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Research article

Investigating string cloud spacetimes with energy-momentum tensor constraints in general relativity

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Abstract: The objective of this study was to investigate the behavior of relativistic $G(QE)_n$ string cloud spacetime endowed with various forms of the string cloud energy-momentum tensor $\mathcal{T}$, which includes the string cloud fluid density α and string tension β. A $G(QE)_n$ string cloud spacetime with a covariantly constant energy-momentum tensor satisfies the equation of state $\frac{\alpha}{\beta} = -1$, and represents either a massive string cloud spacetime or a bulk viscous fluid spacetime. Moreover, when a string cloud spacetime is coupled with a covariantly constant energy-momentum tensor, the equation of state $\frac{\alpha}{\beta} = -1$ coincides with the state equation for a cloud of geometric strings or represents the quintessence era. $G(QE)_n$ string cloud spacetimes with Codazzi-type and pseudo-symmetric energy-momentum tensors were investigated. Furthermore, we characterized $\mathcal{T}$-recurrent, weakly $\mathcal{T}$-symmetric, specially weakly $\mathcal{T}$-symmetric, generalized $\mathcal{T}$-recurrent, semi-generalized $\mathcal{T}$-recurrent, and quadratic Killing-type energy-momentum tensors on a string cloud spacetime with a Killing velocity vector field, and concluded that the string cloud spacetime either represents a massive string cloud spacetime or corresponds to the quintessence era.

Keywords: Lorentzian spacetime manifold; string cloud spacetimes; generalized $\mathcal{T}$-recurrent; semi-generalized $\mathcal{T}$-recurrent tensor; Codazzi-type energy-momentum tensor; geometric strings
Mathematics Subject Classification: 53B30, 53C25, 53C44, 53C50, 53C80

1. Introduction

In 1915, Einstein formulated the general theory of relativity (GTR), providing a framework in which gravity emerges as a consequence of the curvature of spacetime induced by the presence of mass. The spacetime structure is governed by the gravitational field equations, expressed through the renowned Einstein field equations (EFEs) without the inclusion of the cosmological constant, formulated as [1]

$$\mathrm{Ric} - \frac{\tau}{2} g = \kappa \mathcal{T}. \tag{1.1}$$

Here, g represents the semi- (or pseudo-) Riemannian metric, while $\mathcal{T}$ denotes the energy-momentum tensor. The Ricci tensor Ric and the scalar curvature R of spacetime are very important for realizing Einstein's goal of a static universe. The universal gravitational constant, G, is incorporated into the EFEs as κ, the gravitational constant, defined as $\kappa = 8\pi G$. To explore the relationship between string state counting and black hole entropy, Letelier was the first to provide comprehensive solutions for string clouds characterized by spherical symmetry [2]. These solutions were subsequently generalized to incorporate third-order Lovelock gravity [3] and Einstein-Gauss-Bonnet theory in the Letelier spacetime [4]. Many additional extended solutions have also been studied in this context, see [5–7]. An intriguing phenomenon in quantum optics [8] reveals that atomic motion in light radiation, within the framework of optical molasses, resembles the behavior of a particle in a viscous fluid. Drawing an analogy, one could conceptualize gravitational molasses as arising from quantum fluctuations in the spacetime bulk viscosity, which is associated with the cosmological constant (abbreviated as CC). This effect may be attributed to such fluctuations.

An alternative approach, as suggested in the literature, is to interpret the CC as the energy density of quantum vacuum fluctuations, expressed as $\Lambda = \frac{8\pi G H_0}{c^4}$, where H_0 represents the vacuum expectation value. Under this interpretation, a nonzero CC would induce a dissipative process analogous to matter energy, with the bulk viscosity of spacetime determined by a constant scalar curvature. Conversely, a vanishing CC would result in a Ricci-flat spacetime [9].

Within the framework of both the GTR and cosmology, spacetime is modeled as a time-oriented, four-dimensional, connected Lorentzian manifold. This specific category of pseudo-Riemannian manifolds is distinguished by a Lorentzian metric possessing the signature $(-, +, +, +)$ and plays a fundamental role in the GTR [1, 10]. The geometry of Lorentzian manifolds is formulated to characterize the behavior of vectors within these manifolds, rendering them a powerful framework for exploring the GTR. When the Ricci tensor assumes a particular form, such manifolds are termed quasi-Einstein manifolds. In the context of spacetimes, they are specifically referred to as perfect fluid spacetimes [11–14]. The Ricci tensor Ric of type $(0, 2)$ is given by

$$\mathrm{Ric} = \gamma_1 g + \gamma_2 \eta \otimes \eta, \tag{1.2}$$

where γ_1 and γ_2 are scalars, and η is a 1-form corresponding to the velocity vector field ξ, such that $\eta(\xi) = -1$, i.e., the velocity vector field ξ is a unit timelike vector field. A non-flat Riemannian or pseudo-Riemannian manifold $(\mathcal{H}_n, g)$, where $n > 2$, is said to be a generalized quasi-Einstein manifold (briefly, $\mathrm{G(QE)}_n$) if its Ricci tensor is non-zero and satisfies the condition [15]:

$$\mathrm{Ric} = \gamma_1 g + \gamma_2 \eta \otimes \eta + \gamma_3 \theta \otimes \theta, \tag{1.3}$$

where γ_1, γ_2, and γ_3 are non-zero scalars, and η and θ are two non-zero 1-forms defined by

$$g(D, \xi) = \eta(D), \quad g(D, \zeta) = \theta(D),$$

for any vector field $D \in \mathfrak{X}(\mathcal{H}^n, g)$. ξ and ζ are metrically equivalent to the 1-forms η and θ, and $g(\xi, \zeta) = 0$. $(\mathcal{H}^4, g)$ is a perfect fluid spacetime if $\gamma_3 = 0$.

This paper aims to analyze the behavior of relativistic G(QE)$_n$ string cloud spacetime, which is endowed with various forms of the string cloud energy-momentum tensor $\mathcal{T}$. This tensor includes both the string cloud fluid density α and the string tension β, key components in describing the dynamics of string cloud spacetimes. A G(QE)$_n$ string cloud spacetime with a covariantly constant energy-momentum tensor satisfies the equation of state $\frac{\alpha}{\beta} = -1$, a condition that signifies the spacetime either represents a massive string cloud or behaves like a bulk viscous fluid. In the scenario where the string cloud spacetime is coupled with a covariantly constant energy-momentum tensor, the equation of state $\frac{\alpha}{\beta} = -1$ aligns with the state equation for a cloud of geometric strings or can be interpreted as marking the quintessence era in cosmology.

The study also investigates more intricate configurations of G(QE)$_n$ string cloud spacetime, focusing on energy-momentum tensors of Codazzi-type and pseudo-symmetric forms. Furthermore, this paper characterizes various types of energy-momentum tensors, such as $\mathcal{T}$-recurrent, weakly $\mathcal{T}$-symmetric, specially weakly $\mathcal{T}$-symmetric, generalized $\mathcal{T}$-recurrent, semi-generalized $\mathcal{T}$-recurrent, and quadratic Killing-type tensors in the context of a string cloud spacetime equipped with a Killing velocity vector field. The analysis concludes that such string cloud spacetimes either correspond to a massive string cloud configuration or are associated with the cosmological quintessence era. Through these investigations, we establish a comprehensive understanding of the properties and implications of energy-momentum tensors in determining the nature of relativistic string cloud spacetimes.

