AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(9): 21159–21192.
DOI: 10.3934/math.2025945
Received: 05 June 2025
Revised: 25 August 2025
Accepted: 02 September 2025
Published: 15 September 2025

Research article

Qualitative analysis of the fractional Basset problem with boundary conditions via the Caputo–Fabrizio derivative

Shayma Adil Murad[1,*], **Ava Sh. Rafeeq**[2], **Alan M. Omar**[1], **Mohammed O. Mohammed**[2], **Thabet Abdeljawad**[3,4,5,*] **and Manar A. Alqudah**[6]

[1] Department of Mathematics, College of Science, University of Duhok, Duhok 42001, Iraq

[2] Department of Mathematics, College of Science, University of Zakho, Duhok, Iraq

[3] Department of Fundamental Sciences, Faculty of Engineering and Architecture, Istanbul Gelisim University, Avcılar-Istanbul 34310, Turkiye

[4] Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia

[5] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

[6] Department of Mathematical Sciences, College of Sciences, Princess Nourah bint Abdulrahman University P. O. Box 84428, Riyadh 11671, Saudi Arabia

* **Correspondence:** Email: shayma.murad@uod.ac, tabdeljawad@psu.edu.sa.

Abstract: The concept of fixed points serves as an effective and essential tool in analyzing nonlinear phenomena. This study investigates the existence and uniqueness of solutions for a class of Basset-type fractional differential equations with boundary conditions involving the Caputo–Fabrizio fractional derivative. These equations emerge from the generalized Basset force describing the motion of a sphere settling in a viscous fluid. Darbo's fixed point theorem, combined with the measure of noncompactness, is applied to establish the existence of solutions. Uniqueness is ensured via Banach's fixed point theorem. Additionally, stability analysis is performed using Ulam–Hyers and Ulam–Hyers–Rassias concepts. An illustrative example, supported by tables and figures, demonstrates the applicability of the theoretical results.

Keywords: fractional differential equations $\mathcal{FDE}$s; Basset equation; Caputo–Fabrizio ($\mathcal{CF}$) derivative; fixed point theorems; Ulam stability
Mathematics Subject Classification: 34A08, 26A33, 34K20, 34A12, 97M50

1. Introduction

Fractional differential equations ($\mathcal{FDE}$s) have gained substantial popularity in recent decades due to their ability to model a variety of physical, chemical, and engineering processes. $\mathcal{FDE}$s initially emerged within the framework of dynamic systems and control theory, where they are used to describe controllers and regulated systems. Their increasing relevance is also driven by novel applications in modeling physical systems, such as the simulation of rheological materials that exhibit both dissipative and elastic behavior. For additional applications of $\mathcal{FDE}$s, see [1–3]. Fractional differential equations are important for understanding complex physical phenomena. According to Mainardi [4], fractional calculus helps describe linear viscoelastic materials, where the current response of these materials is influenced by their past behavior, a phenomenon known as memory effects.

Recent studies [5–7] have explored advanced mathematical models to analyze fluid flow and wave propagation phenomena, incorporating effects such as thermal delay, electromagnetic fields, and nonlocal elasticity. In the field of control theory, Zhang et al. [8] proposed a robust fault-tolerant scheme for attitude stabilization in spacecraft based on Lyapunov methods, while Wang et al. [9] developed a consensus approach for fuzzy fractional-order multi-agent systems. Collectively, these works reflect the growing interest in modeling complex physical and engineering systems using generalized and memory-based frameworks-challenges also addressed in this study through the fractional modeling of Basset-type equations with Caputo–Fabrizio derivatives.

The Basset equation is fundamental in mathematical physics, and its application appears in the dynamics of a sphere immersed in an incompressible viscous fluid. This is a classical problem with notable applications in material sciences, as well as in the study of geophysical flows. A particularly important aspect is the study of a sphere subjected to gravity, which was first presented by Basset [10] and later extended in [11], where a special hydraulic force known as "Basset's force" was introduced. Mainardi [12, 13] identified the generalized Basset's force by examining Basset's force in terms of a fractional derivative of order $\frac{1}{2}$ of the particle's velocity with respect to the fluid. For more details about the Basset equation, see [14–16].

In modern applied mathematics and engineering, fixed-point methods play a crucial role in solving nonlinear problems, optimizing iterative algorithms, and modeling dynamic systems. Key fixed-point theorems provide rigorous criteria for the existence, uniqueness, and stability of solutions to fractional differential equations, which are essential for applications in control theory, signal processing, and fluid dynamics. Recent advances in this field are well-documented, with significant contributions such as [17–19]. Initial and boundary value problems together with Ulam stability for various fractional derivatives have been examined in [20–22]. The solution and Ulam–Hyers–Rassias stability of boundary value problems involving both Caputo fractional and standard derivatives were investigated by Castro and Silva [23], which ensures important properties around the exact solution.

The Basset problem describes the Lagrangian acceleration of a spherical particle in an unsteady flow, combining viscous forces, gravitational force, buoyancy, virtual mass, and Basset forces. While widely applied, the fundamental solution's properties remained unknown. The authors in [24] established the global existence and uniqueness of solutions for the fractional Basset–Boussinesq–Oseen equation within a partially ordered Banach space, along with an easily implementable approximation method. These results were later improved in [25] by relaxing the initial conditions and verifying a new existence and uniqueness theorem under weaker conditions.

More specifically, the well-posedness of the Basset problem in spaces of smooth functions was demonstrated by Ashyralyev [26]. Cona [27] presented a new result on the existence for the Basset–type initial value problem using the fixed-point iteration method. Bahuguna and Anjali [28] studied a generalized Basset problem in viscoelasticity, with the existence and uniqueness of solutions confirmed via Rothe's method in fractional Banach spaces. A priori estimates were derived, ensuring the well-posedness of the discrete problem.

The Caputo–Fabrizio fractional derivative ($\mathcal{CFFD}$), defined with a nonsingular exponential kernel [29], has been effectively used in modeling various phenomena in fields such as biology and infectious diseases [30–32]. Moreover, considerable attention has been given to the existence and Ulam stability of solutions of the initial and boundary value problems for fractional differential equations with Caputo–Fabrizio fractional derivatives ($\mathcal{CFFD}$s); see [33–35]. Fractional differential equations via the Caputo–Fabrizio derivative are examined in [36–38] , with a focus on their analytical properties and applications. The Hyers–Ulam stability and existence of solutions for weighted fractional differential equations were studied in [39].

Although numerous contributions have addressed problems involving the Caputo–Fabrizio fractional derivative, to the best of our knowledge, the Basset problem, formulated under the Caputo–Fabrizio derivative within a theoretical setting and solved using the modified Caputo–Fabrizio operator, has not been reported previously. This dual-approach methodology enhances the model's ability to capture memory effects and effectively handle boundary conditions. It provides novel theoretical insights and practical significance that have not been explored before. The contributions of this work can be summarized as follows:

(1) The establishment of a new Caputo–Fabrizio fractional Basset equation ($\mathcal{CFFBE}$) capturing memory effects in Basset-type models.

(2) Rigorous validation of the existence of solutions through Darbo's fixed-point theorem, which generalizes both the Banach contraction principle and the Schauder fixed-point theorem by employing the measure of noncompactness; for more details about Darbo's theorem, see [40–42].

(3) The uniqueness of the solution is investigated by applying Banach's fixed-point theorem.

(4) Investigation of the Ulam–Hyers and Ulam–Hyers–Rassias stabilities of the proposed boundary value problem.

(5) Presentation of an illustrative example, supported by figures and tables, to validate the theoretical findings and demonstrate the practical significance of the proposed model.

The remaining parts are structured in the following manner. Section 2 presents the problem formulation and discusses the key qualitative properties of the problem. Section 3 reviews the fundamental mathematical principles, such as the definitions, lemmas, and the fixed-point technique necessary for this study. Section 4 details some auxiliary lemmas, including the conversion of the ($\mathcal{CFFBE}$) into an equivalent integral equation. Section 5 contains the main results, covering the existence and uniqueness of solutions for the $\mathcal{CFFBE}$. Section 6 analyzes the Ulam–Hyers ($\mathcal{UH}$) and Ulam–Hyers–Rassias ($\mathcal{UHR}$) stability results within a general framework. Finally, Section 7 is dedicated to evaluating the effectiveness of the main outcomes.

2. Basic equation

The qualitative behavior of fractional differential equations, including the existence, uniqueness, and stability of solutions, is essential for understanding the complex dynamics of fractional-order systems. Recognized as generalizations of classical integer-order models, these systems require rigorous analysis to ensure their mathematical and physical validity.

Inspired by the significance of the modified Caputo–Fabrizio ($\mathcal{MCF}$) fractional derivative introduced in [43, 44] and the findings in [26–28], this analysis considers a composite fractional differential equation involving two Caputo–Fabrizio derivatives of different orders, given by the boundary value problem

$$ {}^{\mathcal{CF}}\mathcal{D}_0^{\rho}\Phi(\varkappa) + {}^{\mathcal{CF}}\mathcal{D}_0^{\omega}\Phi(\varkappa) + \Phi(\varkappa) = \mathcal{Q}(\varkappa), \qquad \varkappa \in \mathfrak{J} = [0, \Upsilon], \tag{2.1} $$

$$ \Phi(0) = \Phi(\Upsilon), \tag{2.2} $$

where $\mathcal{Q}(\varkappa) = \mathbf{Q}\left(\varkappa, \Phi(\varkappa), {}^{\mathcal{CF}}\mathcal{D}_0^{\rho}\Phi(\varkappa), {}^{\mathcal{CF}}\mathcal{D}_0^{\omega}\Phi(\varkappa)\right)$ with $\mathbf{Q} : \mathfrak{J} \times \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$, ${}^{CF}D_0^{\omega}$ and ${}^{\mathcal{CF}}\mathcal{D}_0^{\rho}$ are Caputo–Fabrizio derivatives, $0 < \rho, \omega < 1$, within this framework, the existence, uniqueness, and stability of solutions will be rigorously examined, as these properties are fundamental to the mathematical and physical validity of the model. Ensuring these qualitative aspects guarantees that the solutions are well-defined and robust under perturbations, which is crucial for applications across various fields such as viscoelasticity, fluid dynamics, control theory, and anomalous diffusion.

