues of our matrix are as follows: 1.754524E+12, 1.560116E+09, 3.966556E+07, 3.313754E+05, 1.515372E+04, 1.007115E+04, 1.198765E+03, 7.172950E+02, 3.086800E+02, 1.146600E+02, and 1.887900E+01. Using these values, the condition number is calculated to be approximately 304854.8, which is significantly higher than the commonly used threshold of 100. This indicates the presence of severe multicollinearity. These findings are further supported by the variance inflation factors (VIFs) of the explanatory variables, which are: 3.126, 1.325, 89.300, 61.937, 3.026, 15.433, 12.742, 24.844, 1.872, and 6.203. We observe that several VIF values exceed the critical value of 10, further confirming the existence of strong correlations among the variables and indicating severe multicollinearity, which may adversely affect the interpretability and reliability of the data's estimates.

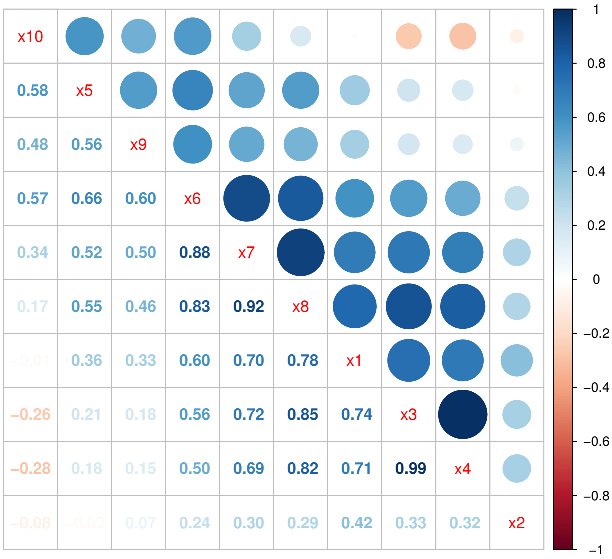

Figure 9. Correlation matrix for independent variables in the breast cancer data.

This diagnostic plot in Figure 10 highlights several concerns with the model, particularly regarding outliers and influential points. The residual distribution and Q-Q plot indicate that the residuals don't follow a normal distribution, especially at the tails, suggesting the presence of outliers. In the residuals vs. fitted values plot, we see a curved pattern and some points that are far from the zero line, which could point to model misspecification and additional outliers. The Cook's distance plot stands out by identifying a few data points, especially one that has an unusually large effect on the model's results. Moreover, the boxplot in Figure 11 of all variables shows clear evidence of extreme values, particularly in variable x4, which has many high outliers well outside the interquartile range. Other variables, like x3 and y, also show mild outliers. These findings strongly suggest the presence of influential observations and indicate that the dataset might benefit from transformation, outlier handling, or robust regression methods.