2. String cloud spacetime

String cloud spacetime (briefly, SCS) is the spacetime filled with the energy-momentum tensor of the string cloud type [16, 17]. String cloud spacetimes are theoretical solutions to Einstein's field equations, inspired by concepts from string theory. They describe a gravitational field influenced by a distribution of strings in a particular region of spacetime. These strings can be considered as a cloud or a collection with specific density and tension properties. The matter content in these models consists of a collection of strings spread over the spacetime. The density of the strings and their orientation impact the metric of the spacetime. The energy-momentum tensor of the string cloud is derived from the dynamics of strings and provides the source term in Einstein's field equations. It typically has anisotropic characteristics due to the tension along the string's length. Solutions to Einstein's equations for string cloud models are obtained under specific assumptions about the distribution of strings. These can lead to different geometries such as static, spherically symmetric, or even cosmological solutions. In standard cosmological models, the universe's material composition is often modeled as a string cloud spacetime, where the energy-momentum tensor associated with the string cloud plays a significant role in shaping the spacetime dynamics [16, 18–20]. A novel type of string cosmological model, both with and without magnetic fields, was presented by the authors in [19], within the context of a spacetime exhibiting G_3 symmetry. To achieve this, an additional term for the magnetic field is incorporated into the standard energy-momentum tensor for cosmic strings using the methods developed by Letelier [18] and Stachel [20]. Within the framework of Rosen's bimetric theory, Sahoo and Mishra [21] investigated plane-symmetric spacetime involving quark matter, which is connected to both the string cloud and the domain wall. They found that, within this theory, string clouds and domain walls do not exist, and bimetric relativity fails to provide an explanation for the early universe. Using both general relativity and Barber's second self-creation theory, Rao and Neelima [22] studied the anisotropic Bianchi type-VI spacetime with strange quark matter attached to the string cloud. They found that the presence of a scalar field modifies the matter distribution but does not alter the spacetime geometry. In conventional cosmological models, the energy-momentum tensor of string clouds holds significant importance, as it is widely believed that the universe's material structure reflects the properties of a spacetime influenced by string clouds [16, 23–25].

The energy-momentum tensor (briefly, EMT) of a string cloud spacetime is given by [16]

$$\mathcal{T}(D, E) = \alpha\eta(D)\eta(E) - \beta\theta(D)\theta(E), \tag{2.1}$$

where $D, E \in \mathfrak{X}(\mathcal{H}^4, g)$, and α, β are the energy density and the string tension for the string cloud fluid. θ is a unit spacelike covector in the direction of the string, and η is a unit timelike covector where $\eta(D) = g(D, \xi)$, $\theta(D) = g(D, \zeta)$, and ξ and ζ are orthogonal vector fields. Taken together, Eqs (1.1), (1.3), and (2.1) imply that

$$\gamma_1 = \frac{\kappa(\alpha + \beta)}{2}, \quad \gamma_2 = \kappa\alpha, \quad \gamma_3 = -\kappa\beta. \tag{2.2}$$

Now, the cloud fluid's energy density α and the string tension β are related by [16, 26]

$$\alpha = \alpha_0 + \beta, \tag{2.3}$$

where α_0 denotes the rest energy density of the particle. In light of Eqs (2.1) and (2.3), the EFEs for a relativistic string cloud spacetime are derived as follows:

$$\mathrm{Ric}(D, E) = \frac{\tau}{2} g(D, E) + \kappa\alpha\eta(D)\eta(E) - \kappa\beta\theta(D)\theta(E). \tag{2.4}$$

3. String cloud spacetimes with covariantly constant EMT

In [27], Chaki and Ray studied general relativistic spacetimes whose energy-momentum tensor is covariantly constant. Motivated by this, we now examine string cloud spacetimes with a covariantly constant energy-momentum tensor.

From the EFEs, we obtain

$$\nabla\mathcal{T} = 0 \implies \nabla\mathrm{Ric} = 0, \tag{3.1}$$

which immediately implies that the scalar curvature τ must be constant.

Next, contracting Eq (1.3), we find

$$\tau = n\gamma_1 - \gamma_2 + \gamma_3. \tag{3.2}$$

Using this relation, we set

$$n\gamma_1 - \gamma_2 + \gamma_3 = \bar{\sigma},$$

where $\bar{\sigma}$ is a constant. Then, substituting the expressions for $\gamma_1, \gamma_2, \gamma_3$ from (2.2), we deduce

$$\frac{1}{2}(n-2)\kappa(\alpha+\beta) = \bar{\sigma}, \quad \kappa(\alpha+\beta) = \sigma, \tag{3.3}$$

where $\sigma = \frac{2\bar{\sigma}}{n-2}$.

In the special case $\sigma = 0$, Eq (3.3) reduces to

$$\frac{\alpha}{\beta} = -1. \tag{3.4}$$

Theorem 3.1. *A $G(QE)_n$ string cloud spacetime with a covariantly constant energy-momentum tensor satisfies the state equation $\alpha = \text{constant} - \beta$. In particular, if the constant vanishes, then α is proportional to β, and hence $\frac{\alpha}{\beta} = -1$.*

Corollary 3.1. *In a $G(QE)_n$ string cloud spacetime with a covariantly constant energy-momentum tensor, the string is a massive string and the string cloud spacetime becomes a massive string cloud spacetime.*

As per Eq (3.3), we state the following outcome:

Corollary 3.2. *If a $G(QE)_n$ string cloud spacetime satisfies the EFEs with a covariantly constant energy-momentum tensor, then the spacetime reduces to that of a bulk viscous fluid spacetime.*

According to [28], the energy density $\mathcal{E}_d$ and the particle density α exhibit a direct relationship with the specific energy $\mathbb{E}_{sp}$ and the volume $\mathcal{V}$ of the fluid. Specifically, the energy density is expressed as

$$\mathcal{E}_d = \alpha\, \mathbb{E}_{sp},$$

where the particle density α is inversely proportional to the volume,

$$\alpha = \frac{1}{\mathcal{V}}.$$

Therefore, in view of Eqs (3.2) and (3.3), we obtain the following outcome:

Corollary 3.3. *If a $G(QE)_n$ string cloud spacetime, obeying the EFEs with a covariantly constant energy-momentum tensor, satisfies the state equation $\alpha = \text{constant} - \beta$, then the specific energy is*

$$\mathbb{E}_{sp} = \frac{2\kappa\mathcal{E}_d}{\kappa\alpha_0 - \sigma},$$

and the volume of the fluid is

$$\mathcal{V} = \frac{2\kappa}{\kappa\alpha_0 - \sigma}.$$

As in [29], the equation of state (EoS) for the Takabayashi string is

$$\alpha = (1+\omega)\beta, \tag{3.5}$$

where ω is a constant. This resembles the EoS connecting matter density and string tension, $v_p = \omega\beta$, which characterizes the Takabayashi string [29]. For $\omega < 0$, geometric strings (Nambu strings [30]) dominate, whereas for $\omega > 0$, particles prevail over strings.

Comparing Eq (3.3), $\alpha = c - \beta$, with Eq (3.5), we observe that the correspondence requires $\omega < 0$. In particular, when the constant $c = 0$, we obtain $\alpha = -\beta$, which coincides with the EoS of a cloud of geometric strings. This leads to the following results:

Theorem 3.2. *If a $G(QE)_n$ string cloud spacetime satisfies the EFEs with a covariantly constant energy-momentum tensor, and if the constant in (3.3) vanishes, then the relation $\alpha = -\beta$ coincides with the state equation of a cloud of geometric strings.*

Corollary 3.4. *In such a spacetime, the relation $\alpha = -\beta$ precisely reproduces the state equation of Nambu strings.*

Corollary 3.5. *If a $G(QE)_n$ string cloud spacetime obeys the EFEs with a covariantly constant energy-momentum tensor and satisfies the state equation $\alpha = -\beta$, then the spacetime necessarily reduces to the quintessence era corresponding to $\omega = -1$.*

Again, if $\nabla\mathrm{Ric} = 0$, then from Eq (1.3), we obtain

$$\begin{aligned} 0 \;=\; & d\gamma_1(D)g(E,F) + d\gamma_2(D)\eta(E)\eta(F) + \gamma_2\left[(\nabla_D\eta)E\,\eta(F) + \eta(E)(\nabla_D\eta)F\right] \\ & + d\gamma_3(D)\theta(E)\theta(F) + \gamma_3\left[(\nabla_D\theta)E\,\theta(F) + \theta(E)(\nabla_D\theta)F\right]. \end{aligned} \tag{3.6}$$

After contracting over E and F, we obtain

$$4d\gamma_1(D) - d\gamma_2(D) + d\gamma_3(D) = 0. \tag{3.7}$$

Substituting $E = \xi$ into Eq (3.6), we deduce

$$d\gamma_1(D)\eta(F) - d\gamma_2(D)\eta(F) - \gamma_2(\nabla_D\eta)F + \gamma_3(\nabla_D\theta)F = 0. \tag{3.8}$$

