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Research article

Generalized weighted composition operators mapping into analytic tent spaces and its closure in the Korenblum space

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Abstract: This paper investigates the boundedness and compactness of generalized weighted composition operators on the closure of analytic tent spaces within the Korenblum space. It further analyzes the boundedness, compactness, and essential norm of these operators when acting from the Korenblum space to analytic tent spaces.

Keywords: Korenblum space; tent space; weighted composition operator; closure
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1. Introduction

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk, and $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$. For $\beta > 0$, the Korenblum space H_β^∞ comprises $f \in H(\mathbb{D})$ with finite norm

$$\|f\|_{H_\beta^\infty} := \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f(z)|,$$

forming a Banach space under this norm. Its closed subspace, the little Korenblum space $H_{\beta,0}^\infty$, consists of $f \in H(\mathbb{D})$ satisfying

$$\lim_{|z|\to 1^-}(1-|z|^2)^\beta|f(z)| = 0.$$

These are also termed Bers-type and little Bers-type spaces, respectively (see [8, 15]). A function f belongs to the Bloch-type space $\mathcal{B}^\beta$ if its derivative $f' \in H_\beta^\infty$. When $\beta = 1$, $\mathcal{B}^\beta$ reduces to the classical Bloch space $\mathcal{B}$.

Let $\zeta > \frac{1}{2}$ and $\eta \in \mathbb{T}$ (the boundary of $\mathbb{D}$). The non-tangential approach region $\Gamma_\zeta(\eta)$ is defined by

$$\Gamma(\eta) := \Gamma_\zeta(\eta) = \left\{ z \in \mathbb{D} : |z - \eta| < \zeta(1 - |z|^2) \right\}.$$

For $0 < p, q < \infty$ and $\alpha > -2$, the tent space $T_p^q(\alpha)$ consists of all measurable functions f on $\mathbb{D}$ satisfying

$$\|f\|_{T_p^q(\alpha)}^q := \int_{\mathbb{T}} \left(\int_{\Gamma(\eta)} |f(z)|^p (1 - |z|^2)^\alpha \, dA(z) \right)^{\frac{q}{p}} |d\eta| < \infty.$$

In $\Gamma_\zeta(\eta)$, we omit the aperture ζ from the notation. For any two choices of ζ, the resulting spaces possess equivalent quasi-norms. Tent spaces, first introduced by Coifman, Meyer, and Stein [4], serve as a framework in harmonic analysis. They unify the study of fundamental function spaces, such as Hardy spaces and Bergman spaces. The analytic tent space $AT_p^q(\alpha)$ is defined as the intersection $T_p^q(\alpha) \cap H(\mathbb{D})$. Notably, when $q = p$, this recovers the weighted Bergman space: $AT_p^p(\alpha) = A_{\alpha+1}^p$. This establishes a critical connection between tent spaces and classical function spaces.

Let $\mathbb{N}$ be the set of positive integers and $S(\mathbb{D})$ the collection of all analytic self-maps of $\mathbb{D}$. For a $\varphi \in S(\mathbb{D})$, $n \in \mathbb{N}$, and $u \in H(\mathbb{D})$, the generalized weighted composition operator (weighted differentiation composition operator) $D_{\varphi,u}^n$ is defined by

$$(D_{\varphi,u}^n f)(z) = u(z) \cdot f^{(n)}(\varphi(z)), \quad z \in \mathbb{D},$$

where $f^{(n)}$ is the n-th derivative of f. This operator was first introduced by the first author in [18]. For more insights and results about generalized weighted composition operators, one can refer to the references [11–13, 18–22]. When $n = 0$, $D_{\varphi,u}^0$ reduces to the classical weighted composition operator uC_φ; with $u = 1$, it becomes the composition operator C_φ. For composition/weighted composition operator studies on Korenblum spaces, see [5,7,8,15]. As Korenblum spaces are limit cases of weighted Bergman spaces, we naturally consider operators between H_β^∞ and A_α^p (or $AT_p^p(\alpha)$). Thus, this work's primary aim is to investigate the operator $D_{\varphi,u}^n : H_\beta^\infty \to AT_p^q(\alpha)$ when $0 < q \le p < \infty$.

Let X be a linear subspace of the normed space Y, and denote by $C_Y(X)$ the closure of X with respect to the Y-norm topology. In [1], Anderson et al. posed an open question about the closure of the H^∞ space within the Bloch space. Later, Ghatage and Zheng [6] investigated the closure of the $BMOA$ space in the Bloch space. Using the established result that H_β^∞ is identical to the space $\mathcal{B}^{\beta+1}$, the following theorem was proved in [3].

Theorem A *Let $0 < p, q, \beta < \infty$ and $\alpha > -2$. The following statements hold.*

(i) If $\beta < \frac{\alpha+2}{p}$, then $C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty) = H_\beta^\infty$.
(ii) If $\beta \ge \frac{\alpha+2}{p} + \frac{1}{q}$, then $C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty) = H_{\beta,0}^\infty$.
(iii) Let $\frac{\alpha+2}{p} \le \beta < \frac{\alpha+2}{p} + \frac{1}{q}$, $p \ge 1$ and $f \in H_\beta^\infty$. Then $f \in C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty)$ if and only if for any $\epsilon > 0$,

$$\int_{\mathbb{T}} \left(\int_{\Gamma(\eta) \cap \Omega_\epsilon^\beta(f)} (1 - |z|^2)^{\alpha - p\beta} dA(z) \right)^{\frac{q}{p}} |d\eta| < \infty.$$

Here

$$\Omega_\epsilon^\beta(f) = \left\{ z \in \mathbb{D} : |f(z)|(1 - |z|^2)^\beta \ge \epsilon \right\}.$$

In [2], Aulaskari and Zhao studied composition operators that map from the Bloch space $\mathcal{B}$ into the closure of specific Möbius-invariant subspaces within $\mathcal{B}$. In [10], Qian and Li investigated composition operators from the logarithmic Bloch space $\mathcal{B}_{\log}$ to the $\mathcal{B}_{\log}$-norm closure of Dirichlet-type spaces $\mathcal{D}^2_\alpha$. It should be emphasized that, in the existing literature, there is no research on generalized weighted composition operators (or even weighted composition operators) acting on norm closures of function spaces. This gap motivates us to study such operator classes on $C_Y(X)$-type structures. The second objective of this work is to investigate the operator $D^n_{\varphi,u}$ as a mapping from $H^\infty_\beta(H^\infty_{\beta,0})$ to $C_{H^\infty_\beta}(AT^q_p(\alpha)\cap H^\infty_\beta)$, and also as a mapping from $C_{H^\infty_\beta}(AT^q_p(\alpha)\cap H^\infty_\beta)$ to itself.

