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Research article

Propagation dynamics of an influenza transmission model with nonlocal dispersal

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Abstract: This paper investigated the existence and nonexistence of traveling wave solutions for a nonlocal dispersal influenza transmission model with human mobility. We established the existence of nonnegative entire solutions by combining the upper-lower solution method with Schauder's fixed point theorem. Appropriate Lyapunov functionals were constructed to determine the asymptotic behavior of solutions at $+\infty$. Due to the influence of the nonlocal dispersal operator, the asymptotic behavior at $-\infty$ for the critical wave speed could not be directly established via the Hartman-Grobman theorem. Through careful analysis of the wave equation, we overcame this difficulty and established the desired asymptotic properties. Finally, we used numerical simulations to verify the existence of the traveling wave solution, and compared the effects of the nonlocal dispersal pattern and the local dispersal pattern on the wave speed.

Keywords: influenza transmission models; nonlocal dispersal; traveling wave solutions; existence; fixed point theorem
Mathematics Subject Classification: 35K57, 35R20, 92D25

1. Introduction

Contact behavior is a critical determinant in influenza transmission dynamics. As infectious diseases propagate, human populations naturally adapt by modifying their mobility patterns to mitigate infection risks (e.g., Balinska and Rizzo [3]). In 2012, Wang [21] introduced a novel mathematical framework that incorporates adaptive mobility behaviors into epidemic modeling. This framework is described by the following system of ordinary differential equations:

$$\begin{cases} \frac{dS}{dt} = \Lambda - \mu S - \frac{\beta m S I}{1+hI}, \\ \frac{dI}{dt} = \frac{\beta m S I}{1+hI} - (\mu + \gamma) I, \\ \frac{dm}{dt} = m\left(b - am - \frac{\alpha I}{1+hI}\right), \end{cases} \tag{1.1}$$

where S and I represent the densities of susceptible and infected individuals at time t; respectively; m denotes the intensity of population mobility at time t, and parameters $\Lambda, \mu, \beta, m, h, \gamma, a, b$ are all positive constants. According to [21], (1.1) admits three equilibria: a semi-trivial equilibrium $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0\right)$ (corresponding to the absence of population movement), a disease-free equilibrium $E_1 = \left(\frac{\Lambda}{\mu}, 0, \frac{b}{a}\right)$, and an endemic equilibrium $E^* = (S^*, I^*, m^*)$, where

$$S^* = \frac{a(\mu+\gamma)(1+hI^*)^2}{\beta\,(b + (bh-\alpha)I^*)}, \qquad m^* = \frac{b + (bh-\alpha)I^*}{a(1+hI^*)},$$

and I^* satisfies

$$(\mu+\gamma)(a\mu h^2 + b\beta h - \alpha\beta)I^{*2} + ((\mu+\gamma)(2ah\mu + b\beta) - \beta(bh-\alpha))\,I^* \\ + a\mu\gamma - \Lambda\beta b + a\mu^2 = 0.$$

The basic reproduction number of (1.1) is

$$R_0 = \frac{\Lambda\beta b}{\mu a(\mu+\gamma)},$$

which plays a critical role in influenza transmission dynamics. Wang [21] proved that E_1 is globally asymptotically stable when $R_0 < 1$, while the unique positive equilibrium E^* is asymptotically stable when $R_0 > 1$. Subsequently, Cai and Wang [4] proved that E^* is globally asymptotically stable for $R_0 > 1$ by constructing an appropriate Lyapunov functional.

When modeling the influence of population spatial dispersal on disease transmission, researchers typically employ diffusion equations—such as reaction-diffusion and discrete diffusion models—to characterize epidemic dynamics (e.g., [1, 14, 16, 20, 22, 24, 28–30]). It is now established that nonlocal dispersal operators provide a more effective framework for capturing long-range species dispersal, including human mobility (Andreu-Vaillo et al. [2]; Fife [6]). Consequently, nonlocal dispersal models have attracted growing research interest, with significant advances documented in [5, 10, 27] and related references.

Traveling wave solutions represent a distinct class of entire solutions characterized by constant propagation speed and invariant profiles during propagation. This concept originates from the seminal works of Fisher [7] and Kolmogorov et al. [11] on reaction-diffusion equations. In particular, the study of traveling wave solutions in nonlocal dispersal systems has stimulated significant research in recent years, one can see [5, 26, 27, 32].

Motivated by [2, 4, 16, 21], we propose the following mobility-dependent epidemic model:

$$\begin{cases} \partial_t S = d_1\,(J * S - S) + \Lambda - \mu S - \dfrac{\beta m S I}{1+hI}, \\ \partial_t I = d_2\,(J * I - I) + \dfrac{\beta m S I}{1+hI} - (\mu+\gamma)\,I, \\ \partial_t m = d_3\,(J * m - m) + m\left(b - am - \dfrac{\alpha I}{1+hI}\right), \end{cases} \tag{1.2}$$

where $S(t,x)$ and $I(t,x)$ denote the densities of susceptible and infected individuals at time $t > 0$ and location $x \in \mathbb{R}$, respectively, and $m(t,x)$ represents the population movement intensity, defined as the time spent in public places per unit time. The positive constants d_1, d_2, d_3 are dispersal coefficients. The nonlocal dispersal operator $J * \cdot - \cdot$ is given by

$$(J * w - w)(t,x) = \int_{\mathbb{R}} J(x-y)w(t,y)dy - w(t,x),$$

where the kernel function $J(\cdot)$ satisfies:

(J) $J \in C^1(\mathbb{R})$, $J(x) = J(-x) \geq 0$, $\int_{\mathbb{R}} J(y)\,dy = 1$, and J satisfies the decay bounds:

$$\int_{\mathbb{R}} J(x)e^{\lambda x}dx < \infty \text{ for any } \lambda \in (0,\infty) \text{ and } \int_{\mathbb{R}} |J'(x)|dx < \infty.$$

It is well-known that the combination of upper-lower solutions with Schauder's fixed point theorem is an established method for proving the existence of traveling wave solutions in diffusive systems. This approach has been successfully applied to reaction-diffusion systems [1, 14, 22, 24], discrete systems [18, 29], and nonlocal dispersal systems [5, 9, 15, 27, 31, 32]. To establish the existence of traveling wave solutions for system (1.2), we implement the following procedure: (i) Auxiliary truncated problem: Construct an auxiliary truncated system and prove the existence of nonnegative solutions using upper-lower solutions coupled with Schauder's fixed point theorem. (ii) Asymptotic behavior at $+\infty$: Establish the boundedness of the solution and apply Lyapunov functionals [12, 29, 32] to demonstrate that solutions converge to the disease-free equilibrium E_1 and endemic equilibrium E^* as $\xi \to +\infty$. (iii) Asymptotic behavior at $-\infty$: Due to the nonhomogeneous nature of the dispersal operator, the Hartman-Grobman theorem [17] cannot be directly applied to determine behavior as $\xi \to -\infty$ at critical wave speed. We overcome this by performing a delicate analysis of the wave equation's specific form, rigorously establishing the asymptotic behavior when $c = c_*$. (iv) Nonexistence: Using asymptotic propagation theory [13], we rigorously demonstrate the nonexistence of traveling wave solutions for $R_0 > 1$ with $0 < c < c_*$, and for $R_0 < 1$ with $c > 0$.

This study significantly advances spatial epidemiology through three key innovations: (1) Developing a mobility-dependent influenza model with nonlocal dispersal that accurately captures modern transmission dynamics—particularly human behavioral adaptations like contact reduction during outbreaks—where classical diffusion models fail due to their fixed-contact assumption. (2) Providing a complete theoretical characterization of traveling waves, including rigorous existence proofs for epidemic waves with speeds $c \geq c_*$ and establishing critical nonexistence results for $0 < c < c_*, R_0 > 1$ and $c > 0, R_0 < 1$. (3)When the kernel function is Gaussian, we derive the relationship between the minimum wave speed and the decay rate of the kernel function, accompanied by numerical verification. These advances bridge theory-practice gaps in epidemic modeling, while our extensible framework provides a transferable toolkit for analyzing wave propagation in spatially structured populations across diseases.

The rest of this paper is organized as follows. In Section 2, we give the preliminaries and main results of this paper. In Section 3, we prove the existence and nonexistence of traveling wave solutions of (1.2). In Section 4, we perform numerical simulations to confirm the existence of traveling wave solutions and to investigate how nonlocal and local dispersal patterns influence the wave speed. Concluding perspectives appear in Section 5.

2. Preliminaries and main results

In this section, the preliminaries and main results are given. Let

$$S(t,x) = S(\xi),\ I(t,x) = I(\xi),\ m(t,x) = m(\xi)$$

with $\xi = x + ct$, then the wave equation corresponding to system (1.2) is as follows:

$$\begin{cases} cS'(\xi) = d_1 (J * S(\xi) - S(\xi)) + \Lambda - \mu S(\xi) - \frac{\beta m(\xi) S(\xi) I(\xi)}{1+hI(\xi)}, \\ cI'(\xi) = d_2 (J * I(\xi) - I(\xi)) + \frac{\beta m(\xi) S(\xi) I(\xi)}{1+hI(\xi)} - (\mu+\gamma) I(\xi), \\ cm'(\xi) = d_3 (J * m(\xi) - m(\xi)) + m(\xi)\left(b - am(\xi) - \frac{\alpha I(\xi)}{1+hI(\xi)}\right). \end{cases} \tag{2.1}$$

We intend to find solutions $(S(\xi), I(\xi), m(\xi))$ of (2.1), which are nonnegative and satisfy the following boundary conditions:

$$(S(-\infty), I(-\infty), m(-\infty)) = \left(\frac{\Lambda}{\mu}, 0, \frac{b}{a}\right), \tag{2.2}$$

and

$$(S(+\infty), I(+\infty), m(+\infty)) = (S^*, I^*, m^*). \tag{2.3}$$

The linearized characteristic equation of I at the disease-free equilibrium E_1 is

$$\Delta(\lambda, c) := d_2 \int_{\mathbb{R}} J(y)(e^{-\lambda y} - 1)dy - c\lambda + \frac{\beta \Lambda b}{\mu a} - (\mu+\gamma) = 0.$$

By a similar analysis in [5, 12], we can get the following results.

Lemma 2.1. *Assume* $R_0 = \frac{\beta\Lambda b}{\mu a(\mu+\gamma)} > 1$. *Then, there exists a positive pair of* (λ_*, c_*) *such that*

$$\Delta(\lambda_*, c_*) = 0, \quad \frac{\partial \Delta(\lambda_*, c_*)}{\partial \lambda} = 0,$$

where c_* *can be defined as*

$$c_* = \inf_{\lambda>0} \left\{ \frac{1}{\lambda} \left[d_2 \left(\int_{\mathbb{R}} J(x) e^{-\lambda x} dx - 1 \right) + \frac{\beta \Lambda b}{\mu a} - (\mu+\gamma) \right] \right\}.$$

Furthermore,

(i) if $c > c_*$, *the equation* $\Delta(\lambda, c) = 0$ *has two positive roots* $\lambda_1 = \lambda_1(c) < \lambda_2(c) = \lambda_2$, *and*

$$\Delta(\lambda, c) \begin{cases} < 0, \ \lambda \in (\lambda_1, \lambda_2), \\ > 0, \ \lambda \in (0, \lambda_1) \cup (\lambda_2, +\infty); \end{cases}$$

(ii) if $0 < c < c_*$, *we have* $\Delta(\lambda, c) > 0$ *for any* $\lambda > 0$.

Subsequently, we present the main results of this paper.

Theorem 2.1. *Suppose **(J)** holds.*

(i) If $R_0 > 1$, then for any $c \geq c_*$, the system (1.2) admits a positive and bounded traveling wave solution $(S(\xi), I(\xi), m(\xi))$ *satisfying* (2.2) *and* (2.3).

(ii) If $R_0 > 1, 0 < c < c_*$ or $R_0 < 1, c > 0$, there exist no nontrivial traveling wave solutions $(S(\xi), I(\xi), m(\xi))$ *of system (1.2) satisfying* (2.2) *and* (2.3).

Remark 2.1. (i) Note that in (1.2), we assume that S, I, and m share the same dispersal kernel function $J(\cdot)$ merely for convenience, while in reality they can be assigned distinct kernel functions J_1, J_2, and J_3 respectively, provided each satisfies condition **(J)**.

