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Research article

Impact of noise in a stochastic coral reefs model

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Abstract: As known, coral reefs have important economic and social value, but they are damaged by various environmental factors worldwide. In this work, a delayed coral reefs system with white noise and L*é*vy jumps are studied. First, we examine the existence problem using Lyapunov analysis methods. Then, from a stochastic analysis technique, we studied the problems for the stochastics persistence and extinction. Furthermore, conditions for stochastic bounded in mean were obtained. Our results indicated that large noise intensity was not conducive to populations. The above results may help us better understand the macroalgae and coral reef dynamics in the fluctuating environments. The most important findings of this article are that we studied a variable coefficient coral reef model under various random disturbances and identified the impact of random perturbation and L*é*vy jump on the dynamic properties of the system. Finally, five numerical simulations are presented to check the obtained results.

Keywords: coral reefs model; delay; noise; dynamics
Mathematics Subject Classification: 34F05, 37H10

1. Introduction

Coral reefs are a type of structure formed by animals of the Caryophyllaceae order. It is widely present in deep and shallow waters, and has important aesthetic and commercial value. Furthermore, it plays a significant role in resisting coastal storms and protecting fisheries. Coral reefs contain abundant oil and gas resources, and the diverse corals can be used as decorative crafts. Many reef areas have been developed as tourist destinations, and many countries rely on it to generate economic income [1–4]. There are two major reasons for the destruction of coral reefs: First, natural disasters, such as hurricanes, tsunamis, earthquakes, and volcanic eruptions; the second is human destruction, such as overfishing, oil and gas development, and pollution emissions. Therefore, many scholars are concerned about the protection and rational development of coral reefs. Thus, the establishment of coral reef models and the study of their dynamic mechanisms have important practical value.

In 2007, Mumby et al. [1] introduced a coral reef model

$$
\begin{aligned}
w'(t) &= w(t)\Big(a - aw(t) + (b-a)z(t) - \frac{c}{1-z(t)}\Big),\\
z'(t) &= z(t)\Big(e - d - (b+e)w(t) - ez(t)\Big),
\end{aligned}
\tag{1.1}
$$

where $w(t)$ denotes coverage of macroalgae, $z(t)$ denotes coverage of corals, and $b < d < a < r < 2a,\ 0 < c < a$,

- a represents speed which large algae grow on grass;
- b represents speed that the corals are overgrown by macroalgae;
- c denotes the rate at which large algae are eaten by parrot fish;
- e denotes the rate that coral reproduces on grass;
- d denotes mortality value of corals.

In [5], the authors studied basic properties of system (1.1) including the nonexistence and the dissipativity of limit cycles. Considering the inherent time delay, system (1.1) can be changed into the following delay model:

$$
\begin{aligned}
w'(t) &= w(t)\Big(a - aw(t) + (b-a)z(t)\Big) - \frac{cw(t-\upsilon)}{1-z(t-\upsilon)},\\
z'(t) &= z(t)\Big(e - d - (b+e)w(t) - ez(t)\Big),
\end{aligned}
\tag{1.2}
$$

here $\upsilon > 0$ is delay. They further discussed the dynamics of the equilibrium in the linearized equation of system (1.2).

Consider uncertainties from the recycle, let $a \to a+\sigma_1 dB(t)$ and $e \to e+\sigma_2 dB(t)$, then system (1.1) is changed into the form:

$$
\begin{aligned}
dw(t) &= w(t)\Big(a - aw(t) + (b-a)z(t) - \frac{c}{1-z(t)}\Big)dt + \sigma_1 w(t)dB(t),\\
dz(t) &= z(t)\Big(e - d - (b+e)w(t) - ez(t)\Big)dt + \sigma_2 z(t)dB(t),
\end{aligned}
\tag{1.3}
$$

where $B(t)$ denotes the 1−dimensional Brownian motion. Huang [6] discussed permanence and ergodicity of system (1.3) using Lie algebra and the geometric analysis method. Stochastic population systems consider only the influence of white noise and cannot accurately reveal the complex dynamic mechanisms of the population. As pointed out by the researchers in [7, 8], using a Lévy process can overcome these limitations in stochastic population systems. In fact, the sudden stochastic environmental disturbances can be described by Lévy noise. In recent years, many results for population ecosystems with the effects of Lévy noise have been obtained, such as a stochastic population system with Allee impact by Lévy noise [9]; stochastic neural network by Lévy noise [10]; stochastic population system with age-dependent [11]; and Lotka-Volterra model with random noises and delay [12]. For more two-dimensional, multi-dimensional and multi-noise problems, see [13–15].

Inspired by the above studies we investigate a non-autonomous random coral reefs model with delays and Lévy noises:

$$
\begin{aligned}
dx =& x(t^-)\Big(\gamma(t) - \gamma(t)x(t^-) - \beta_1(t)y(t^-)\Big)dt - \frac{g(t)x(t^- - \kappa(t))}{1 - y(t^- - \kappa(t))}dt \\
&+ \sigma_1(t)x(t^-)dB_1(t) + \int_{\mathcal{Y}} x(t^-)\gamma_1(t,v)\widetilde{M}(dt,dv), \\
dy =& y(t^-)\Big(\beta_2(t) - \beta_3(t)x(t^-) - r(t)y(t^-)\Big)dt \\
&+ \sigma_2(t)y(t^-)dB_2(t) + \int_{\mathcal{Y}} y(t^-)\gamma_2(t,v)\widetilde{M}(dt,dv),
\end{aligned}
\tag{1.4}
$$

where the parameter functions $\gamma,\ \beta_j,\ g,\ \sigma_i,\ \kappa$, and r are positively continuous functions. $x(t^-)$ denotes the left limit of x, $y(t^-)$ denotes the left limit of y, and M denotes a Poisson numeration measure. The $\widetilde{M}$ and δ are defined on a measurable subset $\mathcal{Y}$ of $(0,\infty)$ with $\delta(\mathcal{Y}) < \infty$, and $M(dt,dv) - \delta(dv)dt = \widetilde{M}(dt,dv)$, $\gamma_i(t,v) : \mathcal{Y} \times \Omega$ $(i = 1,2)$ is continuous and is $\mathbb{B}(\mathcal{Y}) \times G_t$-measurable. In view of biological meanings, the intensities of Lévy noises should be taken as a smaller value. System (1.4) satisfies the following initial condition:

$$
\xi = (x(s), y(s)) \in C([-\kappa^+, 0], \mathbb{R}^2_+),
\tag{1.5}
$$

where $C([-\kappa^+, 0], \mathbb{R}^2_+)$ denotes a continuous function space. For the biologic significance of system (1.4), y and x satisfy conditions:

$$
\Gamma = \{(x,y) :\ x < 1 - y,\ x, y > 0\}.
$$

Furthermore, we need the following assumptions:
(H_1) $1 + \gamma_i(s,v) > 0,\ \int_{\mathcal{Y}} |\ln(1+\gamma_i(s,v))| \vee |\ln(1+\gamma_i(s,v))|^2 \mu(dv) \le c_1$ for $s \ge 0$ and $v \in \mathcal{Y},\ i = 1,2,$ where $c_1 > 0$ is a constant.
(H_2) $\kappa'(s) < 1$ for $s \ge 0$.