3. Preliminaries

This section presents some definitions, lemmas, and foundational concepts that will be used throughout the remainder of the paper. Denote the Banach space of all continuous functions by $C(\mathfrak{J}, \mathfrak{X})$ with the norm

$$ \|\Phi\| = \sup_{\varkappa \in [0,\Upsilon]} |\Phi(\varkappa)|. $$

Let $L^1(\mathfrak{J})$ represent the space of Bochner integrable functions $\Phi : \mathfrak{J} \to \mathfrak{X}$, with the norm

$$ \|\Phi\|_1 = \int_0^{\Upsilon} |\Phi(\varkappa)| d\varkappa. $$

Definition 3.1. *[44] The Caputo–Fabrizio fractional derivative for $\Phi \in L^1(\mathfrak{J})$, $\rho \in (0, 1)$ is defined as*

$$ {}_0^{\mathcal{CF}}D_{\varkappa}^{\rho}(\Phi(\varkappa)) = \frac{\mathcal{M}(\rho)}{1-\rho} \int_0^{\varkappa} \Phi'(\nu) e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)} d\nu. \tag{3.1} $$

To normalize the function $\mathcal{M}(\rho)$, the conditions $\mathcal{M}(0) = \mathcal{M}(1) = 1$ are imposed. Under this normalization, the modified Caputo–Fabrizio fractional integral is defined as follows

$$ {}^{\mathcal{CF}\mathcal{M}}I_0^{\rho}\Phi(\varkappa) = \frac{1-\rho}{\mathcal{M}(\rho)}(\Phi(\varkappa) - \Phi(0)) + \frac{\rho}{\mathcal{M}(\rho)} \int_0^{\varkappa} (\Phi(\nu) - \Phi(0)) d\nu. \tag{3.2} $$

Lemma 3.2. *[44] For $\Phi \in L^1(\mathfrak{J})$ and $\rho \in (0, 1)$, the following results hold*

(1) $({}^{\mathcal{CFM}}\mathcal{I}_0^{\rho}\, {}^{\mathcal{CF}}\mathcal{D}_0^{\rho}\Phi)(\varkappa) = \Phi(\varkappa) - \Phi(0)$

(2) $({}^{\mathcal{CF}}\mathcal{D}_0^{\rho}\Phi\, {}^{\mathcal{CFM}}\mathcal{I}_0^{\rho})(\varkappa) = \Phi(\varkappa) - \Phi(0)$.

Definition 3.3. *[40] Let $\mathfrak{X}$ be a Banach space and $\mathfrak{B}_{H2}$ a bounded subsets of $\mathfrak{X}$. Then, the Hausdorff measure of noncompactness of $\mathfrak{B}_{H2}$ is defined by*

$$F(\mathfrak{B}_{H2}) = \inf\{\tau_t > 0 : \mathfrak{B}_{H2} \textit{ has a finite cover by balls of radius } \tau_t\}.$$

Lemma 3.4. *[40] Let $\mathfrak{B}_{H1}, \mathfrak{B}_{H2} \subset \mathfrak{X}$ be bounded. Then, the Hausdorff measure of noncompactness has the following properties:*

(1) $\mathfrak{B}_{H1} \subset \mathfrak{B}_{H2} \Rightarrow F(\mathfrak{B}_{H1}) \leq F(\mathfrak{B}_{H2})$;

(2) $F(\mathfrak{B}_{H1}) = 0 \Leftrightarrow \mathfrak{B}_{H1}$ *is relatively compact;*

(3) $F(\mathfrak{B}_{H1} \cup \mathfrak{B}_{H2}) = \max\{F(\mathfrak{B}_{H1}), F(\mathfrak{B}_{H2})\}$;

(4) $F(\mathfrak{B}_{H1}) = F(\overline{\mathfrak{B}_{H1}}) = F(conv(\mathfrak{B}_{H1}))$, *where* $\overline{\mathfrak{B}_{H1}}$ *and* $\mathfrak{B}_{H1}$ *represent the closure and the convex hull of* $\mathfrak{B}_{H1}$, *respectively;*

(5) $F(\mathfrak{A}_g + \mathfrak{A}_y) \leq F(\mathfrak{B}_{H1}) + F(\mathfrak{B}_{H2})$, *where* $(\mathfrak{B}_{H1}) + (\mathfrak{B}_{H2}) = \{\mathfrak{l} + \mathfrak{v} : \mathfrak{l} \in \mathfrak{B}_{H1}, \mathfrak{v} \in \mathfrak{B}_{H2}\}$;

(6) $F(\mathfrak{w}\mathfrak{B}_{H1}) \leq |\mathfrak{w}|F(\mathfrak{B}_{H1}) \ \forall \ \mathfrak{w} \in \mathfrak{R}$.

Lemma 3.5. *[40] If $\hat{W} \subseteq C(\mathfrak{J}, \mathfrak{X})$ is bounded and equicontinuous, then $F(\hat{W}(\varkappa))$ is continuous on $\mathfrak{J}$ and*

$$F(\hat{W}) = \sup_{\varkappa \in \mathfrak{J}} F(\hat{W}(\varkappa)).$$

The set $\mathfrak{B}_{H2} \subset L(\mathfrak{J}, \mathfrak{X})$ is (uniformly) bounded if $\exists\, \hat{\rho} \in L^1(\mathfrak{J}, \mathfrak{R}^+)$, such that

$$\|\Phi(\varkappa)\| \leq \hat{\rho}(\varkappa), \ \forall\ \Phi \in \mathfrak{B}_{H2} \quad \textit{and a.e. } \varkappa \in \mathfrak{J}.$$

Lemma 3.6. *[45] If $\{\Phi_{\hat{n}}\}_{\hat{n}=1}^{\infty} \subset L^1(\mathfrak{J}, \mathfrak{X})$ is integrable (uniformly), then $F(\{\Phi_{\hat{n}}\}_{n=1}^{\infty})$ is measurable, and*

$$F\left(\left\{\int_a^{\varkappa} \Phi_{\hat{n}}(\nu)d\nu\right\}_{\hat{n}=1}^{\infty}\right) \leq\ 2\int_a^{\varkappa} F\left(\{\Phi_{\hat{n}}(\nu)\}_{\hat{n}=1}^{\infty}\right) d\nu.$$

Lemma 3.7. *[46] If $\hat{W}$ is bounded, then for each ∇, there is a sequence $\{\Phi_{\hat{n}}\}_{\hat{n}=1}^{\infty} \subset \hat{W}$, such that*

$$F(\hat{W}) \leq 2F\left(\{\Phi_{\hat{n}}\}_{\hat{n}=1}^{\infty}\right) + \nabla.$$

Definition 3.8. *[46] The mapping $\mathcal{G} : \Omega \subset \mathfrak{X} \to \mathfrak{X}$ is said to be a F-contraction, if there exist a positive constant, $\mathfrak{K} < 1$ exist such that*

$$F(\mathcal{G}(\hat{W})) \leq \mathfrak{K}F(\hat{W}), \quad \textit{for all bounded } \ \hat{W} \subset \Omega.$$

Definition 3.9. *[47] A function $\mathfrak{f} : \mathfrak{J} \times \mathfrak{X} \to \mathfrak{X}$ satisfies the Carathéodory conditions if the following hold:*

- $\mathfrak{f}(\varkappa, \Phi)$ *is continuous with respect to* $\Phi \in \mathfrak{X}$ *for* $\varkappa \in \mathfrak{J}$,
- $\mathfrak{f}(\varkappa, \Phi)$ *is measurable with respect to* $\varkappa$ *for* $\Phi \in \mathfrak{X}$.

Theorem 3.10. *[40] Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space $\mathfrak{X}$ and let $\mathcal{G} : \Omega \to \Omega$ be a continuous operator. If $\mathcal{G}$ is a F-contraction, then $\mathcal{G}$ has at least one fixed point.*

4. Some auxiliary lemmas

This section is devoted to demonstrating the equivalence of the boundary value problems (2.1) and (2.2).

Definition 4.1. *[48] Let $n < \omega \le n+1$ and Φ be such that $\Phi^{(n)} \in H^1(a,b)$. Set $\beta = \omega - n$. Then $\beta \in (0,1]$ and we define*

$$\left({}_{a}^{\mathcal{CFC}}\mathcal{D}^{\omega}\Phi\right)(t) = \left({}_{a}^{\mathcal{CFC}}\mathcal{D}^{\beta}\,\Phi^{(n)}\right)(t).$$

In the left Riemann–Liouville sense, this has the following form:

$$\left({}_{a}^{\mathcal{CFR}}\mathcal{D}^{\omega}\Phi\right)(t) = \left({}_{a}^{\mathcal{CFR}}\mathcal{D}^{\beta}\,\Phi^{(n)}\right)(t).$$

The associated fractional integral is

$$\left({}_{a}^{\mathcal{CF}}\mathcal{I}^{\omega}\Phi\right)(t) = \left({}_{a}I^{n}\,{}_{a}^{\mathcal{CF}}\mathcal{I}^{\beta}\Phi\right)(t).$$

Lemma 4.2. *Let n be a natural number and $\omega, \rho \in (0,1)$. Then*

$$\begin{aligned}
{}^{\mathcal{CF}}\mathcal{I}_a^{\rho}\,{}^{\mathcal{CF}}\mathcal{D}_a^{\omega+n}\Phi(\varkappa) = \Bigg[& M_1\sum_{m=0}^{n}\Big(\frac{\omega}{\omega-1}\Big)^{n-m}\Phi^{(m)}(\varkappa) + \Big[\frac{(\rho-\omega)M(\omega)}{(1-\omega)\omega}e^{-\frac{\omega}{1-\omega}(\varkappa-a)} \\
& - \frac{\rho M(\omega)}{\omega}\Big]\sum_{m=0}^{n}\big(\frac{\omega}{\omega-1}\big)^{n-m}\Phi^{(m)}(a) + \Big[\frac{\rho M(\omega)}{\omega}e^{-\frac{\omega}{1-\omega}(a-s)} \\
& - \frac{\rho M(\omega)}{\omega}e^{-\frac{\omega}{1-\omega}(\varkappa-s)}\Big]\sum_{m=0}^{n}\Big(\frac{\omega}{\omega-1}\Big)^{n-m}\Phi^{(m)}(s) + M_1\Big(\frac{\omega}{\omega-1}\Big)^{n}\int_a^{\varkappa} e^{-\frac{\omega}{1-\omega}(\varkappa-s)}\Phi(s)ds \\
& - \Big(\frac{\rho M(\omega)}{\omega}\Big)\Big(\frac{\omega}{\omega-1}\Big)^{n}\int_a^{s} e^{-\frac{\omega}{1-\omega}(\varkappa-\varepsilon)}\Phi(\varepsilon)d\varepsilon - \Big(\frac{\rho M(\omega)}{\omega}\Big)\Big(\frac{\omega}{\omega-1}\Big)^{n}\int_a^{s} e^{-\frac{\omega}{1-\omega}(a-\varepsilon)}\Phi(\varepsilon)d\varepsilon\Bigg]K_{\rho},
\end{aligned}$$

where

$$K_{\rho} = \frac{2}{(2-\rho)M(\rho)}, \qquad M_1 = (1-\rho)\Big(\frac{M(\omega)}{1-\omega}\Big).$$

