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Research article

Contributions to the convergence analysis of nonconvex equilibrium problems

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Abstract: This paper establishes a successive approximation method designed to compute a point belonging simultaneously to the fixed-point set of a given mapping (relatively nonexpansive) and to the solution set of a nonconvex equilibrium problem within a Banach space. Our present findings provide a unifying generalization of numerous previously obtained results, beginning with the transition from Hilbertian structures to the more general Banach framework, and further encompassing the passage from convex analytical settings to their considerably more delicate nonconvex counterparts. We mainly extend the results proved in [27] from convex to nonconvex settings.

Keywords: nonconvex equilibrium problems; V-uniformly prox-regular sets; V-uniformly prox-regular functions; V-proximal trustworthy spaces; relatively nonexpansive mappings; fixed-point set
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1. Introduction

Consider a real Banach space X. One formulation of equilibrium problems commonly found in the literature is:

$$\text{Locate } \bar{u} \in K \text{ so that } ł(\bar{u}, u) \geq 0, \quad \text{for all } u \in K, \tag{1.1}$$

where $K \subset X$ is closed and convex, and $ł : K \times K \to \mathbb{R}$ is convex regarding the 2nd variable and satisfies $ł \equiv 0$ on $K \times K$.

The study of equilibrium problems is primarily motivated by their role as a unifying framework that generalizes several fundamental models, ranging from optimization paradigms and variational inequality frameworks to fixed-point formulations and equilibrium configurations in the sense of Nash.

This generality makes them highly relevant for an extensive range of applications across diverse domains of economic analysis, game theory, engineering, operations research, and physical sciences.

Moreover, extending equilibrium theory beyond convex settings to nonconvex structures in more general spaces, such as reflexive smooth Banach spaces, addresses the complexity of real-world problems where convexity assumptions often fail. These investigations also pave the way for the development of iterative algorithms and provide deeper theoretical insights into existence, uniqueness, and stability properties of solutions, making equilibrium problems both practically significant and mathematically rich. For further comprehensive exposition of the multifaceted applications of equilibrium problems, the reader is referred to [15, 28] together with the extensive bibliography therein.

In [15], the authors obtained a collection of existence results for solving (1.1), relying on the coercivity of the bifunction ł. Since then, significant research has focused on the analysis of convergence of iterative schemes for solving (1.1). In the context of Hilbert spaces, Moudafi [21] and Noor [22] proposed numerical algorithms and proved both weak and strong convergence of these methods for solving (1.1).

In Banach space settings, the works by Li [19], Fan [17], and Liu [20] introduced different approaches to solving (1.1) in this broader setting.

For the *nonconvex case*, Bounkhel and Al-Sinan [14] extended the equilibrium problem to Hilbert spaces with the formulation:

$$\text{Locate } \bar{u} \in K \text{ so that } ł(\bar{u}, u) + \theta\|u - \bar{u}\|^2 \geq 0, \quad \text{for all } u \in K, \tag{1.2}$$

where $K \subset X$ is not necessarily convex, and $ł : K \times K \to \mathbb{R}$ satisfies $ł \equiv 0$ on $K \times K$, which is *not necessarily convex* regarding the second variable. In [14], the authors demonstrated the convergence of numerical algorithms for solving (1.2), by assuming some regularity assumptions of both K and the bifunction ł (regarding the second variable) over K. Subsequent works have explored both the solution existence and the convergence behavior of numerical schemes to solutions of (1.2); see, for instance, [11–13, 23, 24].

The broadening of this problem (1.2) to Banach space settings was proposed in [11, 12] as follows:

$$\text{Locate } \bar{u} \in K \text{ so that } ł(\bar{u}, u) + \theta V(J(\bar{u}); u) \geq 0, \quad \text{for all } u \in K, \tag{1.3}$$

where V is defined by $V(J(\bar{u}); u) = \|u\|^2 - 2\langle J(\bar{u}), u\rangle + \|\bar{u}\|^2$ and J is the duality mapping on X.

Some special cases of (1.3) include:

(1) When $\theta = 0$, problem (1.3) aligns with the classical equilibrium inequality studied in [15].
(2) When X is Hilbert, the proposed problem (1.3) aligns with (1.2).
(3) When $ł(\bar{u}, u) = \langle T(\bar{u}), u - \bar{u}\rangle$, the proposed problem (1.3) coincides with the problem studied in [11].

The chief aim in this work is to investigate the convergence analysis of certain algorithms toward solutions of (1.3) under consideration. Mainly, we extend Proposition 4.1 and Theorem 4.1 in [27] from convex to nonconvex settings. The exposition of the present investigation is structured in the following manner. In Section 2, we state the main definitions and results needed in our research. We also establish new results on uniformly V-prox-regular sets (in short V-*u.p.r.*). Section 3 is focused on our proposed equilibrium problems (1.3) in Banach spaces. In Section 4, we prove in Theorem 4.1 the convergence in the strong topology of a considered iterative scheme toward solutions of (1.3). We also prove the convergence in the weak topology in Theorem 4.3 toward solutions of (1.3).

2. Preliminaries

Throughout the paper, X is a reflexive smooth Banach space, and we write $\mathbb{B}$ and $\mathbb{B}_*$ for the closed unit balls in X and X^*, respectively. By $\langle \cdot, \cdot \rangle$ we denote the duality pairing connecting X and X^*. We also denote J and J^* the duality mappings on X and X^*, respectively. Our assumption on the space X ensures the single-valuedness of J. Indeed, J is single-valued whenever X is a reflexive and smooth Banach space (see [26]). The behavior and the features of J and J^* are extensively documented; for more information, consult [26]. We review several important concepts and definitions below.

Definition 2.1.

- *For a l.s.c. function (lower semi-continuous)* $g : X \to \mathbb{R} \cup \{+\infty\}$ *and a point* $u \in \ dom\, g := \{v \in X : g(v) < \infty\}$*, we define the V-proximal subdifferential (see [8])* $\partial^\pi g$ *of* g *at* u *by the set all vectors* $u^* \in \partial^\pi g(u)$ *for which there are* $\sigma > 0, \delta > 0$ *so that*

$$\langle u^*, u' - u\rangle \leq g(u') - g(u) + \sigma V(J(u), u')), \text{ for all } u' \in u + \delta\mathbb{B}. \tag{2.1}$$

  We notice that $\partial^\pi \mathfrak{l}(\bar{u}) \subset L\mathbb{B}_*$*, whenever* g *is considered to be locally Lipschitz continuous at* $\bar{u}$ *(see [6]).*

- *We define the V-proximal normal cone of a given closed subset* $\emptyset \neq A \subset X$ *at* $u \in A$ *by* $N^\pi(A; u) := \partial^\pi \psi_A(u)$*, where* ψ_A *is the indicator function of* A*, and so we write* $u^* \in N^\pi(A; u)$ *provided that there are* $\sigma > 0, \delta > 0$ *in a way whereby*

$$\langle u^*, u' - u\rangle \leq \sigma V(J(u), u')), \text{ for all } u' \in (u + \delta\mathbb{B}) \cap A. \tag{2.2}$$

  The global formulation of (2.2) was proved in [8] to be an equivalent characterization of $N^\pi(A; u)$ *as* $u^* \in N^\pi(A; u)$ *as long as there exists* $\sigma > 0$ *so that*

$$\langle u^*, u' - u\rangle \leq \sigma V(J(u), u'), \text{ for all } u' \in A. \tag{2.3}$$

  Notice that $N^\pi(A; u)$ *is also described (see [8]) with the help of* π_A *as follows:*

$$u^* \in N^\pi(A; \bar{u}) \Leftrightarrow \exists \alpha > 0, \text{ so that } \bar{u} \in \pi_S(J\bar{u} + \alpha u^*).$$

  We restate the concept of $\pi_A : X^* \to A$ *as follows:*

$$\bar{u} \in \pi_A(u^*) \Leftrightarrow \bar{u} \in A \text{ with } V(u^*; \bar{u}) = \inf_{a \in A} V(u^*; a) =: d_A^V(u^*).$$

- *The subdifferential* $\partial^{L\pi} g(\bar{u})$ *(called the limiting V-proximal) is described by (see [6])*

$$u^* \in \partial^{L\pi} g(\bar{u}) \Leftrightarrow \exists u_n \to^g \bar{u} \text{ and } u_n^* \in \partial^\pi g(u_n) \text{ with } u_n^* \to^w u^*,$$

  where $u_n \to^g \bar{u}$ *means* $u_n \to \bar{u}$ *with* $g(u_n) \to g(\bar{u})$*, and* $u_n^* \to^w u^*$ *stands for the weak convergence of* u_n^* *to* u^**.*

- *The normal cone* $N^{L\pi}(A; \bar{u})$ *(called the limiting V-proximal) is described as follows (see [6]):*

$$u^* \in N^{L\pi}(A; \bar{u}) \Leftrightarrow \exists u_n \to^A \bar{u} \text{ and } u_n^* \in N^\pi(A; u_n) \text{ with } u_n^* \to^w u^*.$$

- *The space X is designated as V-proximal trustworthy (in short V-prox.-trust.) (see [7]) if $\forall\epsilon > 0$, any $g_1, g_2 : X \to \mathbb{R} \cup \{\infty\}$, and $\forall u \in X$ so that g_1 is l.s.c. and g_2 is Lipschitz continuous around u, we have*
$$\partial^\pi(g_1 + g_2)(z) \subset \cup\{\partial^\pi g_1(z_1) + \partial^\pi g_2(z_2) : z_i \in U_{g_i}(z, \epsilon), i = 1, 2\} + \epsilon\mathbb{B}_*.$$
Here, $U_{g_i}(z, \epsilon) := \{x \in z + \epsilon\mathbb{B}$ so that $|g_i(z) - g_i(x)| < \epsilon\}$. It was proven in [7] that L^p spaces ($p > 1$) are V-prox.-trust..

Also, we state without proofs the following three important results required for this investigation. Their proofs can be found in [2, 7, 8], respectively.

Proposition 2.2.

(1) Consider that X is V-prox.-trust., $\bar{u} \in X$, g_1 is l.s.c., and g_2 is locally Lipschitz continuous around $\bar{u}$. Then, we have
$$\partial^{L\pi}(g_1 + g_2)(\bar{u}) \subset \partial^{L\pi} g_1(\bar{u}) + \partial^{L\pi} g_2(\bar{u}).$$
(2) Suppose the boundness of the set K and that X is a q-unif. convex space. Then, $\exists\alpha > 0$ in a way
$$\|u - u'\|^q \le \alpha V(J(u), u'), \quad \text{for all } u, u' \in K.$$
(3) For p-unif. smooth spaces X, we have that J is unif. continuous over bounded sets.