Let $\mathbb{R}^2_+ = \{(\zeta,\eta) \in \mathbb{R}^2 : \zeta,\eta > 0\}$ and $(\Theta, G, \{G_t\}_{t\ge0}, P)$ be a complete probability set with a $\sigma-$field filtration $\{G_t\}_{t\ge0}$ fulfilling the standard conditions. For an integral function ϑ on $[0,\infty)$, denote

$$
\langle\vartheta(u)\rangle = \frac{1}{u}\int_0^u \vartheta(\tau)d\tau,\ \langle\vartheta(u)\rangle^- = \inf \frac{1}{u}\int_0^u \vartheta(\tau)d\tau,\ \langle\vartheta(u)\rangle^+ = \sup \frac{1}{u}\int_0^u \vartheta(\tau)d\tau.
$$

For a function ϑ on $[0,\infty)$, denote $\vartheta^+ = \sup_{\tau\ge0}\vartheta(\tau),\ \vartheta^- = \inf_{\tau\ge0}\vartheta(\tau)$.

The differences between the results of this article and existing results, as well as the innovations of this article, are as follows:
(1) We study for the first time, the dynamic properties of a variable coefficient coral reef model under random disturbances.
(2) We find that the white noises and Lévy noises have an important impact on the population system.
(3) We develop stochastic analysis techniques to study complex systems, and the above methods have good reference value for studying other stochastic dynamical systems.

The arrangement of the following parts of this article is as follows: In Section 2, we give the existence and uniqueness for the global positive solution by using Lyapunov analysis methods. In Section 3, we give the stochastic extinction and persistence in mean. Stochastic bounded in the mean are obtained in Section 4. In Section 5, we give five examples for verifying our results. We draw some conclusions in the final part.

2. Existence

In this part, by the Lyapunov analysis methods [16–18], we obtain some existence results for system (1.4).

Theorem 2.1. Suppose that (H_1)–(H_2) and (1.5) satisfy. Then, system (1.4) has unique solution (x, y) on $t \geq 0$, and the solution will stay on $\mathbb{R}_+^2$ with probability 1.

Proof. Since local Lipschitz condition holds for the coefficients of system (1.4), and $(x(s), y(s)) \in C([-\kappa^+, 0], \mathbb{R}_+^2)$, system (1.4) has the unique positive solution (x, y) for $t \in [-\kappa^+, \kappa_e)$, here κ_e represents the stopping time. Let n_0 be large enough, then initial values $x(s), y(s) \in [\frac{1}{n_0}, n_0]$. For any positive integer $n > n_0$, let the stopping time be

$$\gamma_n = \inf\left\{t \in [-\kappa^+, \kappa_e) : x(t) \notin (\frac{1}{n}, n) \text{ or } y(t) \notin (\frac{1}{n}, n)\right\}.$$

If $\gamma_\infty = \infty$, then $\kappa_e = \infty$. Hence, we should show that $\gamma_\infty = \infty$. Assume that $\gamma_\infty \neq \infty$. For $\varsigma \in (0, 1)$ and $\Gamma > 0$, we have $P(\gamma_\infty \leq \Gamma) > \varsigma$. Then, for $n_1 \geq n_0$, we get

$$P(\gamma_n \leq \Gamma) \geq \varsigma \text{ for all } n \geq n_1. \tag{2.1}$$

Let $\mathcal{V}(x, y) \in C^2(\mathbb{R}_+^2, \mathbb{R}_+)$ be

$$\mathcal{V}(x, y) = x^{\frac{1}{2}} - 1 - 0.5 \ln x + y^{\frac{1}{2}} - 1 - 0.5 \ln y + \frac{0.5}{|1 - \kappa'(t)|^-} \int_{t-\kappa(t)}^{t} \frac{g(t)x(\nu)}{x(t)(1 - y(\nu))} d\nu.$$

$\mathcal{V}$ is nonnegative. By (1.4), we get

$$\begin{aligned} d\mathcal{V}(x, y) =& \mathcal{L}\mathcal{V}dt + 0.5(x^{0.5} - 1)\sigma_1 dB_1(t) \\ &+ 0.5(y^{0.5} - 1)\sigma_2 dB_2(t) \\ &+ \int_{\mathcal{Y}} \left[(\ln(1 + \gamma_1(t, \nu))^{\frac{1}{2}} - 1)x^{\frac{1}{2}} - 0.5 \ln(1 + \gamma_1(t, \nu))\right] \widetilde{M}(dt, d\nu) \\ &+ \int_{\mathcal{Y}} \left[(\ln(1 + \gamma_2(t, \nu))^{\frac{1}{2}} - 1)y^{\frac{1}{2}} - 0.5 \ln(1 + \gamma_2(t, \nu))\right] \widetilde{M}(dt, d\nu), \end{aligned} \tag{2.2}$$

where

$$
\begin{aligned}
\mathcal{L}\mathcal{V} =&\frac{1}{2}(x^{0.5}-1)\Big(\gamma-\gamma x-\beta_1 y-\frac{gx(t-\kappa(t))}{x(1-y(t-\kappa(t)))}\Big)+0.125(-x^{0.5}+2)\sigma_1^2\\
&+\int_{\mathcal{Y}}\Big[((1+\gamma_1(t,v))^{\frac{1}{2}}-1-0.5\gamma_1(t,v))x^{\frac{1}{2}}-0.5(\gamma_1(t,v)-\ln(1+\gamma_1(t,v)))\Big]\delta(dv)\\
&+0.5(y^{0.5}-1)\Big(\beta_2-\beta_3 x-ry\Big)+0.125(-y^{0.5}+2)\sigma_2^2\\
&+\int_{\mathcal{Y}}\Big[((1+\gamma_2(t,v))^{\frac{1}{2}}-1-0.5\gamma_2(t,v))y^{\frac{1}{2}}-0.5(\gamma_2(t,v)-\ln(1+\gamma_2(t,v)))\Big]\delta(dv)\\
&+\frac{0.5}{|1-\kappa'(t)|^-}\frac{g(t)x(t)}{x(t)(1-y(t))}-\frac{0.5(1-\kappa'(t))}{|1-\kappa'(t)|^-}\frac{g(t)x(t-\kappa(t))}{x(t)(1-y(t-\kappa(t)))}\\
\le&-0.5\gamma^-x^{1.5}+0.5\gamma^+x^{0.5}+0.5(\gamma^++\beta_3^+)x+0.5(\beta_1^++r^+)y+0.25(\sigma_1^2)^+\\
&-0.5r^-y^{1.5}+0.5r^+y^{0.5}+0.25(\sigma_2^2)^++\frac{0.5g^+}{|1-\kappa'(t)|^-}x^{-1}\\
&+0.5\int_{\mathcal{Y}}(|\gamma_1(t,v)|+|\ln(1+\gamma_1(t,v))|)\delta(dv)\\
&+0.5\int_{\mathcal{Y}}(|\gamma_2(t,v)|+|\ln(1+\gamma_2(t,v))|)\delta(dv)\\
\le& C,
\end{aligned}
\tag{2.3}
$$