In this paper, we establish characterizations for the boundedness, compactness, and essential norm of the operator $D^n_{\varphi,u}: H^\infty_\beta \to AT^q_p(\alpha)$. Furthermore, we study the boundedness and compactness of the operator $D^n_{\varphi,u}: H^\infty_\beta(H^\infty_{\beta,0}) \to C_{H^\infty_\beta}(AT^q_p(\alpha)\cap H^\infty_\beta)$ and the compactness of the operator $D^n_{\varphi,u}: C_{H^\infty_\beta}(AT^q_p(\alpha)\cap H^\infty_\beta) \to C_{H^\infty_\beta}(AT^q_p(\alpha)\cap H^\infty_\beta)$.

Throughout this paper, we assert that $E \lesssim F$ if there exists a constant C such that $E \le CF$. The notation $E \asymp F$ signifies that both $E \lesssim F$ and $F \lesssim E$.

2. Boundedness, compactness, and essential norm of $D^n_{\varphi,u}: H^\infty_\beta \to AT^q_p(\alpha)$

In this section, we investigate the boundedness, compactness, and essential norm of the operator $D^n_{\varphi,u}: H^\infty_\beta \to AT^q_p(\alpha)$.

The following lemma, which will play a crucial role in the proof of our main theorem, may be found in [9, Lemma 3.1].

Lemma 2.1. *Let $0<\tau<\infty$, $1<\lambda<\infty$, and $e^{-\tau/\lambda}\le r_0<1$. Then there is a positive constant C, depending only on λ, τ and r_0, such that*

$$\sum_{j=1}^{\infty} \lambda^{j\lambda\tau} r^{\lambda^{j+1}} \ge \frac{C}{(1-r^2)^{\lambda\tau}}$$

for all $r_0 \le r < 1$.

Based on the well-established result that $H^\infty_\beta = \mathcal{B}^{\beta+1}$, and making use of the higher-order derivatives characterization of the Bloch type space (as detailed in [17]), we get the following result.

Lemma 2.2. *Let $\beta>0$, $n\in\mathbb{N}$, and $f\in H(\mathbb{D})$. Then $f\in H^\infty_\beta$ if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{n+\beta}|f^{(n)}(z)|<\infty.$$

Moreover, $\|f\|_{H^\infty_\beta}$ is equivalent to $\|f\|_{H^{\infty,n}_\beta}$. Here,

$$\|f\|_{H^{\infty,n}_\beta} = \sum_{j=0}^{n-1}|f^{(j)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{n+\beta}|f^{(n)}(z)|.$$

Theorem 2.3. *Let $0<\beta<\infty$, $0<q\le p<\infty$, $n\in\mathbb{N}\cup\{0\}$, $\alpha>-2$, $u\in H(\mathbb{D})$ and $\varphi\in S(\mathbb{D})$. Then $D^n_{\varphi,u}: H^\infty_\beta \to AT^q_p(\alpha)$ is bounded if and only if*

$$\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)} \frac{|u(z)|^p}{(1-|\varphi(z)|^2)^{p(n+\beta)}}(1-|z|^2)^\alpha dA(z)\right)^{\frac{q}{p}} |d\eta| < \infty. \tag{2.1}$$

Proof. First, we assume that (2.1) holds. Let $f \in H_\beta^\infty$. Using Lemma 2.2, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}} \left(\int_{\Gamma(\eta)} |(D_{\varphi,u}^n f)(z)|^p (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} |d\eta| \\
&= \int_{\mathbb{T}} \left(\int_{\Gamma(\eta)} |u(z)|^p |f^{(n)}(\varphi(z))|^p (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} |d\eta| \\
&\lesssim \|f\|_{H_\beta^{\infty,n}}^q \int_{\mathbb{T}} \left(\int_{\Gamma(\eta)} \frac{|u(z)|^p}{(1-|\varphi(z)|^2)^{p(n+\beta)}} (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} |d\eta| < \infty,
\end{aligned}$$

which implies that $D_{\varphi,u}^n : H_\beta^\infty \to AT_p^q(\alpha)$ is bounded.

Conversely, assume that $D_{\varphi,u}^n : H_\beta^\infty \to AT_p^q(\alpha)$ is bounded. For each $0 \le \theta < 2\pi$ and $0 < t \le 1$, set

$$g_{\theta,t}(z) = \sum_{j=0}^{\infty} 2^{j\beta} (te^{i\theta})^{2^j} z^{2^j}, \quad z \in \mathbb{D}. \tag{2.2}$$

From [14, Lemma 2], we see that $g_{\theta,t} \in H_\beta^\infty$ and $\|g_{\theta,t}\|_{H_\beta^\infty} \lesssim 1$ are independent of θ and t. For simplicity, denote $g_{\theta,1}$ by g_θ. Using Fubini's theorem, Minkowski's inequality, and the assumption that $\frac{p}{q} \ge 1$, we get

$$\begin{aligned}
1 &\gtrsim \int_0^{2\pi} \|D_{\varphi,u}^n g_\theta\|_{AT_p^q(\alpha)}^q \frac{d\theta}{2\pi} \\
&\asymp \int_0^{2\pi} \left(\int_{\mathbb{T}} \left(\int_{\Gamma(\eta)} |(D_{\varphi,u}^n g_\theta)(z)|^p (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} |d\eta| \right) \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \left(\int_{\mathbb{T}} \left(\int_{\Gamma(\eta)} |u(z)|^p |g_\theta^{(n)}(\varphi(z))|^p (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} |d\eta| \right) \frac{d\theta}{2\pi} \\
&= \int_{\mathbb{T}} \left(\int_0^{2\pi} \left(\int_{\Gamma(\eta)} |u(z)|^p |g_\theta^{(n)}(\varphi(z))|^p (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} \frac{d\theta}{2\pi} \right) |d\eta| \\
&\ge \int_{\mathbb{T}} \left(\int_{\Gamma(\eta)} \left(\int_0^{2\pi} |u(z)|^q |g_\theta^{(n)}(\varphi(z))|^q \frac{d\theta}{2\pi} \right)^{\frac{p}{q}} (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} |d\eta| \\
&= \int_{\mathbb{T}} \left(\int_{\Gamma(\eta)} |u(z)|^p \left(\int_0^{2\pi} |g_\theta^{(n)}(\varphi(z))|^q \frac{d\theta}{2\pi} \right)^{\frac{p}{q}} (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} |d\eta|.
\end{aligned}$$