Proof. Form Definition 4.1, it holds that

$$\begin{aligned}
{}^{\mathcal{CF}}\mathcal{I}_a^{\rho}\,{}^{\mathcal{CF}}\mathcal{D}_a^{\omega+n}\Phi(\varkappa) &= {}^{\mathcal{CF}}\mathcal{I}_a^{\rho}\,{}^{\mathcal{CF}}\mathcal{D}_a^{\omega}(\mathcal{D}_a^{n}\Phi(\varkappa)) = {}^{\mathcal{CF}}\mathcal{I}_a^{\rho}\,{}^{\mathcal{CF}}\mathcal{D}_a^{\omega}\,\Phi^{(n)}(\varkappa) \\
&= \frac{2}{(2-\rho)M(\rho)}[(1-\rho)^{\mathcal{CF}}\mathcal{D}_a^{\omega}\,\Phi^{(n)}(\varkappa) + \rho\int_a^{\varkappa}{}^{\mathcal{CF}}\mathcal{D}_a^{\omega}\,\Phi^{(n)}(s)ds],
\end{aligned}$$

where

$$(1-\rho)^{\mathcal{CF}}\mathcal{D}_a^{\omega}\,\Phi^{(n)}(\varkappa) = (1-\rho)\Big(\frac{M(\omega)}{1-\omega}\Big)\int_a^{\varkappa} e^{-\frac{\omega}{1-\omega}(\varkappa-s)}\,\Phi^{(n)}(s)ds.$$

Applying integration by parts, it follows that

$$\begin{aligned}
(1-\rho)\,{}^{\mathcal{CF}}\mathcal{D}_a^{\omega}\Phi^{(n)}(\varkappa) = {} & M_1\sum_{m=0}^{n}\Big(\frac{\omega}{\omega-1}\Big)^{n-m}\Phi^{(m)}(\varkappa) - e^{-\frac{\omega}{1-\omega}(\varkappa-a)}M_1\sum_{m=0}^{n}\Big(\frac{\omega}{\omega-1}\Big)^{n-m}\Phi^{(m)}(a) \\
& + M_1\Big(\frac{\omega}{\omega-1}\Big)^{n}\int_a^{\varkappa} e^{-\frac{\omega}{1-\omega}(\varkappa-s)}\,\Phi(s)ds,
\end{aligned} \tag{4.1}$$

and

$$\rho \int_a^{\varkappa} {}^{C\mathcal{F}}\mathcal{D}_a^{\omega}\Phi^{(n)}(s)ds = \rho\frac{M(\omega)}{1-\omega}\int_a^{\varkappa}\int_a^{s} e^{-\frac{\omega}{1-\omega}(s-\varepsilon)}\Phi^{(n)}(\varepsilon)d\varepsilon ds,$$

$$\rho \int_a^{\varkappa} {}^{C\mathcal{F}}\mathcal{D}_a^{\omega}\Phi^{(n)}(s)ds = \rho\cdot\frac{M(\omega)}{1-\omega}\bigg(\int_a^{\varkappa} e^{-(\frac{\omega}{1-\omega})s}ds\bigg)\bigg(\int_a^{s} e^{(\frac{\omega}{1-\omega})\varepsilon}\Phi^{(n)}(\varepsilon)d\varepsilon\bigg). \tag{4.2}$$

The first integral is directly computed as

$$\int_a^{\varkappa} e^{-(\frac{\omega}{1-\omega})s}ds = -\bigg(\frac{1-\omega}{\omega}\bigg)e^{-(\frac{\omega}{1-\omega})\varkappa} + \bigg(\frac{1-\omega}{\omega}\bigg)e^{-(\frac{\omega}{1-\omega})a}.$$

Applying recursive integration by parts, the second integral takes the following form:

$$\int_a^{s} e^{(\frac{\omega}{1-\omega})\varepsilon}\Phi^{(n)}(\varepsilon)d\varepsilon = e^{-(\frac{\omega}{\omega-1})s}\sum_{m=0}^{n}\bigg(\frac{\omega}{\omega-1}\bigg)^{n-m}\Phi^{(m)}(s) - e^{-(\frac{\omega}{\omega-1})a}\sum_{m=0}^{n}\bigg(\frac{\omega}{\omega-1}\bigg)^{n-m}\Phi^{(m)}(a)$$
$$+\bigg(\frac{\omega}{\omega-1}\bigg)^{n}\int_a^{s}\Phi(\varepsilon)e^{-(\frac{\omega}{\omega-1})\varepsilon}d\varepsilon.$$

Substituting the computed integrals into the original expression in Eq (4.2) yields the final form

$$\rho \int_a^{\varkappa} {}^{C\mathcal{F}}\mathcal{D}_a^{\omega}\Phi^{(n)}(s)ds = -\frac{\rho M(\omega)}{\omega}\Big[e^{-\frac{\omega}{1-\omega}(\varkappa-s)}\sum_{m=0}^{n}\Big(\frac{\omega}{\omega-1}\Big)^{n-m}\Phi^{(m)}(s) - e^{-\frac{\omega}{1-\omega}(\varkappa-a)}\sum_{m=0}^{n}\Big(\frac{\omega}{\omega-1}\Big)^{n-m}\Phi^{(m)}(a)$$
$$+\Big(\frac{\omega}{\omega-1}\Big)^{n}\int_a^{s} e^{-\frac{\omega}{1-\omega}(\varkappa-\varepsilon)}\Phi(\varepsilon)d\varepsilon - e^{-\frac{\omega}{1-\omega}(a-s)}\sum_{m=0}^{n}\Big(\frac{\omega}{\omega-1}\Big)^{n-m}\Phi^{(m)}(s)$$
$$+\sum_{m=0}^{n}\Big(\frac{\omega}{\omega-1}\Big)^{n-m}\Phi^{(m)}(a) + \Big(\frac{\omega}{\omega-1}\Big)^{n}\int_a^{s} e^{-\frac{\omega}{1-\omega}(a-\varepsilon)}\Phi(\varepsilon)d\varepsilon\Big]. \tag{4.3}$$

From Eqs (4.1) and (4.3), it follows that

$$ {}^{C\mathcal{F}}\mathcal{I}_a^{\rho}\,{}^{C\mathcal{F}}\mathcal{D}_a^{\omega+n}\Phi(\varkappa) = \bigg[M_1\sum_{m=0}^{n}\Big(\frac{\omega}{\omega-1}\Big)^{n-m}\Phi^{(m)}(\varkappa) + \Big[\frac{(\rho-\omega)M(\omega)}{(1-\omega)\omega}e^{-\frac{\omega}{1-\omega}(\varkappa-a)}$$
$$-\frac{\rho M(\omega)}{\omega}\Big]\sum_{m=0}^{n}(\frac{\omega}{\omega-1})^{n-m}\Phi^{(m)}(a) + \Big[\frac{\rho M(\omega)}{\omega}e^{-\frac{\omega}{1-\omega}(a-s)}$$
$$-\frac{\rho M(\omega)}{\omega}e^{-\frac{\omega}{1-\omega}(\varkappa-s)}\Big]\sum_{m=0}^{n}\Big(\frac{\omega}{\omega-1}\Big)^{n-m}\Phi^{(m)}(s) + M_1\Big(\frac{\omega}{\omega-1}\Big)^{n}\int_a^{\varkappa} e^{-\frac{\omega}{1-\omega}(\varkappa-s)}\Phi(s)ds$$
$$-\Big(\frac{\rho M(\omega)}{\omega}\Big)\Big(\frac{\omega}{\omega-1}\Big)^{n}\int_a^{s} e^{-\frac{\omega}{1-\omega}(\varkappa-\varepsilon)}\Phi(\varepsilon)d\varepsilon$$
$$-\Big(\frac{\rho M(\omega)}{\omega}\Big)\Big(\frac{\omega}{\omega-1}\Big)^{n}\int_a^{s} e^{-\frac{\omega}{1-\omega}(a-\varepsilon)}\Phi(\varepsilon)d\varepsilon\bigg]K_{\rho}.$$

□

Lemma 4.3. *The function $\Phi \in C([0,\Upsilon],\Re)$ is a solution of the boundary value problems* (2.1) *and* (2.2) *if and only if Φ satisfies the integral equation*

$$
\begin{aligned}
\Phi(\varkappa) &= \frac{1}{\kappa_1+\kappa_3+1}\Big(\frac{(1-\omega)\kappa_2}{\omega}\left(1-e^{\frac{-\omega}{1-\omega}\varkappa}\right)+\kappa_1 e^{\frac{-\omega}{1-\omega}\varkappa}+\frac{\rho}{\mathcal{M}(\rho)}\varkappa+\kappa_3 \\
&\quad +1\Big)\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\Phi(\nu)e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu-\frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\Phi(\nu)d\nu-\frac{\kappa_2}{\kappa_4}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon \\
&\quad +\frac{\kappa_2}{\kappa_4}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon+\frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(Q(\Upsilon)-Q(0))+\frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(Q(\nu)-Q(0))d\nu\Big) \\
&\quad +\frac{\omega\kappa_1}{(1-\omega)(\kappa_1+\kappa_3+1)}\int_0^{\varkappa}\Phi(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu-\frac{\kappa_2}{\kappa_1+\kappa_3+1}\int_0^{\varkappa}\Phi(\nu)d\nu \\
&\quad -\frac{\kappa_2}{\kappa_1+\kappa_3+1}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon+\frac{\kappa_2}{\kappa_1+\kappa_3+1}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon \\
&\quad +\frac{\kappa_3}{\kappa_1+\kappa_3+1}(Q(\varkappa)-Q(0))+\frac{\rho}{(\kappa_1+\kappa_3+1)\mathcal{M}(\rho)}\int_0^{\varkappa}(Q(\nu)-Q(0))d\nu,
\end{aligned} \tag{4.4}
$$

where $\varkappa \in \mathfrak{J}=[0,\Upsilon]$, $0\le\varepsilon\le\nu\le\varkappa\le\Upsilon$,

$$
\begin{aligned}
\kappa_1 &= \frac{(1-\rho)\mathcal{M}(\omega)}{(1-\omega)\mathcal{M}(\rho)}, \quad \kappa_2=\frac{\rho\mathcal{M}(\omega)}{(1-\omega)\mathcal{M}(\rho)}, \quad \kappa_3=\frac{1-\rho}{\mathcal{M}(\rho)}, \\
\kappa_4 &= \Big(\kappa_1-\frac{(1-\omega)\kappa_2}{\omega}\Big)\Big(1-e^{\frac{\omega}{1-\omega}\Upsilon}\Big)-\frac{\rho}{\mathcal{M}(\rho)}\Upsilon.
\end{aligned}
$$