(ii) For the critical wave speed case, the existence of traveling wave solutions can also be proven by constructing upper-lower solutions (see [5]), but this requires the kernel function to have compact support.

Next, taking the Gaussian kernel as an example, we consider the influence of the kernel function's decay rate on the minimal wave speed.

Theorem 2.2. *Let $J(x) = fe^{-lx^2}$ with $f = \sqrt{l/\pi}$. The minimal wave speed c_* satisfies:*

$$c_* = \frac{d_2(e^{z^*} - 1) + \frac{\beta\Lambda b}{\mu a} - (\mu + \gamma)}{2\sqrt{lz^*}},$$

where z^* is the unique solution to the equation:

$$e^z(2z - 1) = \frac{\frac{\beta\Lambda b}{\mu a} - (\mu + \gamma) - d_2}{d_2}, \ z \in (0, \infty).$$

3. Proof of the main results

In this section, we first investigate the existence of traveling wave solutions by employing the method of upper-lower solutions combined with Schauder's fixed-point theorem. Then, by the asymptotic propagation theory, we prove the nonexistence of traveling wave solutions.

3.1. The case: $R_0 > 1$ and $c > c_*$

This section focuses on the supercritical wave speed case, where we begin by constructing appropriate upper-lower solutions.

Lemma 3.1. $S_+(\xi) \equiv \frac{\Lambda}{\mu}$ *satisfies*

$$cS'_+(\xi) = d_1 (J * S_+(\xi) - S_+(\xi)) + \Lambda - \mu S_+(\xi).$$

Lemma 3.2. *The function* $I_+(\xi) = \min\left\{e^{\lambda_1 \xi}, \frac{1}{h}\left(\frac{\beta\Lambda b}{\mu a(\mu+\gamma)} - 1\right)\right\}$ *satisfies*

$$cI'_+(\xi) \geq d_2 (J * I_+(\xi) - I_+(\xi)) + \frac{\beta\Lambda b}{\mu a}\frac{I_+(\xi)}{1 + hI_+(\xi)} - (\mu + \gamma)I_+(\xi), \tag{3.1}$$

where $\xi \neq \frac{1}{\lambda_1}\ln\left[\frac{1}{h}\left(\frac{\beta\Lambda b}{\mu a(\mu+\gamma)} - 1\right)\right] := \xi_1$ *and* $I'_+(\xi_1+) \leq I'_+(\xi_1-)$.

Proof. When $\xi < \xi_1$, we have $I_+(\xi) = e^{\lambda_1 \xi}$. By the definition of λ_1, it follows that

$$\left[d_2 \int_{\mathbb{R}} J(\xi - y)e^{-\lambda_1(\xi - y)}dy - d_2 - c\lambda_1 + \frac{\beta\Lambda b}{\mu a} - (\mu + \gamma)\right]e^{\lambda_1\xi} = 0,$$

which implies

$$cI'_+(\xi) = d_2 \int_{\mathbb{R}} J(\xi - y)e^{\lambda_1 y}dy - d_2 I_+(\xi) + \frac{\beta\Lambda b}{\mu a} I_+(\xi) - (\mu + \gamma)I_+(\xi).$$

Moreover, this leads to the inequality:

$$cI'_+(\xi) \geq d_2 \left(J * I_+(\xi) - I_+(\xi)\right) + \frac{\beta\Lambda b}{\mu a}\frac{I_+(\xi)}{1 + hI_+(\xi)} - (\mu + \gamma)I_+(\xi).$$

For $\xi > \xi_1$, $I_+(\xi) = \frac{1}{h}\left(\frac{\beta\Lambda b}{\mu a(\mu+\gamma)} - 1\right)$ is a constant. Substituting this into the expression yields:

$$\frac{\beta\Lambda b}{\mu a}\frac{I_+(\xi)}{1 + hI_+(\xi)} - (\mu + \gamma)I_+(\xi) = 0.$$

Thus, we obtain

$$cI'_+(\xi) = \frac{\beta\Lambda b}{\mu a}\frac{I_+(\xi)}{1 + hI_+(\xi)} - (\mu + \gamma)I_+(\xi).$$

This implies that

$$cI'_+(\xi) \geq d_2 \left(J * I_+(\xi) - I_+(\xi)\right) + \frac{\beta\Lambda b}{\mu a}\frac{I_+(\xi)}{1 + hI_+(\xi)} - (\mu + \gamma)I_+(\xi).$$

This completes the proof. □

Lemma 3.3. $m_+(\xi) \equiv \frac{b}{a}$ *satisfies*

$$cm'_+(\xi) = d_3 \left(J * m_+(\xi) - m_+(\xi)\right) + m_+(\xi)\left(b - am_+(\xi)\right).$$

Lemma 3.4. *There exist constants $\theta > 0$ and $\rho > 1$ such that the function*

$$S_-(\xi) = \max\left\{\frac{\Lambda}{\mu} - \rho e^{\theta\xi}, \frac{\Lambda ah}{ah\mu + b\beta}\right\}$$

satisfies the inequality

$$cS'_-(\xi) \leq d_1 \left(J * S_-(\xi) - S_-(\xi)\right) + \Lambda - \mu S_-(\xi) - \frac{\beta m_+(\xi)S_-(\xi)I_+(\xi)}{1 + hI_+(\xi)}, \tag{3.2}$$

for all $\xi \neq \frac{1}{\theta}\ln\left(\frac{\Lambda\beta b}{\rho\mu(ah\mu+b\beta)}\right) := \xi_2$, with the condition $S'_-(\xi_2+) \geq S'_-(\xi_2-)$.

Proof. Select $\theta \in (0, \lambda_1)$ satisfying

$$d_1 \int_{\mathbb{R}} J(y)e^{-\theta y}dy - d_1 + c\theta + \mu > 0.$$

Fix this θ and choose sufficiently large $\rho_1 > \frac{\Lambda}{\mu}$ such that $\frac{\Lambda}{\mu} \leq \rho e^{\theta\xi_1}$. Define

$$\rho = \frac{\Lambda\beta b}{\mu a(d_1 \int_{\mathbb{R}} J(y)e^{-\theta y}dy - d_1 + c\theta + \mu)} + \rho_1 + 1,$$

which ensures $\xi_2 < \xi_1$.

For $\xi > \xi_2$, the function $S_-(\xi) = \frac{\Lambda ah}{ah\mu+b\beta}$ is constant. It suffices to confirm that

$$cS'_-(\xi) \leq d_1 \left(J * S_-(\xi) - S_-(\xi)\right) + \Lambda - \mu S_-(\xi) - \frac{\beta b}{ah} S_-(\xi).$$

Since

$$\Lambda - \mu \cdot \frac{\Lambda ah}{ah\mu + b\beta} - \frac{\beta b}{ah} \cdot \frac{\Lambda ah}{ah\mu + b\beta} = 0,$$

the inequality holds, thereby verifying (3.2).

For $\xi < \xi_2$, we have $S_-(\xi) = \frac{\Lambda}{\mu} - \rho e^{\theta\xi}$ and $I_+(\xi) = e^{\lambda_1 \xi}$. Direct computation yields:

$$\begin{aligned}
& d_1 \left(J * S_-(\xi) - S_-(\xi)\right) - cS'_-(\xi) + \Lambda - \mu S_-(\xi) - \frac{\beta m_+(\xi) S_-(\xi) I_+(\xi)}{1 + hI_+(\xi)} \\
=& (d_1 \int_{\mathbb{R}} J(y)e^{-\theta y}dy - d_1 + c\theta + \mu)\rho e^{\theta\xi} - \frac{\beta b}{a}\left(\frac{\Lambda}{\mu} - \rho e^{\theta\xi}\right)\frac{e^{\lambda_1\xi}}{1 + he^{\lambda_1\xi}} \\
>& (d_1 \int_{\mathbb{R}} J(y)e^{-\theta y}dy - d_1 + c\theta + \mu)\rho e^{\theta\xi} - \frac{\beta\Lambda b}{\mu a} e^{\lambda_1\xi} \\
>& (d_1 \int_{\mathbb{R}} J(y)e^{-\theta y}dy - d_1 + c\theta + \mu)\frac{\Lambda\beta b}{\mu a(d_1 \int_{\mathbb{R}} J(y)e^{-\theta y}dy - d_1 + c\theta + \mu)} e^{\theta\xi} - \frac{\beta\Lambda b}{\mu a} e^{\theta\xi} \\
=& 0.
\end{aligned}$$

This confirms that (3.2) is satisfied. □

Lemma 3.5. *There exist constants* $\tilde{\theta} > 0$ *and* $\tilde{\rho} > 1$ *such that the function*

$$m_-(\xi) = \max\left\{\frac{b}{a} - \tilde{\rho}e^{\tilde{\theta}\xi}, 0\right\}$$

satisfies the inequality

$$cm'_-(\xi) \leq d_3 \int_{\mathbb{R}} J(\xi - y)m_-(y)dy - d_3 m_-(\xi) + m_-(\xi)\left(b - am_-(\xi) - \frac{\alpha I_+(\xi)}{1 + hI_+(\xi)}\right), \tag{3.3}$$

for all $\xi \neq \frac{1}{\tilde{\theta}} \ln \frac{b}{\tilde{\rho} a} := \xi_3$*, with the condition* $m'_-(\xi_3+) \geq m'_-(\xi_3-)$.

Proof. Select $\tilde{\theta} \in (0, \lambda_1)$ and $\tilde{\rho} > \tilde{\rho}_1 > \frac{b}{a}$ such that $\frac{b}{a} \leq \tilde{\rho}_1 e^{\tilde{\theta}\xi_1}$ and the inequality

$$(d_3 \int_{\mathbb{R}} J(y)e^{-\theta y}dy - d_3 + c\tilde{\theta} + b)\tilde{\rho} - \frac{b\alpha}{a} - a\tilde{\rho}^2 > 0$$

holds. This guarantees $\xi_3 < \xi_1$.

For $\xi > \xi_3$, the function $m_-(\xi) = 0$ is constant, so (3.3) is trivially satisfied.

For $\xi < \xi_3$, we have $m_-(\xi) = \frac{b}{a} - \tilde{\rho}e^{\tilde{\theta}\xi}$ and $I_+(\xi) = e^{\lambda_1\xi}$. Substituting into the left-hand side of (3.3) yields:

$$\begin{aligned}
&d_3\int_{\mathbb{R}} J(\xi-y)m_-(y)dy - d_3m_-(\xi) - cm'_-(\xi) + m_-(\xi)\left(b - am_-(\xi) - \frac{\alpha I_+(\xi)}{1+hI_+(\xi)}\right)\\
>&(d_3\int_{\mathbb{R}} J(y)e^{-\theta y}dy - d_3 + c\tilde{\theta} + b)\tilde{\rho}e^{\tilde{\theta}\xi} - \frac{b\alpha}{a}\frac{e^{\lambda_1\xi}}{1+he^{\lambda_1\xi}} - a\tilde{\rho}^2e^{2\tilde{\theta}\xi} + \frac{\alpha\tilde{\rho}e^{(\lambda_1+\tilde{\theta})\xi}}{1+he^{\lambda_1\xi}}\\
>&\left[(d_3\int_{\mathbb{R}} J(y)e^{-\theta y}dy - d_3 + c\tilde{\theta} + b)\tilde{\rho} - \frac{b\alpha}{a} - a\tilde{\rho}^2\right]e^{\tilde{\theta}\xi}\\
>&0.
\end{aligned}$$

This confirms that (3.3) holds. □

Lemma 3.6. *Suppose* $0 < \eta < \min\{\lambda_2 - \lambda_1, \theta, \lambda_1, \tilde{\theta}\}$. *There exists a constant* $M > 1$ *such that the function*

$$I_-(\xi) = \max\left\{e^{\lambda_1\xi}(1 - Me^{\eta\xi}), 0\right\}$$

satisfies the inequality

$$cI'_-(\xi) \le d_2I''_-(\xi) + \frac{\beta S_-(\xi)m_-(\xi)I_-(\xi)}{1+hI_-(\xi)} - (\mu+\gamma)I_-(\xi), \tag{3.4}$$

for all $\xi \neq \frac{1}{\eta}\ln\frac{1}{M} := \xi_4$, *with the condition* $I'_-(\xi_4+) \ge I'_-(\xi_4-)$.