By setting $F = \xi$ in Eq (3.8), we obtain

$$-d\gamma_1(D) + d\gamma_2(D) = 0. \tag{3.9}$$

In particular, if γ_3 is constant, then from Eqs (3.7) and (3.9), it follows that γ_1 and γ_2 are also constants. Consequently, Eq (3.8) reduces to

$$\gamma_2(\nabla_D\eta)F = \gamma_3(\nabla_D\theta)F. \tag{3.10}$$

If the 1-form θ is closed, then Eq (3.10) implies that either $\gamma_2 = 0$, or the 1-form η is closed. Furthermore, if $\gamma_2 = 0$, then from Eqs (2.2) and (3.5) we obtain $\omega = -1$, which indicates that the string cloud spacetime corresponds to the quintessence era, or that the velocity vector field ξ is irrotational. Consequently, we present the following result:

Theorem 3.3. *A $G(QE)_n$ string cloud spacetime with a covariantly constant energy-momentum tensor either represents a quintessence era, or the spacetime has zero vorticity.*

It is well known from [31] that on $(\mathcal{H}^4, g)$, we have

$$(\mathrm{div}\,C)(D,E)F = \tfrac{1}{2}\left[f(\nabla_D\mathrm{Ric})(E)F - (\nabla_E\mathrm{Ric})(D)F\right] - \tfrac{1}{2}\left[d\tau(D)g(E,F) - d\tau(E)g(D,F)\right], \tag{3.11}$$

where div denotes divergence, C is the Weyl conformal curvature tensor, and $g(QD,E) = \mathrm{Ric}(D,E)$. This implies

$$\mathrm{Ric} \text{ is of Codazzi-type} \iff \mathrm{div}\,C = 0 \text{ and } \tau \text{ is constant}. \tag{3.12}$$

According to [32], the behavior of a perfect fluid of quark matter, and as stated in [33], an n-dimensional perfect fluid spacetime satisfying $\mathrm{div}\,C = 0$ is a generalized Robertson-Walker (GRW) spacetime. Moreover, the conformal curvature tensor satisfies $C(D,E)\xi = 0$ whenever the flow vector field ξ is irrotational. Furthermore, every GRW spacetime with $n = 4$ is a perfect fluid spacetime if and only if it is a Robertson-Walker (RW) spacetime [34]. Thus, in this case, the spacetime reduces to an RW spacetime. RW spacetimes with constant scalar curvature are further described in [35]. Hence, we state the following:

Theorem 3.4. *A $G(QE)_4$ string cloud spacetime obeying the EFEs with a covariantly constant energy-momentum tensor is necessarily an RW spacetime.*

4. String cloud spacetime endowed with EMT of the Codazzi type

According to [36], a spacetime endowed with a Codazzi-type energy-momentum tensor represents a Yang pure space. Let the energy-momentum tensor be of Codazzi type in a $(SCS)_4$, that is,

$$(\nabla_D \mathcal{T})(E,F) = (\nabla_E \mathcal{T})(D,F). \tag{4.1}$$

In view of (1.1) and (2.4), Eq (4.1) reduces to

$$(\nabla_D \mathrm{Ric})(E,F) - \tfrac{1}{2} d\tau(D) = (\nabla_E \mathrm{Ric})(D,F) - \tfrac{1}{2} d\tau(E). \tag{4.2}$$

Contracting (4.2) along D and E yields $d\tau(F) = 0$, which implies that τ is constant. Therefore, from (4.2) we obtain

$$(\nabla_D \mathrm{Ric})(E,F) = (\nabla_E \mathrm{Ric})(D,F), \tag{4.3}$$

that is, the Ricci tensor Ric is of Codazzi type.

Guilfoyle and Nolan [37] defined a Yang pure space as a Lorentzian manifold $(\mathcal{H}^4, g)$ in which the metric tensor solves Yang's equation:

$$(\nabla_D \mathrm{Ric})(E,F) - (\nabla_E \mathrm{Ric})(D,F) = 0.$$

They further observed that a perfect fluid spacetime with $p + \nu \neq 0$ is a Yang pure space if and only if the spacetime is an RW spacetime. Based on this discussion, we may now state the following result:

Theorem 4.1. *A string cloud spacetime obeying the EFEs coupled with a Codazzi-type energy-momentum tensor is an RW spacetime.*

Again, from Eq (1.3), we have

$$\begin{aligned}
& d\gamma_1(D)g(E,F) + d\gamma_2(D)\eta(E)\eta(F) + d\gamma_3(D)\theta(E)\theta(F) \\
& + \gamma_2 \left[(\nabla_D\eta)E\,\eta(F) + \eta(E)(\nabla_D\eta)F\right] + \gamma_3 \left[(\nabla_D\theta)E\,\theta(F) + \theta(E)(\nabla_D\theta)F\right] \\
= \; & d\gamma_1(E)g(D,F) + d\gamma_2(E)\eta(D)\eta(F) + d\gamma_3(E)\theta(D)\theta(F) \\
& + \gamma_2 \left[(\nabla_E\eta)D\,\eta(F) + \eta(D)(\nabla_E\eta)F\right] + \gamma_3 \left[(\nabla_E\theta)D\,\eta(F) + \theta(D)(\nabla_E\theta)F\right].
\end{aligned} \tag{4.4}$$

After contracting Eq (4.4) over E and F, we obtain

$$\begin{aligned}
& 4\,d\gamma_1(D) - d\gamma_2(D) + d\gamma_3(D) + \gamma_2\left[(\nabla_D\eta)\xi + (\nabla_D\eta)\xi\right] + \gamma_3\left[(\nabla_D\theta)\zeta + (\nabla_D\theta)\zeta\right] \\
= \; & d\gamma_1(D) + d\gamma_2(\xi)\,\eta(D) + d\gamma_3(\zeta)\,\theta(D)
\end{aligned}$$

$$+\gamma_2\left[(\nabla_\xi\eta)D+\eta(D)\operatorname{div}\xi\right]+\gamma_3\left[(\nabla_\zeta\theta)D+\theta(D)\operatorname{div}\zeta\right], \tag{4.5}$$

where the divergences of ξ and ζ are denoted by $\operatorname{div}\xi$ and $\operatorname{div}\zeta$, respectively.

Fixing $D=\xi$ in Eq (4.5), we get

$$3\,d\gamma_1(\xi)+d\gamma_3(\xi)+\gamma_2\operatorname{div}(\xi)=0. \tag{4.6}$$

In particular, if the scalars γ_1 and γ_3 remain invariant under the velocity vector field ξ, Eq (4.6) implies that either $\gamma_2=0$ or $\operatorname{div}\xi=0$.

Now, if $\gamma_2=0$ and $\operatorname{div}\xi\neq 0$, then from [29], an EoS for Takabayashi string (i.e., P-string) is $\omega=-1$. Therefore, the spacetime recovers the quintessence era.

Moreover, for $\gamma_2\neq 0$ and $\operatorname{div}\xi=0$, it means that the expansion scalar vanishes [1]. Hence, we have the following result:

Theorem 4.2. *A $G(QE)_n$ string cloud spacetime coupled with a Codazzi-type energy-momentum tensor represents either a quintessence era or a vanishing expansion scalar, provided that the scalars γ_1 and γ_3 remain invariant under the velocity vector field ξ.*

5. SCS with Ricci semi-symmetric and pseudo-symmetric EMT

Chaki's concept [24] is fundamentally different from the notion of a pseudo-symmetric manifold, as introduced in [14]. We define the endomorphism $(D\wedge_g E)$ by

$$(D\wedge_g E)F=g(E,F)D-g(D,F)E, \tag{5.1}$$

where $D,E,F\in\mathfrak{X}(\mathcal{H})$.