The two nonconvex concepts (see [11, 12]) that will be used in our framework are:

Definition 2.3. *A closed subset $\emptyset \neq K \subset X$ is refereed to as V-uniformly prox-regular (in short V-u.p.r.) with constant $r > 0$ if for all $u \in K$ and all $u^* \in N^\pi(K; u)$, we have*
$$\langle u^*, u' - u\rangle \le \frac{\|u^*\|}{2r} V(J(u), u')), \quad \forall u' \in K. \tag{2.4}$$

Definition 2.4. *A l.s.c. function $g : X \to \mathbb{R} \cup \{+\infty\}$ is referred to as V-uniformly-prox-regular (in short V-u.p.r.) over a nonempty closed subset $K \subset dom g$ as long as for any $u \in K$ and any $u^* \in \partial^\pi g(u)$, we have*
$$\langle u^*, u' - u\rangle \le g(u') - g(u) + \frac{1}{2r} V(J(u), u')), \forall u' \in K. \tag{2.5}$$

Example 2.5. *According to [10], every closed convex set is V-u.p.r. for any constant $r \in (1, +\infty]$, under the convention $\frac{1}{+\infty} = 0$. In addition, Example 4.10 in the same work shows that the union of two mutually disjoint, closed, and convex subsets satisfies the V-u.p.r. property for some positive value $r > 0$. Extending this argument, one can verify that any finite union of pairwise disjoint closed convex sets also enjoys the V-u.p.r. property for an appropriate positive constant.*

Moreover, it was established in [12] that any proper lower semicontinuous convex function is V-u.p.r. on every nonempty closed subset C of its domain with respect to any chosen $r \in (1, +\infty]$. In particular, both the indicator function ψ_C together with the distance function d_C corresponding to a V-u.p.r. set C retain this property on C for the same constant r. Here, the notation d_C stands for the standard metric distance defined by the set C, that is, $d_C(x) := \inf_{s\in C} \|x - s\|$. Furthermore, any function f belonging to the lower-C^2 class (as defined in [5]) over a convex, strongly compact subset $K \subset X$ is also V-u.p.r. on K.

We recall from [9] that *V-u.p.r.* sets K satisfy the equality $T^C(K;u) = \mathbf{K}(K;u)$, $\forall u \in K$. Here, $T^C(K;u)$ denotes the Clarke tangent cone, and $\mathbf{K}(K;u)$ denotes the contingent cone (see [5]). For *V-u.p.r.* sets, we use the notation $T_K(u) := T^C(K;u) = \mathbf{K}(K;u)$, $\forall u \in K$.

The next proposition establishes an intersection rule for V-proximal normal cone which is required in our next proofs.

Proposition 2.6. *Consider a V-prox.-trust. space X and let C_1 and C_2 be two closed nonempty V-u.p.r. sets in X. Let $\bar{u} \in C_1 \cap C_2$ and assume that there are $\delta, \sigma > 0$ so that*

$$d_{C_1\cap C_2}(u) \leq \sigma[d_{C_1}(u) + d_{C_2}(u)], \quad \textit{for all } u \in \bar{u} + \delta\mathbb{B}. \tag{2.6}$$

Then,

$$N^\pi(C_1 \cap C_2; \bar{u}) \subset N^\pi(C_1; \bar{u}) + N^\pi(C_2; \bar{u}).$$

Proof. Let $y^* \in N^\pi(C_1 \cap C_2; \bar{u})$. By Theorem 4.13 in [8], there exists some $\alpha > 0$ so that $y^* = \alpha u^*$ with $u^* \in \partial^\pi d_{C_1\cap C_2}(\bar{u})$. By the definition of ∂^π, there exist $\delta_1 \in (0, \delta)$ and $\sigma_1 > 0$ so that

$$\langle u^*; u - \bar{u}\rangle \leq d_{C_1\cap C_2}(u) - d_{C_1\cap C_2}(\bar{u}) + \sigma_1 V(J(\bar{u}); u), \quad \forall u \in \bar{u} + \delta_1\mathbb{B}. \tag{2.7}$$

Fix any $u \in \bar{u} + \delta_1\mathbb{B}$. Then, by combining (2.6) and (2.7), we obtain

$$\begin{aligned} \langle u^*; u - \bar{u}\rangle &\leq d_{C_1\cap C_2}(u) + \sigma_1 V(J(\bar{u}); u) \\ &\leq \sigma[d_{C_1}(u) + d_{C_2}(u)] + \sigma_1 V(J(\bar{u}); u), \end{aligned}$$

which ensures that $u^* \in \partial^\pi \sigma[d_{C_1} + d_{C_2}](\bar{u}) \subset \partial^{L\pi}\sigma[d_{C_1} + d_{C_2}](\bar{u})$, and by using part (1) in Proposition 2.2, we get

$$u^* \in \partial^\pi \sigma[d_{C_1} + d_{C_2}](\bar{u}) \subset \partial^{L\pi}\sigma[d_{C_1} + d_{C_2}](\bar{u}) \subset \partial^{L\pi}\sigma d_{C_1}(\bar{u}) + \partial^{L\pi}\sigma d_{C_2}(\bar{u}).$$

Thus,

$$y^* = \alpha u^* \in \alpha\sigma\partial^{L\pi} d_{C_1}(\bar{u}) + \alpha\sigma\partial^{L\pi} d_{C_2}(\bar{u}) \subset N^{L\pi}(C_1; u) + N^{L\pi}(C_2; u).$$

Now, we use the fact that the sets C_1 and C_2 are *V-u.p.r.* to use Theorem 2 in [9], and write

$$N^{L\pi}(C_1; u) = N^\pi(C_1; u) \text{ and } N^{L\pi}(C_2; u) = N^\pi(C_2; u).$$

As a result, we obtain

$$y^* \in N^\pi(C_1; u) + N^\pi(C_2; u)$$

and then we are done. □

Remark 2.7. *The condition* (2.6) *has been utilized in several prior works (see [1, 16, 25] and the references therein), and it is called bounded linearly regular in [16], and called linear coherence in [25].*

In the following proposition, we follow the same lines in [1] to establish a sufficient tangential condition for condition (2.6).

Proposition 2.8. *Consider a V-prox.-trust. space X and let C_1 and C_2 be two V-u.p.r. sets in X. Suppose that there are $\beta > 0$ and $\delta > 0$ so that for any $\bar{u} \in C_1 \cap C_2$ we have*

$$\beta\mathbb{B} \subset T_{C_1}(u_1) \cap \mathbb{B} - T_{C_2}(u_2) \cap \mathbb{B}, \tag{2.8}$$

for any $u_1 \in C_1 \cap (\bar{u} + \delta\mathbb{B})$ and any $u_2 \in C_2 \cap (\bar{u} + \delta\mathbb{B})$. Then there are $\delta' > 0$ and $\sigma > 0$ so that

$$d_{C_1 \cap C_2}(u) \leq \sigma[d_{C_1}(x) + d_{C_2}(u)], \quad \forall u \in \bar{u} + \delta'\mathbb{B}.$$

To start proving Proposition 2.8, we require the next proposition from [4].

Proposition 2.9. *Consider two Banach spaces E_1 and E_2 and let $M : E_1 \rightrightarrows E_2$ be with closed graph. Consider $(\bar{u}, \bar{y})$ be an element in the graph of M, i.e., $\bar{y} \in M(\bar{u})$. Suppose that there are $\beta, \delta > 0$ so that*

$$\forall u \in \mathbb{B}, \exists v \in \mathbb{B} : \ (v, \beta u) \in T_{gph\, M}(x, y), \ \forall (x, y) \in [(\bar{u} + \delta\mathbb{B}) \times (\bar{y} + \delta\mathbb{B})] \cap gph\, M. \tag{2.9}$$

Then, there are $\delta_1, \sigma > 0$ in a way that

$$d_{M^{-1}(y)}(x) \leq \sigma d_{M(x)}(y), \quad \forall (x, y) \in [(\bar{u} + \delta_1\mathbb{B}) \times (\bar{y} + \delta_1\mathbb{B})]. \tag{2.10}$$

The coming lemma will also be required.

Lemma 2.10. *Let C_1 and C_2 be two closed V-u.p.r. sets in X. Define $M : E \rightrightarrows E \times E$ as $M(x) := (C_1 - x) \times (C_2 - x)$. Then, $\forall x \in X$ and $\forall (y, z) \in M(x)$,*

$$T_{gph\, M}(x, (y, z)) = \{(u, v, w) \in X^3 : u + v \in T_{C_1}(x + y) \text{ and } u + w \in T_{C_2}(x + z)\}. \tag{2.11}$$

Proof. The proof is is straightforward from the definition of tangent cones and can be found in [5]. □

Proof of Proposition 2.8. Define $M : E \rightrightarrows E \times E$ as in Lemma 2.10, that is, $M(x) = (C_1 - x) \times (C_2 - x)$. Fix any point $\bar{u} \in C_1 \cap C_2$. Let $v, w \in \frac{\beta}{2}\mathbb{B}$, so $v - w \in \beta\mathbb{B}$. Choose any real number $\eta \in (0, \delta)$. Fix any $x \in X$ and any $(y, z) \in M(x)$, (that is, $x + y \in C_1$ and $x + z \in C_2$) with $x \in \bar{u} + \frac{\eta}{2}\mathbb{B}$, $y \in \frac{\eta}{2}\mathbb{B}$, and $z \in \frac{\eta}{2}\mathbb{B}$. Hence,

$$\|x + z - \bar{u}\| \leq \eta \text{ and } \|x + y - \bar{u}\| \leq \eta.$$

Therefore, by the tangential condition (2.8), there exist $u_1 \in T_{C_1}(x + y) \cap \mathbb{B}$ and $u_2 \in T_{C_2}(x + z) \cap \mathbb{B}$ so that $v - w = u_1 - u_2$. Set $u := u_1 - v$. Then, $u + v = u_1$ and $u + w = u_2$, so $u + v \in T_{C_1}(x + y) \cap \mathbb{B}$ and $u + w \in T_{C_2}(x + z) \cap \mathbb{B}$. Through the application of Lemma 2.10, one arrives at $(u, v, w) \in T_{gph\, M}(x, (y, z))$. Observe that $v, w \in \frac{\beta}{2}\mathbb{B}$ and $u \in \frac{2+\beta}{2}\mathbb{B}$. Hence, $\frac{2}{2+\beta}u \in \mathbb{B}$, $\frac{2}{\beta}v \in \mathbb{B}$, and $\frac{2}{\beta}w \in \mathbb{B}$, and so

$$\left([2/(2+\beta)]u, [2/(2+\beta)]\left[\frac{2}{\beta}v\right], [2/(2+\beta)]\left[\frac{2}{\beta}w\right]\right) \in T_{gph\, M}(x, (y, z)).$$

Set $\beta_1 := [2/(2+\beta)]$. Since the choice of (v, w) is arbitrary in $\frac{\beta}{2}\mathbb{B}$, we conclude the following: $\forall (v', w') \in \mathbb{B} \times \mathbb{B}$, $\exists u' := \frac{2}{2+\beta}u \in \mathbb{B}$ so that

$$(u', \beta_1(v', w')) \in T_{gph\, M}(x, (y, z)), \quad \forall (x, (y, z)) \in [(\bar{u} + \frac{\eta}{2}\mathbb{B}) \times (\frac{\eta}{2}\mathbb{B} \times \frac{\eta}{2}\mathbb{B})],$$

that is, the assumption in (2.9) in Proposition 2.9 is satisfied.