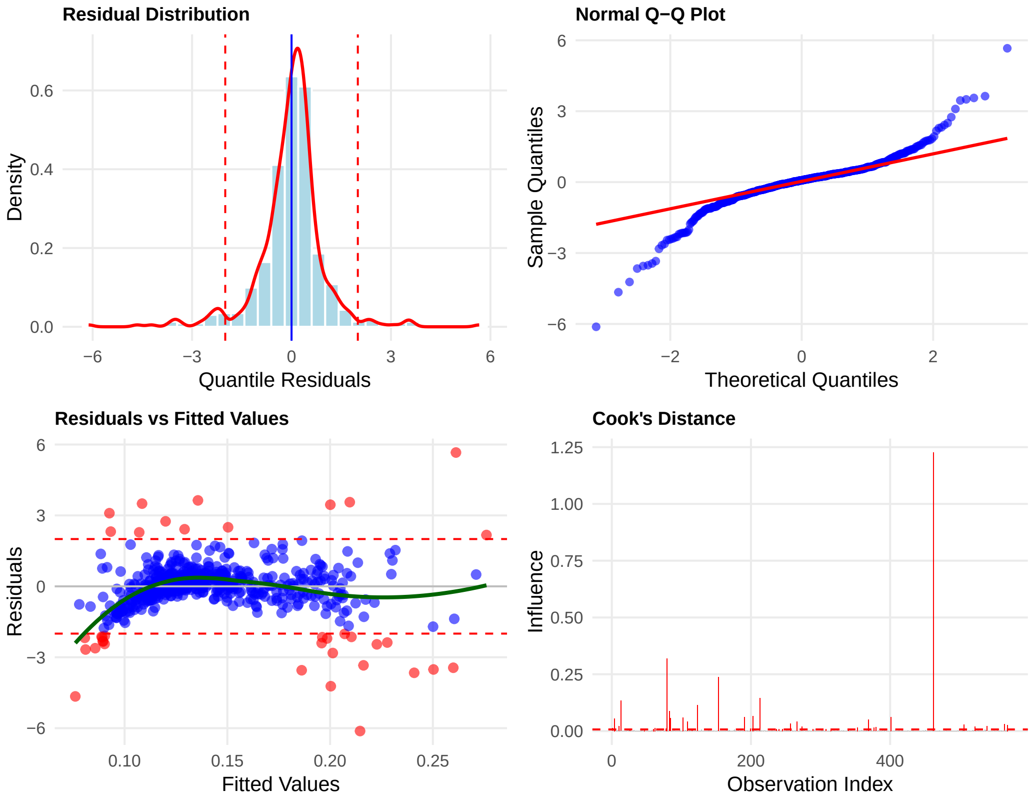

Figure 10. Model diagnostic plots showing residual distribution, Q-Q plot, residuals vs fitted values, and Cook's distance.

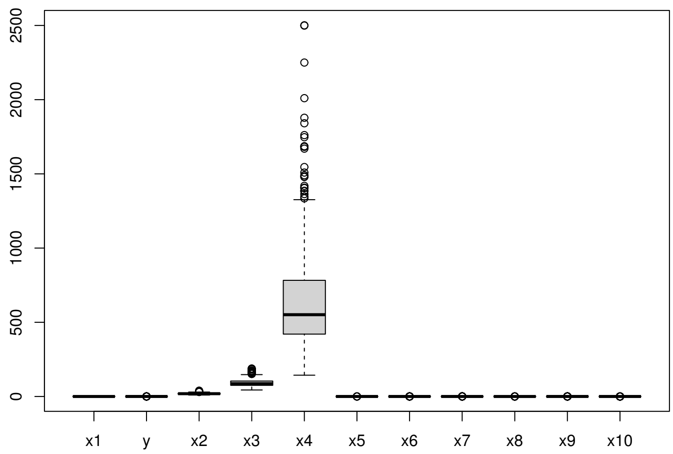

Figure 11. Boxplot of all variables.

After identifying issues with multicollinearity and outliers in the data, we will model the data using the traditional Beta model. This will include BMLE for the basic model, along with unbiased estimators that address multicollinearity alone, such as BRRE and BMRTE. Additionally, we will employ robust estimators that address both multicollinearity and outliers, including: $\text{BRRE}_{\text{LSMLE}}$, $\text{BRRE}_{\text{LMDPDE}}$, $\text{BRRE}_{\text{SMLE}}$, $\text{BRRE}_{\text{MDPDE}}$, $\text{BMRTE}_{\text{LSMLE}}$, $\text{BMRTE}_{\text{LMDPDE}}$, $\text{BMRTE}_{\text{SMLE}}$, $\text{BMRTE}_{\text{MDPDE}}$.

Table 10 presents the estimated coefficients and MSE for both non-robust and robust estimation methods applied to the breast cancer dataset. The traditional BMLE exhibits the highest MSE, indicating poor performance, likely due to the presence of multicollinearity and outliers in the data. Non-robust estimators such as BRRE and BMRTE show noticeable improvement in MSE with more superiority for BMRTE, but the most significant gains are seen with robust variants. Among them, $\text{BMRTE}_{\text{LSMLE}}$ achieves the lowest MSE value of 61.31, closely followed by $\text{BMRTE}_{\text{LMDPDE}}$ and $\text{BMRTE}_{\text{SMLE}}$, which also yield low MSEs. These robust methods are designed to mitigate the impact of both multicollinearity and outliers, and their effectiveness is reflected in the consistent and stable coefficient estimates across different robust techniques. For example, parameters like β_5 and β_6, which show large fluctuations under MLE, become more stable and moderate under robust estimators. Overall, the findings suggest that proposed robust BMRTE techniques, particularly $\text{BMRTE}_{\text{LSMLE}}$, provide more reliable and accurate coefficient estimates in the presence of data contamination and multicollinearity.