Thus, we define the tensors $\mathcal{R}\cdot\mathcal{R}$, $\mathcal{R}\cdot\mathrm{Ric}$, $Q(g,\mathcal{R})$, and $Q(g,\mathrm{Ric})$ as follows:

$$\begin{aligned}(\mathcal{R}(G,H)\cdot\mathcal{R})(D,E)F=&\ \mathcal{R}(G,H)\mathcal{R}(D,E)F-\mathcal{R}(\mathcal{R}(G,H)D,E)F\\&-\mathcal{R}(D,\mathcal{R}(G,H)E)F-\mathcal{R}(D,E)\mathcal{R}(G,H)F,\end{aligned} \tag{5.2}$$

$$(\mathcal{R}(G,H)\cdot\mathrm{Ric})(D,E)=-\mathrm{Ric}(\mathcal{R}(G,H)D,E)-\mathrm{Ric}(D,\mathcal{R}(G,H)E), \tag{5.3}$$

$$\begin{aligned}Q(g,\mathcal{R})(D,E)F=&\ (G\wedge_g H)\mathcal{R}(D,E)F-\mathcal{R}((G\wedge_g H)D,E)F\\&-\mathcal{R}(D,(G\wedge_g H)E)F-\mathcal{R}(D,E)(G\wedge_g H)F,\end{aligned} \tag{5.4}$$

$$Q(g,\mathrm{Ric})(D,E)=-\mathrm{Ric}((G\wedge_g H)D,E)-\mathrm{Ric}(D,(G\wedge_g H)E), \tag{5.5}$$

where $G,H,D,E,F\in\mathfrak{X}(\mathcal{H})$.

A semi-Riemannian manifold is said to be pseudo-symmetric [14] if $\mathcal{R}\cdot\mathcal{R}$ and $Q(g,\mathcal{R})$ are linearly dependent at each point of the manifold. Hence, we have

$$\mathcal{R}\cdot\mathcal{R}=f_{\mathcal{R}}Q(g,\mathcal{R}),$$

for some smooth function $f_{\mathcal{R}}$.

Also, a semi-Riemannian manifold is said to be Ricci pseudo-symmetric [14] if

$$\mathcal{R} \cdot \mathrm{Ric} = f_{\mathrm{Ric}} Q(g, \mathrm{Ric}),$$

which holds on the set

$$A_{\mathrm{Ric}} = \{D \in \mathcal{H} : \mathrm{Ric} \neq \tfrac{\tau}{n} g\},$$

where f_{Ric} is some function on A_{Ric}. Every pseudo-symmetric manifold is Ricci pseudo-symmetric, but the converse is not necessarily true.

In [13], the author investigated spacetimes with pseudo-symmetric energy-momentum tensors in the sense of Chaki [24]. Here, we study Deszcz's notion of pseudo-symmetry.

Let $(\mathcal{H}^4, g)$ admit Ricci semi-symmetry, that is,

$$(\mathcal{R}(G, H))\mathrm{Ric}(D, E) = 0,$$

which implies

$$-\mathrm{Ric}(\mathcal{R}(G, H)D, E) - \mathrm{Ric}(D, \mathcal{R}(G, H)E) = 0. \tag{5.6}$$

Using Eq (1.3) in (5.6), we obtain

$$-\gamma_2\Big[\eta(\mathcal{R}(G, H)D)\eta(E) + \eta(D)\eta(\mathcal{R}(G, H)E)\Big] - \gamma_3\Big[\theta(\mathcal{R}(G, H)D)\theta(E) + \theta(D)\theta(\mathcal{R}(G, H)E)\Big] = 0. \tag{5.7}$$

Putting $E = \xi$ in (5.7), we obtain

$$\gamma_2\, g(\mathcal{R}(G, H)D, \xi) = 0, \tag{5.8}$$

which implies that either $\gamma_2 = 0$ or $\mathrm{Ric}(G, \xi) = 0$.

Therefore, from [29], an EoS for Takabayashi string (i.e., P-string) is $\omega = -1$. Thus, the spacetime corresponds to a quintessence era.

We now state our finding:

Theorem 5.1. *A G(QE)$_n$ pseudo-symmetric string cloud spacetime is a quintessence era.*

Next, we suppose that $(\mathcal{H}^4, g)$ admits a pseudo-symmetric energy-momentum tensor, that is,

$$\mathcal{R} \cdot \mathcal{T} = f Q(g, \mathcal{T}), \tag{5.9}$$

where f is a smooth function. In view of Eqs (1.1) and (2.4), we have

$$\begin{aligned}
\mathcal{R} \cdot \mathrm{Ric} = \kappa\mathcal{R} \cdot T &= f Q(g, T) \\
&= -\kappa\mathcal{T}\,(g(H, D)G - g(G, D)H, E) - \kappa\mathcal{T}\,(D, g(H, E)G - g(G, E)H) \\
&= -g(E, G)\mathrm{Ric}(D, H) + g(D, G)\mathrm{Ric}(E, H) - g(E, H)\mathrm{Ric}(D, G) + g(D, H)\mathrm{Ric}(G, E) \\
&= \mathrm{Ric}\,(g(E, G)D - g(D, G)E, H) - \mathrm{Ric}\,(G, g(E, H)D - g(D, H)E) \\
&= Q(g, \mathrm{Ric}),
\end{aligned} \tag{5.10}$$

which means that

$$\mathrm{Ric} \cdot \mathrm{Ric} = f Q(g, \mathrm{Ric}). \tag{5.11}$$

By virtue of (1.3), Eq (5.11) takes the form

$$\begin{aligned} & -\gamma_2[\eta(\mathcal{R}(D,E)G)\eta(H)+\eta(G)\eta(\mathcal{R}(D,E)H)] \\ & -\gamma_3\left[\theta(\mathcal{R}(D,E)G)\theta(H)+\eta(G)\theta(\mathcal{R}(D,E)H)\right] \\ = & -f[g(E,G)\mathrm{Ric}(D,H)-g(D,G)\mathrm{Ric}(E,H) \\ & +g(E,H)\mathrm{Ric}(D,G)-g(D,H)\mathrm{Ric}(G,E)]. \end{aligned} \tag{5.12}$$

Putting $G=E=\xi$ in (5.12) and using Eq (2.4), we yield

$$f\left[\mathrm{Ric}(D,H)+\left(\frac{\mathcal{R}}{2}-\kappa\right)g(D,H)\right]=0. \tag{5.13}$$

Thus, either $f=0$ or the spacetime is Einstein, which is not possible, as it contradicts the definition of a proper Ricci pseudo-symmetric spacetime. Hence, $\mathcal{R}\cdot\mathcal{T}=0\Rightarrow\mathcal{R}\cdot\mathrm{Ric}=0$, that is, the spacetime is Ricci symmetric. So, our finding is that:

Theorem 5.2. *A G(QE)$_n$ string cloud spacetime coupled with a pseudo-symmetric energy-momentum tensor represents the quintessence era.*

6. String cloud spacetimes with a recurrent EMT

Let $(\mathcal{H}_4, g)$ admit an energy-momentum tensor of recurrent type, that is,

$$(\nabla_D\mathcal{T})(E,F)=\eta(D)\mathcal{T}(E,F). \tag{6.1}$$

From Eqs (1.1) and (6.1), we obtain

$$(\nabla_D\mathrm{Ric})(E,F)-\frac{d\tau(D)}{2}g(E,F)=\eta(D)\mathrm{Ric}(E,F)-\frac{\tau}{2}\eta(D)g(E,F). \tag{6.2}$$

Now, by substituting Eq (1.3) into (6.2), we obtain

$$\begin{aligned} & d\gamma_1(D)g(E,F)+d\gamma_2(D)\eta(E)\eta(F)+\gamma_2\left[(\nabla_D\eta)E\eta(F)+\eta(E)(\nabla_D\eta)F\right] \\ & +d\gamma_3(D)\theta(E)\theta(F)+\gamma_3\left[(\nabla_D\theta)E\theta(F)+\theta(E)(\nabla_D\theta)F\right]-\frac{d\tau(D)}{2}g(E,F) \\ = & \ \eta(D)\mathrm{Ric}(E,F)-\frac{\tau}{2}\eta(D)g(E,F). \end{aligned} \tag{6.3}$$

Contracting over the vector fields E and F, we obtain

$$4d\gamma_1(D)-d\gamma_2(D)-d\gamma_3(D)+2\gamma_2(\nabla_D\eta)\xi+2\gamma_3(\nabla_D\theta)\zeta=\tau\eta(D). \tag{6.4}$$

According to [38], if ξ is a Killing vector field, then $\mathfrak{L}_\xi\alpha=0$ and $\mathfrak{L}_\xi\beta=0$, where $\mathfrak{L}$ denotes the Lie derivative operator. Hence, we deduce

$$d\gamma_1(\xi)=d\gamma_2(\xi)=d\gamma_3(\xi)=0.$$

Moreover, from Eq (3.2), we obtain $d\tau(\xi)=0$. Thus, setting $D=\xi$ in Eq (6.4) and using (3.3), we find

$$\frac{\alpha}{\beta}=-1, \tag{6.5}$$

which shows that the spacetime corresponds to a quintessence era.