Proof. Applying the operator ${}^{CFM}\mathcal{I}_0^{\rho}$ to both sides of Eq (2.1), one obtains

$$
{}^{CFM}\mathcal{I}_0^{\rho\,C\mathcal{F}}\mathcal{D}_0^{\rho}\Phi(\varkappa)+{}^{CFM}\mathcal{I}_0^{\rho\,C\mathcal{F}}\mathcal{D}_0^{\omega}\Phi(\varkappa)+{}^{CFM}\mathcal{I}_0^{\rho}\Phi(\varkappa)={}^{CFM}\mathcal{I}_0^{\rho}Q(\varkappa). \tag{4.5}
$$

After some computations, we have

$$
{}^{CFM}\mathcal{I}_0^{\rho\,C\mathcal{F}}\mathcal{D}_0^{\rho}\Phi(\varkappa)=\Phi(\varkappa)-\Phi(0). \tag{4.6}
$$

Applying the definition of the modified Caputo–Fabrizio fractional integral operator ${}^{CFM}I_0^{\rho}$, as given in Eq (3.2), for the function $f=Q(\varkappa)-\Phi(\varkappa)$, yields

$$
\begin{aligned}
{}^{CFM}\mathcal{I}_0^{\rho}(Q(\varkappa)-\Phi(\varkappa)) &= \frac{1-\rho}{M(\rho)}(Q(\varkappa)-\Phi(\varkappa)-(Q(0)-\Phi(0))) \\
&\quad +\frac{\rho}{M(\rho)}\int_0^{\varkappa}(Q(\nu)-\Phi(\nu)-(Q(0)-\Phi(0)))d\nu.
\end{aligned} \tag{4.7}
$$

By applying Lemma (4.2), it follows that

$$
\begin{aligned}
{}^{CFM}\mathcal{I}_0^{\rho\,C\mathcal{F}}\mathcal{D}_0^{\omega}\Phi(\varkappa) &= \kappa_1\Phi(\varkappa)+\Big(\frac{(1-\omega)\kappa_2}{\omega}\left(1-e^{\frac{\omega}{1-\omega}\varkappa}\right)-\kappa_1\Big)\Phi(0)e^{\frac{-\omega}{1-\omega}\varkappa} \\
&\quad -\frac{\omega\kappa_1}{1-\omega}\int_0^{\varkappa}\Phi(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu+\kappa_2\int_0^{\varkappa}\Phi(\nu)d\nu+\kappa_2\int_0^{\nu}\Phi(\varepsilon)e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon \\
&\quad -\kappa_2\int_0^{\nu}\Phi(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon.
\end{aligned} \tag{4.8}
$$

Now, from (4.6)–(4.8), the Eq (4.5) becomes

$$\begin{aligned}\Phi(\varkappa) &= \frac{1}{\kappa_1+\kappa_3+1}\Big(\frac{(1-\omega)\kappa_2}{\omega}\left(1-e^{\frac{-\omega}{1-\omega}\varkappa}\right)+\kappa_1 e^{\frac{-\omega}{1-\omega}\varkappa}+\frac{\rho}{\mathcal{M}(\rho)}\varkappa+\kappa_3+1\Big)\Phi(0)\\ &+\frac{\omega\kappa_1}{(1-\omega)(\kappa_1+\kappa_3+1)}\int_0^{\varkappa}\Phi(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu-\frac{\kappa_2\mathcal{M}(\rho)+\rho}{(\kappa_1+\kappa_3+1)\mathcal{M}(\rho)}\int_0^{\varkappa}\Phi(\nu)d\nu\\ &-\frac{\kappa_2}{\kappa_1+\kappa_3+1}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon+\frac{\kappa_2}{\kappa_1+\kappa_3+1}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon\\ &+\frac{\kappa_3}{\kappa_1+\kappa_3+1}(\mathcal{Q}(\varkappa)-\mathcal{Q}(0))+\frac{\rho}{(\kappa_1+\kappa_3+1)\mathcal{M}(\rho)}\int_0^{\varkappa}(\mathcal{Q}(\nu)-\mathcal{Q}(0))d\nu.\end{aligned}\tag{4.9}$$

From the boundary condition $\Phi(0)=\Phi(\Upsilon)$, the following holds:

$$\begin{aligned}\Phi(0) &= \frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\Phi(\nu)e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu-\frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\Phi(\nu)d\nu\\ &-\frac{\kappa_2}{\kappa_4}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon+\frac{\kappa_2}{\kappa_4}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon+\frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(\mathcal{Q}(\Upsilon)-\mathcal{Q}(0))\\ &+\frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(\mathcal{Q}(\nu)-\mathcal{Q}(0))d\nu.\end{aligned}\tag{4.10}$$

Substituting Eq (4.10) into Eq (4.9) yields Eq (4.4), which completes the proof. □

For convenience, the following notation is introduced:

$$\begin{aligned}&\mathcal{Z}_0=|\frac{1}{\kappa_1+\kappa_3+1}|,\quad \mathcal{Z}_1=|\frac{\kappa_1}{\kappa_1+\kappa_3+1}|,\quad \mathcal{Z}_2=|\frac{\kappa_2}{\kappa_1+\kappa_3+1}|,\quad \mathcal{Z}_3=|\frac{\kappa_3}{\kappa_1+\kappa_3+1}|,\\ &\mathcal{Z}_4=\frac{1-\omega}{\omega}\mathcal{Z}_2,\quad \mathcal{Z}_5=\frac{\rho}{\mu(\rho)}\mathcal{Z}_0,\quad \mathcal{Z}_6=\mathcal{Z}_0+\mathcal{Z}_3+\mathcal{Z}_4,\quad \mathcal{Z}_7=\frac{\omega}{1-\omega}\mathcal{Z}_1.\end{aligned}$$

The analysis also involves the following constants:

$$\begin{aligned}&\sigma_1(\varkappa)=(1-e^{\frac{-\omega}{1-\omega}\varkappa}),\qquad \sigma_2(\varkappa)=(1-e^{\frac{-\rho}{1-\rho}\varkappa}),\qquad \sigma_3(\varkappa)=(1-e^{-(\frac{\rho}{1-\rho}+\frac{\omega}{1-\omega})\varkappa}),\\ &\widehat{\sigma_1}=\sup\ \sigma_1(\varkappa),\qquad \widehat{\sigma_2}=\sup\ \sigma_2(\varkappa),\qquad \widehat{\sigma_3}=\sup\sigma_3(\varkappa),\\ &\lambda_1(\varkappa)=\left(\mathcal{Z}_4\left(1-e^{\frac{-\omega}{1-\omega}\varkappa}\right)+\mathcal{Z}_1e^{\frac{-\omega}{1-\omega}\varkappa}+\mathcal{Z}_5\Upsilon+\mathcal{Z}_3+\mathcal{Z}_0\right),\qquad \widehat{\lambda_1}=\sup\ \lambda_1(\varkappa),\\ &\mathfrak{S}_1=\left(|\frac{\kappa_1}{\kappa_4}|\,(1-e^{\frac{-\omega}{1-\omega}\Upsilon})+|\frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}|\,\Upsilon+|\frac{\kappa_2(1-\omega)}{\omega\,\kappa_4}|(e^{\frac{\omega}{1-\omega}\Upsilon}-e^{\frac{-\omega}{1-\omega}\Upsilon})\right),\\ &\widehat{\mathscr{F}\mathscr{F}}=\left(|\frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}|(m_0+m_1)+|\frac{\rho}{\kappa_4\mathcal{M}(\rho)}|(m_0+d_0)\Upsilon\right),\\ &\mathfrak{S}_2=\widehat{\lambda_1}\ \widehat{\mathscr{F}\mathscr{F}}+(\mathcal{Z}_5\Upsilon+\mathcal{Z}_3)(m_0+d_0),\\ &\mathfrak{S}_3=\mathcal{Z}_7\ \frac{1-\omega}{\omega}(1-e^{\frac{-\omega}{1-\omega}\varkappa})+\mathcal{Z}_2\ \Upsilon+\mathcal{Z}_2\ \frac{1-\omega}{\omega}(e^{\frac{\omega}{1-\omega}\Upsilon}-e^{\frac{-\omega}{1-\omega}\varkappa}),\\ &\delta_1(\varkappa)=(\mathcal{Z}_2+\mathcal{Z}_7)\frac{\mathcal{M}(\omega)}{1-\omega}e^{\frac{-\omega}{1-\omega}\varkappa}\Upsilon+\mathcal{Z}_5\frac{\mathcal{M}(\omega)}{\omega}\sigma_1(\varkappa),\qquad \widehat{\delta_1}=\sup\delta_1(\varkappa),\end{aligned}$$

$$\delta_2(\varkappa) = |\frac{\mathcal{M}(\rho)(1-\omega)}{(\rho-\omega)}|(\mathcal{Z}_2+\mathcal{Z}_7)(e^{\frac{-\omega}{1-\omega}\varkappa} - e^{\frac{-\rho}{1-\rho}\varkappa}) + \mathcal{Z}_0\sigma_2(\varkappa), \qquad \widehat{\delta_2} = \sup\delta_2(\varkappa),$$

$$\widehat{\mathfrak{F}_1} = \mathcal{Z}_7\ |\frac{\mathcal{M}(\omega)}{\omega}|\widehat{\sigma_1} + \mathcal{Z}_7\,|\frac{\mathcal{M}(\omega)}{1-\omega}|\Big(\frac{(1-\omega)}{\omega}\widehat{\sigma_1} - \Upsilon(1-\widehat{\sigma_1})\Big) + (\mathcal{Z}_2+\mathcal{Z}_5)|\frac{\mathcal{M}(\omega)}{\omega}|\widehat{\sigma_1},$$