where C is a constant. The following proof is similar to those in [19–22]. For the convenience of readers' understanding, we provide detailed proof. From (2.2) and (2.3), we have

$$
\begin{aligned}
\int_0^{\gamma_n\wedge\Gamma}d\mathcal{V}\le\int_0^{\gamma_n\wedge\Gamma}Cdt&+\int_0^{\gamma_n\wedge\Gamma}\Big[0.5(x^{0.5}-1)\sigma_1dB_1(t)+0.5(y^{0.5}-1)\sigma_2dB_2(t)\Big]dt\\
&+\int_0^{\gamma_n\wedge\Gamma}\int_{\mathcal{Y}}\Big[(\ln(1+\gamma_1(s,v))^{\frac{1}{2}}-1)x^{\frac{1}{2}}-0.5\ln(1+\gamma_1(s,v))\Big]\widetilde{M}(ds,dv)dt\\
&+\int_0^{\gamma_n\wedge\Gamma}\int_{\mathcal{Y}}\Big[y^{\frac{1}{2}}(\ln(1+\gamma_2(s,v))^{\frac{1}{2}}-1)-0.5\ln(1+\gamma_2(s,v))\Big]\widetilde{M}(ds,dv)dt.
\end{aligned}
$$

Whence taking the expectation leads to

$$
\begin{aligned}
\mathbb{E}\mathcal{V}(x(\gamma_n\wedge\Gamma),y(\gamma_n\wedge\Gamma))&\le\mathcal{V}(x(0),y(0))+\mathbb{E}\int_0^{\gamma_n\wedge\Gamma}Cdt\\
&\le\widetilde{C}+\Gamma C.
\end{aligned}
\tag{2.4}
$$

For $n\ge n_1$, let $\Omega_n=\{\gamma_n\le\Gamma\}$. Based on (2.1), then $P(\Omega_n)\ge\varsigma$. For each $\chi\in\Omega_n$, $x(\gamma_n,\chi)$ and $y(\gamma_n,\chi)$ equals n or $\frac{1}{n}$. Hence, we have

$$
\mathcal{V}(x(\gamma_n\wedge\Gamma),y(\gamma_n\wedge\Gamma))\ge\Big(n-1-\ln n\Big)\wedge\Big(\frac{1}{n}-1-\ln\frac{1}{n}\Big).
\tag{2.5}
$$

From (2.4) and (2.5), we have

$$
\begin{aligned}
\widetilde{C}+\Gamma C&\ge\mathbb{E}\Big[1_{\Omega_n}\mathcal{V}(x(\gamma_n\wedge\Gamma),y(\gamma_n\wedge\Gamma))\Big]\\
&\ge\varepsilon\Big(n-1-\ln\frac{1}{n}\Big)\wedge\Big(\frac{1}{n}-1-\ln n\Big),
\end{aligned}
$$

where 1_{Ω_n} denotes a indicator mapping for Ω_n. For $n \to \infty$, we get the contradiction $\infty > \widetilde{C} + \Gamma C = \infty$. Hence, $\gamma_\infty = \infty$ almost surely. System (1.4) has a unique positive solution.

3. Persistence and extinction

In this part, we are devoted to considering the stochastic persistence and extinction in the mean for system (1.4). For the convenience of proof, we list the following notations:

$$\Delta_1 = \gamma^+ - \frac{(\sigma_1^2)^-}{2} + \Big[\int_{\mathcal{Y}} (\ln(1+\gamma_1(s,v)) - \gamma_1(s,v))\delta(dv)\Big]^+,$$

$$\Delta_2 = \beta_2^+ - \frac{(\sigma_2^2)^-}{2} + \Big[\int_{\mathcal{Y}} (\ln(1+\gamma_2(s,v)) - \gamma_2(s,v))\delta(dv)\Big]^+,$$

$$\hat{\Delta}_2 = \gamma^- - \frac{(\sigma_2^2)^+}{2} + \Big[\int_{\mathcal{Y}} (\ln(1+\gamma_2(s,v)) - \gamma_2(s,v))\delta(dv)\Big]^-,$$

$$M_i(t) = \int_0^t \sigma_i dB_i(s),\ \Theta_i(t) = \int_0^t \int_{\mathcal{Y}} \ln(1+\gamma_i(s,v))\widetilde{M}(ds,dv), i = 1,2.$$

Definition 3.1. [23, 24]
(1) The $v(\tau)$ is extinct if $\lim_{\tau\to\infty} v(\tau) = 0$ almost surely.
(2) The $v(\tau)$ is strongly persistent in the mean if $\lim_{\tau\to\infty}\langle v(\tau)\rangle^- = 0$ almost surely.
(3) The $v(\tau)$ is weakly persistent in the mean if $\lim_{\tau\to\infty}\langle v(\tau)\rangle^+ = 0$ almost surely.