Proof. Choose M_1 such that $\frac{1}{\eta}\ln\frac{1}{M_1} + 1 = \xi_2$, set $M \ge M_1$, and satisfy $I_-(\xi) < \frac{1}{h}$. For $\xi > \xi_4$, the function $I_-(\xi) = 0$ is constant, so (3.4) is trivially satisfied.

For $\xi < \xi_4 \le \frac{1}{\eta}\ln\frac{1}{M_1} + 1 = \xi_2$, we have $I_-(\xi) = e^{\lambda_1\xi}(1 - Me^{\eta\xi})$ and $S_-(\xi) = \frac{\Lambda}{\mu} - \rho e^{\theta\xi}$. Using the inequality $1 - hI \le \frac{1}{1+hI}$ for $|I| < \frac{1}{h}$ and the relation

$$\int_{\mathbb{R}} J(\xi-y)e^{\lambda_1 y}(1 - Me^{\eta y})dy \le \int_{\mathbb{R}} J(\xi-y)I_-(y)dy,$$

it suffices to verify:

$$\begin{aligned}
&c\left[\lambda_1e^{\lambda_1\xi} - M(\lambda_1+\eta)e^{(\lambda_1+\eta)\xi}\right]\\
\le& d_2\int_{\mathbb{R}} J(\xi-y)(e^{\lambda_1 y} - e^{\lambda_1\xi})dy + Md_2\int_{\mathbb{R}} J(\xi-y)\left[(e^{(\lambda_1+\eta)\xi} - e^{(\lambda_1+\eta)y}\right]dy\\
&- (\mu+\gamma)e^{\lambda_1\xi} + M(\mu+\gamma)e^{(\lambda_1+\eta)\xi} + \beta\left(\frac{b}{a} - \tilde{\rho}e^{\tilde{\theta}\xi}\right)\left(\frac{\Lambda}{\mu} - \rho e^{\theta\xi}\right)e^{\lambda_1\xi}\left(1 - Me^{\eta\xi}\right)\left(1 - he^{\lambda_1\xi}\left(1 - Me^{\eta\xi}\right)\right).
\end{aligned}$$

It follows from $\Delta(\lambda_1, c) = 0$ that we need to prove

$$
\begin{aligned}
&- M\Delta(\lambda_1 + \eta, c)e^{\eta\xi} + \beta\bigg[- \frac{b\Lambda}{a\mu}he^{\lambda_1\xi} + \frac{b\Lambda}{a\mu}Mhe^{(\eta+\lambda_1)\xi} \\
&- \frac{b\rho}{a}e^{\theta\xi} + \frac{b\rho}{a}Me^{(\theta+\eta)\xi} + \frac{b\rho}{a}he^{(\theta+\lambda_1)\xi} - \frac{b\rho}{a}Mhe^{(\theta+\eta+\lambda_1)\xi} \\
&- \frac{\tilde{\rho}\Lambda}{\mu}e^{\tilde{\theta}\xi} + \frac{\tilde{\rho}\Lambda}{\mu}Me^{(\tilde{\theta}+\eta)\xi} + \frac{\tilde{\rho}\Lambda}{\mu}he^{(\tilde{\theta}+\lambda_1)\xi} - \frac{\tilde{\rho}\Lambda}{\mu}Mhe^{(\tilde{\theta}+\eta+\lambda_1)\xi} \\
&+ \tilde{\rho}\rho e^{(\tilde{\theta}+\theta)\xi} - \tilde{\rho}\rho Me^{(\tilde{\theta}+\theta+\eta)\xi} - \tilde{\rho}\rho he^{(\tilde{\theta}+\theta+\lambda_1)\xi} + \tilde{\rho}\rho Mhe^{(\tilde{\theta}+\theta+\eta+\lambda_1)\xi}\bigg] \geq 0.
\end{aligned}
$$

Since $\theta < \min\{\lambda_1, \theta, \tilde{\theta}\}$ and $\Delta(\lambda_1 + \eta, c) < 0$, we can choose sufficiently large M to ensure the inequality holds for all $\xi < \xi_4$. □

For $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)) \in C(\mathbb{R}, \mathbb{R}^3)$, define the Banach space as follows

$$
B_\mu(\mathbb{R}, \mathbb{R}^3) = \left\{\Phi(\cdot) \in C(\mathbb{R}, \mathbb{R}^3) : |\Phi(\cdot)|_\mu < +\infty\right\},
$$

where the norm is defined as

$$
|\Phi(\xi)|_\mu = \max\left\{\sup_{\xi\in\mathbb{R}}\{|\phi_1(\xi)|e^{-\mu|\xi|}\}, \sup_{\xi\in\mathbb{R}}\{|\phi_2(\xi)|e^{-\mu|\xi|}\}, \sup_{\xi\in\mathbb{R}}\{|\phi_3(\xi)|e^{-\mu|\xi|}\}\right\},
$$

with $\mu > 0$ small enough. Define the set

$$
\Gamma = \left\{(S, I, m) \in B_\mu(\mathbb{R}, \mathbb{R}^3) \mid (S_-, I_-, m_-) \leq (S, I, m) \leq (S_+, I_+, m_+)\right\}.
$$

For $(S, I, m) \in \Gamma$ and $r > 0$, define

$$
\begin{aligned}
H_1(S, I, m)(\xi) &= d_1J * S(\xi) - d_1S(\xi) + rS(\xi) + \Lambda - \mu S(\xi) - \frac{\beta S(\xi)m(\xi)I(\xi)}{1 + hI(\xi)}, \\
H_2(S, I, m)(\xi) &= d_2J * I(\xi) - d_2I(\xi) + rI(\xi) + \frac{\beta S(\xi)m(\xi)I(\xi)}{1 + hI(\xi)} - (\mu + \gamma)I(\xi) - d_2I(\xi), \\
H_3(I, m)(\xi) &= d_3J * m(\xi) - d_3m(\xi) + rm(\xi) + m(\xi)\left(b - am(\xi) - \frac{\alpha I(\xi)}{1 + hI(\xi)}\right) - d_3m(\xi).
\end{aligned} \tag{3.5}
$$

It is easy to see that H_1 is decreasing in I and m, H_2 is increasing in S and m, and H_3 is decreasing in I. Choose $r > 0$ such that H_1 is increasing in $S > 0$, H_2 is increasing in $I > 0$, and H_3 is increasing in $m > 0$.

Define the operator $F = (F_1, F_2, F_3) : \Gamma \to C(\mathbb{R}, \mathbb{R}^3)$ as

$$
\begin{aligned}
F_1(S, I, m)(\xi) &= \frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)}H_1(S, I, m)(\eta)d\eta, \\
F_2(S, I, m)(\xi) &= \frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)}H_2(S, I, m)(\eta)d\eta, \\
F_3(I, m)(\xi) &= \frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)}H_3(I, m)(\eta)d\eta,
\end{aligned}
$$

Lemma 3.7. $F : \Gamma \to \Gamma$.

Proof. For any $(S, I, m) \in \Gamma$, one needs to prove that

$$S_-(\xi) \le F_1(S, I, m)(\xi) \le \frac{\Lambda}{\mu}, I_-(\xi) \le F_2(S, I, m)(\xi) \le I_+(\xi),$$

and

$$m_-(\xi) \le F_3(S, I, m)(\xi) \le \frac{b}{a}$$

for any $\xi \in \mathbb{R}$.

For $F_1(S, I, m)(\xi)$, it follows from the monotonicity of H_1 that

$$F_1(S_-, I_+, m_+)(\xi) \le F_1(S, I, m)(\xi) \le F_1(S_+, I_-, m_-)(\xi), \quad \xi \in \mathbb{R}.$$

Thus, it suffices to verify that

$$S_-(\xi) \le F_1(S_-, I_+, m_+)(\xi) \le F_1(S_+, I_-, m_-)(\xi) \le \frac{\Lambda}{\mu}, \quad \xi \in \mathbb{R}. \tag{3.6}$$

It follows from Lemma 3.4 that

$$\begin{aligned}
&F_1(S_-, I_+, m_+)(\xi)\\
=&\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)} H_1(S_-, I_+, m_+)(\eta)d\eta\\
\ge&\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)} (cS'_-(\eta) + rS_-(\eta))d\eta\\
=&S_-(\xi)
\end{aligned}$$

for any ξ. According to Lemma 3.1, there holds

$$\begin{aligned}
&F_1(S_+, I_-, m_-)(\xi)\\
=&\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)} H_1(S_+, I_-, m_-)(\eta)d\eta\\
\le&\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)} (cS'_+(\eta) + rS_+(\eta))d\eta\\
=&S_+(\xi).
\end{aligned}$$

for any ξ. Thus, (3.6) holds.

For $F_2(S, I, m)(\xi)$, by the monotonicity of H_2,we need to get that

$$I_-(\xi) \le F_2(S_-, I_-, m_-)(\xi) \le F_2(S_+, I_+, m_+)(\xi) \le I_+(\xi), \xi \in \mathbb{R}. \tag{3.7}$$

By Lemma 3.6, there holds

$$F_2(S_-, I_-, m_-)(\xi)$$

$$
\begin{aligned}
=&\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)} H_2(S_-, I_-, m_-)(\eta) d\eta \\
\geq&\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)} (cI_-'(\eta) + rI_-(\eta)) d\eta \\
=&I_-(\xi)
\end{aligned}
$$

for any ξ. In view of Lemma 3.2, one has

$$
\begin{aligned}
&F_2(S_+, I_+, m_+)(\xi) \\
=&\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)} F_2(S_+, I_+, m_+)(\eta) d\eta \\
\leq&\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)} (cI_+'(\eta) + rI_+(\eta)) d\eta \\
=&I_+(\xi)
\end{aligned}
$$

for any ξ. Thus, (3.7) holds.

For $F_3(S, I, m)(\xi)$, by the monotonicity of H_3, one needs to prove that

$$
m_-(\xi) \leq F_3(I_+, m_-)(\xi) \leq F_3(I_-, m_+)(\xi) \leq \frac{b}{a}, \xi \in \mathbb{R}. \tag{3.8}
$$

It follows from Lemma 3.5 that

$$
\begin{aligned}
&F_3(I_+, m_-)(\xi) \\
=&\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)} H_3(I_+, m_-)(\eta) d\eta \\
\geq&\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)} (cm_-'(\eta) + rm_-(\eta)) d\eta \\
=&m_-(\xi),
\end{aligned}
$$

for any ξ. By Lemma 3.3, one has

$$
\begin{aligned}
&F_3(I_-, m_+)(\xi) \\
=&\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)} H_3(I_-, m_+)(\eta) d\eta \\
\leq&\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)} (cm_+'(\eta) + rm_+(\eta)) d\eta \\
=&m_+(\xi)
\end{aligned}
$$

for any ξ. Hence, (3.8) holds. This completes the proof. □

Lemma 3.8. *The operator $F : \Gamma \to \Gamma$ is completely continuous in $B_\mu(\mathbb{R}, \mathbb{R}^3)$ under the norm $|\cdot|_\mu$.*

Proof. For any $\Phi_1 = (S_1, I_1, m_1)$ and $\Phi_2 = (S_2, I_2, m_2) \in \Gamma$, one has

$$\begin{aligned}&\left|\frac{\beta S_1 I_1 m_1}{1+hI_1} - \frac{\beta S_2 I_2 m_2}{1+hI_2}\right| \\ \leq &\frac{\beta\Lambda b}{\mu a}|I_1 - I_2| + \frac{2\beta b}{ha}|S_1 - S_2| + \frac{2\beta\Lambda}{h\mu}|m_1 - m_2|,\end{aligned}$$

and

$$|e^{-\mu|\xi|}\int_{\mathbb{R}} J(\xi-\eta)h(\eta)d\eta \leq \int_{\mathbb{R}} J(\xi-\eta)h(\eta)e^{-\mu|\eta|}e^{\mu|\eta+\xi|}d\eta \leq \int_{\mathbb{R}} J(\eta)e^{\mu|\eta|}d\eta|h|_\mu.$$

It then follows that

$$\begin{aligned}&e^{-\mu|\xi|}|F_1(S_1, I_1, m_1)(\xi) - F_1(S_2, I_2, m_2)(\xi)| \\ \leq&\frac{1}{c}e^{-\mu|\xi|}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)}\left[(r-\mu+d_1+\frac{2\beta b}{ha})|S_1 - S_2|(\eta) + \frac{\beta\Lambda b}{\mu a}|I_1 - I_2|(\eta) + \frac{2\beta\Lambda}{h\mu}|m_1 - m_2|(\eta)\right]d\eta \\ &+\frac{1}{c}e^{-\mu|\xi|}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)}\int_{\mathbb{R}} J(\xi-\eta)|S_1 - S_2|(\eta)d\eta\end{aligned}$$