$$\widehat{\mathfrak{F}_2} = \widehat{\delta_1}\widehat{\mathscr{F}\mathscr{F}} + \mathcal{Z}_2\,|\frac{\omega}{1-\omega}| + \mathcal{Z}_3\ |\frac{\mathcal{M}(\omega)}{1-\omega}|\Big(m_0(1-\widehat{\sigma_1}) + d_0(1+\widehat{\sigma_1})\Big) + \mathcal{Z}_5\ (m_0+d_0)|\frac{\mathcal{M}(\omega)}{\omega}|\widehat{\sigma_1},$$

$$\widehat{\mathfrak{U}_1} = \widehat{\delta_2}\widehat{\mathscr{F}\mathscr{F}} + \mathcal{Z}_2\,|\frac{\omega}{1-\omega}| + \mathcal{Z}_3\ |\frac{\mathcal{M}(\rho)}{1-\rho}|\Big(m_0(1-\widehat{\sigma_2}) + d_0(1+\widehat{\sigma_2})\Big) + \mathcal{Z}_0(d_0+m_0)\widehat{\sigma_2},$$

$$\widehat{\mathfrak{U}_2} = \mathcal{Z}_7\ \frac{\mathcal{M}(\rho)}{\rho}\widehat{\sigma_2} + \mathcal{Z}_7\ \frac{\mathcal{M}(\rho)(1-\omega)}{(\omega-2\omega\rho+\rho)}\widehat{\sigma_3} + \mathcal{Z}_7\ \frac{\mathcal{M}(\rho)}{\rho}(\widehat{\sigma_1}-\widehat{\sigma_3}) + \mathcal{Z}_0\ \frac{(\kappa_2\mathcal{M}(\rho)+\rho)}{\rho}\widehat{\sigma_2},$$

$$\mathfrak{W}_1 = \widehat{\lambda_1}\ \mathfrak{S}_1 + (\mathcal{Z}_7+\mathcal{Z}_2)\ \frac{1-\omega}{\omega}\widehat{\sigma_1} + \mathcal{Z}_2\ \Upsilon + \mathcal{Z}_2\ \ \frac{1-\omega}{\omega}(e^{\frac{\omega}{1-\omega}\Upsilon} - 1)$$

$$\mathfrak{W}_2 = \widehat{\lambda_1}\frac{\rho}{\kappa_4\mathcal{M}(\rho)}\Upsilon + \mathcal{Z}_3 + \mathcal{Z}_5\ \Upsilon, \qquad \mathfrak{W}_3 = \widehat{\delta_1}\mathfrak{S}_1 + \widehat{\mathfrak{F}_1},$$

$$\mathfrak{W}_4 = \Big(\widehat{\delta_1}\frac{\rho}{\kappa_4\mathcal{M}(\rho)}\Upsilon + \mathcal{Z}_5\frac{\mathcal{M}(\omega)}{\omega}\widehat{\sigma_1}\Big), \quad \mathfrak{W}_5 = \widehat{\delta_2}\ \mathfrak{S}_1 + \widehat{\mathfrak{U}_2}, \quad \mathfrak{W}_6 = \frac{\widehat{\delta_2}\,\rho}{\kappa_4\mathcal{M}(\rho)}\Upsilon + \mathcal{Z}_0\widehat{\sigma_2},$$

$$\hat{\theta}_1 = \mathfrak{W}_6 + \mathcal{Z}_3\frac{\mathcal{M}(\rho)}{1-\rho}|1-\widehat{\sigma_2}|, \quad \hat{\theta}_2 = \mathfrak{W}_4 + \mathcal{Z}_3\frac{\mathcal{M}(\omega)}{1-\omega}|1-\widehat{\sigma_1}|,$$

$$\mathfrak{Y} = \Big(\mathfrak{W}_2 + \hat{\theta}_1 + \hat{\theta}_2\Big), \quad T_1 = \frac{(1+\rho(\varkappa-1))}{\mathcal{M}(\rho)(1-\mathfrak{W}_1)},$$

$$T_2 = \Big(\sigma_2(\varkappa) + \frac{(1+\rho(\varkappa-1))\mathfrak{W}_5}{\mathcal{M}(\rho)(1-\mathfrak{W}_1)}\Big), \ \ T_3 = \Big(\frac{(1-\mathfrak{W}_1)\mathcal{M}(\omega)\rho\sigma_1(\varkappa) + (1+\rho(x-1))\mathfrak{W}_3\omega}{\mathcal{M}(\rho)\omega(1-\mathfrak{W}_1)}\Big),$$

$$\mathfrak{W}_7 = \daleth\Big(\frac{\mathfrak{W}_2}{1-\mathfrak{W}1} + \frac{\mathfrak{W}_5\mathfrak{W}_2}{1-\mathfrak{W}_1} + \frac{\mathfrak{W}_2\mathfrak{W}_3}{1-\mathfrak{W}_1} + \hat{\theta}_1 + \hat{\theta}_2\Big).$$

5. Main results (existence and uniqueness)

The existence and uniqueness of the solution to Eq (2.1) are established under the following premises:

($\aleph_1$) The function $\mathbf{Q} : \mathfrak{J}\times\mathfrak{R}\times\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$ satisfies Carathéodory conditions.

($\aleph_2$) A positive constants $d_0 > 0$ exists such that

$$\Big|\mathbf{Q}\,(\varkappa, y, z, w)\Big| \le d_0,$$

and $\mathbf{Q}$ is a continuous nondecreasing function satisfying

$$m_0 = \sup_{\varkappa\in\mathfrak{J}}|\mathbf{Q}\,(\varkappa,0,0,0)\,|, \ \ \textit{and}\ \ m_1 = \sup_{\varkappa\in\mathfrak{J}}|\mathbf{Q}(\Upsilon, y(\Upsilon), z(\Upsilon), w(\Upsilon))|,$$

for all $y, z, w \in \mathfrak{R}$ and $\varkappa \in \mathfrak{J}$.

($\aleph_3$) $\mathbf{Q}$ is continuous and satisfies the Lipschitz condition, and a constant $\daleth > 0$ exists such that

$$|\mathbf{Q}\,(\varkappa, y_1, z_1, w_1) - \mathbf{Q}\,(\varkappa, y_2, z_2, w_2)\,| \le \daleth\ (|y_1-y_2| + |z_1-z_2| + |w_1-w_2|),$$

for each $y_1, y_2, z_1, z_2, w_1, w_1 \in \mathfrak{R}$, and $\varkappa \in \mathfrak{J}$.

($\aleph_4$) A positive constant $\mathfrak{K}$ exists such that

$$F(\mathbf{Q}\,(\varkappa, W, \mathcal{D}^{\rho}W, \mathcal{D}^{\omega}W)\,) \le \mathfrak{K}\,F(W),$$

for any bounded set $W \subset \mathfrak{R}$ and $\varkappa \in \mathfrak{J}$.

Theorem 5.1. *Assume that the conditions* $(\aleph_1), (\aleph_2)$*, and* $(\aleph_4)$ *hold. If* $\hat{\theta}_3 < 1$*, then the problems* (2.1) *and* (2.2) *have at least one solution defined on* $\mathfrak{J}$.

Proof. Consider the operator $\mathcal{G} : C(\mathfrak{J}, \mathfrak{X}) \longrightarrow C(\mathfrak{J}, \mathfrak{X})$ defined by

$$\begin{aligned}
\mathcal{G}\Phi(\varkappa) = & \frac{1}{\kappa_1 + \kappa_3 + 1}\Big(\frac{(1-\omega)\kappa_2}{\omega}\left(1 - e^{\frac{-\omega}{1-\omega}\varkappa}\right) + \kappa_1 e^{\frac{-\omega}{1-\omega}\varkappa} + \frac{\rho}{\mathcal{M}(\rho)}\varkappa + \kappa_3 \\
& + 1\Big)\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\Phi(\nu)e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu - \frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\Phi(\nu)d\nu - \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon \\
& + \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon + \frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(Q(\Upsilon) - Q(0)) + \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(Q(\nu) - Q(0))d\nu\Big) \\
& + \frac{\omega\kappa_1}{(1-\omega)(\kappa_1+\kappa_3+1)}\int_0^{\varkappa}\Phi(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu - \frac{\kappa_2}{\kappa_1+\kappa_3+1}\int_0^{\varkappa}\Phi(\nu)d\nu \\
& - \frac{\kappa_2}{\kappa_1+\kappa_3+1}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon + \frac{\kappa_2}{\kappa_1+\kappa_3+1}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon + \frac{\kappa_3}{\kappa_1+\kappa_3+1}(Q(\varkappa) - Q(0)) \\
& + \frac{\rho}{(\kappa_1+\kappa_3+1)\mathcal{M}(\rho)}\int_0^{\varkappa}(Q(\nu) - Q(0))d\nu.
\end{aligned} \tag{5.1}$$

It is evident that $\mathcal{G}$ is well defined by conditions $(\aleph_1)$–$(\aleph_3)$. Consequently, the fractional integral Eqs (2.1) and (2.2) is expressed as follows:

$$\Phi = \mathcal{G}\Phi. \tag{5.2}$$

Subsequently, showing the existence of fixed points for (5.2) is equivalent to the existence of a solution for (4.4). Let $\mathfrak{B}_{\mathcal{A}} = \{\Phi \in C(\mathfrak{J}, \mathfrak{X}) : \|\Phi\|_{\infty} < \mathcal{A}\}$ be a closed convex set with $\mathcal{A} > 0$, such that

$$\mathcal{A} \geq \frac{(\mathfrak{S}_2 + \widehat{\mathfrak{U}_1} + \widehat{\mathfrak{F}_2})}{1 - \Big((\widehat{\lambda_1} + \widehat{\delta_1} + \widehat{\delta_2})\mathfrak{S}_1 + (\mathfrak{S}_3 + \widehat{\mathfrak{F}_1} + \widehat{\mathfrak{U}_2})\Big)}.$$