Theorem 3.1. Suppose that assumptions (H_1)-(H_2) are satisfied. The solution $(x, y) \in \mathbb{R}_+^2$ of (1.4) exists asymptotic behaviors:
(1)

$$\limsup_{\tau\to\infty} \frac{1}{\tau} \ln x(\tau) \le \Delta_1 \text{ almost surely}$$

and

$$\limsup_{\tau\to\infty} \frac{1}{\tau} \ln y(\tau) \le \Delta_2 \text{ almost surely.}$$

(2) If $\Delta_1 < 0$, then

$$\lim_{\tau\to\infty} x(\tau) = 0 \text{ almost surely.}$$

If $\Delta_2 < 0$, then

$$\lim_{\tau\to\infty} y(\tau) = 0 \text{ almost surely.}$$

(3) If $\Delta_1 > 0$, then

$$\lim_{\tau\to\infty}\langle x(\tau)\rangle^+ \le \frac{\Delta_1}{\gamma^-} \text{ almost surely.}$$

If $\Delta_2 > 0$, then

$$\lim_{\tau\to\infty}\langle y(\tau)\rangle^+ \le \frac{\beta_2^+\beta_3^+\Delta_2}{\beta_2^-}.$$

(4) If $\hat{\Delta}_2 > \frac{\beta_3^+\Delta_1}{\gamma^-}$, we obtain that

$$\lim_{\tau\to\infty}\langle y(\tau)\rangle^- \geq \frac{\hat{\Delta}_2\gamma^- - \beta_3^+\Delta_1}{r^+\gamma^-} \text{ almost surely.}$$

Proof. Using the first equation of (1.4), then

$$\begin{aligned}\ln x(t) = \ln x(0) + \int_0^t \Big[\gamma(\tau) - \gamma(\tau)x(\tau) - \beta_1(\tau)y(\tau) - \frac{g(\tau)x(\tau-\kappa(\tau))}{x(\tau)(1-y(\tau-\kappa(\tau)))}\\ - 0.5\sigma_1^2(\tau) + \int_{\mathcal{Y}}(\ln(1+\gamma_1(\tau,v)) - \gamma_1(\tau,v))\delta(dv)\Big]d\tau\\ + \int_0^t \sigma_1 dB_1(s) + \int_0^t\int_{\mathcal{Y}} \ln(1+\gamma_1(\tau,v))\widetilde{M}(d\tau,dv).\end{aligned} \tag{3.1}$$

By (3.1) we have

$$\frac{\ln x(t)/x(0)}{t} \leq \Delta_1 + \frac{M_1(t)+\Theta_1(t)}{t}. \tag{3.2}$$

From the theorem in [25] and (3.2), then

$$\lim_{\tau\to\infty}\frac{M_1(\tau)}{\tau} = 0 \text{ and } \lim_{\tau\to\infty}\frac{\Theta_1(\tau)}{\tau} = 0.$$

Thus, it follows by (3.1) that

$$\frac{1}{t}\ln x(t) \leq \Delta_1 + \frac{\ln x(0) + M_1(t) + \Theta_1(t)}{t}. \tag{3.3}$$

In view of (3.3), then

$$\lim_{\tau\to\infty}\sup\frac{\ln x(\tau)}{\tau} \leq \Delta_1 \text{ almost surely.}$$

If $\Delta_1 < 0$, then

$$\lim_{t\to\infty} x(t) = 0 \text{ almost surely.}$$

If $\Delta_1 > 0$, in view of Lemma 2 in [26], then

$$\lim_{t\to\infty}\langle x(t)\rangle^+ \leq \frac{\Delta_1}{\gamma^-} \text{ almost surely.}$$

From the second equation of (1.4), then

$$\begin{aligned}\ln y(t) = \ln y(0) + \int_0^t \Big[\beta_2(\tau) - \beta_3(\tau)x(\tau) - r(\tau)y(\tau)\\ - 0.5\sigma_2^2(\tau) + \int_{\mathcal{Y}}(\ln(1+\gamma_2(\tau,v)) - \gamma_2(\tau,v))\delta(dv)\Big]d\tau\\ + \int_0^t \sigma_2 dB_2(\tau) + \int_0^t\int_{\mathcal{Y}} \ln(1+\gamma_2(\tau,v))\widetilde{M}(d\tau,dv).\end{aligned} \tag{3.4}$$

By (3.4), then

$$\frac{\ln y(t)/y(0)}{t} \leq \Delta_2 + \frac{M_1(t)+\Theta_1(t)}{t}. \tag{3.5}$$

Thus,

$$\lim_{\tau\to\infty}\frac{M_2(\tau)}{\tau}=0,\quad \lim_{\tau\to\infty}\frac{\Theta_2(\tau)}{\tau}=0.$$

By (3.5), then

$$\frac{1}{t}\ln y\le \Delta_2+\frac{\ln y(0)+M_2(t)+\Theta_2(t)}{t}. \tag{3.6}$$

Taking the limit on both sides of (3.6), we can derive

$$\limsup_{\tau\to\infty}\frac{1}{\tau}\ln y(\tau)\le \Delta_2 \text{ almost surely.}$$

If $\Delta_2<0$, then

$$\lim_{\tau\to\infty} y(\tau)=0.$$

On the other hand, if $\Delta_2>0$, in view of Lemma 2 in [26], then

$$\lim_{t\to\infty}\langle y(t)\rangle^+\le\frac{\beta_2^+\beta_3^+\Delta_2}{\beta_2^-}\text{ almost surely.}$$

By (1.4), then

$$\frac{\ln y/y(0)}{t}\ge\hat{\Delta}_2-\beta_3^+\langle x\rangle^+-r^+\langle y\rangle+\frac{M_2(t)+\Theta_2(t)}{t}. \tag{3.7}$$

From $\hat{\Delta}_2-\frac{\beta_3^+\Delta_1}{\gamma^-}>0$, (3.7), Lemma 2 in [26], then

$$\lim_{\tau\to\infty}\langle y(\tau)\rangle^-\ge\frac{\hat{\Delta}_2\gamma^- -\beta_3^+\Delta_1}{r^+\gamma^-}.$$

4. Stochastic boundedness

Theorem 2.1 gives a good foundation for us to further discuss the stochastivs bounded in mean of solutions.

Theorem 4.1. Assume that assumptions (H_1)-(H_2) satisfy and $p>0$. The solution $(x,y)\in\mathbb{R}_+^2$ of system (1.4) has asymptotic properties:

$$\limsup_{t\to\infty}\mathbb{E}x^p(t)\le L_1(p)$$

and

$$\limsup_{t\to\infty}\mathbb{E}y^p(t)\le L_2(p),$$

where $L_1(p)$ and $L_2(p)$ are positive constants.