Note that

$$\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)}e^{-\mu|\xi|}e^{\mu|\eta|}d\eta \leq \frac{c}{r-c\mu},$$

and

$$\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)}d\eta \leq \frac{c}{r}.$$

Thus,

$$\begin{aligned}&|F_1(S_1, I_1, m_1) - F_1(S_2, I_2, m_2)|_\mu \\ \leq&\frac{1}{r-c\mu}\left(r-\mu+\frac{2\beta b}{ha}+\frac{\beta\Lambda b}{\mu a}+\frac{2\Lambda\beta}{h\mu}+d_1+d_1\int_{\mathbb{R}} J(\eta)e^{\mu|\eta|}d\eta\right)|\Phi_1 - \Phi_2|_\mu \\ :=&M_1|\Phi_1 - \Phi_2|_\mu.\end{aligned}$$

Hence,

$$\sup_{\xi\in\mathbb{R}}|F_1(S_1, I_1, m_1)(\xi) - F_1(S_2, I_2, m_2)(\xi)|\,e^{-\mu|\xi|} \leq M_1|\Phi_1 - \Phi_2|_\mu.$$

Similarly, one can find M_2, M_3 such that

$$\begin{aligned}&\sup_{\xi\in\mathbb{R}}|F_2(S_1, I_1, m_1)(\xi) - F_2(S_2, I_2, m_2)(\xi)|\,e^{-\mu|\xi|} \leq M_2|\Phi_1 - \Phi_2|_\mu, \\ &\sup_{\xi\in\mathbb{R}}|F_3(I_1, m_1)(\xi) - F_3(I_2, m_2)(\xi)|\,e^{-\mu|\xi|} \leq M_3|\Phi_1 - \Phi_2|_\mu.\end{aligned}$$

Therefore, $F : \Gamma \to \Gamma$ is continuous.

In the following, we prove that F is a compact operator. It is easy to see that the derivatives $\left|\frac{d}{d\xi}F_1(S,I,m)(\xi)\right|_\mu$, $\left|\frac{d}{d\xi}F_2(S,I,m)(\xi)\right|_\mu$, and $\left|\frac{d}{d\xi}F_3(S,I,m)(\xi)\right|_\mu$ are bounded. This boundedness implies that $F(\Gamma)$ is uniformly bounded and equicontinuous with respect to the norm $|\cdot|_\mu$.

Further, for any $n \in \mathbb{N}$, define the operator $F^n = (F_1^n, F_2^n, F_3^n)$ as follows:

$$F^n(S,I,m)(\xi) = \begin{cases} F(S,I,m)(-n), & \xi \in (-\infty, -n], \\ F(S,I,m)(\xi), & \xi \in [-n, n], \\ F(S,I,m)(n), & \xi \in [n, +\infty). \end{cases}$$

It is clear that $F^n : \Gamma \to B_\mu(\mathbb{R}, \mathbb{R}^3)$ is continuous. Moreover, $F^n(\Gamma)$ is uniformly bounded and equicontinuous with respect to the norm $|\cdot|_\mu$. By the Arzelà–Ascoli theorem, $F^n : \Gamma \to B_\mu(\mathbb{R}, \mathbb{R}^3)$ is a compact operator.

Since the bound

$$|F_1(S,I,m)(\xi)| \le \frac{1}{c}e^{-\mu|\xi|}\int_{-\infty}^{\xi} e^{\frac{r}{c}(\eta-\xi)}\left[(r-\mu+2d_1+\frac{2\beta b}{ha})S(\eta)+\frac{\beta\Lambda b}{\mu a}I(\eta)+\frac{2\beta\Lambda}{h\mu}m(\eta)\right]d\eta$$

holds, we deduce that

$$|F_1(S,I,m)(\xi)| \le \frac{1}{r-c\mu}\left[(r-\mu+2d_1+\frac{2\beta b}{ha})\frac{\Lambda}{\mu}+\frac{\beta\Lambda b}{\mu a}\frac{1}{h}\left(\frac{\beta\Lambda b}{\mu a(\mu+\gamma)}-1\right)+\frac{2b\beta\Lambda}{ah\mu}\right] := M_4.$$

This leads to the estimate

$$\begin{aligned} &\left|F_1^n(S,I,m)(\xi)-F_1(S,I,m)(\xi)\right|_\mu \\ =&\sup_{\xi\in\mathbb{R}}\left|F_1^n(S,I,m)(\xi)-F_1(S,I,m)(\xi)\right|e^{-\mu|\xi|} \\ =&\sup_{\xi\in(-\infty,-n]\cup[n,+\infty)}\left|F_1^n(S,I,m)(\xi)-F_1(S,I,m)(\xi)\right|e^{-\mu|\xi|} \\ \le& 2M_4e^{-\mu n}\to 0 \quad \text{as } n\to+\infty. \end{aligned}$$

Similarly, we obtain

$$\left|F_2^n(S,I,m)(\xi)-F_2(S,I,m)(\xi)\right|_\mu \to 0 \quad \text{as } n\to+\infty,$$

$$\left|F_3^n(I,m)(\xi)-F_3(I,m)(\xi)\right|_\mu \to 0 \quad \text{as } n\to+\infty.$$

Thus, the convergence

$$|F^n(S,I,m)(\xi)-F(S,I,m)(\xi)|_\mu \to 0 \quad \text{as } n\to+\infty$$

is established. Consequently, the sequence $\{F^n\}_{n=0}^\infty$ converges to F with respect to the norm $|\cdot|_\mu$. Therefore, F is compact in $B_\mu(\mathbb{R},\mathbb{R}^3)$ under the norm $|\cdot|_\mu$. This completes the proof. □

Note that Γ is a bounded and closed convex set in $B_\mu(\mathbb{R}, \mathbb{R}^3)$, and it follows from Schauder's fixed point theorem that there exists $(S(\cdot), I(\cdot), m(\cdot)) \in \Gamma$ such that

$$(S(\xi), I(\xi), m(\xi)) = F(S, I, m)(\xi), \quad \xi \in \mathbb{R}.$$

Furthermore, (S, I, m) satisfies the system (2.1) and

$$S_-(\xi) \leq S(\xi) \leq \frac{\Lambda}{\mu}, I_-(\xi) \leq I(\xi) \leq I_+(\xi), m_-(\xi) \leq m(\xi) \leq \frac{b}{a}. \tag{3.9}$$

For any nonnegative solution (S, I, m) of the system (2.1), S, S', I, I', m, and m' are all uniformly bounded and continuously differentiable. By (3.9), it is easy to verify that (2.2) holds. First, we assert that $\frac{\Lambda ah}{ah\mu+b\beta} \leq S(\xi) < \frac{\Lambda}{\mu}$, $0 < I(\xi) \leq \frac{1}{h}\left(\frac{\beta\Lambda b}{\mu a(\mu+\gamma)} - 1\right)$, $0 < m(\xi) < \frac{b}{a}$, $\forall \xi \in \mathbb{R}$. Indeed, if some $\xi_1 \in \mathbb{R}$ exists so that $S(\xi_1) = \frac{\Lambda}{\mu}$, then $S'(\xi_1) \geq 0$. The first equation of system (2.1) gives that

$$d_1 \int_{\mathbb{R}} J(y)[S(\xi_1 - y) - S(\xi_1)]dy + \frac{\beta S(\xi_1)m(\xi_1)I(\xi_1)}{1 + hI(\xi_1)} < 0,$$

a contradiction. Thus, $S(\xi) < \frac{\Lambda}{\mu}$, $\forall \xi \in \mathbb{R}$. Similarly, we can also prove that $m(\xi) < \frac{b}{a}$, $\forall \xi \in \mathbb{R}$. Additionally, if there exists $\xi_2 \in \mathbb{R}$ such that $I(\xi_2) = 0$, then $I'(\xi_2) = 0$ and $(J * I - I)(\xi_2) \geq 0$, and the equality holds if, and only if, $I \equiv 0$. Furthermore, I has a nontrivial nonnegative continuous lower solution. Thus, $I(\xi) > 0$, $\forall \xi \in \mathbb{R}$. Similarly, $m(\xi) > 0$, $\forall \xi \in \mathbb{R}$.

Next, similar to [27], we can get the following lemma.

Lemma 3.9. *Let* $0 < c_1 \leq c_2$ *be given and* (S, I, m) *be a solution of system* (2.1) *with speed* $c \in [c_1, c_2]$. *Thus, there is* $\tau > 0$ *such that if* $I(\xi) \leq \tau$ *for* $\xi \in \mathbb{R}$, *then* $I'(\xi) > 0$.

Next, we complete the proof of Theorem 2.1 (i) for $R_0 > 1$ and $c > c_*$.

Proof of Theorem 2.1 (i) for $R_0 > 1$ *and* $c > c_*$. From the above discussion, we only need to prove that (S, I, m) satisfies the boundary condition (2.3). Below, we prove it by constructing a suitable Lyapunov functional $V(S, I, m)(\xi)$. Recall that $V(S, I, m)$ is a Liapunov functional on $\mathbb{R}^3_{++}$ if V is continuous on $\mathbb{R}^3_+$, and $\frac{dV(S,I,m)(\xi)}{d\xi} \leq 0$ for any $(S, I, m) \in \mathbb{R}^3_{++}$ (see the definition in [8, Page 316]). Let

$$f(y) = y - 1 - \ln y, \quad \alpha_1(y) = \int_y^{+\infty} J(x)dx, \quad \alpha_2(y) = \int_{-\infty}^{y} J(x)dx.$$

Then, $\lim\limits_{y\to+\infty} \alpha_1(y) = 0$, $\lim\limits_{y\to-\infty} \alpha_2(y) = 0$ from the assumption **(J)**. Set

$$V(S, I, m)(\xi) = cV_1(S, I, m)(\xi) + d_1\tau_1 S^* U_1(S)(\xi) + d_2 I^* U_2(I)(\xi) + d_3\tau_2 m^* U_3(m)(\xi), \tag{3.10}$$

where

$$V_1(S, I, m)(\xi) = \tau_1 S^* f\left(\frac{S(\xi)}{S^*}\right) + I^* f\left(\frac{I(\xi)}{I^*}\right) + \tau_2 m^* f\left(\frac{m(\xi)}{m^*}\right),$$

$$U_1(S)(\xi) = \int_0^{+\infty} \alpha_1(y) f\left(\frac{S(\xi - y)}{S^*}\right)dy - \int_{-\infty}^{0} \alpha_2(y) f\left(\frac{S(\xi - y)}{S^*}\right)dy,$$

$$U_2(I)(\xi) = \int_0^{+\infty} \alpha_1(y) f\left(\frac{I(\xi - y)}{I^*}\right) dy - \int_{-\infty}^{0} \alpha_2(y) f\left(\frac{I(\xi - y)}{I^*}\right) dy,$$

$$U_3(m)(\xi) = \int_0^{+\infty} \alpha_1(y) f\left(\frac{m(\xi - y)}{m^*}\right) dy - \int_{-\infty}^{0} \alpha_2(y) f\left(\frac{m(\xi - y)}{m^*}\right) dy,$$

and

$$\tau_1 = 1 + hI^*, \quad \tau_2 = \frac{\beta\Lambda(1 + hI^*)}{\alpha\mu}.$$

Due to the boundedness of $S(\xi), I(\xi)$, and $m(\xi)$, we have that $V_1(\xi), U_1(\xi)$, and $U_3(\xi)$ are well-defined and bounded from below. According to the construction of the lower solution, there is a $\xi_0 < 0$ such that $I(\xi) \geq e^{\lambda_1 \xi}\left(1 - Me^{\eta\xi}\right) > 0$ for any $\xi \leq \xi_0 < 0$. Fix $\xi \in \mathbb{R}$, and there exists $C_1 > 0$ such that

$$\begin{aligned}
&\int_0^{+\infty} \alpha_1(y) f\left(\frac{I(\xi - y)}{I^*}\right) dy \\
&= \int_0^{\xi-\xi_0} \alpha_1(y) f\left(\frac{I(\xi - y)}{I^*}\right) dy + \int_{\xi-\xi_0}^{+\infty} \alpha_1(y) f\left(\frac{I(\xi - y)}{I^*}\right) dy \\
&\leq \int_0^{\xi-\xi_0} \alpha_1(y) f\left(\frac{I(\xi - y)}{I^*}\right) dy \\
&\quad + \int_{\xi-\xi_0}^{+\infty} \alpha_1(y) \left(\frac{C_1}{I^*} - 1 + \ln I^* - \ln\left(1 - Me^{\eta(\xi - y)}\right) - \lambda_1(\xi - y)\right) dy \\
&< +\infty.
\end{aligned}$$