Step 1. To establish that $\mathcal{G}\mathfrak{B}_{\mathcal{A}} \subset \mathfrak{B}_{\mathcal{A}}$, the condition $(\aleph_2)$ is applied, which yields the following:

$$\begin{aligned}
|\mathcal{G}\Phi(\varkappa)| \leq & \Big(\mathcal{Z}_4\left(1 - e^{\frac{-\omega}{1-\omega}\varkappa}\right) + \mathcal{Z}_1 e^{\frac{-\omega}{1-\omega}\varkappa} + \mathcal{Z}_5\Upsilon + \mathcal{Z}_3 + \mathcal{Z}_0\Big)\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}|\Phi(\nu)|e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu \\
& + \frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}|\Phi(\nu)|d\nu + \frac{\kappa_2}{\kappa_4}\int_0^{\nu}|\Phi(\varepsilon)|e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon + \frac{\kappa_2}{\kappa_4}\int_0^{\nu}|\Phi(\varepsilon)|e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon \\
& + \frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(|Q(\Upsilon)| + |Q(0)|) + \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(|Q(\nu)| + |Q(0)|)d\nu\Big) + \mathcal{Z}_7\int_0^{\varkappa}|\Phi(\nu)|e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu \\
& + \mathcal{Z}_2\int_0^{\varkappa}|\Phi(\nu)|d\nu + \mathcal{Z}_2\int_0^{\nu}|\Phi(\varepsilon)|e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon + \mathcal{Z}_2\int_0^{\nu}|\Phi(\varepsilon)|e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon + \mathcal{Z}_3(|Q(\varkappa)| + |Q(0)|) \\
& + \mathcal{Z}_5\int_0^{\varkappa}(|Q(\nu)| + |Q(0)|)d\nu.
\end{aligned} \tag{5.3}$$

Using the same approach with condition $\aleph_2$, and performing the necessary calculations, Eq (5.3) becomes as follows:

$$
\begin{aligned}
&\leq \mathcal{A}\,\widehat{\lambda_1}\left(\frac{\kappa_1}{\kappa_4}(1-e^{\frac{-\omega}{1-\omega}\Upsilon})+\frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\Upsilon+\frac{\kappa_2(1-\omega)}{\omega\,\kappa_4}(1-e^{\frac{-\omega}{1-\omega}\Upsilon})+\frac{\kappa_2(1-\omega)}{\omega\,\kappa_4}(e^{\frac{\omega}{1-\omega}\Upsilon}-1)\right)\\
&+\widehat{\lambda_1}\,\frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(m_0+m_1)+\widehat{\lambda_1}\,\frac{\rho}{\kappa_4\mathcal{M}(\rho)}(m_0+d_0)\Upsilon+\mathcal{Z}_7\,\mathcal{A}\,\frac{1-\omega}{\omega}(1-e^{\frac{-\omega}{1-\omega}\varkappa})+\mathcal{Z}_2\,\mathcal{A}\Upsilon\\
&+\mathcal{Z}_2\,\mathcal{A}\,\frac{1-\omega}{\omega}(e^{\frac{\omega}{1-\omega}\Upsilon}-e^{\frac{-\omega}{1-\omega}\varkappa})+(\mathcal{Z}_3+\mathcal{Z}_5\,\Upsilon)(m_0+d_0).
\end{aligned}
$$

Hence, for all $\Phi\in\mathfrak{B}_{\mathcal{A}}$, the result is

$$\|\mathcal{G}\Phi\|\leq(\widehat{\lambda_1}\,\mathfrak{S}_1+\mathfrak{S}_3)\mathcal{A}+\mathfrak{S}_2. \tag{5.4}$$

Similarly, the next result is derived as follows:

$$
\begin{aligned}
|\mathcal{G}^{\mathcal{CF}}\mathcal{D}^{\rho}\Phi(\varkappa)|&\leq\Big(\frac{\mathcal{M}(\rho)(1-\omega)}{(\rho-\omega)}(\mathcal{Z}_2+\mathcal{Z}_7)(e^{\frac{-\omega}{1-\omega}\varkappa}-e^{\frac{-\rho}{1-\rho}\varkappa})+\mathcal{Z}_0\sigma_2(\varkappa)\Big)\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}|\Phi(\nu)|e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu\\
&+\frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}|\Phi(\nu)|d\nu+\frac{\kappa_2}{\kappa_4}\int_0^{\nu}|\Phi(\varepsilon)|e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon\\
&+\frac{\kappa_2}{\kappa_4}\int_0^{\nu}|\Phi(\varepsilon)|e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon+\frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(|Q(\Upsilon)|+|Q(0)|)+\frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(|Q(\nu)|+|Q(0)|)d\nu\Big)\\
&+\mathcal{Z}_7\frac{\mathcal{M}(\rho)}{1-\rho}\int_0^{\varkappa}|\Phi(\nu)|e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu+\mathcal{Z}_7\frac{\omega\mathcal{M}(\rho)}{(1-\omega)(1-\rho)}\int_0^{\varkappa}\int_0^{\nu}e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}|\Phi(\varepsilon)|d\varepsilon d\nu\\
&+\mathcal{Z}_0\frac{(\kappa_2\mathcal{M}(\rho)+\rho)}{(1-\rho)}\int_0^{\varkappa}|\Phi(\nu)|e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu+\mathcal{Z}_2\,\frac{\omega}{1-\omega}+\mathcal{Z}_3\frac{\mathcal{M}(\rho)}{(1-\rho)}\int_0^{\varkappa}|Q'(\nu)|e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu\\
&+\mathcal{Z}_0\frac{\rho}{(1-\rho)}\int_0^{\varkappa}|Q(\nu)|e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu+\mathcal{Z}_0(1-e^{\frac{-\rho}{1-\rho}\varkappa})|Q(0)|,\\
&\leq\widehat{\delta_2}\,\mathfrak{S}_1\mathcal{A}+\widehat{\delta_2}\left(\frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(m_0+m_1)+\frac{\rho}{\kappa_4\mathcal{M}(\rho)}(m_0+d_0)\Upsilon\right)+\mathcal{Z}_7\mathcal{A}\,\frac{\mathcal{M}(\rho)}{\rho}(1-e^{-\frac{\rho}{1-\rho}\varkappa})\\
&+\mathcal{Z}_7\,\mathcal{A}\,\frac{\mathcal{M}(\rho)}{1-\rho}e^{-(\frac{\rho}{1-\rho}+\frac{\omega}{1-\omega})\varkappa}\left(\frac{1}{(\frac{\rho}{1-\rho}+\frac{\omega}{1-\omega})}(e^{(\frac{\rho}{1-\rho}+\frac{\omega}{1-\omega})\varkappa}-1)-\frac{(1-\rho)}{\rho}(e^{(\frac{\rho}{1-\rho})\varkappa}-1)\right)\\
&+\mathcal{Z}_0\mathcal{A}\,\frac{(\kappa_2\mathcal{M}(\rho)+\rho)}{\rho}(1-e^{-\frac{\rho}{1-\rho}\varkappa})+\mathcal{Z}_2\frac{\omega}{1-\omega}\\
&+\mathcal{Z}_3\,\frac{\mathcal{M}(\rho)}{1-\rho}\Big(m_0e^{-\frac{\rho}{1-\rho}\varkappa}+d_0(1+(1-e^{\frac{-\rho}{1-\rho}\varkappa}))\Big)+\mathcal{Z}_0(d_0+m_0)(1-e^{-\frac{\rho}{1-\rho}\varkappa}).
\end{aligned}
$$

Therefore, for all $\Phi\in\mathfrak{B}_{\mathcal{A}}$, one obtains

$$\|\mathcal{G}^{\mathcal{CF}}\mathcal{D}^{\rho}\Phi\|\leq(\widehat{\delta_2}\,\mathfrak{S}_1+\widehat{\mathfrak{U}_2})\mathcal{A}+\widehat{\mathfrak{U}_1}. \tag{5.5}$$

At this point, the norm $\|\mathcal{G}^{\mathcal{CF}}\mathcal{D}^{\omega}\Phi\|$ is evaluated using the same analytical technique

$$
\begin{aligned}
|\mathcal{G}^{\mathcal{CF}}\mathcal{D}^{\omega}\Phi(\varkappa)| \leq & \Big((\mathcal{Z}_2+\mathcal{Z}_7)\frac{\mathcal{M}(\omega)}{1-\omega}e^{\frac{-\omega}{1-\omega}\varkappa}\Upsilon+\sigma_1(\varkappa)\mathcal{Z}_5\frac{\mathcal{M}(\omega)}{\omega}\Big)\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}|\Phi(\nu)|e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu \\
& +\frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}|\Phi(\nu)|d\nu+\frac{\kappa_2}{\kappa_4}\int_0^{\nu}|\Phi(\varepsilon)|e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon+\frac{\kappa_2}{\kappa_4}\int_0^{\nu}|\Phi(\varepsilon)|e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon \\
& +\frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(|\mathcal{Q}(\Upsilon)|+|\mathcal{Q}(0)|)+\frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(|\mathcal{Q}(\nu)|+|\mathcal{Q}(0)|)d\nu\Big)+\mathcal{Z}_7\frac{\mathcal{M}(\omega)}{1-\omega}\int_0^{\varkappa}|\Phi(\nu)|e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu \\
& +\mathcal{Z}_7\frac{\omega\mathcal{M}(\omega)}{(1-\omega)^2}\int_0^{\varkappa}\int_0^{\nu}|\Phi(\varepsilon)|e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon d\nu+\mathcal{Z}_5\frac{\mathcal{M}(\omega)}{\omega}(1-e^{\frac{-\omega}{1-\omega}\varkappa})|\mathcal{Q}(0)| \\
& +\frac{(\mathcal{Z}_2+\mathcal{Z}_5)\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}|\Phi(\nu)|e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu+\frac{\mathcal{Z}_2\,\omega}{1-\omega} \\
& +\frac{(\mathcal{Z}_3+\mathcal{Z}_5)\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}(|\mathcal{Q}'(\nu)|+|\mathcal{Q}(\nu)|)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu.
\end{aligned}
\tag{5.6}
$$

Under condition ($\aleph_2$), and after performing the necessary computations, Eq (5.6) simplifies to

$$
\|\mathcal{G}^{\mathcal{CF}}\mathcal{D}^{\omega}\Phi\| \leq (\widehat{\delta_1}\ \mathfrak{S}_1+\widehat{\mathfrak{F}_1})\mathcal{A}+\widehat{\mathfrak{F}_2}. \tag{5.7}
$$

Combining the inequalities (5.4), (5.5), and (5.7) leads to

$$
\|\mathcal{G}\Phi\|_{\mathfrak{X}} \leq \Big((\widehat{\lambda_1}\ +\widehat{\delta_1}+\widehat{\delta_2})\mathfrak{S}_1+(\mathfrak{S}_3+\widehat{\mathfrak{F}_1}+\widehat{\mathfrak{U}_2})\Big)\mathcal{A}+(\mathfrak{S}_2+\widehat{\mathfrak{U}_1}+\widehat{\mathfrak{F}_2}) \leq \mathcal{A}.
$$

$$
\|\mathcal{G}\Phi\|_{\mathfrak{X}} \leq \mathcal{A}. \quad \text{That is ,} \qquad \mathcal{G}\mathfrak{B}_{\mathcal{A}} \subset \mathfrak{B}_{\mathcal{A}}.
$$