Theorem 6.1. *A $\mathcal{T}$-recurrent G(QE)$_n$ string cloud spacetime represents a quintessence era, provided that the velocity vector field is Killing.*

7. String cloud spacetimes admitting a weakly symmetric EMT

We assume that the energy-momentum tensor of $(SCS)_4$ is of weakly symmetric type, that is,

$$(\nabla_D \mathcal{T})(E,F) = \eta(D)\mathcal{T}(E,F) + \theta(E)\mathcal{T}(D,F) + \delta(F)\mathcal{T}(E,D). \tag{7.1}$$

Using (1.1) and (1.3), Eq (7.1) leads to

$$\begin{aligned} & d\gamma_1(D)g(E,F) + d\gamma_2(D)\eta(E)\eta(F) + \gamma_2\left[(\nabla_D\eta)E\eta(F) + \eta(E)(\nabla_D\eta)F\right] \\ & + d\gamma_3(D)\theta(E)\theta(F) + \gamma_3\left[(\nabla_D\theta)E\theta(F) + \theta(E)(\nabla_D\theta)F\right] - \frac{d\tau(D)}{2}g(E,F) \\ = \; & \eta(D)\mathrm{Ric}(E,F) - \frac{\tau}{2}\eta(D)g(E,F) + \theta(E)\mathrm{Ric}(D,F) - \frac{\tau}{2}\eta(E)g(D,F) \\ & + \delta(F)\mathrm{Ric}(E,D) - \frac{\tau}{2}\eta(F)g(E,D). \end{aligned} \tag{7.2}$$

Contracting (7.2) over E and F, and using (2.4), we obtain

$$\begin{aligned} & 4d\gamma_1(D) - d\gamma_2(D) + 2\gamma_2(\nabla_D\eta)\xi + 2\gamma_3(\nabla_D\theta)\zeta - 2d\mathcal{R}(D) \\ = \; & -\eta(D)\tau + \left(\tfrac{\tau}{2} - \kappa\alpha\right)\theta(D) + \left(\tfrac{\tau}{2} - \kappa\beta\right)\eta(D) - \tfrac{\tau}{2}(\theta(D) + \delta(D)). \end{aligned} \tag{7.3}$$

On the other hand, if ξ is a Killing vector field, then from [40] we have

$$d\gamma_1(\xi) = d\gamma_2(\xi) = 0 \quad \Rightarrow \quad d\tau(\xi) = 0. \tag{7.4}$$

Substituting $D = \xi$ into (7.3) and applying (7.4), we obtain

$$\tau = -2\kappa\beta. \tag{7.5}$$

Moreover, from (2.2) and (3.2), it follows that

$$\tau = \kappa(\alpha + \beta). \tag{7.6}$$

Combining (7.5) and (7.6), we arrive at

$$\kappa(\alpha + 3\beta) = 0, \tag{7.7}$$

which implies that $\kappa \neq 0$ and consequently $\frac{\alpha}{\beta} = -3 < 0$. This leads to $\omega < 0$, which coincides with the equation of state for a cloud of geometric strings (Nambu strings [30]). Hence, we obtain the desired result.

Theorem 7.1. *A weakly $\mathcal{T}$-symmetric $G(QE)_n$ string cloud spacetime that satisfies the EFEs corresponds to the equation of state for a cloud of geometric strings, provided the velocity vector field is a Killing vector field.*

Corollary 7.1. *If a $G(QE)_n$ string cloud spacetime satisfying the EFEs is weakly $\mathcal{T}$-symmetric, then the equation of state $\alpha = -3\beta$ reduces to that of Nambu strings, provided the velocity vector field is a Killing vector field.*

Corollary 7.2. *If a $G(QE)_n$ string cloud spacetime satisfying the EFEs is weakly $\mathcal{T}$-symmetric, then the string cloud represents a massive string, provided the velocity vector field is a Killing vector field.*

8. String cloud spacetimes with a special weakly symmetric EMT

The concept of a special weakly Ricci symmetric manifold was introduced and analyzed by Singh and Quddus [39]. In a similar vein, we now define and examine a special weakly $\mathcal{T}$-symmetric (SWTS) manifold, that is,

$$(\nabla_D \mathcal{T})(E,F) = 2\eta(D)\mathcal{T}(E,F) + \eta(E)\mathcal{T}(D,F) + \eta(F)\mathcal{T}(D,E), \tag{8.1}$$

where η is a 1-form defined by $\eta(D) = g(D,\xi)$, with ξ the associated vector field.

Using (1.1) and (1.3) in (8.1), we get

$$\begin{aligned}(\nabla_D \mathrm{Ric})(E,F) - \frac{\tau}{2}g(E,F) = 2\eta(D)\Big[\mathrm{Ric}(E,F) - \frac{\tau}{2}g(E,F)\Big] + \eta(E)\Big[\mathrm{Ric}(D,F) - \frac{\tau}{2}g(D,F)\Big] \\ + \eta(F)\Big[\mathrm{Ric}(D,E) - \frac{\tau}{2}g(D,E)\Big].\end{aligned} \tag{8.2}$$

Contracting over E and F, we obtain

$$\begin{aligned}&4\,d\gamma_1(D) - d\gamma_2(D) + d\gamma_3(D) + 2\gamma_2(\nabla_D\eta)\xi + 2\gamma_3(\nabla_D\theta)\zeta - 2\,d\tau(D) \\ &= 2\eta(D)\tau - 4\eta(D)\tau + 2\Big[\mathrm{Ric}(D,\xi) - \tfrac{\tau}{2}\eta(D)\Big].\end{aligned} \tag{8.3}$$

If ξ is Killing, then from [40] we have

$$d\gamma_1(\xi) = d\gamma_2(\xi) = d\gamma_3(\xi) = 0 \quad \Rightarrow \quad d\tau(\xi) = 0. \tag{8.4}$$

Putting $D = \xi$ in (8.3) and using (8.4), we obtain

$$\kappa(\alpha + \beta) = 0,$$

which implies $\kappa \neq 0$ and hence

$$\frac{\alpha}{\beta} = -1,$$

that is, $\omega < 0$, which coincides with the state equation for the cloud of geometric strings (Nambu strings [30]). Thus, we arrive at the desired conclusion.

Theorem 8.1. *A special weakly $\mathcal{T}$-symmetric $G(QE)_n$ string cloud spacetime that satisfies the EFEs represents the state equation for a cloud of geometric strings, provided the velocity vector field is Killing.*

Corollary 8.1. *If a $G(QE)_n$ string cloud spacetime satisfying the EFEs is special weakly $\mathcal{T}$-symmetric, then the state equation $\alpha = -\beta$ reduces to the state equation of Nambu strings, provided the velocity vector field is Killing.*

Corollary 8.2. *If a $G(QE)_n$ string cloud spacetime satisfying the EFEs is special weakly $\mathcal{T}$-symmetric, then the string represents a massive string, provided the velocity vector field is Killing.*

9. Generalized $\mathcal{T}$-recurrent string cloud spacetimes

Let $(\mathcal{H}_4, g)$ admit an energy-momentum tensor of generalized $\mathcal{T}$-recurrent type, that is,

$$(\nabla_D\mathcal{T})(E,F) = \eta(D)\mathcal{T}(E,F) + \theta(D)\mathcal{T}(E,F). \tag{9.1}$$

So, from Eqs (1.1) and (9.1), we have

$$\begin{aligned}&(\nabla_D\mathrm{Ric})(E,F) - \tfrac{d\tau(D)}{2}g(E,F)\\ &= \eta(D)\mathrm{Ric}(E,F) - \tfrac{\tau}{2}\eta(D)g(E,F) + \theta(D)\mathrm{Ric}(E,F) - \tfrac{\tau}{2}\theta(D)g(E,F).\end{aligned} \tag{9.2}$$