Let $C_\xi = \max\limits_{s \in [\xi, +\infty)} f(I(s)) > 0$, then

$$\int_{-\infty}^{0} \alpha_2(y) f\left(\frac{I(\xi - y)}{I^*}\right) dy < \int_{-\infty}^{0} \alpha_2(y) C_\xi dy < +\infty.$$

Thus, $U_2(\xi)$ is well-defined. It is easy to see that C_ξ is bounded from below when ξ is large enough. Therefore, the Lyapunov functional $V(S, I, m)(\xi)$ is well-defined and bounded from below when ξ is large enough. According to

$$\frac{d\alpha_1(y)}{dy} = -J(y), \ \frac{d\alpha_2(y)}{dy} = J(y), \ \alpha_1(0) = \alpha_2(0) = \frac{1}{2},$$

one can get

$$\begin{aligned}
\frac{dU_1}{d\xi} &= \frac{d}{d\xi} \int_0^{+\infty} \alpha_1(y) f\left(\frac{S(\xi - y)}{S^*}\right) dy - \frac{d}{d\xi} \int_{-\infty}^{0} \alpha_2(y) f\left(\frac{S(\xi - y)}{S^*}\right) dy \\
&= \int_0^{+\infty} \alpha_1(y) \frac{d}{d\xi} f\left(\frac{S(\xi - y)}{S^*}\right) dy - \int_{-\infty}^{0} \alpha_2(y) \frac{d}{d\xi} f\left(\frac{S(\xi - y)}{S^*}\right) dy \\
&= -\int_0^{+\infty} \alpha_1(y) \frac{d}{dy} f\left(\frac{S(\xi - y)}{S^*}\right) dy + \int_{-\infty}^{0} \alpha_2(y) \frac{d}{dy} f\left(\frac{S(\xi - y)}{S^*}\right) dy \\
&= f\left(\frac{S(\xi)}{S^*}\right) - \int_{-\infty}^{+\infty} J(y) f\left(\frac{S(\xi - y)}{S^*}\right) dy.
\end{aligned}$$

Similarly, one has

$$
\begin{aligned}
\frac{dU_2}{d\xi} &= f\left(\frac{I(\xi)}{I^*}\right) - \int_{-\infty}^{+\infty} J(y) f\left(\frac{I(\xi - y)}{I^*}\right) dy, \\
\frac{dU_3}{d\xi} &= f\left(\frac{m(\xi)}{m^*}\right) - \int_{-\infty}^{+\infty} J(y) f\left(\frac{m(\xi - y)}{m^*}\right) dy.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\frac{dV}{d\xi} =& \tau_1 \left(1 - \frac{S*}{S}\right)(cS'(\xi)) + \left(1 - \frac{I*}{I}\right)(cI'(\xi)) + \tau_2 \left(1 - \frac{m*}{m}\right)(cm'(\xi)) \\
& + d_1\tau_1 S^* \frac{dU_1}{d\xi} + d_2 I^* \frac{dU_2}{d\xi} + d_3\tau_2 m^* \frac{dU_3}{d\xi} \\
=& \tau_1 \left(1 - \frac{S^*}{S}\right)\left[d_1(J * S - S) + \Lambda - \mu S - \frac{\beta S m I}{1 + hI}\right] \\
& + \left(1 - \frac{I^*}{I}\right)\left[d_2(J * I - I) + \frac{\beta S m I}{1 + hI} - (\mu + \gamma) I\right] \\
& + \tau_2 \left(1 - \frac{m^*}{m}\right)\left[d_3(J * m - m) + m\left(b - am - \frac{\alpha I}{1 + hI}\right)\right] \\
& + d_1\tau_1 S^* \frac{dU_1}{d\xi} + d_2 I^* \frac{dU_2}{d\xi} + d_3\tau_2 m^* \frac{dU_3}{d\xi} \\
=& \tau_1 \left(1 - \frac{S^*}{S}\right)\left(\mu S^* + \frac{\beta S^* m^* I^*}{1 + hI^*} - \mu S - \frac{\beta S m I}{1 + hI}\right) \\
& + \left(1 - \frac{I^*}{I}\right) I \left(\frac{\beta S m}{1 + hI} - \frac{\beta S^* m^*}{1 + hI^*}\right) \\
& + \tau_2 \left(1 - \frac{m^*}{m}\right) m \left(am^* + \frac{\alpha I^*}{1 + hI^*} - am - \frac{\alpha I}{1 + hI}\right) \\
& + d_1\tau_1 \left(1 - \frac{S^*}{S}\right)(J * S - S) + d_1\tau_1 S^* \frac{dU_1}{d\xi} \\
& + d_2 \left(1 - \frac{I^*}{I}\right)(J * I - I) + d_2 I^* \frac{dU_2}{d\xi} \\
& + d_3\tau_2 \left(1 - \frac{m^*}{m}\right)(J * m - m) + d_3\tau_2 m^* \frac{dU_3}{d\xi} \\
=& \frac{-\tau_1\mu(S - S^*)^2}{S} + \tau_1 \left(\frac{\beta S^* m^* I^*}{1 + hI^*} - \frac{\beta S m I}{1 + hI} - \frac{S^*}{S}\frac{\beta S^* m^* I^*}{1 + hI^*} + \frac{\beta S^* m I}{1 + hI}\right) \\
& + \frac{\beta S m (I - I^*)}{1 + hI} - \frac{\beta S^* m^* (I - I^*)}{1 + hI^*} - \tau_2 a (m - m^*)^2 - \frac{\tau_2 \alpha (m - m^*)(I - I^*)}{(1 + hI)(1 + hI^*)} \\
& + d_1\tau_1 \left(1 - \frac{S^*}{S}\right)(J * S - S) + d_1\tau_1 S^* \frac{dU_1}{d\xi} \\
& + d_2 \left(1 - \frac{I^*}{I}\right)(J * I - I) + d_2 I^* \frac{dU_2}{d\xi} \\
& + d_3\tau_2 \left(1 - \frac{m^*}{m}\right)(J * m - m) + d_3\tau_2 m^* \frac{dU_3}{d\xi}
\end{aligned}
$$

$$:= B_1 + B_2 + B_3 + B_4,$$

where

$$\begin{aligned}
B_1 =& \frac{-\tau_1\mu(S-S^*)^2}{S} + \tau_1\left(\frac{\beta S^* m^* I^*}{1+hI^*} - \frac{\beta S m I}{1+hI} - \frac{S^*}{S}\frac{\beta S^* m^* I^*}{1+hI^*} + \frac{\beta S^* m I}{1+hI}\right)\\
&+ \frac{\beta S m(I-I^*)}{1+hI} - \frac{\beta S^* m^*(I-I^*)}{1+hI^*} - \tau_2 a(m-m^*)^2 - \frac{\tau_2 a(m-m^*)(I-I^*)}{(1+hI)(1+hI^*)}\\
=& \frac{-\tau_1\mu(S-S^*)^2}{S} - \tau_2 a(m-m^*)^2 - \frac{\tau_1\beta(S-S^*)^2 m^* I^*}{S(1+hI)(1+hI^*)} - - \frac{\beta(I-I^*)^2 h S^* m^*}{(1+hI)(1+hI^*)}\\
&- \beta(S-S^*)(m-m^*)I^* - \frac{\beta\left(\frac{\Lambda}{\mu} - S^*\right)(m-m^*)(I-I^*)}{1+hI}.
\end{aligned}$$

Since $\left|\frac{\beta\left(\frac{\Lambda}{\mu}-S^*\right)}{1+hI}\right| \le \frac{b\beta^2\Lambda}{ah\mu^2}$, by Young's inequality with ϵ, we have

$$-\beta I^*(S-S^*)(m-m^*) \le \beta I^*\left[\frac{\epsilon(S-S^*)^2}{2} + \frac{(m-m^*)^2}{2\epsilon}\right],$$

and for $p \in \mathbb{R}$,

$$-\frac{\beta\left(\frac{\Lambda}{\mu} - S^*\right)(I-I^*)(m-m^*)}{1+hI} \le \frac{\epsilon\Lambda\beta}{2\mu}\left(\frac{b\beta}{ah\mu}\right)^p (I-I^*)^2 + \frac{\beta\Lambda}{2\epsilon\mu}\left(\frac{b\beta}{ah\mu}\right)^{2-p}(m-m^*)^2.$$

Then

$$\begin{aligned}
B_1 \le& -\frac{\mu^2(1+hI^*)(S-S^*)^2}{\Lambda} - \frac{\beta\Lambda a(1+hI^*)(m-m^*)^2}{\alpha\mu} - \frac{\beta\mu m^* I^*(S-S^*)^2}{\Lambda}\\
&- \frac{b\beta^2 S^* m^*(I-I^*)^2 h}{a\mu(1+hI^*)} + \frac{\epsilon\beta I^*(S-S^*)^2}{2} + \frac{\beta I^*(m-m^*)^2}{2\epsilon}\\
&+ \frac{\epsilon\Lambda\beta}{2\mu}\left(\frac{b\beta}{ah\mu}\right)^p (I-I^*)^2 + \frac{\beta\Lambda}{2\epsilon\mu}\left(\frac{b\beta}{ah\mu}\right)^{2-p}(m-m^*)^2\\
=& -\left[\frac{\mu^2(1+hI^*) + \beta\mu m^* I^*}{\Lambda} - \frac{\epsilon\beta I^*}{2}\right](S-S^*)^2\\
&- \left[\frac{b\beta^2 S^* m^*}{a\mu(1+hI^*)} - \frac{\epsilon\beta\left(\frac{b\beta}{ah\mu}\right)^p}{2\mu}\right](I-I^*)^2\\
&- \left[\frac{\beta\Lambda a(1+hI^*)}{\alpha\mu} - \frac{\beta I^*}{2\epsilon} - \frac{\beta\Lambda\left(\frac{b\beta}{ah\mu}\right)^{2-p}}{2\epsilon\mu}\right](m-m^*)^2.
\end{aligned}$$

Choose ε sufficiently small and an appropriate p such that

$$\frac{\mu^2(1+hI^*) + \beta\mu m^* I^*}{\Lambda} > \frac{\epsilon\beta I^*}{2},\ \frac{b\beta^2 S^* m^*}{a\mu(1+hI^*)} > \frac{\epsilon\beta\left(\frac{b\beta}{ah\mu}\right)^p}{2\mu}$$

and

$$\frac{\beta\Lambda a(1+hI^*)}{\alpha\mu} > \frac{\beta\mu I^* - \beta\Lambda\left(\frac{b\beta}{ah\mu}\right)^{2-p}}{2\epsilon\mu}.$$

Thus, $B_1 \leq 0$, and then,

$$\begin{aligned}
B_2 =& d_1\tau_1\left(1-\frac{S^*}{S}\right)(J*S-S) + d_1\tau_1 S^*\frac{dU_1}{d\xi}\\
=& d_1\tau_1\left[J*S-S-\frac{S^*}{S}(J*S-S)+S^*\frac{dU_1}{d\xi}\right]\\
=& d_1\tau_1\left[\int_{-\infty}^{+\infty} J(y)S(\xi-y)\,dy - S - \frac{S^*}{S}\int_{-\infty}^{+\infty} J(y)S(\xi-y)\,dy\right]\\
&+ d_1\tau_1\left[S^*+S^*f\left(\frac{S(\xi)}{S^*}\right)-S^*\int_{-\infty}^{+\infty} J(y)f\left(\frac{S(\xi-y)}{S^*}\right)dy\right]\\
=& d_1\tau_1 S^*\int_{-\infty}^{+\infty} J(y)\left(\frac{S(\xi-y)}{S^*}-\frac{S(\xi-y)}{S}-\ln\frac{S(\xi)}{S^*}\right)dy\\
&- d_1\tau_1 S^*\int_{-\infty}^{+\infty} J(y)f\left(\frac{S(\xi-y)}{S^*}\right)dy\\
=& d_1\tau_1 S^*\int_{-\infty}^{+\infty} J(y)\left(\frac{S(\xi-y)}{S^*}-1-\ln\frac{S(\xi-y)}{S^*}\right)dy\\
&- d_1\tau_1 S^*\int_{-\infty}^{+\infty} J(y)\left(\frac{S(\xi-y)}{S}-1-\ln\frac{S(\xi-y)}{S}\right)dy\\
&- d_1\tau_1 S^*\int_{-\infty}^{+\infty} J(y)f\left(\frac{S(\xi-y)}{S^*}\right)dy\\
=& -d_1\tau_1 S^*\int_{-\infty}^{+\infty} J(y)f\left(\frac{S(\xi-y)}{S}\right)dy \leq 0.
\end{aligned}$$

Similarly,

$$B_3 = -d_2 I^*\int_{-\infty}^{+\infty} J(y)f\left(\frac{I(\xi-y)}{I}\right)\mathrm{d}y \leq 0,$$

and

$$B_4 = -d_3\tau_2 m^*\int_{-\infty}^{+\infty} J(y)f\left(\frac{m(\xi-y)}{m}\right)\mathrm{d}y \leq 0.$$

Therefore, $\frac{\mathrm{d}V}{\mathrm{d}\xi} \leq 0$, which implies that V is monotonically decreasing in ξ and has a lower bound.