Step 2. The operator $\mathcal{G}$ is continuous. Let Φ_n be a sequence in $\mathfrak{B}_{\mathcal{A}}$ such that $\Phi_n \to \Phi$ as $n \to \infty$. Since Φ satisfies ($\aleph_1$), it follows that for each $\varkappa \in \mathfrak{J}$, the following expression holds:

$$
\begin{aligned}
|\mathcal{G}\Phi_n(\varkappa)-\mathcal{G}\Phi(\varkappa)| \leq & \widehat{\lambda_1}\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}|\Phi_n(\nu)-\Phi(\nu)|e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu \\
& +\frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}|\Phi_n(\nu)-\Phi(\nu)|d\nu+\frac{\kappa_2}{\kappa_4}\int_0^{\nu}|\Phi_n(\varepsilon)-\Phi(\varepsilon)|e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon \\
& +\frac{\kappa_2}{\kappa_4}\int_0^{\nu}|\Phi_n(\varepsilon)-\Phi(\varepsilon)|e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon \\
& +\frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}|\mathbf{Q}(\nu,\Phi_n(\nu),\mathcal{D}^{\rho}\Phi_n(\nu),\mathcal{D}^{\omega}\Phi_n(\nu))-\mathbf{Q}(\nu,\Phi(\nu),\mathcal{D}^{\rho}\Phi(\nu),\mathcal{D}^{\omega}\Phi(\nu))|d\nu\Big) \\
& +\mathcal{Z}_7\int_0^{\varkappa}|\Phi_n(\nu)-\Phi(\nu)|e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu+\mathcal{Z}_2\int_0^{\varkappa}|\Phi_n(\nu)-\Phi(\nu)|d\nu \\
& +\mathcal{Z}_2\int_0^{\nu}|\Phi_n(\varepsilon)-\Phi(\varepsilon)|e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon+\mathcal{Z}_2\int_0^{\nu}|\Phi_n(\varepsilon)-\Phi(\varepsilon)|e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon \\
& +\mathcal{Z}_3|\mathbf{Q}(\varkappa,\Phi_n(\varkappa),\mathcal{D}^{\rho}\Phi_n(\varkappa),\mathcal{D}^{\omega}\Phi_n(\varkappa))-\mathbf{Q}(\varkappa,\Phi(\varkappa),\mathcal{D}^{\rho}\Phi(\varkappa),\mathcal{D}^{\omega}\Phi(\varkappa))| \\
& +\mathcal{Z}_5\int_0^{\varkappa}|\mathbf{Q}(\nu,\Phi_n(\nu),\mathcal{D}^{\rho}\Phi_n(\nu),\mathcal{D}^{\omega}\Phi_n(\nu))-\mathbf{Q}(\nu,\Phi(\nu),\mathcal{D}^{\rho}\Phi(\nu),\mathcal{D}^{\omega}\Phi(\nu))|d\nu,
\end{aligned}
$$

and

$$\|\mathcal{G}\Phi_n - \mathcal{G}\Phi\| \leq \mathfrak{W}_1\|\Phi_n(.) - \Phi(.)\| \\ + \mathfrak{W}_2\|\mathbf{Q}\,(., \Phi_n(.), \mathcal{D}^\rho\Phi_n(.), \mathcal{D}^\omega\Phi_n(.)) - \mathbf{Q}(., \Phi(.), \mathcal{D}^\rho\Phi(.), \mathcal{D}^\omega\Phi(.))\|. \tag{5.8}$$

In the same manner, it follows that

$$|\mathcal{G}^{C\mathcal{F}}\mathcal{D}^\rho\Phi_n(\varkappa) - \mathcal{G}^{C\mathcal{F}}\mathcal{D}^\rho\Phi(\varkappa)| \leq \widehat{\delta_2}\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^\Upsilon|\Phi_n(\nu) - \Phi(\nu)|e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu \\ + \frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^\Upsilon|\Phi_n(\nu) - \Phi(\nu)|d\nu + \frac{\kappa_2}{\kappa_4}\int_0^\nu|\Phi_n(\varepsilon) - \Phi(\varepsilon)|e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon + \frac{\kappa_2}{\kappa_4}\int_0^\nu|\Phi_n(\varepsilon) \\ - \Phi(\varepsilon)|e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon + \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^\Upsilon|\mathbf{Q}\,(\nu, \Phi_n(\nu), \mathcal{D}^\rho\Phi_n(\nu), \mathcal{D}^\omega\Phi_n(\nu)) \\ - \mathbf{Q}\,(\nu, \Phi(\nu), \mathcal{D}^\rho\Phi(\nu), \mathcal{D}^\omega\Phi(\nu))\,|d\nu\Big) + (\mathcal{Z}_7\frac{\mathcal{M}(\rho)}{1-\rho} + \mathcal{Z}_0\frac{(\kappa_2\mathcal{M}(\rho)+\rho)}{(1-\rho)})\int_0^\varkappa|\Phi_n(\nu) \\ - \Phi(\nu)|e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu + \mathcal{Z}_7\frac{\omega\mathcal{M}(\rho)}{(1-\omega)(1-\rho)}\int_0^\varkappa\int_0^\nu e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}|\Phi_n(\varepsilon) - \Phi(\varepsilon)|d\varepsilon d\nu \\ + \mathcal{Z}_3\frac{\mathcal{M}(\rho)}{(1-\rho)}\int_0^\varkappa|\mathbf{Q}'\,(\nu, \Phi_n(\nu), \mathcal{D}^\rho\Phi_n(\nu), \mathcal{D}^\omega\Phi_n(\nu)) - \mathbf{Q}'\,(\nu, \Phi(\nu), \mathcal{D}^\rho\Phi(\nu), \mathcal{D}^\omega\Phi(\nu))\,|e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu \\ + \mathcal{Z}_0\frac{\rho}{(1-\rho)}\int_0^\varkappa|\mathbf{Q}\,(\nu, \Phi_n(\nu), \mathcal{D}^\rho\Phi_n(\nu), \mathcal{D}^\omega\Phi_n(\nu)) - \mathbf{Q}\,(\nu, \Phi(\nu), \mathcal{D}^\rho\Phi(\nu), \mathcal{D}^\omega\Phi(\nu))\,|e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu,$$

and

$$\|\mathcal{G}^{C\mathcal{F}}\mathcal{D}^\rho\Phi_n - \mathcal{G}^{C\mathcal{F}}\mathcal{D}^\rho\Phi\| \leq \mathfrak{W}_5\|\Phi_n(.) - \Phi(.)\| \\ + \mathcal{Z}_3\;\frac{\mathcal{M}(\rho)}{\rho}\widehat{\sigma_2}\|\mathbf{Q}'\,(., \Phi_n(.), \mathcal{D}^\rho\Phi_n(.), \mathcal{D}^\omega\Phi_n(.)) - \mathbf{Q}'\,(., \Phi(.), \mathcal{D}^\rho\Phi(.), \mathcal{D}^\omega\Phi(.))\,\| \\ + \mathfrak{W}_6|\mathbf{Q}\,(., \Phi_n(.), \mathcal{D}^\rho\Phi_n(.), \mathcal{D}^\omega\Phi_n(.)) - \mathbf{Q}\,(., \Phi(.), \mathcal{D}^\rho\Phi(.), \mathcal{D}^\omega\Phi(.))\,\|. \tag{5.9}$$

Furthermore, applying the same technique to each $\Phi_n \in \mathfrak{B}_{\mathcal{A}}$, one obtains

$$\|\mathcal{G}^{C\mathcal{F}}\mathcal{D}^\omega\Phi_n - \mathcal{G}^{C\mathcal{F}}\mathcal{D}^\omega\Phi\| \leq \mathfrak{W}_3\|\Phi_n(.) - \Phi(.)\| \\ + \frac{\mathcal{Z}_3\mathcal{M}(\omega)\widehat{\sigma_1}}{\omega}\|\mathbf{Q}'\,(., \Phi_n(.), \mathcal{D}^\rho\Phi_n(.), \mathcal{D}^\omega\Phi_n(.)) - \mathbf{Q}'(., \Phi(.), \mathcal{D}^\rho\Phi(.), \mathcal{D}^\omega\Phi(.))\| \\ + \mathfrak{W}_4\|\mathbf{Q}\,(., \Phi_n(.), \mathcal{D}^\rho\Phi_n(.), \mathcal{D}^\omega\Phi_n(.)) - \mathbf{Q}(., \Phi(.), \mathcal{D}^\rho\Phi(.), \mathcal{D}^\omega\Phi(.))\|. \tag{5.10}$$

From Eqs (5.8)–(5.10), the following result is obtained:

$$\|\mathcal{G}\Phi_n - \mathcal{G}\Phi\|_{\mathfrak{X}} \leq (\mathfrak{W}_1 + \mathfrak{W}_3 + \mathfrak{W}_5)\|\Phi_n(.) - \Phi(.)\| + \mathcal{Z}_3(\frac{\mathcal{M}(\omega)}{\omega}\widehat{\sigma_1} + \frac{\mathcal{M}(\rho)}{\rho}\widehat{\sigma_2}) \\ \|\mathbf{Q}'\,(., \Phi_n(.), \mathcal{D}^\rho\Phi_n(.), \mathcal{D}^\omega\Phi_n(.)) - \mathbf{Q}'(., \Phi(.), \mathcal{D}^\rho\Phi(.), \mathcal{D}^\omega\Phi(.))\| \\ + (\mathfrak{W}_2 + \mathfrak{W}_4 + \mathfrak{W}_6)\|\mathbf{Q}\,(., \Phi_n(.), \mathcal{D}^\rho\Phi_n(.), \mathcal{D}^\omega\Phi_n(.)) - \mathbf{Q}(., \Phi(.), \mathcal{D}^\rho\Phi(.), \mathcal{D}^\omega\Phi(.))\|.$$

From the inequality above, when $\Phi_n \to \Phi$, $\|\mathcal{G}\Phi_n - \mathcal{G}\Phi\|_{\mathfrak{X}} \to 0$, that is, $\mathcal{G}$ is continuous on $\mathfrak{X}$.