Figure 12 shows the results of a comparison of the performance of three estimators, BMILE, BRRE, and BMRTE, based on their MSE and Bias across different values of the shrinkage parameter k. Among the three, BMRTE consistently delivers the best performance, achieving the lowest MSE while keeping the bias well-controlled. This advantage is especially noticeable at intermediate values of k, where the estimator effectively balances the trade-off between bias and variance. These results highlight BMRTE's strong robustness in handling multicollinearity and outliers, making it a reliable choice across various scenarios.

Through the empirical validation presented in Table 11, all theoretical conditions derived in Theorems 1–7 are rigorously confirmed using real breast cancer data. The results unequivocally support the superiority of the proposed estimator $\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}}$ over a range of competing estimators under various multicollinear and outlier data. Below is a detailed itemization of the validations:

- In Theorem 1, the necessary condition $\hat{\phi}\lambda_j^2 + \lambda_j k^2(d+1)^2\hat{\alpha}^2_{\text{LSMLE},j} - \hat{\phi}(\lambda_j + k(d+1))^2 < 0$, for all $j = 1, \ldots, p+1$, is satisfied for $j = 1, \ldots, 11$. This confirms the superiority of the estimator $\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}}$ over $\hat{\boldsymbol{\beta}}_{\text{BMLE}}$.
- In Theorem 2, the necessary condition $(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\text{LSMLE}j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\text{MLE}j}) < 0,$ for all $j = 1, ..., p+1$, is satisfied for $j = 1, \ldots, 11$. This confirms the superiority of the estimator $\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}}$ over $\hat{\boldsymbol{\beta}}_{\text{BRRE}}$.
- In Theorem 3, the necessary condition $k^2(d+1)^2\hat{\alpha}^2{}_{\text{LSMLE}j} - k^2(d+1)^2\hat{\alpha}^2{}_{\text{MLE}j} < 0$, for all j =1, ..., p+1, is satisfied for $j = 1, \ldots, 11$. This confirms the superiority of the estimator $\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}}$ over $\hat{\boldsymbol{\beta}}_{\text{BMRTE}}$.
- In Theorem 4, the necessary condition $d\lambda_j\Big((k^2\hat{\alpha}^2_{\text{LSMLE},j}(d+2) - 2\hat{\phi}k)\lambda_j + k^2(2k\hat{\alpha}^2_{\text{LSMLE},j}(d+1) - \hat{\phi}(d+2))\Big) < 0,$ for all $j = 1, ..., p+1$, is satisfied for $j = 1, \ldots, 11$. This confirms the superiority of the estimator $\hat{\boldsymbol{\beta}}_{\text{BMRTE}_{\text{LSMLE}}}$ over $\hat{\boldsymbol{\beta}}_{\text{BRRE}_{\text{LSMLE}}}$.
- In Theorem 5, the necessary condition $(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\text{LSMLE}j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j +$

$k^2\hat{\alpha}^2{}_{\mathrm{LMDPDE}\,j}) < 0$, for all $j = 1, ..., p+1$, is satisfied for $j = 1, \ldots, 11$. This confirms the superiority of the estimator $\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}$ over $\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{LMDPDE}}}$.

- In Theorem 6, the necessary condition $(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\mathrm{SMLE}\,j}) < 0$, for all $j = 1, ..., p+1$, is satisfied for $j = 1, \ldots, 11$. This confirms the superiority of the estimator $\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}$ over $\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{SMLE}}}$.
- In Theorem 7, the necessary condition $(\lambda_j + k)^2(\hat{\phi}\lambda_j + k^2(d+1)^2\hat{\alpha}^2{}_{\mathrm{LSMLE}\,j}) - (\lambda_j + k(d+1))^2(\hat{\phi}\lambda_j + k^2\hat{\alpha}^2{}_{\mathrm{MDPDE}\,j}) < 0$, for all $j = 1, ..., p+1$, is satisfied for $j = 1, \ldots, 11$. This confirms the superiority of the estimator $\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}$ over $\hat{\boldsymbol{\beta}}_{\mathrm{BRRE}_{\mathrm{MDPDE}}}$.