Proof. Define the function $V_1(x)=x^p$. By (1.4), we drive

$$\mathbb{E}(e^t x^p)=V_1(x(0))+\mathbb{E}\int_0^t e^\tau\Big[V_1(x(\tau))+\mathcal{L}V_1(x(\tau))\Big]d\tau,$$

here

$$\begin{aligned}\mathcal{L}V_1 =& px^p\Big(\gamma - \gamma x - \beta_1 y - \frac{gx(t-\kappa(t))}{x(1-y(t-\kappa(t)))} - \frac{1-p}{2}\sigma_1^2\Big)\\ &+ \int_{\mathbb{Y}}\Big[(1+\gamma_1(t,v))^p - 1 - p\gamma_1(t,v)\Big]\delta(dv)x^p.\end{aligned}$$

Thus,

$$V_1(x(\tau)) + \mathcal{L}V_1(x(\tau)) \le -p\gamma(\tau)x^{p+1} + \Big[p\gamma(\tau) + 1 + \frac{p(p-1)}{2}\sigma_1^2(\tau)\Big]x^p \le L_1(p).$$

Further, we have

$$\mathbb{E}(e^t x^p) \le V_1(x(0)) + \mathbb{E}\int_0^t e^s L_1(p)ds = V_1(x(0)) + L_1(p)(e^t - 1),$$

which generates the assertion

$$\limsup_{t\to\infty} \mathbb{E}x^p(t) \le L_1(p).$$

Define the function $V_2(y) = y^p$. By (1.4), then

$$\mathbb{E}(e^t y^p) = V_2(y(0)) + \mathbb{E}\int_0^t e^\tau\Big[V_2(y(\tau)) + \mathcal{L}V_2(y(\tau))\Big]d\tau,$$

where

$$\begin{aligned}\mathcal{L}V_2 =& py^p\Big(\beta_2 - \beta_3 x - ry - \frac{1-p}{2}\sigma_2^2\Big)\\ &+ \int_{\mathbb{Y}}\Big[(1+\gamma_2(t,v))^p - 1 - p\gamma_2(t,v)\Big]\delta(dv)y^p.\end{aligned}$$

Thus,

$$V_2(y(\tau)) + \mathcal{L}V_2(y(\tau)) \le -pr(\tau)y^{p+1} + \Big[p\beta_2(\tau) + 1 + 0.5p(p-1)\sigma_2^2(\tau)\Big]y^p \le L_2(p).$$

Furthermore, we have

$$\mathbb{E}(e^t y^p) \le V_2(y(0)) + \mathbb{E}\int_0^t e^s L_2(p)ds = V_2(y(0)) + L_2(p)(e^t - 1),$$

which generates the assertion

$$\limsup_{t\to\infty} \mathbb{E}y^p(t) \le L_2(p).$$

5. Numerical examples

In this part, we use the method provided in [27] to provide numerical solutions for system (1.4). System (1.4) can be discretized into the following system

$$
\begin{aligned}
x^{k+1} - x^k =& x^k\Big(\gamma(k\Delta\tau) - \gamma(k\Delta\tau)x^k - \beta_1(k\Delta\tau)y^k\Big)\Delta\tau - \frac{g(k\Delta\tau)x^k(k\Delta\tau - \kappa(k\Delta\tau))}{1 - y^k(k\Delta\tau - \kappa(k\Delta\tau))}\Delta\tau \\
&+ \sigma_1(k\Delta\tau)x^k(\Delta\tau)^{\frac{1}{2}}\xi_k + \frac{\sigma_1^2(k\Delta\tau)}{2}(x^k)^2(\xi_k^2\Delta t - \Delta\tau), \\
x^{k+1} - x^k =& x^k \sum_{j=N(t_k)+1}^{N(t_k+1)} h_1(\mathcal{Y}_{1j}), \\
y^{k+1} - y^k =& y^k\Big(\beta_2(k\Delta\tau) - \beta_3(k\Delta\tau)x^k - r(k\Delta\tau)y^k\Big)\Delta\tau \\
&+ \sigma_1(k\Delta\tau)y^k(\Delta\tau)^{\frac{1}{2}}\xi_k + \frac{\sigma_2^2(k\Delta\tau)}{2}(y^k)^2(\xi_k^2\Delta\tau - \Delta\tau), \\
y^{k+1} - y^k =& +y^k \sum_{j=N(t_k)+1}^{N(t_k+1)} h_2(\mathcal{Y}_{2j}),
\end{aligned}
$$

where $k = 1, 2, \cdots$, ξ_k is a Gaussian random variable with satisfying standard distribution $N(0, 1)$,and γ_i is Lévy noise intensity. Here, we let $\ln \mathcal{Y}_{ij} = y_{ij} \sim \gamma_i N(0, 1)$ with $h_i(\mathcal{Y}_{ij}) = \mathcal{Y}_{ij} - 1$, $N(t_k)$ are counting Poisson course, $M(t)$ is equivalent to $P(\delta\Delta t)$. In the system (1.4), taking the coefficients as follows:

$$
\begin{gathered}
\gamma = 0.5 + 0.1\sin\frac{t}{3},\ \beta_1 = 0.3 + 0.01\cos\frac{\pi t}{2},\ \beta_2 = 0.5 + 0.1\cos\frac{\pi t}{3}, \\
\beta_3 = 0.2 + 0.01\sin\frac{\pi t}{4},\ r = 0.7 + 0.001\cos\frac{\pi t}{5},\ g = 0.5 + 0.01\sin\frac{\pi t}{2}, \\
\kappa(t) = t - 0.1\sin t,\ \mathcal{Y} = (0, \infty),\ \delta(\mathcal{Y}) = 1.
\end{gathered}
$$

We have

$$
\gamma^- = 0.4,\ \gamma^+ = 0.6,\ \ \beta_2^+ = 0.6,\ \beta_2^- = 0.4,\ \beta_3^+ = 0.21.
$$

Example 5.1. Let $\sigma_1 = 0.25+0.05\sin\frac{t}{2}$, $\sigma_2 = 6.5+0.02\sin\frac{t}{3}$, $\gamma_1 = \gamma_2 = 0.12$. By simple computation, we have $\sigma_1^- = 0.2$, $\sigma_2^- = 6.48$. Thus,

$$
\Delta_1 = \gamma^+ - 0.5(\sigma_1^2)^- + \Big[\int_{\mathcal{Y}}(\ln(1 + \gamma_1(s, v)) - \gamma_1(s, v))\delta(dv)\Big]^+ \approx 0.46
$$

and

$$
\Delta_2 = \beta_2^+ - 0.5(\sigma_2^2)^- + \Big[\int_{\mathcal{Y}}(\ln(1 + \gamma_2(s, v)) - \gamma_2(s, v))\delta(dv)\Big]^+ \approx -20.51 < 0.
$$

It follows by Theorem 3.1 that the solution (x, y) of (1.4) has the asymptotic property:

$$
\lim_{\tau\to\infty}\langle x\rangle^+ \leq \frac{\Delta_1}{\gamma^-} = 1.15,\ \ \lim_{\tau\to\infty} y = 0.
$$

Figure 1a,b verifies the above results.