Using the following inequality (see [23] or [9, Lemma F])

$$\prod_{k=0}^{n-1} (2^j - k) \ge \frac{2^{nj}}{n!}, \quad n \in \mathbb{N},\ j \in \mathbb{N},\ 2^j \ge n,$$

we have

$$\int_0^{2\pi} |g_\theta^{(n)}(\varphi(z))|^q \frac{d\theta}{2\pi} \ge \int_0^{2\pi} \left| \sum_{j \ge [\log_2 n]}^{\infty} \prod_{k=0}^{n-1} (2^j - k) 2^{j\beta} \varphi(z)^{2^j - n} e^{i\theta 2^j} \right|^q \frac{d\theta}{2\pi}$$

$$\gtrsim \left(\sum_{j \geq [\log_2 n]}^{\infty} \left(\prod_{k=0}^{n-1} (2^j - k) \right)^2 2^{2j\beta} |\varphi(z)|^{2(2^j - n)} \right)^{\frac{q}{2}}$$

$$\gtrsim \left(\sum_{j \geq [\log_2 n]}^{\infty} 2^{2j(n+\beta)} |\varphi(z)|^{2(2^j - n)} \right)^{\frac{q}{2}},$$

where $[x] = \inf\{n \in \mathbb{N} : n \geq x\}$.

Using Lemma 2.1 with the choice $r_0 = e^{-\tau/2}$, we deduce the existence of a positive constant C, depending only on τ and n, such that for all z satisfying $e^{-\tau/2} \leq |\varphi(z)| < 1$, the inequality

$$\sum_{j=1}^{\infty} 2^{2j(n+\beta)} |\varphi(z)|^{2^{j+1}} \geq \frac{C}{(1 - |\varphi(z)|^2)^{2(n+\beta)}}$$

holds. Given $n + \beta > 0$, there exists a $\varrho \in [e^{-\tau/2}, 1)$ for all $|\varphi(z)| \in [\varrho, 1)$, the partial sum satisfies

$$\sum_{1 \leq j < [\log_2 n]} 2^{2j(n+\beta)} |\varphi(z)|^{2^{j+1}} \leq \frac{C}{2(1 - |\varphi(z)|^2)^{2(n+\beta)}}.$$

Hence, for all $|\varphi(z)| \in [\varrho, 1)$,

$$\sum_{j \geq [\log_2 n]}^{\infty} 2^{2j(n+\beta)} |\varphi(z)|^{2(2^j - n)} \geq \frac{C}{2(1 - |\varphi(z)|^2)^{2(n+\beta)}},$$

which implies that

$$\int_{\mathbb{T}} \left(\int_{\Gamma(\eta) \cap \{|\varphi(z)| \geq \varrho\}} \frac{|u(z)|^p}{(1 - |\varphi(z)|^2)^{p(n+\beta)}} (1 - |z|^2)^{\alpha} dA(z) \right)^{\frac{q}{p}} |d\eta| \lesssim 1.$$

Noting that $u \in AT_p^q(\alpha)$, we obtain

$$\int_{\mathbb{T}} \left(\int_{\Gamma(\eta) \cap \{|\varphi(z)| < \varrho\}} \frac{|u(z)|^p}{(1 - |\varphi(z)|^2)^{p(n+\beta)}} (1 - |z|^2)^{\alpha} dA(z) \right)^{\frac{q}{p}} |d\eta| \lesssim \|u\|_{AT_p^q(\alpha)}^q.$$

Therefore,

$$\int_{\mathbb{T}} \left(\int_{\Gamma(\eta)} \frac{|u(z)|^p}{(1 - |\varphi(z)|^2)^{p(n+\beta)}} (1 - |z|^2)^{\alpha} dA(z) \right)^{\frac{q}{p}} |d\eta| < \infty.$$

The proof is complete.

□

From Theorem 2.3 and the fact that $AT_p^p(\alpha) = A_{\alpha+1}^p$, we immediately obtain the following corollary.

Corollary 2.4. *Let* $0 < p, \beta < \infty$, $n \in \mathbb{N} \cup \{0\}$, $\alpha > -1$, $u \in H(\mathbb{D})$, *and* $\varphi \in S(\mathbb{D})$. *Then* $D_{\varphi,u}^n : H_\beta^\infty \to A_\alpha^p$ *is bounded if and only if*

$$\int_{\mathbb{D}} \frac{|u(z)|^p}{(1 - |\varphi(z)|^2)^{p(n+\beta)}} (1 - |z|^2)^{\alpha} dA(z) < \infty.$$

Next, we consider the compactness and essential norm of the operator $D^n_{\varphi,u} : H^\infty_\beta \to AT^q_p(\alpha)$. For a bounded linear operator $T : X \to Y$ between Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, recall that the essential norm is given by the infimum

$$\|T\|_{e,X\to Y} = \inf_K \{\|T - K\|_{X\to Y} : K \in \mathcal{K}(X,Y)\},$$

where $\mathcal{K}(X,Y)$ denotes the space of compact operators from X to Y. It is clear that T is compact if and only if $\|T\|_{e,X\to Y} = 0$.

For each $m \in \mathbb{N}$, define

$$h^m_{\theta,t}(z) = z^m g_{\theta,t}(z),$$

where $g_{\theta,t}$ denotes the test function specified in (2.2). Given that the inclusion $z^m H^\infty_{\beta,0} \subseteq H^\infty_{\beta,0}$, it follows that the family $\{h^m_{\theta,t}\}_{m\in\mathbb{N}}$ is contained within $H^\infty_{\beta,0}$. Furthermore, the H^∞_β-norm $\|h^m_{\theta,t}\|_{H^\infty_\beta}$ is uniformly bounded with respect to θ, t, and m. Using Lemma 4 in [14], for every bounded linear functional $\Lambda \in (H^\infty_\beta)^*$, we get

$$\sup_{\theta,t} |\Lambda(h^m_{\theta,t})| \to 0 \quad \text{as} \quad m \to \infty.$$

This establishes the weak convergence of the sequence $\{h^m_{\theta,t}\}_{m\in\mathbb{N}}$ to 0 in H^∞_β. Using the complete continuity property of compact operators, the subsequent lemma is derived. The detailed proof is omitted for conciseness.