Now, utilizing Eq (1.3) in (9.2), we get

$$\begin{aligned}&d\gamma_1(D)\, g(E,F) + d\gamma_2(D)\, \eta(E)\eta(F) + \gamma_2[(\nabla_D\eta)(E)\, \eta(F) + \eta(E)(\nabla_D\eta)(F)]\\ &+ d\gamma_3(D)\, \theta(E)\theta(F) + \gamma_3[(\nabla_D\theta)(E)\, \theta(F) + \theta(E)(\nabla_D\theta)(F)] - \tfrac{d\tau(D)}{2}g(E,F)\\ &= \eta(D)\mathrm{Ric}(E,F) - \tfrac{\tau}{2}\eta(D)g(E,F) + \theta(D)\mathrm{Ric}(E,F) - \tfrac{\tau}{2}\theta(D)g(E,F).\end{aligned} \tag{9.3}$$

After contracting over the vector fields E and F, we obtain

$$4d\gamma_1(D) - d\gamma_2(D) - d\gamma_3(D) + 2\gamma_2(\nabla_D\eta)\xi + 2\gamma_3(\nabla_D\theta)\zeta - 2d\tau(D) = -\tau[\eta(D) + \theta(D)]. \tag{9.4}$$

If ξ is a Killing vector field, then from [40] we have

$$d\gamma_1(\xi) = d\gamma_2(\xi) = d\gamma_3(\xi) = 0 \quad \Rightarrow \quad d\tau(\xi) = 0. \tag{9.5}$$

Substituting $D = \xi$ into (9.4), and using (3.2) together with (9.5), we obtain

$$\frac{\alpha}{\beta} = -1, \tag{9.6}$$

which shows that the spacetime corresponds to a quintessence era.

Theorem 9.1. *A $G(QE)_n$ generalized $\mathcal{T}$-recurrent string cloud spacetime represents a quintessence era, provided the velocity vector field is Killing.*

10. String cloud spacetimes with a semi-generalized recurrent EMT

A Riemannian manifold $\mathcal{H}$ is said to be a semi-generalized $\mathcal{T}$-recurrent if

$$(\nabla_D\mathcal{T})(E,F) = 2\eta(D)\mathcal{T}(E,F) + 3\theta(D)g(E,F), \tag{10.1}$$

where η and θ are two 1-forms, θ is non-zero, and ξ and ζ are two vector fields such that $\eta(D) = g(D,\xi)$, $\theta(D) = g(D,\zeta)$.

Using (1.1) and (1.3) in (10.1), we yield

$$\begin{aligned}&d\gamma_1(D)g(E,F) + d\gamma_2(D)\eta(E)\eta(F) + \gamma_2\left[(\nabla_D\eta)E\eta(F) + \eta(E)(\nabla_D\eta)F\right]\\ &+ d\gamma_3(D)\theta(E)\theta(F) + \gamma_3\left[(\nabla_D\theta)E\theta(F) + \theta(E)(\nabla_D\theta)F\right] - \frac{d\tau(D)}{2}g(E,F)\end{aligned}$$

$$= \eta(D)\mathrm{Ric}(E,F) - \frac{\tau}{2}\eta(D)g(E,F) + 3\theta(D)g(E,F). \tag{10.2}$$

Contraction (10.2) over E and F, we reflect

$$\begin{aligned} & 4d\gamma_1(D) - d\gamma_2(D) + d\gamma_3(D) + 2\gamma_2(\nabla_D\eta)\xi + 2\gamma_3(\nabla_D\theta)\zeta - 2d\tau(D) \\ = \; & -2\eta(D)\tau + 12\theta(D). \end{aligned} \tag{10.3}$$

If ξ is Killing, then from [40] we have

$$d\gamma_1(\xi) = d\gamma_2(\xi) = d\gamma_3(\xi) = 0 \Rightarrow d\tau(\xi) = 0. \tag{10.4}$$

Putting $D = \xi$ in (10.3) and using (10.4) yields

$$2\kappa(\alpha + \beta) = 0, \tag{10.5}$$

which implies that $\kappa \neq 0$ and $\frac{\alpha}{\beta} = -1$, that is, $\omega < 0$, which coincides with the state equation for the cloud of geometric strings (Nambu strings [30]). We arrive at the outcome.

Theorem 10.1. *A semi-generalized $\mathcal{T}$-recurrent $G(QE)_n$ string cloud spacetime satisfying the EFEs represents the state equation for a cloud of geometric strings, provided the velocity vector field is Killing.*

Corollary 10.1. *If a $G(QE)_n$ string cloud spacetime satisfying the EFEs is semi-generalized $\mathcal{T}$-recurrent, then the state equation $\alpha = -\beta$ corresponds to that of Nambu strings, provided the velocity vector field is Killing.*

Corollary 10.2. *If a $G(QE)_n$ string cloud spacetime satisfying the EFEs is semi-generalized $\mathcal{T}$-recurrent, then the string represents a massive string, provided the velocity vector field is Killing.*

11. Almost pseudosymmetric EMT

The notion of an almost pseudo-Ricci symmetric non-flat Lorentzian manifold ($n > 3$) was introduced in [41]. In contrast, in our study we consider the almost pseudosymmetric energy-momentum tensor. A non-flat Lorentzian manifold is said to be an almost pseudosymmetric energy-momentum spacetime if its energy-momentum tensor $\mathcal{T}$ of type $(0,2)$ is not identically zero and satisfies the following condition:

$$(\nabla_D\mathcal{T})(E,F) = [\eta(D) + \theta(D)]\mathcal{T}(E,F) + \eta(E)\mathcal{T}(D,F) + \eta(F)\mathcal{T}(D,E). \tag{11.1}$$

Using (1.1) in (11.1), we obtain

$$\begin{aligned} (\nabla_D\mathrm{Ric})(E,F) = {} & [\eta(D) + \theta(D)]\mathrm{Ric}(E,F) + \eta(E)\mathrm{Ric}(D,F) + \eta(F)\mathrm{Ric}(D,E) \\ & - \frac{\tau}{2}\Big([\eta(D) + \theta(D)]g(E,F) + \eta(E)g(D,F) + \eta(F)g(D,E)\Big) \\ & + \tfrac{1}{2}d\tau(D)\, g(E,F). \end{aligned} \tag{11.2}$$

Again, using (1.3) in (11.2), we obtain

$$\begin{aligned}
& d\gamma_1(D)g(E,F) + d\gamma_2(D)\eta(E)\eta(F) + \gamma_2\left[(\nabla_D\eta)E\eta(F) + \eta(E)(\nabla_D\eta)F\right] \\
& + d\gamma_3(D)\theta(E)\theta(F) + \gamma_3\left[(\nabla_D\theta)E\theta(F) + \theta(E)(\nabla_D\theta)F\right] - \frac{d\tau(D)}{2}g(E,F) \\
= \; & [\eta(D) + \theta(D)]\mathrm{Ric}(E,F) + \eta(E)\mathrm{Ric}(D,F) + \eta(F)\mathrm{Ric}(D,E) \\
& - \frac{\tau}{2}[(\eta(D) + \theta(D))g(E,F) + \eta(E)g(F,D) + \eta(F)g(E,D)] + \frac{d\tau(D)}{2}g(E,F).
\end{aligned} \tag{11.3}$$

After contracting over E and F, and taking a frame field, we get

$$\begin{aligned}
& 4d\gamma_1(D) - d\gamma_2(D) + d\gamma_3(D) + 2\gamma_2(\nabla_D\eta)\xi + 2\gamma_3(\nabla_D\theta)\zeta \\
= \; & -[\eta(D) + \theta(D)]\tau + 2\mathrm{Ric}(D,\xi) - \tau\eta(D) - 2d\tau(D).
\end{aligned} \tag{11.4}$$

As per [38], if ξ is a Killing vector field, then $\mathfrak{L}_\xi\alpha = 0$ and $\mathfrak{L}_\xi\beta = 0$, where $\mathfrak{L}$ denotes the Lie derivative operator. Hence, we get

$$d\gamma_1(\xi) = d\gamma_2(\xi) = d\gamma_3(\xi) = 0.$$

Moreover, from Eq (3.2), we obtain $d\mathcal{R}(\xi) = 0$. Thus, for $D = \xi$ in Eq (11.4), we yield

$$\kappa(\alpha + \beta) = 0, \tag{11.5}$$

which implies that $\kappa \neq 0$ and $\frac{\alpha}{\beta} = -1$, that is, $\omega < 0$, which coincides with the state equation for the cloud of geometric strings (Nambu strings [30]). We arrive at the outcome.