Example 5.2. Let $\sigma_1 = 5.22+0.01\sin\frac{t}{2}$, $\sigma_2 = 0.2+0.03\sin\frac{t}{3}$, $\gamma_1 = \gamma_2 = 0.15$. By simple computation, we have $\sigma_1^- = 5.21$, $\sigma_2^- = 0.17$. Thus,

$$\Delta_1 = \gamma^+ - 0.5(\sigma_1^2)^- + \left[\int_{\mathcal{Y}} (\ln(1+\gamma_1(s,u)) - \gamma_1(s,u))\delta(du)\right]^+ \approx -13.12 < 0$$

and

$$\Delta_2 = \beta_2^+ - 0.5(\sigma_2^2)^- + \left[\int_{\mathcal{Y}} (\ln(1+\gamma_2(s,u)) - \gamma_2(s,u))\delta(du)\right]^+ \approx 4.48 > 0.$$

It follows by Theorem 3.1 that the solution (x, y) of (1.4) has the asymptotic behaviors:

$$\lim_{t\to\infty}\langle y(t)\rangle^+ \le \frac{\beta_2^+\beta_3^+\Delta_2}{\beta_2^-} = 1.41, \quad \lim_{\tau\to\infty} x(\tau) = 0.$$

Figure 2a,b verifies the above results.

Example 5.3. Let $\sigma_1 = 4 + 0.05\sin\frac{t}{2}$, $\sigma_2 = 5 + 0.02\sin\frac{t}{3}$, $\gamma_1 = \gamma_2 = 0.5$. By simple computation, we have $\sigma_1^- = 3.95$, $\sigma_2^- = 4.98$. Thus,

$$\Delta_1 = \gamma^+ - 0.5(\sigma_1^2)^- + \left[\int_{\mathcal{Y}} (\ln(1+\gamma_1(s,u)) - \gamma_1(s,u))\delta(du)\right]^+ \approx -7.8 < 0$$

and

$$\Delta_2 = \beta_2^+ - 0.5(\sigma_2^2)^- + \left[\int_{\mathcal{Y}} (\ln(1+\gamma_2(s,u)) - \gamma_2(s,u))\delta(du)\right]^+ \approx -12.3 < 0.$$

It follows by Theorem 3.1 that the solution $(x(t), y(t))$ of system (1.4) has the asymptotic property:

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0 \text{ almost surely.}$$

Figure 3a,b verifies the above results.

From Figures 1–3, when the external environment is harsh (the densities of white noises and Lévy noise are very high), it will go extinct. Hence, the density of external disturbance has a direct impact on the survival of populations. In order for coral reefs to survive for a long time, humans should create a favorable environment.

Example 5.4. Let $\sigma_1 = 0.02 + 0.01\sin\frac{t}{5}$ and $\sigma_2 = 0.03 + 0.02\sin\frac{t}{2}$. Furthermore, choose Lévy noises $\gamma_1 = \gamma_2 = 0.01$. It can be seen that all conditions of Theorem 4.1 hold. Based on Theorem 4.1, (1.4) has a non-trivial solution that is stochastic bounded in mean. Figure 4a,b verifies the above results.

Example 5.5. Let $\sigma_1 = 0.03 + 0.01\sin\frac{t}{3}$ and $\sigma_2 = 0.04 + 0.02\sin\frac{t}{2}$. Furthermore, choose Lévy noises $\gamma_1 = \gamma_2 = 0$. All conditions of Theorem 4.1 hold. Based on Theorem 4.1, (1.4) has a non-trivial solution that is stochastic bounded in the mean. Figure 5a,b verifies the above results.

Remark 5.1. In Figure 1a, when the rate γ of macroalgae spread is high, σ_1 and γ_1 are low, the macroalgae x is surviving. In Figure 1b, when the rate r of corals recruit is low, σ_2 and γ_2 are high, the corals y are extinct. In Figure 2a, when the rate γ of macroalgae spread is low, σ_1 and γ_1 are high, the macroalgae x is extinct. In Figure 2b, when the rate r of corals recruit is high, σ_2 and γ_2 are low, the corals y are surviving. In Figure 3a, when the rate γ of macroalgae spread is low, σ_1 and γ_1 are high, the macroalgae x is extinct. In Figure 3b, when the rate r of corals recruit is low, density σ_2 of white noise and Lévy noise γ_2 are high, the corals y are extinct. In Figure 4a,b, when σ_1, σ_2, γ_1, and γ_2 are all sufficiently small, all assumptions of Theorem 4.1 are satisfied, and the macroalgae x and the corals y are stochastic bounded in mean. In Figure 5a,b, the dynamical mechanism of solution to (1.4) without Lévy noise can be found.

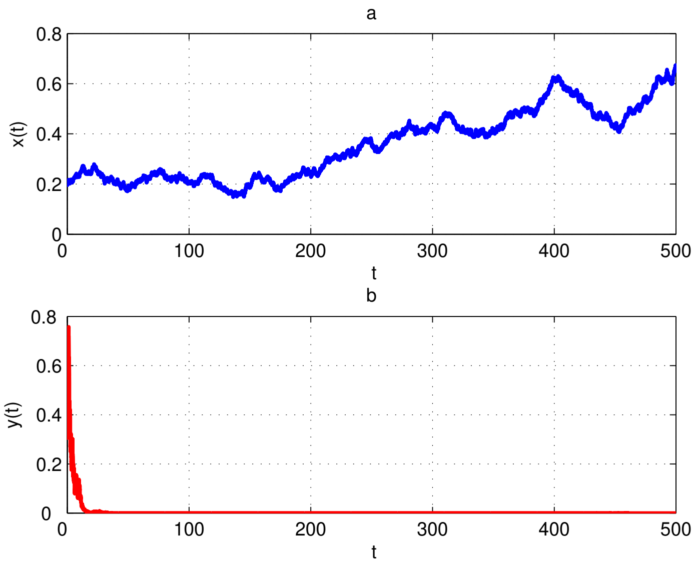

Figure 1. The numerical simulation of the stochastic path of system (1.4) using the parameters of Example 5.1.

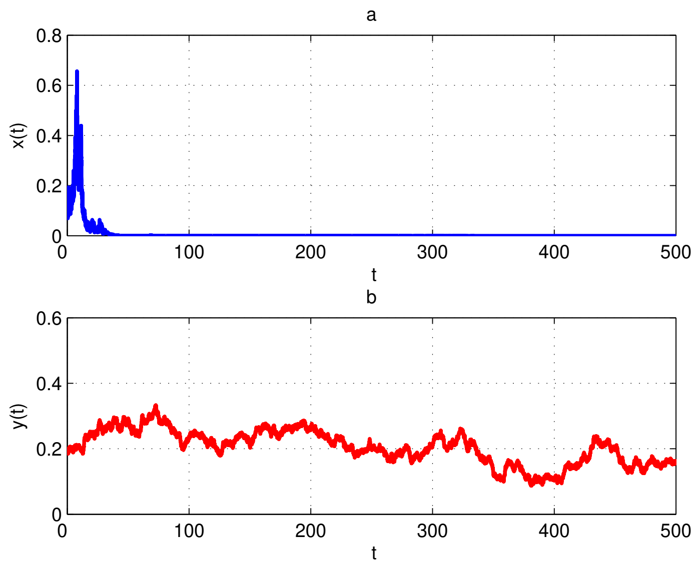

Figure 2. The numerical simulation of the stochastic path of system (1.4) using the parameters of Example 5.2.