Lemma 2.5. *Let $0 < \beta < \infty$, $0 < q \le p < \infty$, $\alpha > -2$. For any compact operator $T : H^\infty_\beta \to AT^q_p(\alpha)$, it holds that*

$$\lim_{m\to\infty} \sup_{\theta,t} \|T h^m_{\theta,t}\|_{AT^q_p(\alpha)} = 0,$$

where the supremum is taken over all $0 \le \theta < 2\pi$ and $0 < t < 1$.

The following compactness criterion holds for the operator $D^n_{\varphi,u}$ mapping from H^∞_β to $AT^q_p(\alpha)$, as established in [5, Proposition 3.11].

Lemma 2.6. *Let $0 < \beta < \infty$, $0 < q \le p < \infty$, $n \in \mathbb{N} \cup \{0\}$, $\alpha > -2$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $D^n_{\varphi,u} : H^\infty_\beta \to AT^q_p(\alpha)$ is bounded. Then $D^n_{\varphi,u} : H^\infty_\beta \to AT^q_p(\alpha)$ is compact if and only if for every bounded sequence $\{f_j\}$ in H^∞_β that converges to 0 uniformly on compact subsets of $\mathbb{D}$, we have*

$$\lim_{j\to\infty} \|D^n_{\varphi,u} f_j\|_{AT^q_p(\alpha)} = 0.$$

Theorem 2.7. *Let $0 < \beta < \infty$, $0 < q \le p < \infty$, $n \in \mathbb{N} \cup \{0\}$, $\alpha > -2$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Suppose that $D^n_{\varphi,u} : H^\infty_\beta \to AT^q_p(\alpha)$ is bounded. Then it holds that*

$$\|D^n_{\varphi,u}\|^q_{e,H^\infty_\beta\to AT^q_p(\alpha)} \asymp \limsup_{r\to 1} \int_{\mathbb{T}} \left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|>r\}} \frac{|u(z)|^p}{(1-|\varphi(z)|^2)^{p(n+\beta)}} (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} |d\eta|. \tag{2.3}$$

Proof. First, we establish the upper bounds in (2.3). Consider an arbitrary positive integer j and a function $f \in H(\mathbb{D})$. Define the linear operator C_j by

$$C_j f(z) = f\left(\frac{jz}{j+1}\right).$$

It is easy to check that C_j is bounded on the space H_β^∞. Now, using Lemma 2.6 with the constant function $u \equiv 1$ and the analytic self-map $\varphi(z) = \frac{jz}{j+1}$ of $\mathbb{D}$, we deduce that C_j is compact on H_β^∞. So,

$$\begin{aligned}\|D_{\varphi,u}^n\|_{e,H_\beta^\infty\to AT_p^q(\alpha)} &\le \liminf_{j\to\infty}\|D_{\varphi,u}^n - D_{\varphi,u}^n C_j\| \\ &= \liminf_{j\to\infty}\sup_{\|f\|_{H_\beta^\infty}\le 1}\|D_{\varphi,u}^n(id - C_j)f\|_{AT_p^q(\alpha)},\end{aligned}$$

where id denotes the identity operator on H_β^∞. Fix a positive integer j and an $f \in H_\beta^\infty$ with $\|f\|_{H_\beta^\infty} \le 1$, we have

$$\begin{aligned}&\|D_{\varphi,u}^n(id - C_j)f\|_{AT_p^q(\alpha)}^q \\ \lesssim &\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|\ge r\}}|u(z)|^p\left|f^{(n)}(\varphi(z)) - f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|^p(1-|z|^2)^\alpha dA(z)\right)^{\frac{q}{p}}|d\eta| \\ &+\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|\le r\}}|u(z)|^p\left|f^{(n)}(\varphi(z)) - f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|^p(1-|z|^2)^\alpha dA(z)\right)^{\frac{q}{p}}|d\eta|\end{aligned}$$

for any $r \in (0,1)$. By Lemma 2.2, we deduce that

$$\left|f^{(n)}(\varphi(z)) - f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right| \le C\cdot\frac{\|f\|_{H_\beta^\infty}}{(1-|\varphi(z)|^2)^{n+\beta}},$$

where C denotes a positive constant dependent only on n and β. This implies that, for any $r \in (0,1)$,

$$\begin{aligned}&\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|\ge r\}}|u(z)|^p\left|f^{(n)}(\varphi(z)) - f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|^p(1-|z|^2)^\alpha dA(z)\right)^{\frac{q}{p}}|d\eta| \\ \le &C\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|\ge r\}}\frac{|u(z)|^p}{(1-|\varphi(z)|^2)^{p(n+\beta)}}(1-|z|^2)^\alpha dA(z)\right)^{\frac{q}{p}}|d\eta|.\end{aligned}$$

It should be noted that this estimate is independent of j.

We now proceed to establish the following limit for an arbitrary $r \in (0,1)$:

$$\lim_{j\to\infty}\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|\le r\}}|u(z)|^p\left|f^{(n)}(\varphi(z)) - f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|^p(1-|z|^2)^\alpha\, dA(z)\right)^{\frac{q}{p}}|d\eta| = 0.$$

Let $\rho = \varphi(z)$ and denote the radial segment by $\left[\frac{j\rho}{j+1},\rho\right]$. By integrating $f^{(n+1)}$ along this segment, we have

$$\left|f^{(n)}(\rho) - f^{(n)}\left(\frac{j\rho}{j+1}\right)\right| \le \frac{|\rho|}{j+1}\cdot\left|f^{(n+1)}(\xi(\rho))\right|$$

for some intermediate point $\xi(\rho) \in \left[\frac{j\rho}{j+1},\rho\right]$. Applying Cauchy's integral estimate to $f^{(n+1)}$ on the circle centered at $\xi(\rho)$ with radius $R \in (0, 1-r)$, we obtain

$$|f^{(n+1)}(\xi(\rho))| \le \frac{1}{R}\max_{|\zeta|=R+r}|f^{(n)}(\zeta)|.$$

Hence,

$$\begin{aligned}&\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|\le r\}}|u(z)|^p\left|f^{(n)}(\varphi(z))-f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|^p(1-|z|^2)^\alpha dA(z)\right)^{\frac{q}{p}}|d\eta|\\ \lesssim&\frac{r^q}{R^q(j+1)^q}\frac{1}{[1-(R+r)^2]^{q(n+\beta)}}\|u\|^q_{AT^q_p(\alpha)}.\end{aligned}$$