Theorem 11.1. *If a $G(QE)_n$ string cloud spacetime satisfying the EFEs admits an almost pseudosymmetric energy-momentum tensor, then it corresponds to the EoS for a cloud of geometric strings, provided the velocity vector field is Killing.*

Corollary 11.1. *If a $G(QE)_n$ string cloud spacetime satisfying the EFEs admits an almost pseudosymmetric energy-momentum tensor, then the EoS $\alpha = -\beta$ corresponds to that of Nambu strings, provided the velocity vector field is Killing.*

Corollary 11.2. *If a $G(QE)_n$ string cloud spacetime satisfying the EFEs admits an almost pseudosymmetric energy-momentum tensor, then the string corresponds to a massive string, provided the velocity vector field is Killing.*

12. String cloud spacetimes with quadratic Killing EMT

The energy-momentum tensor of a string cloud spacetime is quadratic Killing if $\mathcal{T}$ satisfies the condition:

$$(\nabla_D\mathcal{T})(E,F) + (\nabla_E\mathcal{T})(D,F) + (\nabla_F\mathcal{T})(D,E) = 0. \tag{12.1}$$

Using (1.1) in (12.1), we get

$$\begin{aligned}
0 = \; & (\nabla_D\mathrm{Ric})(E,F) - \frac{d\tau(D)}{2}g(E,F) + (\nabla_E\mathrm{Ric})(D,F) - \frac{d\tau(D)}{2}g(D,F) \\
& + (\nabla_F\mathrm{Ric})(E,D) - \frac{d\tau(D)}{2}g(D,E).
\end{aligned} \tag{12.2}$$

Again, using (2.3) in (12.2), we have

$$
\begin{aligned}
0 \;=\;& d\gamma_1(D)g(E,F) + d\gamma_2(D)\eta(E)\eta(F) + \gamma_2\left[(\nabla_D\eta)E\eta(F) + \eta(E)(\nabla_D\eta)F\right] \\
&+ d\gamma_3(D)\theta(E)\theta(F) + \gamma_3\left[(\nabla_D\theta)E\theta(F) + \theta(E)(\nabla_D\theta)F\right] - \frac{d\tau(D)}{2}g(E,F) \\
&+ d\gamma_1(E)g(D,F) + d\gamma_2(E)\eta(D)\eta(F) + \gamma_2\left[(\nabla_E\eta)D\eta(F) + \eta(E)(\nabla_E\eta)F\right] \\
&+ d\gamma_3(E)\theta(D)\theta(F) + \gamma_3\left[(\nabla_E\theta)D\theta(F) + \theta(D)(\nabla_E\theta)F\right] - \frac{d\tau(E)}{2}g(D,F) \\
&+ d\gamma_1(F)g(D,E) + d\gamma_2(F)\eta(D)\eta(E) + \gamma_2\left[(\nabla_F\eta)D\eta(E) + \eta(D)(\nabla_F\eta)E\right] \\
&+ d\gamma_3(F)\theta(E)\theta(D) + \gamma_3\left[(\nabla_F\theta)E\theta(D) + \theta(E)(\nabla_F\theta)E\right] - \frac{d\tau(F)}{2}g(E,D). \qquad (12.3)
\end{aligned}
$$

Contracting over E and F, we acquire

$$
\begin{aligned}
0 \;=\;& 5d\gamma_1(D) - d\gamma_2(D) + d\gamma_3(D) + 2\gamma_2(\nabla_D\eta)\xi + 2\gamma_3(\nabla_D\theta)\zeta - 2d\tau(D) \\
&+ d\gamma_2(\xi)\eta(D) + \gamma_2[(\nabla_\xi\eta)D + (\nabla_\xi\eta)\xi\eta(D)] \\
&+ d\gamma_3(\zeta)\theta(D) + \gamma_3[(\nabla_\zeta\theta)D + \mathrm{div}\zeta\theta(D)] - \frac{d\tau(\tau)}{2} \\
&+ d\gamma_1(D) + d\gamma_2(\xi)\eta(D) + \gamma_2[(\nabla_\xi\eta)(D) + \mathrm{div}\xi\eta(D)] \\
&+ d\gamma_3(\zeta)\theta(D) + \gamma_3[(\nabla_\zeta\theta)D + diH\zeta\theta(D)] - \frac{d\tau(D)}{2}, \qquad (12.4)
\end{aligned}
$$

where divξ and divζ denote the divergence of ξ and ζ, respectively. According to [38], if ξ is a Killing vector field, then $\mathfrak{L}_\xi\alpha = 0$ and $\mathfrak{L}_\xi\beta = 0$, where $\mathfrak{L}$ denotes the Lie derivative operator. Hence, we get

$$d\gamma_1(\xi) = d\gamma_2(\xi) = d\gamma_3(\xi) = 0.$$

Moreover, from Eq (3.2), we obtain $d\tau(\xi) = 0$. Thus, for $D = \xi$ in Eq (12.4), we have

$$\alpha\,\mathrm{div}(\xi) = 0. \qquad (12.5)$$

We infer from (12.5) that either $\alpha = 0$ or div$\xi = 0$. Hence, from (3.5), we conclude that the string cloud spacetime either recovers the quintessence era or the expansion scalar vanishes [1]. Therefore, we state the following:

Theorem 12.1. *A G(QE)$_n$ string cloud spacetime endowed with a quadratic Killing-type energy-momentum tensor either represents the quintessence era or exhibits a vanishing expansion scalar, provided the velocity vector field is Killing.*

13. Conclusions

In this work, we investigate the behavior of relativistic G(QE)$_n$ string cloud spacetimes endowed with various forms of the string cloud energy-momentum tensor $\mathcal{T}$, which incorporates the string cloud fluid density α and string tension β. For example, we demonstrate that a G(QE)$_n$ string cloud spacetime with a covariantly constant energy-momentum tensor satisfies the EoS $\frac{\alpha}{\beta} = -1$, which corresponds to either a massive string cloud spacetime or a bulk viscous fluid spacetime.

Additionally, when coupled with a covariantly constant energy-momentum tensor, this EoS coincides with the state equation for a cloud of geometric strings or signifies the quintessence era. The paper also explores $G(QE)_n$ string cloud spacetimes with Codazzi-type and pseudo-symmetric energy-momentum tensors, providing further insights into their geometric properties. Moreover, we analyze various energy-momentum tensors, such as $\mathcal{T}$-recurrent, weakly $\mathcal{T}$-symmetric, special weakly $\mathcal{T}$-symmetric, generalized $\mathcal{T}$-recurrent, semi-generalized $\mathcal{T}$-recurrent, and quadratic Killing-type, on string cloud spacetimes with a Killing velocity vector field. Our results show that such spacetimes either represent a massive string cloud or align with the quintessence era, providing a deeper understanding of the possible structures of $G(QE)_n$ string cloud spacetimes in cosmological models. Thus, we conclude that the analysis of these different energy-momentum tensors leads to significant insights into the nature of $G(QE)_n$ string cloud spacetimes, with important implications for both theoretical and cosmological studies.