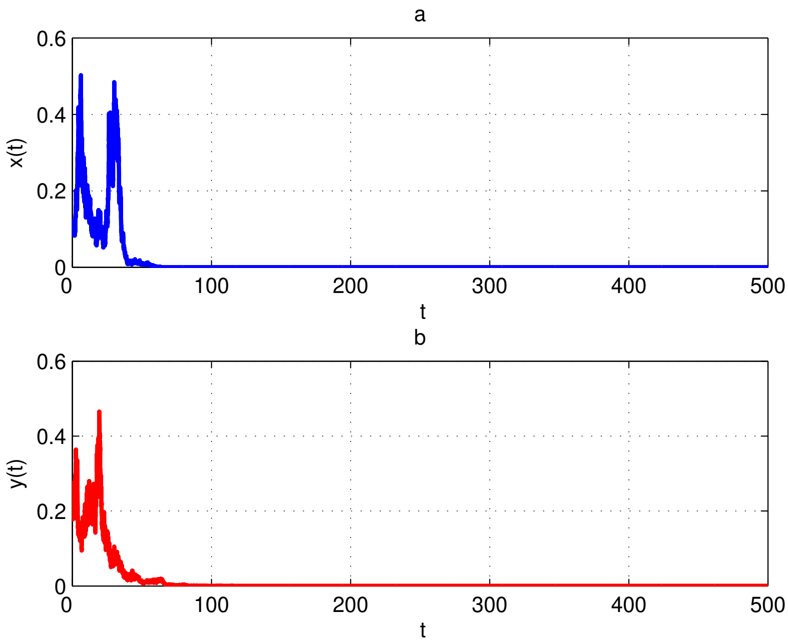

Figure 3. The numerical simulation of the stochastic path of system (1.4) using the parameters of Example 5.3.

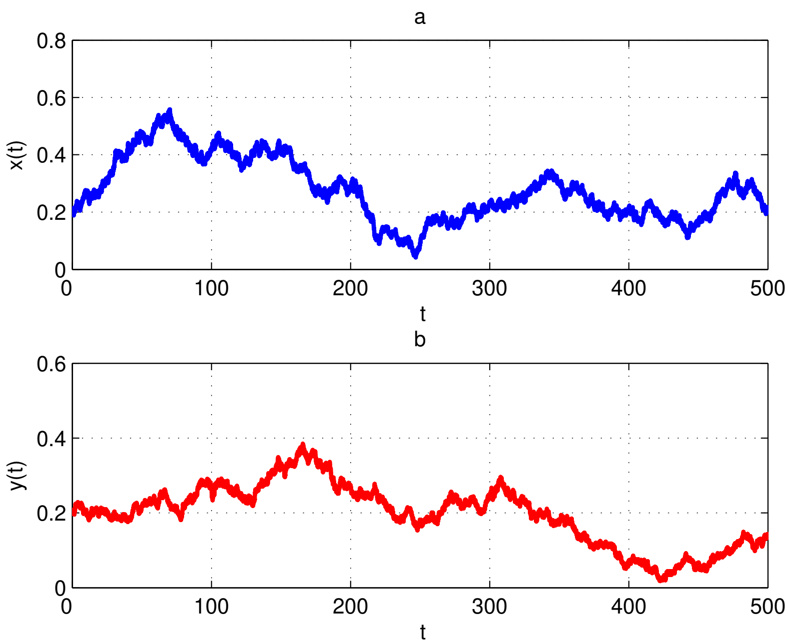

Figure 4. The numerical simulation of the stochastic path of system (1.4) using the parameters of Example 5.4.

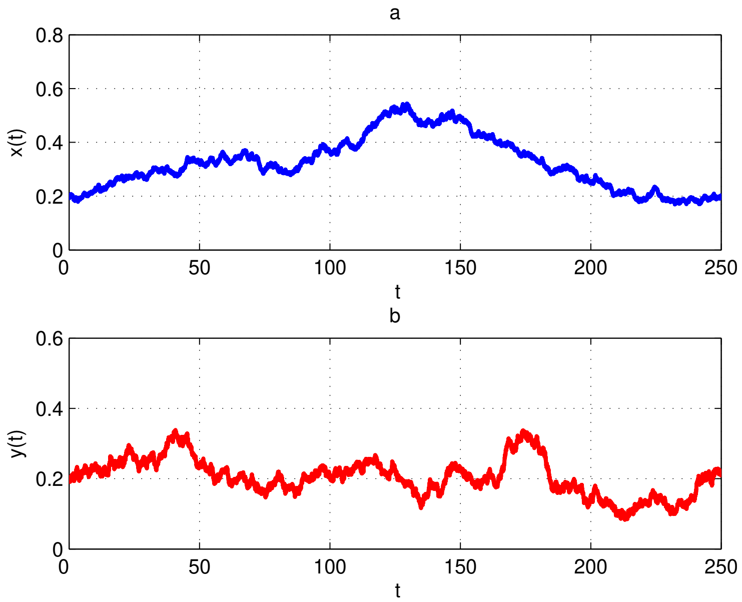

Figure 5. The numerical simulation of the stochastic path of system (1.4) using the parameters of Example 5.5.

6. Conclusions

In this paper, an extension of the delayed coral reefs model with stochastic disturbances originally developed in [3] was derived to explore the impacts of environmental noises on the populations dynamic mechanism. In spite of population systems with Lévy noises were reported in the papers, a delayed coral reef model taking into account the impacts of stochastic disturbances has been rarely considered. Therefore, we study the influences of stochastic disturbances on the dynamic mechanism of a coral reef model. We first show the existence results. Next, we study the stochastic extinction, persistence, and stochastic bounded in mean, which are of great significance for the coral reef model. The numerical examples indicate that white noises and Lévy noises have a great influence on population dynamics in the aquatic environments.

In fact, when system (1.4) with only white noises or only Lévy noise or both, it can be found that the persistence of the coral reef model remains unchanged with the weak noise intensities ([5, 6]). For the strong noise intensities, coral reefs and macroalgae are the stochastic extinction ([2, 3]). Our results indicate that the weak white noises or Lévy noises could not affect the stochastic persistence of the system. Although we obtained rich dynamic properties of system (1.4), there are no results on the system's ergodicity and system (1.4) with Markov processes. In a future study, we will conduct research on these aspects.