Now, we are ready to use Proposition 2.9 to conclude the existence of $\sigma > 0$ and $\bar{\delta} > 0$ so that

$$d_{M^{-1}(y,z)}(x) \leq \sigma d_{M(x)}(y,z), \quad \forall x \in \bar{u} + \frac{\eta}{2}\mathbb{B}, \forall y, z \in \frac{\eta}{2}\mathbb{B}.$$

Specifically, for every $x \in \bar{u} + \frac{\eta}{2}\mathbb{B}$ and for $(y,z) = (0,0)$, we have

$$d_{M^{-1}(0,0)}(x) \leq \sigma d_{M(x)}(0,0).$$

On the other hand, we have

$$\begin{aligned} M^{-1}(0,0) &= \{v \in X : \text{ so that } (0,0) \in M(v) = (C_1 - v) \times (C_2 - v)\} \\ &= \{v \in X : \text{ so that } v \in C_1 \text{ and } v \in C_1\} = C_1 \cap C_2, \end{aligned}$$

and

$$\begin{aligned} d_{M(x)}(0,0) &= \inf_{(y,z) \in M(x)} \|(y,z) - (0,0)\| \\ &= \inf_{(y,z) \in (C_1 - x) \times (C_2 - x)} \|(y,z)\| \\ &= \inf_{y \in C_1 - x} \|y\| + \inf_{z \in C_2 - x} \|z\| \\ &= \inf_{s_1 \in C_1} \|s_1 - x\| + \inf_{s_2 \in C_2} \|s_2 - x\| \\ &= d_{C_1}(x) + d_{C_2}(x). \end{aligned}$$

Therefore, we obtain

$$d_{C_1 \cap C_2}(x) \leq \sigma[d_{C_1}(x) + d_{C_2}(x)], \quad \text{ for all } x \in \bar{u} + \frac{\eta}{2}\mathbb{B},$$

and so the demonstration of Proposition 2.8 is finished. □

We use Propositions 2.6 and 2.8 to demonstrate the next stability result of *V-u.p.r.* under the intersection operation. It will be needed in our proofs.

Proposition 2.11. *Consider a V-prox.-trust. space X and consider two closed nonempty V-u.p.r. sets C_1 and C_2 with ratios $r_1 > 0$ and $r_2 > 0$, respectively. Assume that*

$$\beta\mathbb{B} \subset T_{C_1}(x_1) \cap \mathbb{B} - T_{C_2}(x_2) \cap \mathbb{B},$$

for any $x_1 \in C_1 \cap (\bar{u} + \delta\mathbb{B})$ and any $x_2 \in C_2 \cap (\bar{u} + \delta\mathbb{B})$. Then, the intersection $C_1 \cap C_2$ is a closed V-u.p.r. set with the ratio $r := \frac{\beta \min\{r_1, r_2\}}{2}$.

Proof. Assume that C_1 and C_2 are *V-u.p.r.* with the ratios $r_1 > 0$ and $r_2 > 0$, respectively. Let $x \in C_1 \cap C_2$ and $x^* \in N^\pi(C_1 \cap C_2; x)$. Then, by Proposition 2.6 and Proposition 2.8, there are $x_1^* \in N^\pi(C_1; x)$ and $x_2^* \in N^\pi(C_2; x)$ so that $x^* = x_1^* + x_2^*$. So, by virtue of Definition 2.3, one may concisely articulate

$$\langle x_1^*, y - x \rangle \leq \frac{\|x_1^*\|}{2r_1} V(J(x), y)), \quad \forall y \in C_1 \tag{2.12}$$

and

$$\langle x_2^*, y - x\rangle \leq \frac{\|x_2^*\|}{2r_2} V(J(x), y)), \quad \forall y \in C_2. \tag{2.13}$$

On the other side, by the tangential condition, we have that for any $v \in \mathbb{B}$, there exist $v_1 \in T_{C_1}(\bar{u}) \cap \mathbb{B}$ and $v_2 \in T_{C_2}(\bar{u}) \cap \mathbb{B}$ so that $\beta v = v_1 - v_2$. Thus,

$$\begin{array}{rcl}\langle \beta x_1^*; v\rangle & = & \langle x_1^*; \beta v\rangle = \langle x_1^*; v_1 - v_2\rangle = \langle x_1^*; v_1\rangle - \langle x_1^*; v_2\rangle = \langle x_1^*; v_1\rangle - \langle x^* - x_2^*; v_2\rangle \\ & = & \langle x_1^*; v_1\rangle + \langle x_2^*; v_2\rangle + \langle x^*; -v_2\rangle.\end{array}$$

We take into consideration that $x_i^* \in N^\pi(C_i; \bar{u})$ and $v_i \in T_{C_i}(\bar{u})$, $i = 1, 2$, to write (see [5])

$$\langle x_i^*; v_i\rangle \leq 0,\ i = 1, 2.$$

Therefore, we get

$$\langle \beta x_1^*; v\rangle \leq \langle x^*; -v_2\rangle \leq \|x^*\|, \quad \forall v \in \mathbb{B},$$

which ensures that $\beta\|x_1^*\| \leq \|x^*\|$. Similarly, we obtain $\beta\|x_2^*\| \leq \|x^*\|$. Consequently, combining (2.12) and (2.13), we obtain for any $y \in C_1 \cap C_2$ that

$$\begin{array}{rcl}\langle x^*, y - x\rangle & = & \langle x_1^*, y - x\rangle + \langle x_2^*, y - x\rangle \\ & \leq & \dfrac{\|x_1^*\|}{2r_1} V(J(x), y)) + \dfrac{\|x_2^*\|}{2r_2} V(J(x), y)) \\ & \leq & \dfrac{\|x^*\|}{2\beta r_1} V(J(x), y)) + \dfrac{\|x^*\|}{2\beta r_2} V(J(x), y)) \\ & \leq & \dfrac{\|x^*\|}{2r} V(J(x), y)),\end{array}$$

with $r = \frac{\beta \min\{r_1, r_2\}}{2}$. Thus, $\forall x \in C_1 \cap C_2$ and $\forall x^* \in N^\pi(C_1 \cap C_2; x)$, we have

$$\langle x^*, y - x\rangle \leq \frac{\|x^*\|}{2r} V(J(x), y)), \quad \forall y \in C_1 \cap C_2.$$

This means that $C_1 \cap C_2$ is V-*u.p.r.* with $r = \frac{\beta \min\{r_1, r_2\}}{2}$, and then the proof is finished. □

We restate from [27] the definition of relatively nonexpansive mappings (in short *relativ.nonexp.*) and firmly nonexpansive mappings (in short *firm. nonexp.*). For more details and examples, we refer to [27].

Definition 2.12. *A given $R : X \to X$ is known as relativ. nonexp. from K to itself as long as $F_K(R) \neq \emptyset$, $\hat{F}_K(R) = F_K(R)$, and for any $u \in K$ and any $u' \in F_K(R)$, we have $V(J(R(u)); u') \leq V(J(u); u')$. Here, $F_K(R)$ denotes the fixed-point collection of R on K, and $F_X(R)$ stands for the fixed-point set of R on X, and $\hat{F}_K(R)$ stands for the family of asymptotic fixed points of R on K, that is, the collection of points $\hat{x} \in K$ for which there is $\{x_n\} \subset K$ that is weakly convergent to $\hat{x}$ and satisfies* $\lim_{n\to\infty} \|x_n - R(x_n)\| = 0$.

Definition 2.13. *A given $R : X \to K$ is known as firm. nonexp. as long as*

$$\langle J(R(x)) - J(R(y)), R(x) - R(y)\rangle \leq \langle J(x) - J(y), R(x) - R(y)\rangle,\ \textit{for all } x, y \in \textit{dom}\, R.$$

We start by extending Lemma 2.3 in [27] from the convex sets to V-*u.p.r.* sets.

Theorem 2.14. *Consider a V-prox.-trust. space X and let $\emptyset \neq K \subset X$ be a closed set, and let $R : X \to X$ be a relativ. nonexp. map. on X. Suppose that K is V-u.p.r. with constant $r > 0$. Assume that the following tangential condition is verified: There are $\beta > 0$ and $\delta > 0$ so that for any $\bar{u} \in F_K(R)$, we have*

$$\beta\mathbb{B} \subset T_K(u_1) \cap \mathbb{B} - T_{F_X(R)}(u_2) \cap \mathbb{B}, \tag{2.14}$$

$\forall u_1 \in K \cap (\bar{u} + \delta\mathbb{B})$ and $\forall u_2 \in F_X(R) \cap (\bar{u} + \delta\mathbb{B})$. Then, $F_K(R)$ is closed and V-u.p.r..

Proof. 1- We demonstrate first that $F_K(R)$ is a closed set. Consider any sequence $\{x_n\}$ in $F_K(R)$ (i.e. $R(x_n) = x_n$) converging to some point $\bar{u}$, and we shall show that $R(\bar{u}) = \bar{u}$. Obviously, $\bar{u} \in K$. By virtue of the fact that R is *relativ. nonexp.*, we derive

$$V(J(R(\bar{u})); x_n) \leq V(J(\bar{u}); x_n), \quad \forall n \geq 1.$$

So, since J and V are continuous, we deduce that

$$V(J(R(\bar{u})); \bar{u}) = \lim_n V(J(R(\bar{u})); x_n) \leq \lim_n V(J(\bar{u}); x_n) = V(J(\bar{u}); \bar{u}) = 0.$$

This guarantees that $R(\bar{u}) = \bar{u}$, and the demonstration of the first part is finished.