Table 10. MSE and coefficients of all non-robust and robust estimation methods for breast cancer data.

Coefficient	Traditional	Non-robust	Robust estimator				Non-robust	Robust estimator			
	BMLE	BRRE	BRRE (LSMLE)	BRRE (LMDPDE)	BRRE (SMLE)	BRRE (MDPDE)	BMRTE	BMRTE (LSMLE)	BMRTE (LMDPDE)	BMRTE (SMLE)	BMRTE (MDPDE)
β_0	-3.29027	-3.1096	-3.15181	-3.09896	-3.1096	-3.1096	-3.02363	-3.06139	-3.00913	-3.02363	-3.02363
β_1	0.00883	0.01897	0.0119	0.01551	0.01897	0.01897	0.02566	0.01934	0.02269	0.02566	0.02566
β_2	-0.00017	-0.00083	-7e-04	-0.00093	-0.00083	-0.00083	-0.00123	-0.00114	-0.00135	-0.00123	-0.00123
β_3	0.02086	0.0187	0.01953	0.01863	0.0187	0.0187	0.01742	0.01815	0.01725	0.01742	0.01742
β_4	-0.00059	-0.00048	-0.00053	-0.00047	-0.00048	-0.00048	-4e-04	-0.00045	-0.00039	-4e-04	-4e-04
β_5	0.42024	-0.12534	-0.20148	-0.15103	-0.12534	-0.12534	-0.20987	-0.25832	-0.22735	-0.20987	-0.20987
β_6	-0.64411	-0.42385	-0.38255	-0.39538	-0.42385	-0.42385	-0.33685	-0.2962	-0.30778	-0.33685	-0.33685
β_7	-0.03893	-0.14092	-0.1577	-0.10243	-0.14092	-0.14092	-0.15314	-0.16021	-0.11585	-0.15314	-0.15314
β_8	-0.34447	-0.03446	0.04821	-0.03019	-0.03446	-0.03446	-0.00751	0.04684	-0.00341	-0.00751	-0.00751
β_9	-0.03807	-0.17306	-0.16847	-0.20542	-0.17306	-0.17306	-0.24105	-0.24124	-0.27108	-0.24105	-0.24105
β_{10}	-0.3492	-0.3127	-0.29783	-0.31109	-0.3127	-0.3127	-0.29947	-0.29038	-0.29625	-0.29947	-0.29947
MSE	2530.11147	118.46066	108.45916	108.44035	118.46066	118.46066	67.81113	61.30519	61.64094	67.81113	67.81113

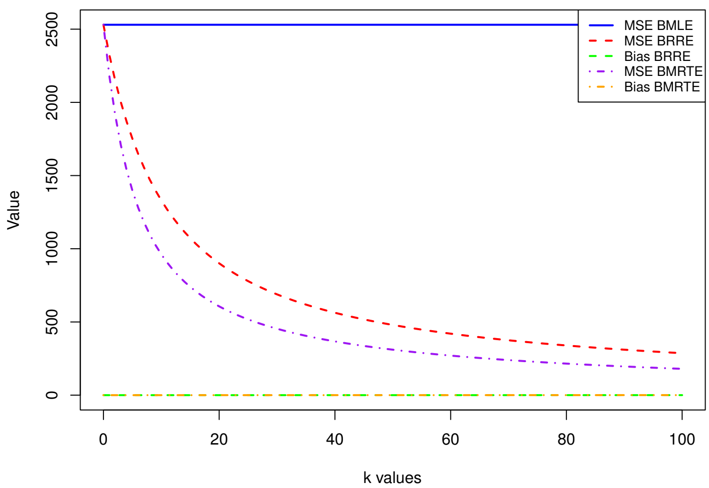

Figure 12. Comparison of MSE and Bias for different estimators across different parameter values.