In the following, we will use the properties of the Lyapunov functional V to describe the asymptotic behavior of the traveling wave solution (S, I, m) as it approaches positive infinity. Now, we choose an increasing sequence $\{\xi_n\}_{n\geq 0}$ such that $\xi_n \to +\infty$ as $n \to +\infty$, and let

$$S_n(\xi) = S(\xi+\xi_n)_{n\geq 0},\ I_n(\xi) = I(\xi+\xi_n)_{n\geq 0},\ m_n(\xi) = m(\xi+\xi_n)_{n\geq 0}.$$

Since S_n, I_n, and m_n are uniformly bounded in $C^{1,1}(\mathbb{R})$, by passing to a subsequence, we can assume that S_n, I_n, and m_n converge to nonnegative functions $S_{+\infty}$, $I_{+\infty}$, and $m_{+\infty}$, respectively. Furthermore, for sufficiently large n, there exists a constant C_1 such that

$$C_1 \leq V(S_n, I_n, m_n)(\xi) = V(S, I, m)(\xi + \xi_n) \leq V(S, I, m)(\xi).$$

Therefore, there exists $\nu \in \mathbb{R}$ such that for any $\xi \in \mathbb{R}$, $\lim_{n\to+\infty} V(S_n, I_n, m_n)(\xi) = \nu$. By (3.10) and Lebesgue's dominated convergence theorem, there is

$$\lim_{n\to+\infty} V(S_n, I_n, m_n)(\xi) = V(S_{+\infty}, I_{+\infty}, m_{+\infty})(\xi),\ \xi \in \mathbb{R}.$$

Thus, $V(S_{+\infty}, I_{+\infty}, m_{+\infty})(\xi) \equiv \nu$. On the other hand, $\frac{dV}{d\xi} = 0$ if and only if $S(\xi) \equiv S^*$, $I(\xi) \equiv I^*$, $m(\xi) \equiv m^*$. Therefore,

$$S(+\infty) = S^*,\ I(+\infty) = I^*,\ m(+\infty) = m^*.$$

This completes the proof. □

3.2. The case: $R_0 > 1$ and $c = c_*$

This section establishes the existence of traveling wave solutions with the critical wave speed through a limiting argument approach. It should be noted that since the construction of upper-lower solutions for $c > c_*$ depends on the selection of c, the asymptotic behavior at $\xi \to -\infty$ requires separate analysis.

Proof of Theorem 2.1 (i) for $R_0 > 1$ and $c = c_*$. Let $c_n \subset (c_*, c_* + 1]$ be a decreasing sequence such that $\lim_{n\to+\infty} c_n = c_*$. There exists a traveling wave solution $(S_n(\cdot), I_n(\cdot), m_n(\cdot))$ with wave speed c_n, satisfying (2.1)–(2.3). From Lemma 3.9, we get that for any $(S_n(\cdot), I_n(\cdot), m_n(\cdot))$ of system (2.1) with speed $c_n \in (c_*, c_* + 1]$, there exists $\delta > 0$ such that $I_n'(\xi) > 0$ as $I_n(\xi) \leq \delta$ for any $\xi \in \mathbb{R}$. Now, we prove that there exists a subsequence of (S_n, I_n, m_n) that converges to $(S(x + c_*t), I(x + c_*t), m(x + c_*t))$, which satisfies (2.1). By the Arzelà-Ascoli theorem and selecting a diagonal subsequence, we can find a subsequence of (S_n, I_n, m_n), still denoted as (S_n, I_n, m_n), such that (S_n, I_n, m_n) and (S_n', I_n', m_n') uniformly converge to (S, I, m) and (S', I', m') on each bounded interval. By the Lebesgue dominated convergence theorem, we obtain $\lim_{n\to+\infty} J * S_n = J * S$, $\lim_{n\to+\infty} J * I_n = J * I$, $\lim_{n\to+\infty} J * m_n = J * m$ on each bounded interval. Therefore, (S, I, m) satisfies (2.1). For $c = c_*$, the Lyapunov functional constructed with $c > c_*$ can also be used to prove the boundary conditions of the traveling wave solution at $+\infty$. Therefore, for $c = c_*$, we can also obtain $\lim_{\xi\to+\infty} S(\xi) = S^*$, $\lim_{\xi\to+\infty} I(\xi) = I^*$, $\lim_{\xi\to+\infty} m(\xi) = m^*$. By the choice of δ (similar to [27, Theorem 3.19]), we get

$$I(0) = \delta,\ I(\xi) \leq \delta, I'(\xi) > 0,\ \text{for } \xi < 0. \tag{3.11}$$

Thus, the limit $I(-\infty) := \lim_{\xi\to-\infty} I(\xi)$ exists and $I(-\infty) \in [0, \delta)$. If $I(-\infty) > 0$, then because the system (1.1) admits a unique endemic equilibrium E^*, there is $I(-\infty) = I^* \geq \delta$. A contradiction happens because $I(-\infty) < \delta$. This implies that $I(-\infty) = 0$.

Next, we prove that $\lim_{\xi\to-\infty} S(\xi) = \frac{\Lambda}{\mu}$. If $\underline{S} < \bar{S}$, then there exist sequences $\{x_n\}$ and $\{y_n\}$ with $x_n, y_n \to -\infty$ as $n \to +\infty$ such that

$$\lim_{n\to+\infty} S(x_n) = \underline{S},\ \lim_{n\to+\infty} S(y_n) = \bar{S},\ S'(x_n) = S'(y_n) = 0.$$

By the Fatou lemma, there is

$$\liminf_{n\to+\infty} J * S(x_n) \geq \underline{S}, \ \limsup_{n\to+\infty} J * S(y_n) \leq \bar{S}.$$

In the S-equation of (2.1), set $\xi = x_n$ and $\xi = y_n$, respectively. Since $\lim_{\xi\to-\infty} I(\xi) = 0$, letting $n \to +\infty$, it follows that

$$\begin{cases} \Lambda - \mu\underline{S} \leq 0, \\ \Lambda - \mu\bar{S} \geq 0. \end{cases}$$

Therefore, $\bar{S} \leq \frac{\Lambda}{\mu} \leq \underline{S}$, which contradicts the assumption. Hence, the limit $\lim_{\xi\to-\infty} S(\xi)$ exists. Taking $\xi \to -\infty$ in the first equation of (2.1), we obtain $\lim_{\xi\to-\infty} S(\xi) = \frac{\Lambda}{\mu}$.

Finally, we prove $\lim_{\xi\to-\infty} m(\xi) = \frac{b}{a}$. Indeed, we only need to show $\liminf_{\xi\to-\infty} m(\xi) := m^- = \frac{b}{a}$. There exist a sequence $\{\xi_n^*\}$ satisfying $\xi_n^* \to -\infty$ as $n \to +\infty$ so that

$$m'(\xi_n^*) = 0, \ m(\xi_n^*) \to m^- \quad \text{as} \quad n \to +\infty.$$

According to the m-equation of (2.1), one has

$$0 = c_* m'(\xi_n^*) = d_3 \int_{\mathbb{R}} J(y) m(\xi_n^* - y) dy - d_3 m(\xi_n^*) + m(\xi_n^*) \left(b - a m(\xi_n^*) - \frac{\alpha I(\xi_n^*)}{1 + h I(\xi_n^*)} \right).$$

Taking $n \to \infty$, one has

$$d_3 m^- - m^-(b - a m^-) = d_3 \lim_{n\to\infty} \int_{\mathbb{R}} J(y) m(\xi_n^* - y) dy \geq d_3 m^-,$$

that is,

$$m^-(b - a m^-) \leq 0.$$

Hence, either $m^- = 0$ or $m^- \geq \frac{a}{b}$ happen. If $m^- \geq \frac{a}{b}$, the conclusion holds. If $m^- = 0$, it follows from [32, Corollary 3.15 (c)] that

$$\lim_{\xi\to-\infty} m(\xi) = \frac{b}{a}.$$

This completes the proof. □

3.3. *The case:* $R_0 > 1$ *and* $0 < c < c_*$

In this section, we show the nonexistence of traveling wave solutions for $R_0 > 1$ and $0 < c < c_*$. Before proceeding with this task, we first present the propagation properties concerning the single Kolmogorov–Petrovsky–Piskunov equation (hereafter referred to as the KPP equation). Consider

$$\begin{cases} \frac{\partial w(t,x)}{\partial t} = d(J * w(t,x) - w(t,x)) + f(w), & t > 0, x \in \mathbb{R}, \\ w(0,x) = w_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.12}$$

where $d > 0$, J satisfies (J), and $f \in C^1[0,\infty)$ satisfying $f(0) = f(M) = 0, f(w) > 0$ for $w \in (0, M)$, and $f(u)/u$ is decreasing in $u \in [0, M]$ with $M > 0$. It follows [13] that the following result holds.

Lemma 3.10. *Equation* (3.12) *admits a spreading speed* c_1^* *in the sense that for any* $w_0 \in C(\mathbb{R}, \mathbb{R}_+)$ *with* $w_0 \not\equiv 0$*, one has*

$$\liminf_{t\to\infty} \inf_{|x|\leqslant ct} (w(t,x;w_0) - M) = 0, \ \forall c < c_1^*$$

and

$$\limsup_{t\to\infty} \sup_{|x|\geqslant ct} w(t,x;w_0) = 0, \ \forall c > c_1^*,$$

where $w(t,x;w_0)$ *is the solution of (3.12) with initial value* $w(0,x;w_0) = w_0(x)$*. Furthermore,*

$$c_1^* = \inf_{\lambda>0} \left\{ \frac{1}{\lambda} \left[d\left(\int_{\mathbb{R}} J(x)e^{-\lambda x}dx - 1 \right) + f'(0) \right] \right\}$$

Proof of Theorem 2.1 (ii) for $R_0 > 1$ *and* $0 < c < c_*$. Assume there exists $c_1 \in (0, c_*)$ such that (2.1)–(2.3) admits a nontrivial bounded positive solution $(S(x+c_1t), I(x+c_1t), m(x+c_1t))$. By (2.2) and (2.3), we choose $M_\epsilon > 0$ sufficiently large such that for any $\xi < -M_\epsilon$,

$$\frac{\Lambda}{\mu} - \epsilon \leq S(\xi) < \frac{\Lambda}{\mu}, \quad \frac{b}{a} - \epsilon \leq m(\xi) < \frac{b}{a}.$$

Then, for $\xi < -M_\epsilon$,

$$cI'(\xi) \geq d_2(J * I(\xi) - I(\xi)) + \frac{\beta\left(\frac{\Lambda}{\mu} - \epsilon\right)\left(\frac{b}{a} - \epsilon\right) I(\xi)}{1 + hI(\xi)} - (\mu+\gamma)I(\xi). \tag{3.13}$$

Based on the boundedness, there exists a sufficiently large positive constant l such that

$$\frac{\beta I(\xi)}{[1 + hI(\xi)]^{l+1}} \leq \frac{\beta m(\xi)S(\xi)I(\xi)}{1 + hI(\xi)}, \ \xi \geq -M_\epsilon.$$