We shall prove that $F_K(R)$ is V-*u.p.r.*. It is worth noting that $F_K(R) = K \cap \{x \in X : R(x) = x\}$. Let us establish the convexity of $F_X(R) := \{x \in X : R(x) = x\}$. Fix any $u_1, u_2 \in F_X(R)$ and any $\lambda \in [0, 1]$. Set $z_\lambda := \lambda u_1 + (1 - \lambda)u_2$. Then,

$$\begin{aligned}
V(J(R(z_\lambda)); z_\lambda) &= \|R(z_\lambda)\|^2 - 2\langle J(R(z_\lambda)); z_\lambda\rangle + \|z_\lambda\|^2 \\
&= \|R(z_\lambda)\|^2 - 2\langle J(R(z_\lambda)); \lambda u_1 + (1 - \lambda)u_2\rangle + \|z_\lambda\|^2 \\
&= \|R(z_\lambda)\|^2 - 2\lambda\langle J(R(z_\lambda)); u_1\rangle - 2(1 - \lambda)\langle J(R(z_\lambda)); u_2\rangle + \|z_\lambda\|^2 \\
&= \|R(z_\lambda)\|^2 + \lambda V(J(R(z_\lambda)); u_1) - \lambda\|u_1\|^2 - \lambda\|J(R(z_\lambda))\|^2 \\
&\quad +(1 - \lambda)V(J(R(z_\lambda)); u_2) - (1 - \lambda)\|u_2\|^2 - (1 - \lambda)\|R(z_\lambda)\|^2 + \|z_\lambda\|^2.
\end{aligned}$$

By the relative nonexpansive property of R on X, we have

$$\begin{aligned}
V(J(R(z_\lambda)); z_\lambda) &\leq \|R(z_\lambda)\|^2 + \lambda V(J(z_\lambda); u_1) - \lambda\|u_1\|^2 - \lambda\|R(z_\lambda)\|^2 \\
&\quad +(1 - \lambda)V(J(z_\lambda); u_2) - (1 - \lambda)\|u_2\|^2 - (1 - \lambda)\|R(z_\lambda)\|^2 + \|z_\lambda\|^2 \\
&\leq \lambda V(J(z_\lambda); u_1) - \lambda\|u_1\|^2 + (1 - \lambda)V(J(z_\lambda); u_2) - (1 - \lambda)\|u_2\|^2 + \|z_\lambda\|^2 \\
&\leq \lambda\left[\|z_\lambda\|^2 - 2\langle J(z_\lambda); u_1\rangle + \|u_1\|^2\right] - \lambda\|u_1\|^2 \\
&\quad +(1 - \lambda)\left[\|z_\lambda\|^2 - 2\langle J(z_\lambda); u_2\rangle + \|u_2\|^2\right] - (1 - \lambda)\|u_2\|^2 + \|z_\lambda\|^2 \\
&\leq -2\lambda\langle J(z_\lambda); u_1\rangle - 2(1 - \lambda)\langle J(z_\lambda); u_2\rangle + 2\|z_\lambda\|^2 \\
&\leq -2\langle J(z_\lambda); \lambda u_1 + (1 - \lambda)u_2\rangle + 2\|z_\lambda\|^2 = -2\langle J(z_\lambda); z_\lambda\rangle + 2\|z_\lambda\|^2 = 0.
\end{aligned}$$

This ensures that $R(z_\lambda) = z_\lambda$, that is, $z_\lambda \in F_X(R), \forall u_1, u_2 \in F_X(R), \forall \lambda \in [0, 1]$, and so the set $F_X(R)$ is convex. Now, using the tangential condition (2.14) and the fact, proved in Proposition 2.11, that the intersection of a convex set with a V-*u.p.r.* set is V-*u.p.r.*, we conclude the proof. □

The coming lemma extends the result in Alber [3] from convex to nonconvex settings within Banach spaces.

Lemma 2.15. *Consider a V-prox.-trust. space X which is smooth reflexive and strictly convex, and consider a closed V-u.p.r. subset* $\emptyset \neq K \subset X$ *with constant* $r > 0$*. Then,*

$$(1 - \frac{1}{r})V(J(\pi_K(J(u_2))); u_1) + V(J(u_2); \pi_K(J(u_2))) \leq V(J(u_2); u_1), \tag{2.15}$$

$\forall u_1 \in K$ *and* $\forall u_2 \in X$ *with* $\pi_K(J(u_2)) \neq \emptyset$*.*

Proof. Fix any $u_2 \in X$ with $\pi_K(J(u_2)) \neq \emptyset$. Let $\bar{u} := \pi_K(J(u_2))$. Then, for any $u_1 \in K$ we have

$$\begin{aligned} V(J(\bar{u}); u_1) + V(J(u_2); \bar{u}) - V(J(u_2); u_1) &= -2\langle J(\bar{u}); u_1\rangle - 2\langle J(u_2); \bar{u}\rangle + 2\langle J(u_2); u_1\rangle \\ &= 2\langle J(u_2) - J(\bar{u}); u_1 - \bar{u}\rangle. \end{aligned}$$

According to the definition of $N^\pi(K; \bar{u})$, we have $J(u_2) - J(\bar{u}) \in N^\pi(K; \bar{u})$, and so by the *V-u.p.r.* of the set K, we can write

$$\langle J(u_2) - J(\bar{u}); u_1 - \bar{u}\rangle \leq \frac{1}{2r}V(J(\bar{u}); u_1).$$

Thus,

$$V(J(\bar{u}); u_1) + V(J(u_2); \bar{u}) - V(J(u_2); u_1) \leq \frac{1}{r}V(J(\bar{u}); u_1).$$

and so

$$(1 - \frac{1}{r})V(J(\bar{u}); u_1) + V(J(u_2); \bar{u}) \leq V(J(u_2); u_1).$$

This achieves the end of the proof. □

Remark 2.16. *By taking* $r = +\infty$*, we get the convex case and so the conclusion of Lemma 2.15 coincides with Lemma 2.4 in Alber [3] and Proposition 5 in [18].*

The following proposition states different characterizations of the *V-u.p.r.*.

Proposition 2.17. *Consider a closed set* $K \neq \emptyset$ *in a V-prox.-trust. space X. The assertions listed below are equivalent:*

(1) K *is V-u.p.r. with constant* $r > 0$*;*
(2) $\forall u \in K, \forall u^* \in N^\pi(K; u)$ *with* $\|u^*\| < 1$*, we have* $u \in \pi_K(J(u) + ru^*)$*;*
(3) $\forall u \in K, \forall u^* \in N^\pi(K; u)$ *with* $u^* \neq 0$*, we have* $u \in \pi_K(J(u) + r\frac{u^*}{\|u^*\|})$*;*
(4) $\forall u \in K, \forall u^* \in N^\pi(K; u)$ *with* $\|u^*\| < 1$*, we have* $u \in \pi_K(J(u)+r'u^*)$ *for any* $r' \in [0, r]$*. Furthermore,* $\pi_K(J(u) + r'u^*) = \{u\}$ $\forall r' \in (0, r)$*.*

Proof. The equivalences (2.17) $\Longleftrightarrow$ (2.17) $\Longleftrightarrow$(2.17) have been proved in Proposition 4.2 in [10]. We have to prove (2.17) $\Longleftrightarrow$ (2.17). Assume that (2.17) holds, that is, K is *V-u.p.r.* with constant $r > 0$. Let $u \in K$ and $u^* \in N^\pi(K; u)$ with $\|u^*\| < 1$. By virtue of Definition 2.3, we have

$$\langle u^*; u' - u\rangle \leq \frac{\|u^*\|}{2r}V(J(u); u') \leq \frac{1}{2r}V(J(u); u'), \quad \forall u' \in K. \tag{2.16}$$

Using the equality

$$V(J(u) + ru^*; u) - V(J(u) + ru^*; u') = -V(J(u); u') - 2r\langle u^*; u - u'\rangle. \tag{2.17}$$

We write

$$2r\langle u^*; u' - u\rangle = V(J(u) + ru^*; u) - V(J(u) + ru^*; u') + V(J(u); u'). \quad (2.18)$$

Combining (2.16) and (2.18), we obtain

$$V(J(u) + ru^*; u) \leq V(J(u) + ru^*; u'), \quad \forall u' \in K, \quad (2.19)$$

that is, $u \in \pi_K(J(u) + ru^*)$. Conversely, assume that (2.17) holds, that is, $\forall u \in K$ and $\forall u^* \in N^\pi(K; u)$ with $\|u^*\| < 1$, we have $u \in \pi_K(J(u) + ru^*)$. Fix any $u \in K$ and any $u^* \in N^\pi(K; u)$. If $u^* = 0$, then (2.16) holds, and so we assume that $u^* \neq 0$. Fix any $\epsilon > 0$. Then, $\frac{u^*}{\|u^*\|+\epsilon} \in N^\pi(K; u)$ with $\|\frac{u^*}{\|u^*\|+\epsilon}\| < 1$. So, by Part (2.17), we have $u \in \pi_K(J(u) + r\frac{u^*}{\|u^*\|+\epsilon})$. Therefore, by exploiting the definition of π_K, we derive

$$V(J(u) + r\frac{u^*}{\|u^*\| + \epsilon}; u) \leq V(J(u) + r\frac{u^*}{\|u^*\| + \epsilon}; u'), \quad \forall u' \in K. \quad (2.20)$$

Using (2.18), we write

$$V(J(u) + r\frac{u^*}{\|u^*\| + \epsilon}; u) - V(J(u) + r\frac{u^*}{\|u^*\| + \epsilon}; u') = 2r\langle\frac{u^*}{\|u^*\| + \epsilon}; u' - u\rangle - V(J(u); u').$$

Combining this inequality with the previous one (2.20), we obtain

$$2r\langle\frac{u^*}{\|u^*\| + \epsilon}; u' - u\rangle - V(J(u); u') \leq 0, \quad \forall u' \in K,$$

which ensures

$$\langle u^*; u' - u\rangle \leq \frac{\|u^*\| + \epsilon}{2r} V(J(u); u'), \quad \forall u' \in K, \forall \epsilon > 0.$$

Taking $\epsilon \to 0$ yields

$$\langle u^*; u' - u\rangle \leq \frac{\|u^*\|}{2r} V(J(u); u'), \quad \forall u' \in K.$$

Thus, the demonstration is finished. □

We will make use of the following results. Their proofs are available in the references [10, 28], respectively.

Theorem 2.18. *Consider a q-unif. conv. space X with smooth norm. Suppose that K is V-u.p.r. with constant $r > 0$. Then, there is $\beta_0 > 0$ so that $\forall \beta \geq \beta_0$ and $\forall r' \in (0, r)$, we have π_K is Hölder continuous single-valued over $\{u^* \in X^* : \|u^*\| < \beta$ and $d_K^V(x^*) < r'^2\}$.*

Lemma 2.19. *Consider a q-unif. conv. space X and consider $\beta > 0$. Then, there is a convex cont. function $g : [0, 2\alpha] \to \mathbb{R}$ which is strictly increasing with $g(0) = 0$, and*

$$\|tu_1 + (1 - t)u_2\|^2 \leq t\|u_1\|^2 + (1 - t)\|u_2\|^2 - t(1 - t)g(\|u_1 - u_2\|),$$

$\forall u_1, u_2 \in \beta\mathbb{B}$, and $\forall t \in [0, 1]$.