The consistent negativity of all values across $j = 1, \ldots, 11$ for each theorem, as summarized in Table 11, provides strong empirical evidence that the proposed estimator $\hat{\boldsymbol{\beta}}_{\mathrm{BMRTE}_{\mathrm{LSMLE}}}$ is not only theoretically sound but also statistically effective in addressing multicollinearity and outlier. This makes it a robust and reliable choice for practical applications in complex data settings.

Table 11. Verification of the conditions outlined in the theorems was carried out using breast cancer data.

Value	Theorem 1	Theorem 2	Theorem 3	Theorem 4	Theorem 5	Theorem 6	Theorem 7
j=1	-1.435E-17	-7.196E-18	-1.343E-18	-6.471E-18	-6.455E-18	-7.196E-18	-7.196E-18
j=2	-1.815E-11	-9.100E-12	-1.699E-12	-8.184E-12	-8.164E-12	-9.100E-12	-9.100E-12
j=3	-2.807E-08	-1.408E-08	-2.628E-09	-1.266E-08	-1.263E-08	-1.408E-08	-1.408E-08
j=4	-4.007E-04	-2.004E-04	-3.668E-05	-1.801E-04	-1.797E-04	-2.004E-04	-2.004E-04
j=5	-1.753E-01	-8.055E-02	-4.713E-03	-7.097E-02	-7.082E-02	-8.055E-02	-8.055E-02
j=6	-3.831E-01	-1.707E-01	-3.747E-03	-1.496E-01	-1.494E-01	-1.707E-01	-1.707E-01
j=7	-1.721E+01	-6.101E+00	-8.740E-01	-5.316E+00	-5.302E+00	-6.101E+00	-6.101E+00
j=8	-3.645E+01	-1.042E+01	-1.208E+00	-8.930E+00	-8.899E+00	-1.042E+01	-1.042E+01
j=9	-1.072E+02	-1.751E+01	-1.830E+00	-1.456E+01	-1.451E+01	-1.751E+01	-1.751E+01
j=10	-3.188E+02	-1.658E+01	-8.067E-01	-1.344E+01	-1.310E+01	-1.658E+01	-1.658E+01
j=11	-1.989E+03	-5.277E+00	-3.133E-01	-3.981E+00	-4.038E+00	-5.277E+00	-5.277E+00
Satisfied or not	Satisfied	Satisfied	Satisfied	Satisfied	Satisfied	Satisfied	Satisfied

6. Conclusions

The BRM is widely employed for modeling proportional data bounded within the unit interval. However, the BMLE is highly sensitive to multicollinearity and outliers, frequently resulting in unstable estimates and inflated standard errors. To overcome these challenges, we introduce a new class of robust modified ridge-type estimators. The most effective among them, the $\text{BMRTE}_{\text{LSMLE}}$, merges shrinkage and robustness methods to deliver stable estimates even when severe multicollinearity and outliers are present. We support this development with thorough theoretical analysis and extensive numerical tests. Through Monte Carlo simulations spanning a wide range of conditions and a real-world application to breast cancer data, our proposed estimators consistently outperform alternatives. The $\text{BMRTE}_{\text{LSMLE}}$ proves especially powerful in scenarios with high multicollinearity and extreme outliers. Our work underscores the critical need for robust methodologies in beta regression. By providing a solution that is both practical and does not add significant complexity, these estimators make beta regression more reliable and accessible for scientific fields like biomedicine. Despite its contributions, this study has limitations, which highlight promising avenues for future work. First, the computational burden of estimating the MSE for both robust and non-robust estimators grows considerably with the dimensionality of the predictor space (p), raising challenges for high-dimensional applications. Second, the effectiveness of the proposed estimators is contingent on the careful choice of tuning parameters. Thus, the development of automated, data-driven procedures for jointly optimizing shrinkage and robustness represents a valuable direction for methodological advancement. Moreover, the theoretical and numerical investigations presented here are confined to cross-sectional data, leaving unexplored the behavior of these estimators in more complex structures such as time-series, longitudinal, or spatially correlated BRMs. Future research could also integrate tailored outlier detection strategies and robust estimation methods for the response variable, thereby enhancing the overall applicability and robustness of BRMs.

Author contributions

All authors have worked equally to write and review the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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