This is equivalent to

$$\frac{1}{[1 + hI(\xi)]^{l}} \leq m(\xi)S(\xi), \ \xi \geq -M_\epsilon.$$

When l is sufficiently large, the above inequality holds. Therefore,

$$cI'(\xi) \geq d_2(J * I(\xi) - I(\xi)) + \frac{\beta I(\xi)}{[1 + hI(\xi)]^{l+1}} - (\mu+\gamma)I(\xi), \ \xi \geq -M_\epsilon. \tag{3.14}$$

Define

$$g(u) = \inf \frac{\beta\left(\frac{\Lambda}{\mu} - \epsilon\right)\left(\frac{b}{a} - \epsilon\right) v}{(1 + hv)^{l+1}}, \text{ for any } v \in \left(u, \frac{1}{h}\left(\frac{\Lambda\beta b}{a\mu(\mu+\gamma)} - 1\right)\right). \tag{3.15}$$

According to (3.13)–(3.15), $u(t,x) = I(x + c_1t)$ satisfies

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} \geq d_2(J * u(t,x) - u(t,x)) + g(u(t,x)) - (\mu+\gamma)u(t,x), \ x \in \mathbb{R}, \ t > 0, \\ u(0,x) = I(x) > 0, \ x \in \mathbb{R}. \end{cases}$$

By the comparison principle, $u(t,x)$ is an upper solution of the following initial value problem:

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = d_2(J * w(t,x) - w(t,x)) + g(w(t,x)) - (\mu+\gamma)w(t,x), \ x \in \mathbb{R}, \ t > 0, \\ w(0,x) = I(x) > 0, \ x \in \mathbb{R}. \end{cases}$$

Applying the asymptotic propagation theory (Lemma 3.10) with $d = d_2, f(w) = g(w) - (\mu + \gamma)w$, there exists c_ϵ^* that

$$\lim_{t\to+\infty} \inf_{|x|\le\tilde{c}t} w(t,x) > 0,\ \forall \tilde{c} \in (0, c_\epsilon^*),$$

where

$$c_\epsilon^* = \inf_{\lambda>0} \left\{ \frac{1}{\lambda} \left[d_2 \left(\int_{\mathbb{R}} J(x) e^{-\lambda x} dx - 1 \right) + \beta \left(\frac{\Lambda}{\mu} - \epsilon \right) \left(\frac{b}{a} - \epsilon \right) - (\mu + \gamma) \right] \right\}.$$

Since $\lim_{\epsilon\to0} c_\epsilon^* = c_*$, one can take ϵ satisfying $c_1 < c_\epsilon^*$. Choose $\tilde{c} \in (c_1, c_*)$ and set $x = -\tilde{c}t$. Then, $\lim_{t\to+\infty} \inf u(t, -\tilde{c}t) > 0$. On the other hand, $u(t, -\tilde{c}t) = I((c_1 - \tilde{c})t) \to 0$ as $t \to +\infty$, which leads to a contradiction. This completes the proof. □

3.4. *The case:* $R_0 < 1$ *and* $c > 0$

In this section, we prove the nonexistence of traveling wave solutions for $R_0 < 1$ and $c > 0$.

Proof of Theorem 2.1 (ii) for $R_0 < 1$ *and* $c > 0$. Assume that there exists such a nontrivial positive solution $(S(x + ct), I(x + ct), m(x + ct))$ of system (2.1). We have

$$S(\xi) < \frac{\Lambda}{\mu},\ m(\xi) < \frac{b}{a},\ I(\xi) > 0,\ \xi \in \mathbb{R}.$$

Then,

$$\begin{aligned} cI'(\xi) &= d_2 (J * I(\xi) - I(\xi)) + \frac{\beta S(\xi) m(\xi) I(\xi)}{1 + hI(\xi)} - (\mu + \gamma) I(\xi) \\ &\le d_2 (J * I(\xi) - I(\xi)) + \frac{\beta\Lambda b}{\mu a} I(\xi) - (\mu + \gamma) I(\xi). \end{aligned}$$

Thus, $u(t, x) = I(x + ct)$ satifies

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} \le d_2 (J * u - u)(t, x) + \dfrac{\beta\Lambda b}{\mu a} u(t, x) - (\mu + \gamma) u(t, x),\ x \in \mathbb{R},\ t > 0, \\ u(0, x) = I(x) > 0,\ x \in \mathbb{R}. \end{cases}$$

Let $w_0 = \sup_{\xi\in\mathbb{R}} I(\xi)$, then $w_0 > 0$. We consider the following initial value problem

$$\begin{cases} \dfrac{dw}{dt} = \dfrac{\beta\Lambda b}{\mu a} w(t) - (\mu + \gamma) w(t), \\ w(0) = w_0. \end{cases} \tag{3.16}$$

By the comparison principle,

$$0 \le I(t, x) \le w_0 e^{\lambda t},\ t > 0,$$

where $\lambda = \frac{\beta\Lambda b}{\mu a} - (\mu + \gamma)$. Since $R_0 = \frac{\beta\Lambda b}{\mu a(\mu+\gamma)} < 1$, we have $\lambda < 0$. Therefore, it follows form (3.16) that $I(\xi) \equiv 0$ for any $\xi \in \mathbb{R}$, which contradicts (2.3). This completes the proof. □

Proof of Theorem 2.2. Substitute the Gaussian kernel $J(x)$ into the integral:

$$\int_{\mathbb{R}} J(x)e^{-\lambda x}\,\mathrm{d}x = e^{\lambda^2/(4l)}.$$

The formula for c_* becomes:

$$c_* = \inf_{\lambda>0}\left\{\frac{1}{\lambda}\left[d_2\left(e^{\lambda^2/(4l)}-1\right)+\frac{\beta\Lambda b}{\mu a}-(\mu+\gamma)\right]\right\}.$$

Define $g(\lambda)=\frac{d_2}{\lambda}(e^{\lambda^2/(4l)}-1)+\frac{\frac{\beta\Lambda b}{\mu a}-(\mu+\gamma)}{\lambda}$. To find the infimum, compute the derivative:

$$g'(\lambda)=\frac{d_2}{2l}\lambda e^{\lambda^2/(4l)}-\frac{d_2}{\lambda^2}\left(e^{\lambda^2/(4l)}-1\right)-\frac{\frac{\beta\Lambda b}{\mu a}-(\mu+\gamma)}{\lambda^2}.$$

Setting $g'(\lambda)=0$ leads to:

$$d_2e^{\lambda^2/(4l)}\left(\frac{\lambda^2}{2l}-1\right)+d_2-\frac{\beta\Lambda b}{\mu a}-(\mu+\gamma)=0.$$

Let $z=\lambda^2/(4l)$ and $B=\frac{\beta\Lambda b}{\mu a}-(\mu+\gamma)-d_2$. The equation simplifies to:

$$e^z(2z-1)=\frac{B}{d_2}.$$

The function $h(z)=e^z(2z-1)$ is strictly increasing for $z>0$ with $h(0)=-1<\frac{B}{d_2}$ and $h(z)\to+\infty$ as $z\to+\infty$. Thus, there exists a unique solution $z^*>0$. Substituting $\lambda^*=2\sqrt{lz^*}$ back into $g(\lambda)$ yields:

$$c_*=\frac{d_2(e^{z^*}-1)+\frac{\beta\Lambda b}{\mu a}-(\mu+\gamma)}{2\sqrt{lz^*}}.$$

This ends the proof. □

4. Numerical experiments

In this section, we present numerical simulations to validate our theoretical findings on traveling wave solutions. The time and space unit is taken as day and km, respectively.

The parameters are fixed in Table 1. Inspired by [25], we set

$$J_1(x)=J_2(x)=J_3(x)=J(x)=\frac{1}{\sqrt{4\pi}}e^{-x^2/4}.$$

Table 1. List of parameters.

Parameter	Definition	Value	References
d_1	Dispersal coefficient of susceptible individuals	0.2 Day^{-1}	[19]
d_2	Dispersal coefficient of infected individuals	0.3 Day^{-1}	[19]
d_3	Dispersal coefficient of the intensity of population mobility	0.1 Day^{-1}	[19]
Λ	Input rate of susceptible individuals	0.5 $(\text{km}^2\text{Day})^{-1}$	[21]
μ	Output rate of susceptible individuals	0.05 Day^{-1}	Assumed
γ	Recovery from infection	0.2 Day^{-1}	[21]
a	Input rate of the intensity of population mobility	1 $\text{km}^2\text{Day}^{-1}$	[21]
b	Behavioral response dynamics	2 Day^{-1}	[21]
α	Risk-sensitive mobility	0.1 $\text{km}^2\text{Day}^{-1}$	[21]
h	Saturation of incidence	0.01 km^2	[21]
β	Infection rate	0.025 $\text{km}^4\text{Day}^{-1}$	Assumed

In this setting, the disease-free equilibrium is $E_1 = (10, 0, 2)$, and the endemic equilibrium is $E^* = (5.295, 0.941, 1.907)$. The basic reproduction number is computed as $R_0 = 2$, and the corresponding minimal wave speed is $c_* = 0.6268$. For numerical simulations of the traveling wave solutions to system (1.2), we consider a finite spatial domain $[-L, L]$ with $L = 200$, and impose Neumann boundary conditions at both ends:

$$\begin{cases} \frac{\partial S(t,x)}{\partial t} = d_1 \int_{-L}^{L} J_1(x-y)(S(t,y) - S(t,x))dy + \Lambda - \mu S(t,x) - \frac{\beta m(t,x)S(t,x)I(t,x)}{1+hI(t,x)}, \\ \frac{\partial I(t,x)}{\partial t} = d_2 \int_{-L}^{L} J_2(x-y)(I(t,y) - I(t,x))dy + \frac{\beta m(t,x)S(t,x)I(t,x)}{1+hI(t,x)} - (\mu+\gamma)I(t,x), \\ \frac{\partial m(t,x)}{\partial t} = d_3 \int_{-L}^{L} J_3(x-y)(m(t,y) - m(t,x))dy + m(t,x)\left(b - am(t,x) - \frac{\alpha I(t,x)}{1+hI(t,x)}\right), \end{cases} \tag{4.1}$$

where $t > 0$ and $x \in (-200, 200)$.

To discretize system (4.1), we define the temporal and spatial grids with parameters $T = 200$, $N = 801$, $s = T/(N-1)$, $M = 401$, $f = 200/(M-1)$, initializing at $t_0 = 0$ and $x_0 = -200$. The grid points are generated via $t_{j+1} = t_j + s$ for $j = 1, 2, \ldots, N-1$ (temporal discretization) and $x_{i+1} = x_i + f$ for $i = 1, 2, \ldots, M$ (spatial discretization). Using the notation $u_i^j = u(t_j, x_i)$ where u represents S, I, or m, we obtain

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{u_i^{j+1} - u_i^j}{s} + O(s), \\ \frac{\partial^2 u(t,x)}{\partial x^2} = \frac{u_{i+1}^j - 2u_i^j - u_{i-1}^j}{f^2} + O(f^2), \end{cases}$$

and then system (4.1) has the following difference equation:

$$\begin{cases} \frac{S_i^{j+1} - S_i^j}{s} = d_1 \sum_{r=1}^{M-1} J(f(j-r))(S_r^j - S_i^j) + \Lambda - \mu S_i^j - \frac{\beta m_i^j S_i^j I_i^j}{1+hI_i^j}, \\ \frac{I_i^{j+1} - I_i^j}{s} = d_2 \sum_{r=1}^{M-1} J(f(j-r))(S_r^j - S_i^j) + \frac{\beta m_i^j S_i^j I_i^j}{1+hI_i^j} - (\mu+\gamma)I_i^j, \qquad j \geq 1, 2 \leq i \leq 800, \\ \frac{m_i^{j+1} - m_i^j}{s} = d_3 \sum_{r=1}^{M-1} J(f(j-r))(S_r^j - S_i^j) + m_i^j\left(b - am_i^j - \frac{\alpha I_i^j}{1+hI_i^j}\right). \end{cases} \tag{4.2}$$

The discretization of the boundary conditions can be formulated as follows:

$$u(j+1, 0) = u(j+1, 1), u(j+1,) = u(j+1, M-1), u = S, I, m, \ j \geq 1.$$

Now we take the following initial value to show the existence of the traveling wave solution of discrete system (4.2),

$$S=\begin{cases}10, & 1\le j\leqslant 501\\ 5.295, & 502\le j\leqslant 601\end{cases},\ I=\begin{cases}0, & 1\le j\le 501\\ 0.941, & 502\le j\le 601\end{cases},\ m=\begin{cases}2, & 1\le j\le 501\\ 1.907, & 502\le j\le 601\end{cases}. \tag{4.3}$$

Using the initial conditions specified in (4.3), we present numerical simulations of system (4.2) in Figure 1. The solution profiles shown in Figure 1 demonstrate characteristics consistent with traveling wave solutions.