3. Nonconvex equilibrium problems in Banach spaces

We say that X satisfies the assumption $(\mathcal{A})$ if it is a *V-prox.-trust.* space, which is q-unif. conv. and p-unif. smooth. We have to mention that it has been established in [7] that L^p spaces $(p > 1)$ satisfy this assumption $(\mathcal{A})$. We propose the following nonconvex equilibrium problem:

$$\text{Locate } \bar{u} \in K \text{ so that } \text{ł}(\bar{u}, u) + \theta V(J(\bar{u}); u) \geq 0, \quad \forall u \in K. \tag{3.1}$$

Let us denote by $EP(\text{ł}, \theta)$ the solution set of (3.1). For solving (3.1), we require the assumptions on ł and K.

(H1) $\text{ł} \equiv 0$ on $K \times K$;
(H2) For all $y \in K$, the function $\text{ł}(\cdot, y)$ is u.s.c. (upper semicontinuity) on K;
(H3) ł is strictly monotone with ratio $\sigma \geq 0$, i.e.,

$$\text{ł}(u, u') + \text{ł}(u', u) \leq -\sigma \left[V(J(u); u') + V(J(u'); u) \right], \quad \forall u, u' \in K;$$

(H4) $\forall u \in K$, the function $\text{ł}(u, \cdot)$ is l.s.c. on K;
(H5) The reals t, θ, and σ verify the inequality

$$0 \leq \theta < \sigma + \frac{1}{2t}.$$

For $t > 0$ and a nonnegative number $\theta \geq 0$, we introduce the mapping $T_{t,\theta} : X \to K$ as:

$$T_{t,\theta}(u) = \{z \in K : \text{ł}(z, y) + \theta V(J(z); y) + \frac{1}{t}\langle J(z) - J(u); y - z\rangle \geq 0, \quad \forall y \in K\}.$$

We need the following additional assumption on the mapping $T_{t,\theta}$: There is some $\beta > 0$ and $\delta > 0$ so that for any $\bar{u} \in F_K(T_{t,\theta})$ we have

$$\beta \mathbb{B} \subset T_K(u_1) \cap \mathbb{B} - T_{F_X(T_{t,\theta})}(u_2) \cap \mathbb{B}, \tag{3.2}$$

for any $u_1 \in K \cap (\bar{u} + \delta \mathbb{B})$ and any $u_2 \in F_X(T_{t,\theta}) \cap (\bar{u} + \delta \mathbb{B})$.

First, we start by establishing the main properties of the mapping $T_{t,\theta}$.

Proposition 3.1. *Consider X satisfying the assumption $(\mathcal{A})$. Let $K \subset X$ be a closed V-u.p.r. set, and let $\text{ł} : K \times K \to \mathbb{R}$, satisfying (H1)–(H5). Suppose that* (3.2) *is fulfilled. Then, the listed properties are valid:*

(1) $T_{t,\theta}$ is single-valued on its domain;
(2) For all $u, u' \in \operatorname{dom} T_{t,\theta}$

$$\langle T_{t,\theta}(u) - T_{t,\theta}(u'), J(T_{t,\theta}(u)) - J(T_{t,\theta}(u'))\rangle \leq \langle T_{t,\theta}(u) - T_{t,\theta}(u'), J(u) - J(u')\rangle,$$

i.e., $T_{t,\theta}$ is a firm. nonexp. mapping;
(3) $F_K(T_{t,\theta}) = EP(\text{ł}, \theta)$;
(4) For any $x \in \operatorname{dom} T_{t,\theta}$ and any $u \in F_K(T_{t,\theta})$,

$$V(J(T_{t,\theta}(x)); u) \leq V(J(x); u).$$

(5) If $dom\, T_{t,\theta}$ *is closed, then* $EP(\text{ł}, \theta)$ *is closed.*

(6) $EP(\text{ł}, \theta)$ *is a V-u.p.r. set in X.*

Proof. (1) Take t, θ, and σ satisfy $t(\theta - \sigma) < \frac{1}{2}$. Let $u \in dom\, T_{t,\theta}$ and let $z_1, z_2 \in T_{t,\theta}(u)$. Then,

$$\text{ł}(z_1, u') + \theta V(J(z_1); u') + \frac{1}{t}\langle J(z_1) - J(u); u' - z_1\rangle \geq 0, \quad \forall u' \in K$$

and

$$\text{ł}(z_2, u') + \theta V(J(z_2); u') + \frac{1}{t}\langle J(z_2) - J(u); u' - z_2\rangle \geq 0, \quad \forall u' \in K.$$

It follows that

$$\text{ł}(z_2, z_1) + \theta V(J(z_2); z_1) + \frac{1}{t}\langle J(z_2) - J(u); z_1 - z_2\rangle \geq 0$$

and

$$\text{ł}(z_1, z_2) + \theta V(J(z_1); z_2) + \frac{1}{t}\langle J(z_1) - J(u); z_2 - z_1\rangle \geq 0.$$

Adding these two inequalities yields

$$\text{ł}(z_1, z_2) + \text{ł}(z_2, z_1) + \theta V(J(z_1); z_2) + \theta V(J(z_2); z_1) + \frac{1}{t}\langle J(z_1) - J(z_2); z_2 - z_1\rangle \geq 0.$$

The strong monotonicity of ł implies

$$(\theta - \sigma)[V(J(z_1); z_2) + V(J(z_2); z_1)] + \frac{1}{t}\langle J(z_1) - J(z_2); z_2 - z_1\rangle \geq$$
$$\text{ł}(z_1, z_2) + \text{ł}(z_2, z_1) + \theta[V(J(z_1); z_2) + V(J(z_2); z_1)] + \frac{1}{t}\langle J(z_1) - J(z_2); z_2 - z_1\rangle \geq 0$$

and hence

$$\langle J(z_2) - J(z_1); z_2 - z_1\rangle \leq t(\theta - \sigma)[V(J(z_1); z_2) + V(J(z_2); z_1)].$$

Observe that

$$V(J(z_1); z_2) + V(J(z_2); z_1) = 2\langle J(z_2) - J(z_1); z_2 - z_1\rangle. \tag{3.3}$$

Thus,

$$(1 - 2t(\theta - \sigma))\langle J(z_2) - J(z_1); z_2 - z_1\rangle \leq 0.$$

On the other hand, we have by (H5) the inequality $\theta < \sigma + \frac{1}{2t}$, which implies $(1 - 2t(\theta - \sigma)) > 0$, and hence

$$\langle J(z_2) - J(z_1); z_2 - z_1\rangle \leq 0.$$

This ensures by the strict convexity of X, that $z_1 = z_2$, that is, $T_{t,\theta}(u)$ is a singleton and so (1) is proved.

(2) Let $u, u' \in dom\, T_{t,\theta}$. By (1), the sets $T_{t,\theta}(u)$ and $T_{t,\theta}(u')$ are singleton. Then, according to the definition of $T_{t,\theta}$ we have $T_{t,\theta}(u)$, $T_{t,\theta}(u') \in K$ and so for any $z \in K$ we have

$$\text{ł}(T_{t,\theta}(u), z) + \theta V(J(T_{t,\theta}(u)); z) + \frac{1}{t}\langle J(T_{t,\theta}(u)) - J(u); z - T_{t,\theta}(u)\rangle \geq 0$$

and

$$\mathfrak{f}(T_{t,\theta}(u'), z) + \theta V(J(T_{t,\theta}(u')); z) + \frac{1}{t}\langle J(T_{t,\theta}(u')) - J(u'); z - T_{t,\theta}(u')\rangle \geq 0.$$

Substituting $z = T_{t,\theta}(u')$ in the first inequality and $z = T_{t,\theta}(u)$ in the second inequality, we obtain

$$\mathfrak{f}(T_{t,\theta}(u), T_{t,\theta}(u')) + \theta V(J(T_{t,\theta}(u)); T_{t,\theta}(u')) + \frac{1}{t}\langle J(T_{t,\theta}(u)) - J(u); T_{t,\theta}(u') - T_{t,\theta}(u)\rangle \geq 0$$

and

$$\mathfrak{f}(T_{t,\theta}(u'), T_{t,\theta}(u)) + \theta V(J(T_{t,\theta}(u')); T_{t,\theta}(u)) + \frac{1}{t}\langle J(T_{t,\theta}(u')) - J(u'); T_{t,\theta}(u) - T_{t,\theta}(u')\rangle \geq 0.$$

By virtue of the strong monotonicity of f, we deduce

$$-\sigma[V(J(T_{t,\theta}(u')); T_{t,\theta}(u)) + V(J(T_{t,\theta}(u)); T_{t,\theta}(u'))] \geq \mathfrak{f}(T_{t,\theta}(u'), T_{t,\theta}(u)) + \mathfrak{f}(T_{t,\theta}(u), T_{t,\theta}(u')).$$

Adding these three inequalities gives

$$(\theta - \sigma)[V(J(T_{t,\theta}(u')); T_{t,\theta}(u)) + V(J(T_{t,\theta}(u)); T_{t,\theta}(u'))] +$$

$$\frac{1}{t}\langle J(T_{t,\theta}(u)) - J(u) + J(u') - J(T_{t,\theta}(u')); T_{t,\theta}(u') - T_{t,\theta}(u)\rangle \geq 0.$$

Thus, by (3.3) we obtain

$$\begin{aligned} \langle J(u) - J(u'); T_{t,\theta}(u) - T_{t,\theta}(u')\rangle &\geq [1 + 2t(\sigma - \theta)]\langle J(T_{t,\theta}(u)) - J(T_{t,\theta}(u')); T_{t,\theta}(u) - T_{t,\theta}(u')\rangle \\ &\geq \langle J(T_{t,\theta}(u)) - J(T_{t,\theta}(u')); T_{t,\theta}(u) - T_{t,\theta}(u')\rangle. \end{aligned}$$

The last inequality is a consequence of $1 + 2t(\sigma - \theta) > 0$, which is ensured by (H5) and the fact that $\langle J(T_{t,\theta}(u)) - J(T_{t,\theta}(u')); T_{t,\theta}(u) - T_{t,\theta}(u')\rangle > 0$. Therefore, $\forall u, u' \in dom\, T_{t,\theta}$, we have

$$\langle J(T_{t,\theta}(u)) - J(T_{t,\theta}(u')); T_{t,\theta}(u) - T_{t,\theta}(u')\rangle \leq \langle J(u) - J(u'); T_{t,\theta}(u) - T_{t,\theta}(u')\rangle,$$

that is, $T_{t,\theta}$ is *firm. nonexp.*.