By the assumption, we see that $u\in AT^q_p(\alpha)$. Hence,

$$\lim_{j\to\infty}\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|\le r\}}|u(z)|^p\left|f^{(n)}(\varphi(z))-f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|^p(1-|z|^2)^\alpha dA(z)\right)^{\frac{q}{p}}|d\eta|=0.$$

Therefore,

$$\|D^n_{\varphi,u}\|^q_{e,H^\infty_\beta\to AT^q_p(\alpha)}\lesssim\limsup_{r\to1}\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|>r\}}\frac{|u(z)|^p}{(1-|\varphi(z)|^2)^{p(n+\beta)}}(1-|z|^2)^\alpha dA(z)\right)^{\frac{q}{p}}|d\eta|.$$

We now establish the lower bounds in (2.3). First, observe that the norm $\|h^m_{\theta,t}\|_{H^\infty_\beta}$ is uniformly bounded with respect to θ, t and m. For an arbitrary compact operator $K: H^\infty_\beta\to AT^q_p(\alpha)$, we get

$$\|D^n_{\varphi,u}-K\|\gtrsim\|(D^n_{\varphi,u}-K)h^m_{\theta,t}\|_{AT^q_p(\alpha)}\ge\|D^n_{\varphi,u}h^m_{\theta,t}\|_{AT^q_p(\alpha)}-\|Kh^m_{\theta,t}\|_{AT^q_p(\alpha)}$$

valid for all θ, t and m. By Fatou's lemma we have that

$$\begin{aligned}&\sup_{\theta,t}\|D^n_{\varphi,u}h^m_{\theta,t}\|^q_{AT^q_p(\alpha)}\\ \ge&\liminf_{t\to1}\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)}|u(z)|^p|(h^m_{\theta,t})^{(n)}(\varphi(z))|^p(1-|z|^2)^\alpha dA(z)\right)^{\frac{q}{p}}|d\eta|\\ \ge&\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|>r\}}|u(z)|^p|\varphi(z)|^{pm}|g^{(n)}_\theta(\varphi(z))|^p(1-|z|^2)^\alpha dA(z)\right)^{\frac{q}{p}}|d\eta|\end{aligned}$$

for any $r\in(0,1)$. By integrating these inequalities with respect to θ from 0 to 2π, using Fubini's theorem and Minkowski's inequality, we get

$$\begin{aligned}&\int_0^{2\pi}\sup_{\theta,t}\|D^n_{\varphi,u}h^m_{\theta,t}\|^q_{AT^q_p(\alpha)}\frac{d\theta}{2\pi}\\ \ge&\int_{\mathbb{T}}\int_0^{2\pi}\left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|>r\}}|u(z)|^p|\varphi(z)|^{pm}|g^{(n)}_\theta(\varphi(z))|^p(1-|z|^2)^\alpha dA(z)\right)^{\frac{q}{p}}\frac{d\theta}{2\pi}|d\eta|\\ \ge&\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|>r\}}|u(z)|^p|\varphi(z)|^{pm}\left(\int_0^{2\pi}|g^{(n)}_\theta(\varphi(z))|^q\frac{d\theta}{2\pi}\right)^{\frac{p}{q}}(1-|z|^2)^\alpha dA(z)\right)^{\frac{q}{p}}|d\eta|.\end{aligned}$$

From the proof of Theorem 2.3, for an arbitrary $z\in\mathbb{D}$ satisfying $|\varphi(z)|\ge\varrho$, we have

$$\int_0^{2\pi}\left|g^{(n)}_\theta(\varphi(z))\right|^q\frac{d\theta}{2\pi}\gtrsim\frac{1}{(1-|\varphi(z)|^2)^{q(n+\beta)}}.$$

Hence,

$$\int_0^{2\pi} \sup_{\theta,t} \|D^n_{\varphi,u} g^m_{\theta,t}\|^q_{AT^q_p(\alpha)} \frac{d\theta}{2\pi}$$
$$\gtrsim \int_{\mathbb{T}} \left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|>r\}} \frac{|u(z)|^p}{(1-|\varphi(z)|^2)^{p(n+\beta)}} (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} |d\eta|$$

for any $r \in (\varrho, 1)$. Letting $r \to 1$, we get

$$\int_0^{2\pi} \sup_{\theta,t} \|D^n_{\varphi,u} g^m_{\theta,t}\|^q_{AT^q_p(\alpha)} \frac{d\theta}{2\pi}$$
$$\gtrsim \limsup_{r\to 1} \int_{\mathbb{T}} \left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|>r\}} \frac{|u(z)|^p}{(1-|\varphi(z)|^2)^{p(n+\beta)}} (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} |d\eta|.$$

It is worthy of note that this estimate does not depend on m. In light of Lemma 2.5, for an arbitrary compact operator $K : H^\infty_\beta \to AT^q_p(\alpha)$, we obtain

$$\lim_{m\to\infty} \sup_{\theta,t} \|Kh^m_{\theta,t}\|_{AT^q_p(\alpha)} = 0.$$

Therefore,

$$\|D^n_{\varphi,u}\|^q_{e,H^\infty_\beta\to AT^q_p(\alpha)} \gtrsim \limsup_{r\to 1} \int_{\mathbb{T}} \left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|>r\}} \frac{|u(z)|^p}{(1-|\varphi(z)|^2)^{p(n+\beta)}} (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} |d\eta|.$$

The proof is complete. □

From the last theorem, we immediately get the following corollary.