Author contributions

Mei He: Writing-review, conceptualization; Bo Du: Writing-original draft, methodology. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. P. Mumby, A. Hastings, H. Edwards, Thresholds and the resilience of Caribbean coral reefs, *Nature*, **450** (2007), 98–101. https://doi.org/10.1038/nature06252
2. P. Mumby, The impact of exploiting grazers (Scaridae) on the dynamics of Caribbean coral reefs, *Ecol. Appl.*, **16** (2006), 747–769. https://doi.org/10.1890/1051-0761(2006)016[0747:TIOEGS]2.0.CO;2
3. T. Hughes, Catastrophes, phase shifts, and large-scale degradation of a Caribbean coral reefs, *Science*, **265** (1994), 1547–1551. https://doi.org/10.1126/science.265.5178.1547
4. J. Blackwood, A. Hastings, P. Mumby, The effect of fishing on hysteresis in Caribbean coral reefs, *Theor. Ecol.*, **5** (2012), 105–114. https://doi.org/10.1007/s12080-010-0102-0
5. X. Li, H. Wang, Z. Zhang, A. Hastings, Mathematical analysis of coral reef models, *J. Math. Anal. Appl.*, **416** (2014), 352–373. https://doi.org/10.1016/j.jmaa.2014.02.053
6. Z. Huang, Positive recurrent of stochastic coral reefs model, *Physica A*, **486** (2017), 751–761. https://doi.org/10.1016/J.PHYSA.2017.05.073
7. J. Bao, X. Mao, G. Yin, C. Yuan, Competitive lotka-volterra population dynamics with jumps, *Nonlinear Anal. Theor.*, **74** (2011), 6601–6616. https://doi.org/10.1016/j.na.2011.06.043
8. Q. Liu, Q. Chen, Z. Liu, Analysis on stochastic delay lotka-volterra systems driven by Lévy noise, *Appl. Math. Comput.*, **235** (2014), 261–271. https://doi.org/10.1016/j.amc.2014.03.011
9. Q. Zhang, D. Jiang, Y. Zhao, D. O'Regan, Asymptotic behavior of a stochastic population model with Allee effect by Lévy jumps, *Nonlinear Anal. Hybri.*, **24** (2017), 1–12. https://doi.org/10.1016/j.nahs.2016.10.005
10. R. Ikram, A. Khan, A. Raezah, Impact of supervise neural network on a stochastic epidemic model with Lévy noise, *AIMS Mathematics*, **9** (2024), 21273–21293. https://doi.org/10.3934/math.20241033
11. Y. Du, X. Li, M. Ye, Q. Zhang, The numerical algorithm for stochastic age-dependent population system with Lévy noise in a polluted environment, *J. Differ. Equ. Appl.*, **28** (2022), 1214–1263. https://doi.org/10.1080/10236198.2022.2132856
12. Q. Liu, Q. Chen, Dynamics of stochastic delay Lotka-Volterra systems with impulsive toxicant input and Lévy noise in polluted environments, *Appl. Math. Comput.*, **256** (2015), 52–67. https://doi.org/10.1016/j.amc.2015.01.009

13. P. Zhu, Analytical equivalent transformation method for nonlinear stochastic dynamics with multiple noises in high dimensions, *Chaos Soliton Fract.*, **176** (2023), 114134. https://doi.org/10.1016/j.chaos.2023.114134
14. P. Zhu, An equivalent analytical method to deal with cross-correlated exponential type noises in the nonlinear dynamic system, *Chaos Soliton Fract.*, **150** (2021), 111124. https://doi.org/10.1016/j.chaos.2021.111124
15. I. A. Surazhevsky, V. A. Demin, A. I. Ilyasov, A. V. Emelyanov, K. E. Nikiruy, V. V. Rylkov, et al., Noise-assisted persistence and recovery of memory state in a memristive spiking neuromorphic network, *Chaos Soliton Fract.*, **146** (2021), 110890. https://doi.org/10.1016/j.chaos.2021.110890
16. R. Khasminskii, *Stochastic stability of differential equations*, Berlin, Heidelberg: Springer, 2012. https://doi.org/10.1007/978-3-642-23280-0
17. D. Jiang, C. Ji, N. Shi, J. Yu, The long time behavior of DI SIR epidemic model with stochastic perturbation, *J. Math. Anal. Appl.*, **372** (2010), 162–180. https://doi.org/10.1016/j.jmaa.2010.06.003
18. X. Mao, C. Yuan, *Stochastic differential equations with Markovian switching*, World Scientific, 2006. https://doi.org/10.1142/p473
19. X. Mao, G. Marion, E. Renshaw, Environmental noise suppresses explosion in population dynamics, *Stoch. Proc. Appl.*, **97** (2002), 95–110. https://doi.org/10.1016/S0304-4149(01)00126-0
20. X. Zhang, K. Wang, Asymptotic behavior of stochastic Gilpin-Ayala mutualism model with jumps, *Electr. J. Differ. Equ.*, **2013** (2013), 1–17.
21. X. Zhang, K. Wang, Stability analysis of a stochastic Gilpin-Ayala model driven by Lévy noise, *Commun. Nonlinear Sci. Numer. Simul.*, **19** (2014), 1391–1399. https://doi.org/10.1016/j.cnsns.2013.09.013
22. M. Liu, K. Wang, Persistence and extinction of a single-species population system in a polluted environment with random perturbations and impulsive toxicant input, *Chaos Soliton Fract.*, **45** (2012), 1541–1550. https://doi.org/10.1016/j.chaos.2012.08.011
23. C. Zhu, G. Yin, On competitive Lotka-Volterra model in random environments, *J. Math. Anal. Appl.*, **357** (2009), 154–170. https://doi.org/10.1016/j.jmaa.2009.03.066
24. L. Wan, Q. Zhou, Stochastic Lotka-Volterra model with infinite delay, *Statist. Probab. Lett.*, **79** (2009), 698–706. https://doi.org/10.1016/j.spl.2008.10.016
25. R. Lipster, A strong law of large numbers for local martingales, *Stochastic*, **3** (1980), 217–228. https://doi.org/10.1080/17442508008833146
26. M. Liu, K. Wang, Stochastic Lotka-Volterra systems with Lévy noise, *J. Math. Anal. Appl.*, **410** (2014), 750–763. https://doi.org/10.1016/j.jmaa.2013.07.078
27. D. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.*, **43** (2001), 525–546. https://doi.org/10.1137/S0036144500378302

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