(3) $F_K(T_{t,\theta}) = EP(\mathfrak{f}, \theta)$? Observe that

$$u \in F_K(T_{t,\theta}) \Leftrightarrow u = T_{t,\theta}(u) \Leftrightarrow \mathfrak{f}(u, u') + \theta V(J(u); u') \geq 0,\ \forall u' \in K \Leftrightarrow u \in EP(\mathfrak{f}, \theta).$$

(4) Let $u, u' \in dom\, T_{t,\theta}$. From part (2), we have

$$\langle J(T_{t,\theta}(u)) - J(T_{t,\theta}(u')), T_{t,\theta}(u) - T_{t,\theta}(u')\rangle \leq \langle J(u) - J(u'), T_{t,\theta}(u) - T_{t,\theta}(u')\rangle;$$

Let $w \in F_K(T_{t,\theta})$, that is, $T_{t,\theta}(w) = w$. Then,

$$\langle J(T_{t,\theta}(u)) - J(w), T_{t,\theta}(u) - w\rangle \leq \langle J(u) - J(w), T_{t,\theta}(u) - w\rangle$$

and so

$$\langle J(T_{t,\theta}(u)) - J(u), T_{t,\theta}(u) - w\rangle \leq 0.$$

Thus,

$$V(J(T_{t,\theta}(u)); w) - V(J(u); w) = \|T_{t,\theta}(u)\|^2 - 2\langle J(T_{t,\theta}(u)) - J(u); w\rangle - \|x\|^2$$

$$= \|T_{t,\theta}(u)\|^2 + 2\langle J(T_{t,\theta}(u)) - J(u); T_{t,\theta}(u) - w\rangle - \|x\|^2 - 2\langle J(T_{t,\theta}(u)) - J(u); T_{t,\theta}(u)\rangle$$

$$= \|T_{t,\theta}(u)\|^2 + 2\langle J(T_{t,\theta}(u)) - J(u); T_{t,\theta}(u) - w\rangle - \|x\|^2 - 2\|T_{t,\theta}(u)\|^2 + 2\langle J(u); T_{t,\theta}(u)\rangle$$

$$= 2\langle J(T_{t,\theta}(u)) - J(u); T_{t,\theta}(u) - w\rangle - \|x\|^2 - \|T_{t,\theta}(u)\|^2 + 2\langle J(u); T_{t,\theta}(u)\rangle$$

$$= 2\langle J(T_{t,\theta}(u)) - J(u); T_{t,\theta}(u) - w\rangle - V(J(u); T_{t,\theta}(u)) \leq 0.$$

This ensures that

$$V(J(T_{t,\theta}(u)); w) \leq V(J(u); w), \quad \forall u \in dom\, T_{t,\theta}, \forall w \in F_K(T_{t,\theta}).$$

(5) If $dom\, T_{t,\theta}$ is closed, then is $EP(\text{ł}, \theta)$ closed in X? By (3), we have to prove that $F_K(T_{t,\theta})$ is closed in X. Let $\{u_n\}$ be a sequence in $F_K(T_{t,\theta})$ (i.e. $T_{t,\theta}(u_n) = u_n$) converging to some point $\bar{u}$, and we have to demonstrate that $T_{t,\theta}(\bar{u}) = \bar{u}$. Obviously, $\bar{u} \in dom\, T_{t,\theta}$ since $dom\, T_{t,\theta}$ is closed. By (4), we get

$$V(J(T_{t,\theta}(\bar{u})); u_n) \leq V(J(\bar{u}); u_n), \quad \forall n \geq 1.$$

So, by continuity of J and V, we deduce that

$$V(J(T_{t,\theta}(\bar{u})); \bar{u}) = \lim_n V(J(T_{t,\theta}(\bar{u})); u_n) \leq \lim_n V(J(\bar{u}); u_n) = V(J(\bar{u}); \bar{u}) = 0.$$

This ensures that $T_{t,\theta}(\bar{u}) = \bar{u}$, that is, $\bar{u} \in F_K(T_{t,\theta})$.

(6) Is $EP(\text{ł}, \theta)$ V-*u.p.r.* in X? Once again, using (3) it is enough to prove that $F_K(T_{t,\theta})$ is V-*u.p.r.*. Thus, (6) follows from Lemma 2.14 and our assumptions and the part (1) of this proposition, and hence the demonstration is finished. □

To demonstrate the first corollary, we require the following technical lemma. Its proof results from the definition of V and simple computations.

Lemma 3.2. *The following equality is always true:*

$$2\langle J(u_3) - J(u_1), u_3 - u_2\rangle = V(J(u_3); u_2) + V(J(u_1); u_3) - V(J(u_1); u_2), \forall u_1, u_2, u_3 \in X. \tag{3.4}$$

Now, we extend Lemma 2.9 in [27] from convex to V-*u.p.r.* settings.

Corollary 3.3. *Consider X satisfying the assumption* ($\mathcal{A}$). *Let* $K \subset X$ *be a closed* V-*u.p.r. and consider* $\text{ł} : K \times K \to \mathbb{R}$ *satisfying (H1)-(H5) and* (3.2), *and let* $t > 0$. *Then,* $\forall u \in dom\, T_{t,\theta}$ *and* $\forall u' \in F_K(T_{t,\theta})$ *we get*

$$V(J(T_{t,\theta}(u)); u') + V(J(u); T_{t,\theta}(u)) \leq V(J(u), u').$$

Proof. From Proposition 3.1 part (2), we have

$$\langle J(T_{t,\theta}(u)) - J(T_{t,\theta}(u')), T_{t,\theta}(u) - T_{t,\theta}(u')\rangle \leq \langle J(u) - J(u'), T_{t,\theta}(u) - T_{t,\theta}(u')\rangle,\ \forall u, u' \in dom\, T_{t,\theta}.$$

Let $u' \in F_K(T_{t,\theta})$. Then, $T_{t,\theta}(u') = u'$, and so we have

$$\langle J(T_{t,\theta}(u)) - J(u'), T_{t,\theta}(u) - u'\rangle \leq \langle J(u) - J(u'), T_{t,\theta}(u) - u'\rangle,$$

which ensures

$$\langle J(T_{t,\theta}(u)) - J(u), T_{t,\theta}(u) - u'\rangle \leq 0.$$

Using Lemma 3.2 with $z = T_{t,\theta}(u)$, we may write

$$2\langle J(T_{t,\theta}(u)) - J(u), T_{t,\theta}(u) - u'\rangle = V(J(T_{t,\theta}(u)); u') + V(J(u); T_{t,\theta}(u)) - V(J(u); u').$$

Therefore,

$$V(J(T_{t,\theta}(u)); u') + V(J(u); T_{t,\theta}(u)) - V(J(u); u') \leq 0, \quad \forall u \in dom\, T_{t,\theta} \text{ and } u' \in F_K(T_{t,\theta}),$$

which ends the proof. □

4. Convergence results

We assume that X satisfies assumption $(\mathcal{A})$, and we rigorously establish a result characterizing strong convergence for locating a mutual point between the fixed-point collection of the *relativ. nonexp. map.* and the solution collection of our considered nonconvex problem.

Theorem 4.1. *Consider X verifying assumption $(\mathcal{A})$. Let $K \subset X$ be a closed V-u.p.r. with constant $r > 1$ and let $\ell : K \times K \to \mathbb{R}$ verifying (H1)–(H5). Let S be a relativ. nonexp. map from K into K so that $F_K(S) \cap EP(\ell, \theta) \neq \emptyset$. Assume that $dom\, T_{t,\theta} = X$ and that (3.2) is satisfied. Let $x_0 \in K$ with $d^V_{F_K(S)\cap EP(\ell,\theta)}(J(x_0)) < r^2$. Define the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ for $n \geq 0$ as follows:*

$$\begin{cases} u_{n+1} = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n) J(S(x_n))), \\ x_{n+1} = T_{r_n,\theta}(u_{n+1}) \text{ and } y_{n+1} = \pi_{F_K(S)\cap EP(\ell,\theta)}(J(x_{n+1})). \end{cases}$$

Assume that $\alpha_n \in [0, 1]$ and $r_n \geq 0$, $\forall n \geq 0$. Then, $\{y_n\}$ strongly converges to $\bar{y} \in F_K(S) \cap EP(\ell, \theta)$.

Proof. Let $u \in F_K(S) \cap EP(\ell, \theta) \neq \emptyset$. Since the set $F_K(S) \cap EP(\ell, \theta)$ is not necessarily convex, the generalized projection $\pi_{F_K(S)\cap EP(\ell,\theta)}(x_n)$ does not exist necessarily. So, we have to show that $\pi_{F_K(S)\cap EP(\ell,\theta)}(x_n)$ is well-defined. In order to do so, we invoke a recent theorem proved in [10] on the generalized projection onto V-*u.p.r.* sets and recalled in Theorem 2.18. We start for $n = 0$. We have by assumption $d^V_{F_K(S)\cap EP(\ell,\theta)}(J(x_0)) < r^2$, that is, $J(x_0) \in U^V_{F_K(S)\cap EP(\ell,\theta)}(r)$, which ensures, by Theorem 2.18, the existence of the generalized projection $y_0 := \pi_{F_K(S)\cap EP(\ell,\theta)}(J(x_0))$. For $n = 1$, we use the fact that J is one-to-one to define $u_1 := J^{-1}(\alpha_0 J(x_0) + (1 - \alpha_0) J(S(x_0)))$, and since $dom\, T_{r_0,\theta} = X$ we have $T_{r_0,\theta}(u_1) \neq \emptyset$, and so by Proposition 3.1 the set $T_{r_0,\theta}(u_1)$ is a singleton and so we can define $x_1 := T_{r_0,\theta}(u_1)$. Assume now by induction that the sequences $\{x_k\}$, $\{y_k\}$, and $\{u_k\}$ are introduced for $k \in \{0, \cdots, n\}$, and we shall prove that u_{n+1}, x_{n+1}, and y_{n+1} are well-defined. Obviously, u_{n+1} is well defined since J is one-to-one. By assumption, $dom\, T_{r_{n+1},\theta} = X$, so $T_{r_{n+1},\theta}(u_{n+1}) \neq \emptyset$, and so by

Proposition 3.1 the set $T_{r_{n+1},\theta}(u_{n+1})$ is a singleton and so we can define $x_{n+1} := T_{r_{n+1},\theta}(u_{n+1})$. On the other hand, we have from Proposition 3.1, the operator $T_{r_{n+1},\theta}$ is *relativ. nonexp.*. Also, by assumption, we have that S is *relativ. nonexp.*. Then, we have for any $u \in F_K(S) \cap EP(\text{ł}, \theta) = F_K(S) \cap F_K(T_{r_n,\theta})$ that

$$\begin{aligned}
V(J(x_{n+1}); u) &= V(J(T_{r_{n+1},\theta}(u_{n+1})); u) \\
&\leq V(J(u_{n+1}); u) \\
&\leq V(\alpha_n J(x_n) + (1-\alpha_n)J(S(x_n)); u) \\
&\leq \|\alpha_n J(x_n) + (1-\alpha_n)J(S(x_n))\|^2 + \|u\|^2 \\
&\quad -2\langle \alpha_n J(x_n) + (1-\alpha_n)J(S(x_n)); u\rangle \\
&\leq \alpha_n\|x_n\|^2 + (1-\alpha_n)\|S(x_n)\|^2 + \|u\|^2 \\
&\quad -2\langle \alpha_n J(x_n) + (1-\alpha_n)J(S(x_n)); u\rangle \\
&\leq \alpha_n\|x_n\|^2 + (1-\alpha_n)\|S(x_n)\|^2 + \alpha_n\|u\|^2 + (1-\alpha_n)\|u\|^2 \\
&\quad -2\alpha_n\langle J(x_n); u\rangle - 2(1-\alpha_n)\langle J(S(x_n)); u\rangle \\
&= \alpha_n\left[\|x_n\|^2 - 2\langle J(x_n); u\rangle + \|u\|^2\right] \\
&+ (1-\alpha_n)\left[\|S(x_n)\|^2 - 2\langle J(S(x_{n+1})); u\rangle + \|u\|^2\right] \\
&\leq \alpha_n V(J(x_n); u) + (1-\alpha_n)V(J(S(x_n)); u) \\
&\leq \alpha_n V(J(x_n); u) + (1-\alpha_n)V(J(x_n); u) = V(J(x_n); u). \qquad (4.1)
\end{aligned}$$