Corollary 2.8. *Let* $0 < \beta < \infty$, $0 < q \le p < \infty$, $n \in \mathbb{N} \cup \{0\}$, $\alpha > -2$, $u \in H(\mathbb{D})$, *and* $\varphi \in S(\mathbb{D})$ *such that* $D^n_{\varphi,u} : H^\infty_\beta \to AT^q_p(\alpha)$ *is bounded. Then* $D^n_{\varphi,u} : H^\infty_\beta \to AT^q_p(\alpha)$ *is compact if and only if*

$$\limsup_{r\to 1} \int_{\mathbb{T}} \left(\int_{\Gamma(\eta)\cap\{|\varphi(z)|>r\}} \frac{|u(z)|^p}{(1-|\varphi(z)|^2)^{p(n+\beta)}} (1-|z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} |d\eta| = 0.$$

From Theorem 2.7, Corollary 2.8 and the fact that $AT^p_p(\alpha) = A^p_{\alpha+1}$, we also have the following corollary.

Corollary 2.9. *Let* $0 < p, \beta < \infty$, $n \in \mathbb{N} \cup \{0\}$, $\alpha > -1$, $u \in H(\mathbb{D})$, *and* $\varphi \in S(\mathbb{D})$. *Then the following statements hold.*

(i) Suppose that $D^n_{\varphi,u} : H^\infty_\beta \to A^p_\alpha$ *is bounded. Then it holds that*

$$\|D^n_{\varphi,u}\|^p_{e,H^\infty_\beta\to A^p_\alpha} \asymp \limsup_{r\to 1} \int_{|\varphi(z)|>r} \frac{|u(z)|^p}{(1-|\varphi(z)|^2)^{p(n+\beta)}} (1-|z|^2)^\alpha dA(z).$$

(ii) The operator $D^n_{\varphi,u} : H^\infty_\beta \to A^p_\alpha$ *is compact if and only if* $D^n_{\varphi,u} : H^\infty_\beta \to A^p_\alpha$ *is bounded and*

$$\limsup_{r\to 1} \int_{|\varphi(z)|>r} \frac{|u(z)|^p}{(1-|\varphi(z)|^2)^{p(n+\beta)}} (1-|z|^2)^\alpha dA(z) = 0.$$

3. Boundedness and compactness of $D^n_{\varphi,u} : H^\infty_\beta \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$

In this section, we study the boundedness and compactness of $D^n_{\varphi,u} : H^\infty_\beta(H^\infty_{\beta,0}) \to C_{H^\infty_\beta}(AT^q_p(\alpha)\cap H^\infty_\beta)$ and $D^n_{\varphi,u} : C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta) \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$.

Theorem 3.1. *Let $n \in \mathbb{N} \cup \{0\}$, $0 < q,\beta < \infty$, $1 \le p < \infty$, $\alpha > -2$ such that $\frac{\alpha+2}{p} \le \beta < \frac{\alpha+2}{p} + \frac{1}{q}$. Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $D^n_{\varphi,u} : H^\infty_\beta \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ is bounded if and only if for any $\epsilon > 0$,*

$$\int_{\mathbb{T}} \left(\int_{\Gamma(\eta)\cap\{|\varphi^{u,n}_\beta(z)|\ge\epsilon\}} (1 - |z|^2)^{\alpha-p\beta} dA(z) \right)^{\frac{q}{p}} |d\eta| < \infty. \tag{3.1}$$

Here

$$\varphi^{u,n}_\beta(z) := \frac{u(z)(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{n+\beta}}.$$

Proof. First, we assume that (3.1) holds. Let $f \in H^\infty_\beta$. Consider the following inequality for the operator $D^n_{\varphi,u}$:

$$\begin{aligned} |(D^n_{\varphi,u} f)(z)|(1 - |z|^2)^\beta =& |f^{(n)}(\varphi(z))|(1 - |\varphi(z)|^2)^{n+\beta} |\varphi^{u,n}_\beta(z)| \\ \le& \|f\|_{H^{\infty,n}_\beta} |\varphi^{u,n}_\beta(z)|. \end{aligned}$$

For an arbitrary $\delta > 0$, if the inequality

$$|(D^n_{\varphi,u} f)(z)|(1 - |z|^2)^\beta > \delta$$

holds, then it must be that

$$|\varphi^{u,n}_\beta(z)| \ge \epsilon, \quad \text{where } \epsilon := \frac{\delta}{\|f\|_{H^{\infty,n}_\beta}}.$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{T}} \left(\int_{\Gamma(\eta)\cap\Omega^\beta_\delta(D^n_{\varphi,u} f)} (1 - |z|^2)^{\alpha-p\beta} dA(z) \right)^{\frac{q}{p}} |d\eta| \\ \lesssim &\int_{\mathbb{T}} \left(\int_{\Gamma(\eta)\cap\{|\varphi^{u,n}_\beta(z)|\ge\epsilon\}} (1 - |z|^2)^{\alpha-p\beta} dA(z) \right)^{\frac{q}{p}} |d\eta| < \infty. \end{aligned}$$

Using Theorem A, one deduces that $D^n_{\varphi,u} f \in C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$. Consequently, $D^n_{\varphi,u} : H^\infty_\beta \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ is bounded and

$$\|D^n_{\varphi,u} f\|_{C_{H^\infty_\beta}(AT^q_p(\alpha)\cap H^\infty_\beta)} \le C\|f\|_{H^\infty_\beta}$$

holds for some constant $C > 0$ independent of f.

Conversely, assume that $D^n_{\varphi,u} : H^\infty_\beta \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ is bounded. By Theorem 2.5 in [16], there exist functions $f_1, f_2 \in H^\infty_\beta$ satisfying the inequality

$$|f_1(z)| + |f_2(z)| \ge \frac{1}{(1 - |z|^2)^\beta}, \quad z \in \mathbb{D}.$$

Define $g_1(z) := f_1(z) - zf_1(0)$ and $g_2(z) := f_2(z) - zf_2(0)$. Utilizing the asymptotic relation

$$(1-|z|^2)^{\beta+1}|f'(z)| + |f(0)| \asymp (1-|z|^2)^{\beta}|f(z)|,$$

we deduce that $g_1, g_2 \in H^\infty_\beta$ and

$$|g_1'(z)| + |g_2'(z)| \geq \frac{1}{(1-|z|^2)^{\beta+1}}, \quad z \in \mathbb{D}.$$