Author contributions

S. K. Yadav, S. Shenawy and N. B. Turki: Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Visualization; S. K. Yadav and S. Shenawy: Writing–review and editing; N. B. Turki: Funding acquisition, Project administration; Y. L. Li: Conceptualization, Formal analysis, Investigation, Methodology, Writing–review and editing, Validation (critical evaluation of results) of the revised version.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, 1983.
2. P. S. Letelier, Clouds of strings in general relativity, *Phys. Rev. D*, **20** (1979), 1294. https://doi.org/10.1103/PhysRevD.20.1294
3. S. G. Ghosh, U. Papnoi, S. D. Maharaj, Cloud of strings in third order Lovelock gravity, *Phys. Rev. D*, **90** (2014), 044068. https://doi.org/10.1103/PhysRevD.90.044068
4. E. Herscovich, M. G. Richarte, Black holes in Einstein-Gauss-Bonnet gravity with a string cloud background, *Phys. Lett. B*, **689** (2010), 192–200. https://doi.org/10.1016/j.physletb.2010.04.065

5. A. Ganguly, S. G. Ghosh, S. D. Maharaj, Accretion onto a black hole in a string cloud background, *Phys. Rev. D*, **90** (2014), 064037. https://doi.org/10.1103/PhysRevD.90.064037

6. M. G. Richarte, C. Simeone, Traversable wormholes in a string cloud, *Int. J. Mod. Phys. D*, **17** (2008), 1179–1196. https://doi.org/10.1142/S0218271808012759

7. A. K. Yadav, V. K. Yadav, L. Yadav, Cylindrically symmetric inhomogeneous universes with a cloud of strings, *Int. J. Theor. Phys.*, **48** (2009), 568–578. https://doi.org/10.1007/s10773-008-9832-9

8. L. Mandel, E. Wolf, *Optical coherence and quantum optics*, Cambridge University Press, 1995.

9. J. Satish, R. Venkateswarlu, Bulk viscous fluid cosmological models in $f(R,T)$ gravity, *Chin. J. Phys.*, **54** (2016), 830–838. https://doi.org/10.1016/j.cjph.2016.08.008

10. M. Novello, M. J. Rebouças, The stability of a rotating universe, *Astrophys. J.*, **225** (1978), 719–724. https://doi.org/10.1086/156530

11. U. C. De, C. A. Mantica, S. Shenawy, B. Unal, Ricci solitons on singly warped product manifolds and applications, *J. Geom. Phys.*, **166** (2021), 104257. https://doi.org/10.1016/j.geomphys.2021.104257

12. U. C. De, S. K. Chaubey, S. Shenawy, Perfect fluid spacetimes and Yamabe solitons, *J. Math. Phys.*, **62** (2021), 032501. https://doi.org/10.1063/5.0033967

13. S. Guler, S. A. Demirbag, A study of generalized quasi Einstein spacetimes with applications in general relativity, *Int. J. Theor. Phys.*, **55** (2016), 548–562. https://doi.org/10.1007/s10773-015-2692-1

14. R. Jackiw, V. P. Nair, S. Y. Pi, A. P. Polychronakos, Perfect fluid theory and its extensions, *J. Phys. A Math. Gen.*, **37** (2004), R327. https://doi.org/10.1088/0305-4470/37/42/R01

15. M. C. Chaki, On generalized quasi Einstein manifolds, *Publ. Math. Debrecen*, **58** (2001), 683–691.

16. K. A. Bronnikov, S. W. Kim, M. V. Skvortsova, The Birkhoff theorem and string clouds, *Class. Quantum Grav.*, **33** (2016), 195006. https://doi.org/10.1088/0264-9381/33/19/195006

17. M. D. Siddiqi, M. A. Khan, I. Al-Dayel, K. Masood, Geometrization of string cloud spacetime in general relativity, *AIMS Math.*, **8** (2023), 29042–29057. https://doi.org/10.3934/math.20231487

18. P. S. Letelier, String cosmologies, *Phys. Rev. D*, **28** (1983), 2414. https://doi.org/10.1103/PhysRevD.28.2414

19. G. P. Singh, T. Singh, String cosmological models with magnetic field, *Gen. Relativity Gravitation*, **31** (1999), 371–378. https://doi.org/10.1023/A:1026644828215

20. J. Stachel, Thickening the string. I. The string perfect dust, *Phys. Rev. D*, **21** (1980), 2171. https://doi.org/10.1103/PhysRevD.21.2171

21. P. K. Sahoo, B. Mishra, String cloud and domain walls with quark matter for plane symmetric cosmological model in bimetric theory, *J. Theor. Appl. Phys.*, **7** (2013), 1–5.

22. V. U. M. Rao, D. Neelima, Dark energy cosmological model coupled with perfect fluid in a theory of gravitation, *Afr. Rev. Phys.*, **8** (2013), 429–436.

23. A. M. Blaga, Solitons and geometrical structures in a perfect fluid spacetime, *Rocky Mountain J. Math.*, **50** (2020), 41–53. https://doi.org/10.1216/rmj.2020.50.41

24. A. M. Blaga, Remarks on almost Riemann solitons with gradient or torse-forming vector field, *Bull. Malays. Math. Sci. Soc.*, **44** (2021), 3215–3227. https://doi.org/10.1007/s40840-021-01108-9

25. H. Sotani, K. Kohri, T. Harada, Restriction quark matter models by gravitational wave observation, *Phys. Rev. D*, **69** (2004), 084008. https://doi.org/10.1103/PhysRevD.69.084008

26. S. H. H. Tye, Brane inflation: string theory viewed from the cosmos, In: *String theory and fundamental interactions*, Springer, 2008, 949–974. https://doi.org/10.1007/978-3-540-74233-3_28

27. M. C. Chaki, S. Ray, Space-times with covariant-constant energy-momentum tensor, *Int. J. Theor. Phys.*, **35** (1996), 1027–1032. https://doi.org/10.1007/BF02302387

28. S. Weinberg, *Gravitation and cosmology: principles and applications of the general theory of relativity*, New York: John Wiley and Sons, 1972.

29. T. Takabayasi, Theory of one-dimensional relativistic elastic continuum for the model of particles and resonances, In: *Quantum mechanical determinism, causality and particles*, Dordrecht: Springer, 1976, 179–216. https://doi.org/10.1007/978-94-010-1440-3_13

30. Y. Nambu, Quark model and the factorization of the Veneziano amplitude, In: *Proceedings of the International Conference on Symmetries and Quark Models*, 1969, 269–277.

31. L. P. Eisenhart, *Riemannian geometry*, Princeton University Press, 1950. https://doi.org/10.1515/9781400884216

32. PHENIX Collaboration, K. Adcox, S. S. Adler, S. Afanasiev, C. Aidala, N. N. Ajitanand, et al., Formation of dense partonic matter in relativistic nucleus-nucleus collisions at RHIC: experimental evaluation by the PHENIX Collaboration, *Nucl. Phys. A*, **757** (2005), 184–283. https://doi.org/10.1016/j.nuclphysa.2005.03.086

33. C. A. Mantica, L. G. Molinari, U. C. De, A condition for a perfect-fluid space-time to be a generalized Robertson-Walker space-time, *J. Math. Phys.*, **57** (2016), 022508. https://doi.org/10.1063/1.4941942

34. M. Gutiérrez, B. Olea, Global decomposition of a Lorentzian manifold as a generalized Robertson-Walker space, *Differ. Geom. Appl.*, **27** (2009), 146–156. https://doi.org/10.1016/j.difgeo.2008.06.015

35. F. Melia, Cosmological redshift in Friedmann-Robertson-Walker metrics with constant space-time curvature, *Mon. Not. R. Astron. Soc.*, **422** (2012), 1418–1424. https://doi.org/10.1111/j.1365-2966.2012.20714.x

36. S. Mallick, U. C. De, Y. J. Suh, Spacetimes with different forms of energy-momentum tensor, *J. Geom. Phys.*, **151** (2020), 103622. https://doi.org/10.1016/j.geomphys.2020.103622

37. B. S. Guilfoyle, B. C. Nolan, Yang's gravitational theory, *Gen. Relativity Gravitation*, **30** (1998), 473–495. https://doi.org/10.1023/A:1018815027071

38. U. C. De, A. Sardar, K. De, Ricci-Yamabe solitons and 3-dimensional Riemannian manifolds, *Turkish J. Math.*, **46** (2022), 1078–1088. https://doi.org/10.55730/1300-0098.3143

39. H. Singh, Q. Khan, On special weakly symmetric Riemannian manifolds, *Publ. Math. Debrecen*, **58** (2001), 523–536.

40. K. L. Duggal, R. Sharma, *Symmetries of spacetimes and Riemannian manifolds*, Dordrecht: Kluwer Academic Publishers, 1999.

41. U. C. De, A. K. Gazi, On almost pseudo symmetric manifolds, *Ann. Univ. Sci. Budapest. Sect. Math.*, **51** (2008), 53–68.