Therefore, for all $u \in F_K(S) \cap EP(\text{ł}, \theta)$ and any $n \geq 0$, we derive

$$V(J(x_{n+1}); u) \leq V(J(x_0); u),$$

which yields

$$\begin{aligned}
d^V_{F_K(S)\cap EP(\text{ł},\theta)}(J(x_{n+1})) &= \inf_{u\in F_K(S)\cap EP(\text{ł},\theta)} V(J(x_{n+1}); u) \\
&\leq \inf_{u\in F_K(S)\cap EP(\text{ł},\theta)} V(J(x_0); u) \\
&= d^V_{F_K(S)\cap EP(\text{ł},\theta)}(J(x_0)) < r^2,
\end{aligned}$$

which ensures that $J(x_{n+1}) \in U^V_{F_K(S)\cap EP(\text{ł},\theta)}(r)$, and so by Theorem 2.18, the existence of the generalized projection $y_{n+1} := \pi_{F_K(S)\cap EP(\text{ł},\theta)}(J(x_{n+1}))$. Thus, we obtained that the sequences $\{u_{n+1}\}$, $\{x_{n+1}\}$, and $\{y_{n+1}\}$ are well-defined. Now we proceed to the proof of the convergence regarding the strong topology of y_n to $\bar{y} \in F_K(S) \cap EP(\text{ł}, \theta)$. We substitute y_n instead of u in inequality (4.1), we deduce

$$V(J(x_{n+1}); y_n) \leq V(J(x_n); y_n), \quad \forall n \geq 0. \qquad (4.2)$$

On the other hand, by Lemma 2.15 and by the fact that $r > 1$ that

$$\begin{aligned}
V(J(x_{n+1}); y_{n+1}) &= V(J(x_{n+1}); \pi_{F_K(S)\cap EP(\text{ł},\theta)}(J(x_{n+1}))) \\
&\leq V(J(x_n); z) - (1 - \frac{1}{r})V(J(y_{n+1}); z), \\
&\leq V(J(x_n); z), \quad \forall z \in F_K(S) \cap EP(\text{ł}, \theta).
\end{aligned}$$

By taking $z = y_n$ in the previous inequality and by using (4.2) we obtain

$$V(J(x_{n+1}); y_{n+1}) \leq V(J(x_{n+1}); y_n)$$

$$\leq \quad V(J(x_n); y_n), \quad \forall n \geq 0.$$

Fix now any $m \geq n \geq 0$. Then, by induction from (4.1), we get

$$V(J(x_m); y_n) \leq V(J(x_{m-1}); y_n) \leq \cdots \leq V(J(x_n); y_n). \tag{4.3}$$

Once again, we use Lemma 2.15 and the assumption $r > 1$ to write

$$\begin{aligned} V(J(x_m); y_m) &= V(J(x_m); \pi_{F_K(S)\cap EP(\text{ł},\theta)}(J(x_m))) \\ &\leq V(J(x_m); z) - (1 - \frac{1}{r})V(J(y_m); z), \end{aligned}$$

$\forall z \in F_K(S) \cap EP(\text{ł}, \theta)$. By taking $z = y_n$ in the previous inequality, we get

$$V(J(x_m); y_m) \leq V(J(x_m); y_n) - (1 - \frac{1}{r})V(J(y_m); y_n),$$

and so by inequality (4.3) we have

$$\begin{aligned} (1 - \frac{1}{r})V(J(y_m); y_n) &\leq V(J(x_m); y_n) - V(J(x_m); y_m) \\ &\leq V(J(x_n); y_n) - V(J(x_m); y_m) = \Phi_n - \Phi_m, \end{aligned}$$

where $\Phi_n := V(J(x_n); y_n)$. Due to the fact that $\{x_n\}$ and $\{y_n\}$ are both bounded, we have by part (2) in Proposition 2.2 and the fact that X is q-unif. convex with smooth norm

$$\|y_n - y_m\|^q \leq \bar{K}V(J(y_m); y_n) \leq \frac{r\bar{c}}{r-1}[\Phi_n - \Phi_m].$$

Observe that the numerical sequence Φ_n is convergent (since it is decreasing and nonnegative). Taking $n, m \to \infty$, we obtain $\lim_{n,m\to\infty} \|y_n - y_m\| = 0$, that is, the sequence $\{y_n\}$ has the Cauchy property in $F_K(S) \cap EP(\text{ł}, \theta)$, so it is convergent to some limit $\bar{y} := \lim y_n$, and since the set $F_K(S) \cap EP(\text{ł}, \theta)$ is closed, we obtain $\bar{y} \in F_K(S) \cap EP(\text{ł}, \theta)$. □

Remark 4.2. *It can be observed that in the demonstration of Theorem 4.1, we have demonstrated only the convergence of* $\{y_n\}$ *to some point* $\bar{y}$ *in the set* $F_K(S) \cap EP(\text{ł}, \theta)$*. However, we did not get the convergence of* $\{x_n\}$*. In the next theorem, with additional assumptions either on the set K or on the duality mapping J, we prove that* $\{x_n\}$ *weakly converges to the same limit* $\bar{y}$ *of the sequence* $\{y_n\}$ *obtained in Theorem 4.1.*

Theorem 4.3. *Consider X verifying assumption* ($\mathcal{A}$)*. Let* $K \subset X$ *be closed V-u.p.r., and let* $\text{ł} : K \times K \to \mathbb{R}$ *satisfy (H1)–(H5). Consider a relativ. nonexp. mapping S defined from K into K so that* $F_K(S) \cap EP(\text{ł}, \theta) \neq \emptyset$*. Assume that dom* $T_{t,\theta} = X$ *and that* (3.2) *is satisfied. Let* $x_0 \in K$ *with* $d^V_{F_K(S)\cap EP(\text{ł},\theta)}(J(x_0)) < r^2$*. Define the sequences* $\{x_n\}$, $\{y_n\}$*, and* $\{u_n\}$ *for* $n \geq 0$ *as follows:*

$$\begin{cases} u_{n+1} = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n)J(S(x_n))), \\ x_{n+1} = T_{r_n,\theta}(u_{n+1}) \text{ and } y_{n+1} = \pi_{F_K(S)\cap EP(\text{ł},\theta)}(J(x_{n+1})). \end{cases}$$

Suppose that $\alpha_n \in [0, 1]$ *with* $\liminf_n \alpha_n(1 - \alpha_n) > 0$ *and* $r_n \geq a > 0$*,* $\forall n \geq 0$*. Then, the two assertions are true:*

(1) If K is ball compact, then there are subsequences of $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ which all strongly converge to some limit $\bar{y} \in \text{ł}(S) \cap EP(\text{ł}, \theta)$.

(2) If K is weakly closed and f is weakly u.s.c. regarding the first variable over K, and J is weakly sequentially cont., then there is a subseq. of $\{x_n\}$ that is weakly convergent to $\bar{y} \in \text{ł}(S) \cap EP(\text{ł}, \theta)$ obtained in Theorem 4.1.

Proof. Proceeding analogously to the demonstration of Theorem 4.1, we derive the well definedness of the sequences $\{u_n\}$, $\{x_n\}$, and $\{y_n\}$ and the boundedness of $\{x_n\}$ by some constant M and the following inequality:

$$\begin{aligned} V(J(x_{n+1}); u) &= V(J(T_{r_{n+1},\theta}(u_{n+1})); u) \\ &\leq V(J(u_{n+1}); u) \\ &\leq V(\alpha_n J(x_n) + (1-\alpha_n) J(S(x_n)); u) \\ &\leq \|\alpha_n J(x_n) + (1-\alpha_n) J(S(x_n))\|^2 + \|u\|^2 \\ &- 2\langle \alpha_n J(x_n) + (1-\alpha_n) J(S(x_n)); u\rangle, \end{aligned} \tag{4.4}$$

for any $u \in F_K(S) \cap EP(\text{ł}, \theta)$. From another side, we take advantage of the fact that the dual space X^* is q-unif. convex (since X is q'-unif. smooth), and by Lemma 2.19 to write

$$\|\alpha u^* + (1-\lambda) v^*\|^2 \leq \lambda \|u^*\|^2 + (1-\lambda)\|v^*\|^2 - \lambda(1-\lambda) g(\|u^* - v^*\|),$$

for any $u^*, v^* \in M\mathbb{B}_*$ and any $\lambda \in [0, 1]$ and for some continuous convex function g which is increasing and with $g(0) = 0$. Therefore, we obtain for all $u \in F_K(S) \cap EP(\text{ł}, \theta)$ that

$$\begin{aligned} V(J(x_{n+1}); u) &= \|\alpha_n J(x_n) + (1-\alpha_n) J(S(x_n))\|^2 + \|u\|^2 \\ &- 2\langle \alpha_n J(x_n) + (1-\alpha_n) J(S(x_n)); u\rangle \\ &\leq \alpha_n \|x_n\|^2 - \alpha_n(1-\alpha_n) g(\|J(x_n) - J(S(x_n))\|) + (1-\alpha_n)\|S(x_n)\|^2 \\ &\quad +\alpha_n \|u\|^2 + (1-\alpha_n)\|u\|^2 - 2\alpha_n \langle J(x_n); u\rangle - 2(1-\alpha_n)\langle J(S(x_n)); u\rangle \\ &= \alpha_n V(J(x_n); u) + (1-\alpha_n) V(J(S(x_n)); u) \\ &- \alpha_n(1-\alpha_n) g(\|J(x_n) - J(S(x_n))\|) \\ &\leq \alpha_n V(J(x_n); u) + (1-\alpha_n) V(J(x_n); u) \\ &- \alpha_n(1-\alpha_n) g(\|J(x_n) - J(S(x_n))\|) \\ &= V(J(x_n); u) - \alpha_n(1-\alpha_n) g(\|J(x_n) - J(S(x_n))\|), \end{aligned}$$

and so for any $u \in F_K(S) \cap EP(\text{ł}, \theta)$, we have

$$\alpha_n(1-\alpha_n) g(\|J(x_n) - J(S(x_n))\| \leq V(J(x_n); u) - V(J(x_{n+1}); u).$$