According to this rule, it can be deduced that there exist $h_1, h_2 \in H^\infty_\beta$ satisfying the inequality

$$|h_1^{(n)}(z)| + |h_2^{(n)}(z)| \geq \frac{1}{(1-|z|^2)^{n+\beta}}, \quad z \in \mathbb{D}.$$

By assumption, we have that $D^n_{\varphi,u}h_1, D^n_{\varphi,u}h_2 \in C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$. Thus, for an arbitrarily $\epsilon > 0$, applying Theorem A yields the integrability conditions

$$\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\Omega^\beta_{\epsilon/2}(D^n_{\varphi,u}h_1)}(1-|z|^2)^{\alpha-p\beta}\,dA(z)\right)^{q/p}|d\eta| < \infty$$

and

$$\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\Omega^\beta_{\epsilon/2}(D^n_{\varphi,u}h_2))}(1-|z|^2)^{\alpha-p\beta}\,dA(z)\right)^{q/p}|d\eta| < \infty.$$

Furthermore, in the case where $|\varphi^{u,n}_\beta(z)| \geq \epsilon$, we obtain

$$\begin{aligned}&\Big(|(D^n_{\varphi,u}h_1)(z)| + |(D^n_{\varphi,u}h_2)(z)|\Big)(1-|z|^2)^\beta\\ =&\Big(|h_1^{(n)}(\varphi(z))| + |h_2^{(n)}(\varphi(z))|\Big)||u(z)||(1-|z|^2)^\beta\\ \geq&\frac{|u(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{n+\beta}} \geq \epsilon,\end{aligned}$$

which implies that at least one of the following two inequalities holds:

$$|(D^n_{\varphi,u}h_1)(z)|(1-|z|^2)^\beta \geq \frac{\epsilon}{2}$$

or

$$|(D^n_{\varphi,u}h_2)(z)|(1-|z|^2)^\beta \geq \frac{\epsilon}{2}.$$

Therefore, for any $\epsilon > 0$,

$$\begin{aligned}&\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\{|\varphi^{u,n}_\beta(z)|\geq\epsilon\}}(1-|z|^2)^{\alpha-p\beta}dA(z)\right)^{\frac{q}{p}}|d\eta|\\ \leq&\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\Omega^\beta_{\epsilon/2}(D^n_{\varphi,u}h_1))}(1-|z|^2)^{\alpha-p\beta}dA(z)\right)^{\frac{q}{p}}|d\eta|\\ &+\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\Omega^\beta_{\epsilon/2}(D^n_{\varphi,u}h_2))}(1-|z|^2)^{\alpha-p\beta}dA(z)\right)^{\frac{q}{p}}|d\eta|\\ <&\infty.\end{aligned}$$

The proof is complete. □

Theorem 3.2. *Let $n \in \mathbb{N} \cup \{0\}$, $0 < q, \beta < \infty$, $1 \le p < \infty$, $\alpha > -2$ such that $\frac{\alpha+2}{p} \le \beta < \frac{\alpha+2}{p} + \frac{1}{q}$. Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $D^n_{\varphi,u} : H^\infty_{\beta,0} \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ is bounded if and only if $u \in C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ and*

$$\sup_{z\in\mathbb{D}} |\varphi^{u,n}_\beta(z)| < \infty.$$

Proof. Suppose that $u \in C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ and

$$\sup_{z\in\mathbb{D}} |\varphi^{u,n}_\beta(z)| \le C < \infty.$$

Consider a function $f \in H^\infty_{\beta,0}$. For an arbitrary $\epsilon > 0$, there exists a constant r $(0 < r < 1)$ such that the estimate

$$|f^{(n)}(z)|(1-|z|^2)^{n+\beta} < \frac{\epsilon}{C}$$

is satisfied whenever $|z| > r$. Let $z \in \Omega^\beta_\epsilon(D^n_{\varphi,u} f)$. Then,

$$\begin{aligned}\epsilon \le &|(D^n_{\varphi,u} f)(z)|(1-|z|^2)^\beta \\ =&|f^{(n)}(\varphi(z))|(1-|\varphi(z)|^2)^{n+\beta}\frac{|u(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{n+\beta}} \\ \le& C|f^{(n)}(\varphi(z))|(1-|\varphi(z)|^2)^{n+\beta}.\end{aligned}$$

This inequality directly implies $|\varphi(z)| \le r$. Hence,

$$\epsilon \le |f^{(n)}(\varphi(z))||u(z)|(1-|z|^2)^\beta \le \frac{\|f\|_{H^{\infty,n}_\beta}}{(1-r^2)^{n+\beta}}|u(z)|(1-|z|^2)^\beta.$$

Let $\delta = \frac{\epsilon(1-r^2)^{n+\beta}}{\|f\|_{H^{\infty,n}_\beta}}$. Then $|u(z)|(1-|z|^2)^\beta \ge \delta$. Therefore,

$$\Omega^\beta_\epsilon(D^n_{\varphi,u} f) \subseteq \Omega^\beta_\delta(u).$$

By the fact that $u \in C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$, we get

$$\begin{aligned}&\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\Omega^\beta_\epsilon(D^n_{\varphi,u} f)}(1-|z|^2)^{\alpha-p\beta}dA(z)\right)^{\frac{q}{p}}|d\eta| \\ \le&\int_{\mathbb{T}}\left(\int_{\Gamma(\eta)\cap\Omega^\beta_\delta(u)}(1-|z|^2)^{\alpha-p\beta}dA(z)\right)^{\frac{q}{p}}|d\eta| < \infty.\end{aligned}$$

An application of Theorem A yields that $D^n_{\varphi,u} f \in C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$. Consequently, the operator $D^n_{\varphi,u} : H^\infty_{\beta,0} \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ is bounded, as desired.

Conversely, suppose the operator $D^n_{\varphi,u} : H^\infty_{\beta,0} \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ is bounded. Consider the function $f(z) = z^n$, which belongs to $H^\infty_{\beta,0}$. By the boundedness of $D^n_{\varphi,u}$, we have

$$u = D^n_{\varphi,u} f \in C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta).$$

Since $C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty)$ is a subspace of H_β^∞, the boundedness of $D_{\varphi,u}^n : H_{\beta,0}^\infty \to C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty)$ immediately implies its boundedness as an operator from $H_{\beta,0}^\infty$ to H_β^∞.