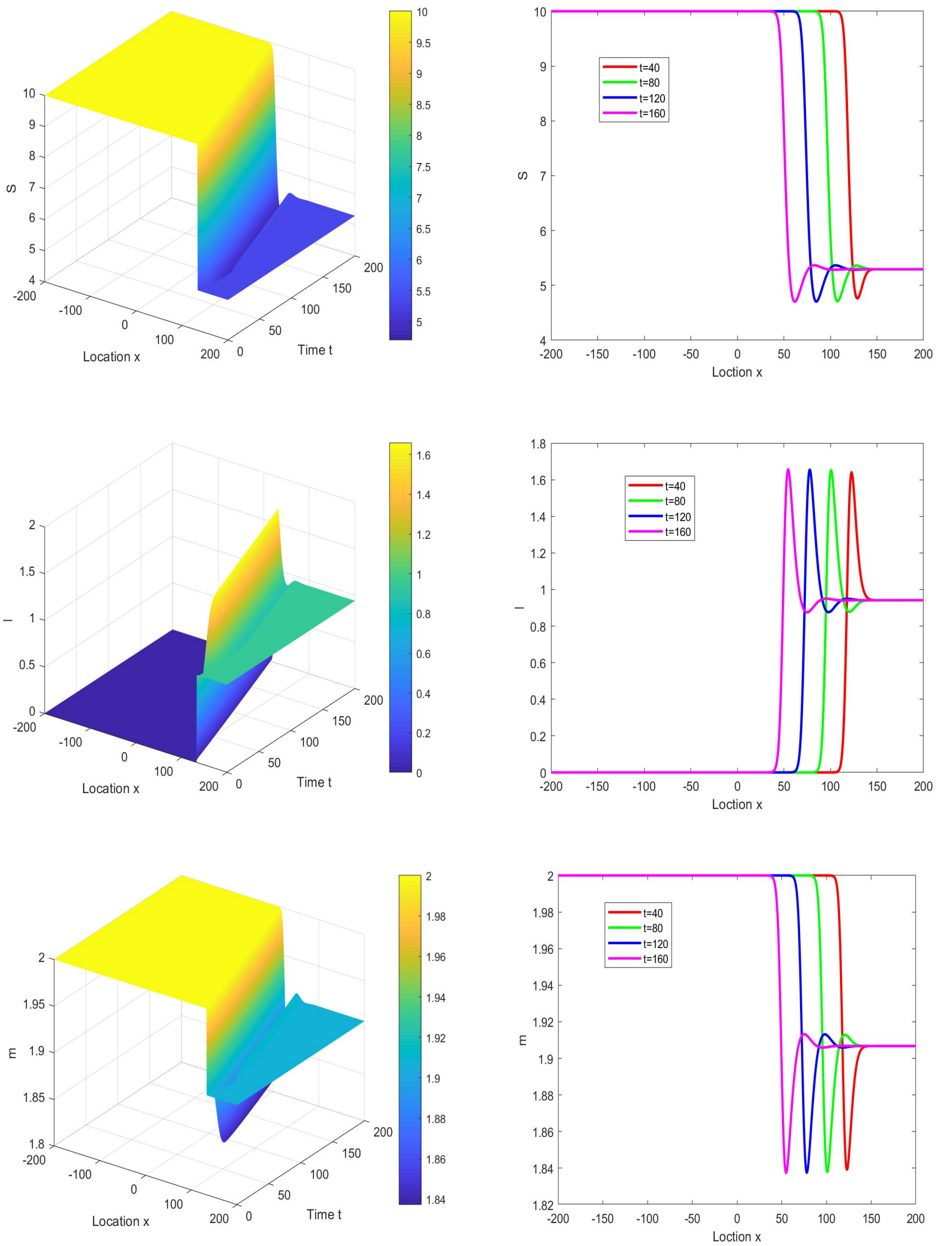

Figure 1. The spread of S, I, and m.

Below we compare the differences between local and nonlocal dispersal. We now replace the nonlocal dispersal operator $J * u$ in system (1.2) with the local dispersal operator Δu, resulting in system (1.2). We have

$$\begin{cases} \frac{\partial S(t,x)}{\partial t} = d_1 \frac{\partial^2 S(t,x)}{\partial x^2} + \Lambda - \mu S(t,x) - \frac{\beta m(t,x) S(t,x) I(t,x)}{1+hI(t,x)}, \\ \frac{\partial I(t,x)}{\partial t} = d_2 \frac{\partial^2 I(t,x)}{\partial x^2} + \frac{\beta m(t,x) S(t,x) I(t,x)}{1+hI(t,x)} - (\mu+\gamma) I(t,x), \qquad t > 0, x \in \mathbb{R}. \\ \frac{\partial m(t,x)}{\partial t} = d_3 \frac{\partial^2 m(t,x)}{\partial x^2} + m(t,x)\left(b - am(t,x) - \frac{\alpha I(t,x)}{1+hI(t,x)}\right), \end{cases} \tag{4.4}$$

According to the computational results, the minimal wave speed for system (4.4) is $c_* = 0.548$. This suggests that the nonlocal dispersal mechanism accelerates the spread, as also illustrated in Figure 2. However, when the kernel function is specified as $J(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$, the minimal wave speed increases to $c_* = 0.3134$, indicating that the nonlocal dispersal in this case slows down the propagation , which is consistent with the observation in Figure 3. These results suggest that the propagation speed decreases when the kernel function exhibits faster decay in its tail, and increases when the decay is slower. This is exactly what is described in Theorem 2.2.

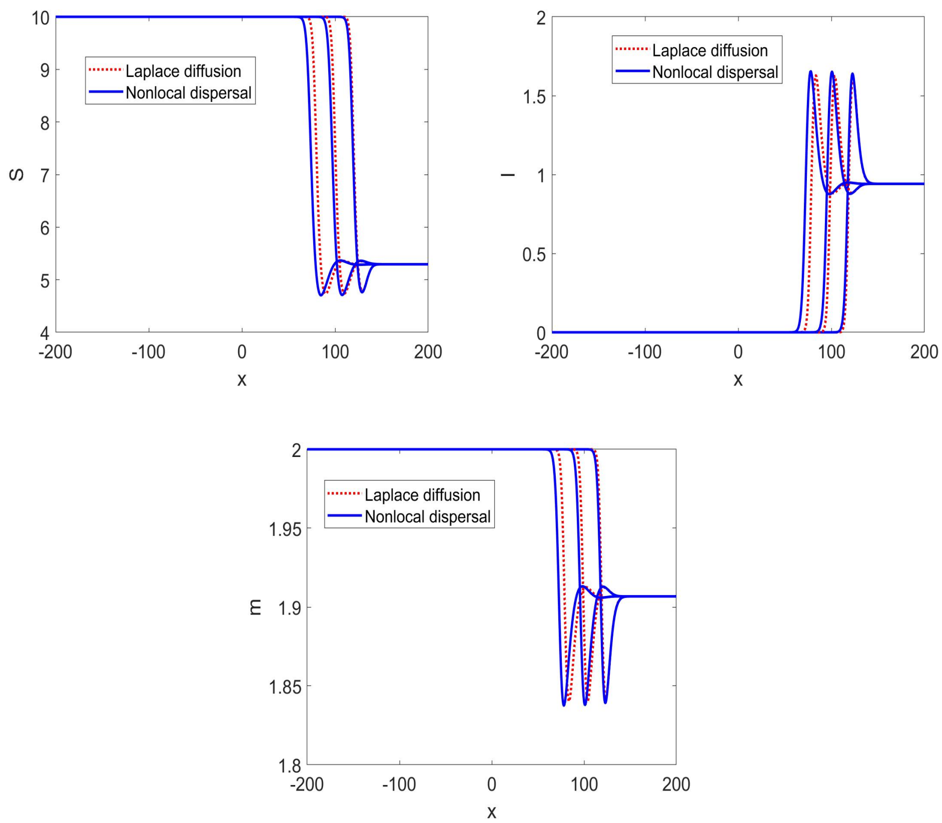

Figure 2. Nonlocal dispersal VS Local dispersal with $J(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}$.

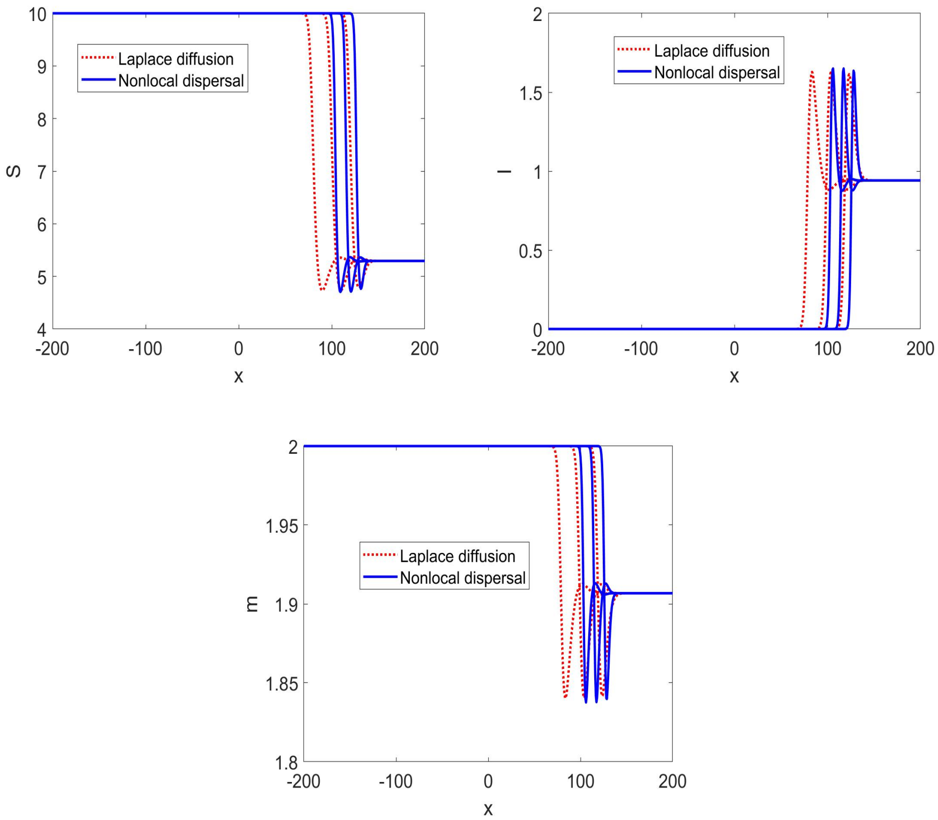

Figure 3. Nonlocal dispersal VS Local dispersal with $J(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$.

5. Conclusions

Classical nonlocal dispersal epidemic models (see, e.g., [12, 19, 23, 26]) typically assume fixed contact rates, overlooking voluntary contact reduction behaviors in response to mobility-linked epidemic risks. To address this gap, we develop a mobility-dependent epidemic model that explicitly incorporates human behavioral responses in influenza transmission dynamics. Our primary contributions establish rigorous mathematical criteria for the existence and nonexistence of traveling wave solutions, which capture essential disease invasion patterns. Crucially, our analytical framework and theoretical results remain applicable to general nonlocal dispersal epidemic models beyond this specific formulation. Furthermore, we employ comprehensive numerical simulations to quantify how nonlocal and local dispersal patterns modulate wave propagation speeds. These integrated advances bridge critical theoretical gaps in spatial epidemiology while providing practical tools for predicting and controlling geographically spreading diseases.

Author contributions

Xuerui Li: Conceptualization, methodology, numerical experiments, writing-review and editing; Boyi Wang: Conceptual framework, formal analysis, writing-review and editing; Yuanyuan Wu: Conceptualization, methodology, writing-original draft. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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