From the above reasoning the numerical sequence $\Phi_n := V(J(x_n); u)$ is convergent in $\mathbb{R}$, and since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we obtain

$$\lim_{n\to\infty} g(\|J(x_n) - J(S(x_n)\|) = 0.$$

This ensures by the specifications of g that $\lim_{n\to\infty} \|J(x_n) - J(S(x_n)\| = 0$, and so by the Holder continuity of J recalled in part (3) in Proposition 2.2, we can derive

$$\lim_{n\to\infty} \|x_n - S(x_n)\| = 0. \tag{4.5}$$

(1) Now, we use the assumption that K is ball compact to get $\{x_{n_k}\}$ converging strongly to $\bar{u} \in K$ and by continuity of J and $\pi_{F_K(S)\cap EP(\mathfrak{t},\theta)}$ on $U^V_{F_K(S)\cap EP(\mathfrak{t},\theta)}(r)$ proved in Theorem 2.18, we obtain

$$\bar{y} = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} \pi_{F_K(S)\cap EP(\mathfrak{t},\theta)}(J(x_{n_k})) = \pi_{F_K(S)\cap EP(\mathfrak{t},\theta)}(J(\bar{u})). \tag{4.6}$$

Also, by (4.5) we have

$$\|\bar{u} - S(\bar{u})\| = \lim_{k\to\infty} \|x_{n_k} - S(x_{n_k})\| = 0,$$

and hence $S(\bar{u}) = \bar{u}$ and so $\bar{u} \in F_K(S)$.

Now we demonstrate the convergence of the subsequence $\{u_{n_k}\}$ to $\bar{u}$. Indeed, since both sequences $\{x_n\}$ and $\{u_n\}$ are both bounded, we have by part (2) in Proposition 2.2 and by virtue of the q-unif. conv. of X that

$$\|x_n - u_n\|^q \leq \bar{K} V(J(u_n); x_n). \tag{4.7}$$

By Corollary 3.3, we also have for all $z \in F_K(S) \cap EP(\mathfrak{t}, \theta)$ that

$$\begin{aligned} V(J(u_n); x_n) &= V(J(u_n); T_{r_{n-1},\theta}(u_n)) \\ &\leq V(J(u_n); z) - V(J(T_{r_{n-1},\theta}(u_n)); z) \\ &\leq V(J(u_n); z) - V(J(x_n); z). \end{aligned} \tag{4.8}$$

Utilizing the relative nonexpansiveness of S, we get

$$\begin{aligned} V(J(u_n); z) &= V(\alpha_n J(x_{n-1}) + (1-\alpha_n) J(S(x_{n-1})); z) \\ &\leq \alpha_n V(J(x_{n-1}); z) + (1-\alpha_n) V(J(S(x_{n-1})); z) \\ &\leq \alpha_n V(J(x_{n-1}); z) + (1-\alpha_n) V(J(x_{n-1}); z) \\ &= V(J(x_{n-1}); z). \end{aligned} \tag{4.9}$$

Combining inequalities (4.7)–(4.9), we obtain

$$\begin{aligned} \|x_n - u_n\|^q &\leq \bar{K} V(J(u_n); x_n) \\ &\leq \bar{K} \left[V(J(u_n); z) - V(J(x_n); z)\right] \\ &\leq \bar{K} \left[V(J(x_{n-1}); z) - V(J(x_n); z)\right] = \bar{K} \left[\Phi_{n-1} - \Phi_n\right]. \end{aligned}$$

Since the sequence $\{\Phi_n\}$ is convergent, we obtain by taking $n \to +\infty$ that

$$\lim_{n\to+\infty} \|x_n - u_n\| = 0. \tag{4.10}$$

Therefore, since the subsequence $\{x_{n_k}\}$ is strongly convergent to $\bar{u}$, by taking $k \to +\infty$, we get

$$\lim_{k\to+\infty} \|u_{n_k} - \bar{u}\| \leq \lim_{k\to+\infty} \|u_{n_k} - x_{n_k}\| + \lim_{k\to+\infty} \|x_{n_k} - \bar{u}\| = 0$$

and hence the subsequence $\{u_{n_k}\}$ converges strongly to $\bar{u}$.

Set $x^*_{n_k} := \frac{J(u_{n_k}) - J(x_{n_k})}{r_{n_k}}$. Then, by our assumption on r_n and by the cont. of J, we obtain

$$\|x^*_{n_k}\| = \frac{1}{r_{n_k}} \|J(y_{n_k}) - J(x_{n_k})\| \leq \frac{1}{a} \|J(y_{n_k}) - J(x_{n_k})\| \to 0 \text{ as } k \to \infty.$$

On the other hand, by the way the sequence $\{x_n\} = \{T_{r_n,\theta}(u_n)\}$ is constructed and by the definition of the mapping $T_{r_n,\theta}$, it follows that for all $n \geq 0$,

$$\ell(x_n, y) + \theta V(J(x_n); y) + \frac{1}{r_n}\langle J(x_n) - J(u_n); y - x_n\rangle \geq 0, \quad \text{for all } y \in K$$

and so the subsequences $\{u_{n_k}\}$ and $\{x_{n_k}\}$ satisfy for all $k \geq 0$ that

$$\ell(x_{n_k}, y) + \theta V(J(x_{n_k}); y) \geq \frac{1}{r_{n_k}}\langle J(u_{n_k}) - J(x_{n_k}); y - x_{n_k}\rangle = \langle x^*_{n_k}; y - x_{n_k}\rangle, \quad \forall y \in C.$$

Now, by taking the limit in this inequality and by bearing in mind the u.s.c. of f regarding the 1st variable and the continuity of both V and J, we conclude

$$\begin{aligned} \ell(\bar{u}, y) + \theta V(J(\bar{u}); y) &\geq \limsup_{k\to+\infty} [\ell(x_{n_k}, y) + \theta V(J(x_{n_k}); y)] \\ &\geq \limsup_{k\to+\infty} \langle x^*_{n_k}; y - x_{n_k}\rangle = 0, \quad \forall y \in K, \end{aligned}$$

and so we obtain

$$\ell(\bar{u}, y) + \theta V(J(\bar{u}; y) \geq 0, \quad \forall y \in K,$$

that is, $\bar{u} \in EP(\ell, \theta)$. Consequently, we deduce that $\bar{u} \in F_K(S) \cap EP(\ell, \theta)$. Using equality (4.6), we get $\bar{u} = \bar{y}$, and so the proof under the ball compactness of K is complete.

(2) Assume now that K is weakly closed, l is weakly u.s.c. regarding the first variable, and J is weakly sequentially continuous. Given that $\{x_n\}$ is bounded, there is $\{x_{n_k}\}$ weakly converging to $\tilde{x} \in K$. From (4.5) and Definition 2.12, we have $\tilde{x} \in \hat{F}_K(S) = F_K(S)$. We have to prove that the limit $\tilde{x} \in EP(\ell, \theta)$. From (4.10), we have

$$\lim_{n\to+\infty} \|x_n - u_n\| = 0.$$

By exploiting the unif. cont. of J over bounded sets, we get

$$\lim_{n\to+\infty} \|J(x_n) - J(u_n)\| = 0.$$

Utilizing the same reasoning above, we obtain

$$\lim_{n\to\infty} \|x^*_n\| = \lim_{n\to\infty} \frac{1}{r_n}\|J(u_n) - J(x_n)\| \leq \frac{1}{a}\|J(u_n) - J(x_n)\| = 0,$$

that is, x^*_n strongly converges to 0 in X^*. Consequently, we obtain by the boundedness of $\{x_n\}$ that

$$\lim_{n\to+\infty} \langle x^*_n; y - x_n\rangle = 0, \quad \forall y \in C.$$

Since $x_{n_k} = T_{r_{n_k},\theta}(u_{n_k})$, we deduce that for all $k \geq 0$,

$$\ell(x_{n_k}, y) + \theta V(J(x_{n_k}); y) \geq \frac{1}{r_{n_k}}\langle J(u_{n_k}) - J(x_{n_k}); y - x_{n_k}\rangle \geq \langle x^*_{n_k}; y - x_{n_k}\rangle, \quad \forall y \in C.$$

It follows from the weak sequential cont. of J that $J(x_{n_k})$ strongly converges to $J(\tilde{x})$, and hence, by the weak u.s.c. of f regarding the first variable, we obtain

$$\limsup_{k\to+\infty} \text{ł}(x_{n_k}, y) \leq \text{ł}(\tilde{x}, y), \quad \forall y \in C.$$

Therefore, the continuity of V concludes the following:

$$\begin{aligned}
\text{ł}(\tilde{x}, y) + \theta V(J(\tilde{x}; y) &\geq \limsup_{k\to+\infty} \text{ł}(x_{n_k}, y) + \theta \lim_{k\to+\infty} V(J(x_{n_k}); y) \\
&\geq \limsup_{k\to+\infty} [\text{ł}(x_{n_k}, y) + \theta V(J(x_{n_k}); y)] \\
&\geq \limsup_{k\to+\infty} \langle x^*_{n_k}; y - x_{n_k}\rangle \\
&\geq 0, \quad \forall y \in K,
\end{aligned}$$

which entails that $\tilde{x} \in EP(\text{ł}, \theta)$, and hence we found a subseq. of $\{x_n\}$ weakly converging to $\tilde{x} \in F_K(S) \cap EP(\text{ł}, \theta)$. By continuity of $\pi_{F_K(S)\cap EP(\text{ł},\theta)}$ on $U^V_{F_K(S)\cap EP(\text{ł},\theta)}(r)$ proved in Theorem 2.18 and the convergence regarding the strong topology of $J(x_{n_k})$ to $J(\tilde{x})$, we may write

$$\lim_{k\to\infty} \pi_{F_K(S)\cap EP(\text{ł},\theta)}(J(x_{n_k})) = \pi_{F_K(S)\cap EP(\text{ł},\theta)}(\lim_{k\to\infty} J(x_{n_k})) = \pi_{F_K(S)\cap EP(\text{ł},\theta)}(J(\tilde{x})).$$

and from the demonstration of Theorem 4.1, we see that $\{y_n\}$ is convergent to $\bar{y}$, that is,

$$\bar{y} = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \pi_{F_K(S)\cap EP(\text{ł},\theta)}(J(x_n)).$$

Hence, $\bar{y} = \pi_{F_K(S)\cap EP(\text{ł},\theta)}(J(\tilde{x})) = \tilde{x}$. Consequently, the argument is fully substantiated, and the proof attains its conclusion. □

Use of AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

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