For an arbitrary $a \in \mathbb{D}$, consider the test function

$$f_a(z) = \frac{1-|a|^2}{(1-\overline{a}z)^{\beta+1}}, \quad z \in \mathbb{D},$$

which is in the space H_β^∞. Using the fact that $u \in H_\beta^\infty$, after a calculation, we see that $\sup_{z\in\mathbb{D}} |\varphi_\beta^{u,n}(z)| < \infty$. The proof is complete. □

The boundedness of the operator $D_{\varphi,u}^n : H_{\beta,0}^\infty \to C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty)$ always implies that $u \in C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty)$. Consequently, throughout the remainder of this work, we shall assume $u \in C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty)$.

Theorem 3.3. *Let $n \in \mathbb{N} \cup \{0\}$, $0 < q, \beta < \infty$, $1 \le p < \infty$, $\alpha > -2$ such that $\frac{\alpha+2}{p} \le \beta < \frac{\alpha+2}{p} + \frac{1}{q}$. Let $u \in C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty)$ and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent.*

(i) $D_{\varphi,u}^n : H_\beta^\infty \to H_\beta^\infty$ *is compact.*
(ii) $D_{\varphi,u}^n : H_{\beta,0}^\infty \to H_\beta^\infty$ *is compact.*
(iii)

$$\lim_{|\varphi(z)|\to 1} |\varphi_\beta^{u,n}(z)| = 0. \tag{3.2}$$

(iv) $D_{\varphi,u}^n : H_\beta^\infty \to C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty)$ *is compact.*
(v) $D_{\varphi,u}^n : C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty) \to C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty)$ *is compact.*
(vi) $D_{\varphi,u}^n : H_{\beta,0}^\infty \to C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty)$ *is compact.*

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$. The proof is similar to the proof of Theorem 2.3 in [20]. For the sake of conciseness, we omit the details.

$(iii) \Rightarrow (iv)$. By hypothesis, there exists a real number $r \in (0,1)$ such that for all w, the inequality

$$|\varphi_\beta^{u,n}(w)| < \frac{\epsilon}{2}$$

holds whenever $|\varphi(w)| \ge r$. Let $z \in \mathbb{D}$ satisfy $|\varphi_\beta^{u,n}(z)| \ge \epsilon$. This implies that $|\varphi(z)| < r$. Hence,

$$\epsilon \le |\varphi_\beta^{u,n}(z)| \le \frac{|u(z)|(1-|z|^2)^\beta}{(1-r^2)^{n+\beta}},$$

which implies that

$$\epsilon(1-r^2)^{n+\beta} \le |u(z)|(1-|z|^2)^\beta.$$

Let $\delta = \epsilon(1-r^2)^{n+\beta}$. Then $z \in \Omega_\delta^\beta(u)$. Given that $u \in C_{H_\beta^\infty}(AT_p^q(\alpha) \cap H_\beta^\infty)$, by Theorem A, it follows that

$$\begin{aligned}
&\int_{\mathbb{T}} \left(\int_{\Gamma(\eta)\cap\{|\varphi_\beta^{u,n}(z)|\ge\epsilon\}} (1-|z|^2)^{\alpha-p\beta} dA(z) \right)^{\frac{q}{p}} |d\eta| \\
&\lesssim \int_{\mathbb{T}} \left(\int_{\Gamma(\eta)\cap\Omega_\delta^\beta(u)} (1-|z|^2)^{\alpha-p\beta} dA(z) \right)^{\frac{q}{p}} |d\eta| < \infty,
\end{aligned}$$

which implies that $D^n_{\varphi,u} : H^\infty_\beta \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ is bounded by Theorem 3.1. In addition, we have that $D^n_{\varphi,u} : H^\infty_\beta \to H^\infty_\beta$ is compact. Combining these two results, the compactness of the operator $D^n_{\varphi,u} : H^\infty_\beta \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ follows.

$(iv) \Rightarrow (v)$. Given the inclusion

$$C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta) \subseteq H^\infty_\beta,$$

by the assumption it follows that $D^n_{\varphi,u} : C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta) \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ is compact.

$(v) \Rightarrow (vi)$. Suppose $D^n_{\varphi,u} : C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta) \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ is compact. Since $H^\infty_{\beta,0}$ is the closure of all polynomials in H^∞_β and the space $AT^q_p(\alpha)$ contains all polynomials, we obtain that $D^n_{\varphi,u} : H^\infty_{\beta,0} \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ is compact.

$(vi) \Rightarrow (ii)$. Assume that the operator $D^n_{\varphi,u} : H^\infty_{\beta,0} \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ is compact. Then $D^n_{\varphi,u} : H^\infty_{\beta,0} \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ is bounded. Since $C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta) \subseteq H^\infty_\beta$, it follows that $D^n_{\varphi,u}$ is compact from $H^\infty_{\beta,0}$ to H^∞_β. The proof is complete.

□

Remark 3.4. *From the above conclusions, we see that the boundedness of $D^n_{\varphi,u} : C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta) \to C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ has not been resolved. At present, we have no idea to solve this problem. This is left as an open problem for interested readers.*

Open Problem. How to characterize those generalized weighted composition operators that are bounded on $C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$?

Remark 3.5. *From the above results and the identity $AT^p_p(\alpha) = A^p_{\alpha+1}$, we directly derive the corresponding results for $C_{H^\infty_\beta}(A^p_\alpha \cap H^\infty_\beta)$. To maintain conciseness, we omit the details.*

4. Conclusions

This paper focuses on exploring the boundedness, compactness, and essential norm of the generalized weighted composition operator $D^n_{\varphi,u}$ from the Korenblum space H^∞_β into the analytic tent space $AT^p_q(\alpha)$. Furthermore, the paper also analyzes the boundedness and compactness of the operator $D^n_{\varphi,u}$ from H^∞_β into the space $C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$, the closure of the analytic tent space within the Korenblum space. We also pointed out that the boundedness of $D^n_{\varphi,u}$ on the space $C_{H^\infty_\beta}(AT^q_p(\alpha) \cap H^\infty_\beta)$ itself remains open.

Author contributions

Xiangling Zhu, Rong Yang and Songxiao Li: Contributed to the conceptualization, visualization, resources, writing—review and editing, formal analysis, project administration, validation, and investigation; Xiangling Zhu: Secured funding. All authors declare that they have contributed equally to this paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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