Step 3. The operator $\mathcal{G}$ is equicontinuous. For any $\varkappa_1, \varkappa_2 \in \mathfrak{I}$, we have

$$
\begin{aligned}
|\mathcal{G}\Phi(\varkappa_2) - \mathcal{G}\Phi(\varkappa_1)| &\leq \Big((\mathcal{Z}_4 + \mathcal{Z}_1)|e^{\frac{-\omega}{1-\omega}\varkappa_2} - e^{\frac{-\omega}{1-\omega}\varkappa_1}| + \mathcal{Z}_5|\varkappa_2 - \varkappa_1|\Big)\mathfrak{S}_1\, \mathcal{A} + \Big(\mathcal{Z}_5|\varkappa_2 - \varkappa_1| \\
&+ (\mathcal{Z}_4 + \mathcal{Z}_1)|e^{\frac{-\omega}{1-\omega}\varkappa_2} - e^{\frac{-\omega}{1-\omega}\varkappa_1}|\Big)\Big(\frac{1-\rho}{\kappa_4 \mathcal{M}(\rho)}(m_0 + m_1) + \frac{\rho}{\kappa_4 \mathcal{M}(\rho)}\int_0^{\Upsilon}(|\mathcal{Q}(\nu)| + m_0)d\nu\Big) \\
&+ \mathcal{Z}_7\mathcal{A}\int_0^{\varkappa_1}(e^{\frac{-\omega}{1-\omega}(\varkappa_2-\nu)} - e^{\frac{-\omega}{1-\omega}(\varkappa_1-\nu)})d\nu + \mathcal{Z}_7\mathcal{A}\int_{\varkappa_1}^{\varkappa_2} e^{\frac{-\omega}{1-\omega}(\varkappa_2-\nu)}d\nu + \mathcal{Z}_2\int_{\varkappa_1}^{\varkappa_2}\mathcal{A}d\nu \\
&+ \mathcal{Z}_2\, \mathcal{A}\int_0^{\nu}(e^{\frac{-\omega}{1-\omega}(\varkappa_2-\varepsilon)} - e^{\frac{-\omega}{1-\omega}(\varkappa_1-\varepsilon)})d\varepsilon + \mathcal{Z}_3|\mathcal{Q}(\varkappa_2) - \mathcal{Q}(\varkappa_1)| + \mathcal{Z}_5\int_{\varkappa_1}^{\varkappa_2}(|\mathcal{Q}(\nu)| + m_0)d\nu.
\end{aligned}
\tag{5.11}
$$

On the basis of condition ($\aleph_2$), Eq (5.11) can be rewritten as

$$
\begin{aligned}
&\leq \Big((\mathcal{Z}_4 + \mathcal{Z}_1)|e^{\frac{-\omega}{1-\omega}\varkappa_2} - e^{\frac{-\omega}{1-\omega}\varkappa_1}| + \mathcal{Z}_5|\varkappa_2 - \varkappa_1|\Big)\Big(\mathfrak{S}_1\, \mathcal{A} + \frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(m_0 + m_1) \\
&+ \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\Upsilon(d_0 + m_0)\Big) + \mathcal{Z}_7\mathcal{A}\frac{1-\omega}{\omega}|e^{\frac{-\omega}{1-\omega}\varkappa_2} - e^{\frac{-\omega}{1-\omega}\varkappa_1}|(e^{\frac{\omega}{1-\omega}\varkappa_1} - 1) \\
&+ \mathcal{Z}_7\mathcal{A}\frac{1-\omega}{\omega}e^{\frac{-\omega}{1-\omega}\varkappa_2}|e^{\frac{\omega}{1-\omega}\varkappa_2} - e^{\frac{\omega}{1-\omega}\varkappa_1}| + \mathcal{Z}_2\,|\varkappa_2 - \varkappa_1|\mathcal{A} + \mathcal{Z}_2\,\mathcal{A}\,\frac{1-\omega}{\omega}|e^{\frac{-\omega}{1-\omega}\varkappa_2} - e^{\frac{-\omega}{1-\omega}\varkappa_1}|(e^{\frac{\omega}{1-\omega}\Upsilon} - 1) \\
&+ \mathcal{Z}_3(|\mathcal{Q}(\varkappa_2) - \mathcal{Q}(\varkappa_1)|) + \mathcal{Z}_5|\varkappa_2 - \varkappa_1|(d_0 + m_0)\Upsilon.
\end{aligned}
\tag{5.12}
$$

Applying the same procedure yields the following result:

$$
\begin{aligned}
|\mathcal{G}^{\mathcal{CF}}\mathcal{D}^{\rho}\Phi(\varkappa_2) - \mathcal{G}^{\mathcal{CF}}\mathcal{D}^{\rho}\Phi(\varkappa_1)| &\leq \Big(\frac{\mathcal{M}(\rho)(1-\omega)}{(\rho-\omega)}(\mathcal{Z}_2 + \mathcal{Z}_7)\Big(|\sigma_1(\varkappa_2) - \sigma_1(\varkappa_1)| - |\sigma_2(\varkappa_2) - \sigma_2(\varkappa_1)|\Big) \\
&+ \mathcal{Z}_0|\sigma_2(\varkappa_2) - \sigma_2(\varkappa_1)|\Big)\Big(\mathfrak{S}_1\mathcal{A} + \frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(m_0 + m_1) + \frac{\rho}{\kappa_4\mathcal{M}(\rho)}(d_0 + m_0)\Upsilon\Big) \\
&+ \mathcal{Z}_7\mathcal{A}\frac{\mathcal{M}(\rho)}{\rho}|\sigma_2(\varkappa_2) - \sigma_2(\varkappa_1)|(e^{\frac{\rho}{1-\rho}\varkappa_1} - 1) + \mathcal{Z}_7\,\mathcal{A}\frac{\mathcal{M}(\rho)}{\rho}e^{\frac{-\rho}{1-\rho}\varkappa_2}|e^{\frac{\rho}{1-\rho}\varkappa_2} - e^{\frac{\rho}{1-\rho}\varkappa_1}| \\
&+ \mathcal{Z}_7\mathcal{A}|\sigma_3(\varkappa_2) - \sigma_3(\varkappa_1)|\Big(\frac{\mathcal{M}(\rho)(1-\omega)}{(\rho - 2\rho\omega + \omega)}(e^{(\frac{\rho}{1-\rho}+\frac{\omega}{1-\omega})\varkappa_1} - 1) - \frac{\mathcal{M}(\rho)}{\rho}(e^{\frac{\rho}{1-\rho}\varkappa_1} - 1)\Big) \\
&+ \mathcal{Z}_7\mathcal{A}e^{-(\frac{\rho}{1-\rho}+\frac{\omega}{1-\omega})\varkappa_2}\Big(\frac{\mathcal{M}(\rho)(1-\omega)}{(\rho-2\rho\omega+\omega)}(e^{(\frac{\rho}{1-\rho}+\frac{\omega}{1-\omega})\varkappa_2} - e^{(\frac{\rho}{1-\rho}+\frac{\omega}{1-\omega})\varkappa_1}) - \frac{\mathcal{M}(\rho)}{\rho}(e^{\frac{\rho}{1-\rho}\varkappa_2} - e^{\frac{\rho}{1-\rho}\varkappa_1})\Big) \\
&+ \mathcal{Z}_0\mathcal{A}\frac{(\kappa_2\mathcal{M}(\rho) + \rho)}{\rho}|\sigma_2(\varkappa_2) - \sigma_2(\varkappa_1)|(e^{\frac{\rho}{1-\rho}\varkappa_1} - 1) + \mathcal{Z}_0\mathcal{A}\frac{(\kappa_2\mathcal{M}(\rho)+\rho)}{\rho}e^{\frac{-\rho}{1-\rho}\varkappa_2}(e^{\frac{\rho}{1-\rho}\varkappa_2} - e^{\frac{\rho}{1-\rho}\varkappa_1}) \\
&+ \mathcal{Z}_0|\sigma_2(\varkappa_2) - \sigma_2(\varkappa_1)|\,m_0 + \mathcal{Z}_3\frac{\mathcal{M}(\rho)}{1-\rho}|\sigma_2(\varkappa_2) - \sigma_2(\varkappa_1)|\Big(|\mathcal{Q}(\varkappa_1)|e^{\frac{\rho}{1-\rho}\varkappa_1} + m_0 + d_0(e^{\frac{\rho}{1-\rho}\varkappa_1} - 1)\Big) \\
&+ \mathcal{Z}_3\frac{\mathcal{M}(\rho)}{1-\rho}e^{\frac{-\rho}{1-\rho}\varkappa_2}\Big(|\mathcal{Q}(\varkappa_2)e^{\frac{\rho}{1-\rho}\varkappa_2} - \mathcal{Q}(\varkappa_1)e^{\frac{\rho}{1-\rho}\varkappa_1}| + d_0|e^{\frac{\rho}{1-\rho}\varkappa_2} - e^{\frac{\rho}{1-\rho}\varkappa_1}|\Big) \\
&+ \mathcal{Z}_0 d_0|\sigma_2(\varkappa_2) - \sigma_2(\varkappa_1)|(e^{\frac{\rho}{1-\rho}\varkappa_1} - 1) + \mathcal{Z}_0 d_0 e^{\frac{-\rho}{1-\rho}\varkappa_2}|e^{\frac{\rho}{1-\rho}\varkappa_2} - e^{\frac{\rho}{1-\rho}\varkappa_1}|.
\end{aligned}
\tag{5.13}
$$

Similarly, the following estimate is derived:

$$
\begin{aligned}
&|\mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\omega}\Phi(\varkappa_2) - \mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\omega}\Phi(\varkappa_1)| \leq \Big((\mathcal{Z}_2+\mathcal{Z}_7)\frac{\mathcal{M}(\omega)}{1-\omega}|e^{\frac{-\omega}{1-\omega}\varkappa_2}\varkappa_2 - e^{\frac{-\omega}{1-\omega}\varkappa_1}\varkappa_1| \\
&+ \mathcal{Z}_5\frac{\mathcal{M}(\omega)}{\omega}|\sigma_1(\varkappa_2)-\sigma_1(\varkappa_1)|\Big)\Big(\mathfrak{S}_1\,\mathcal{A} + \frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(m_0+m_1) + \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\Upsilon(d_0+m_0)\Big) \\
&+ \mathcal{Z}_7\mathcal{A}\frac{\mathcal{M}(\omega)}{\omega}\Big(|e^{\frac{-\omega}{1-\omega}\varkappa_2} - e^{\frac{-\omega}{1-\omega}\varkappa_1}|(e^{\frac{\omega}{1-\omega}\varkappa_1}-1) + (1-e^{-\frac{\omega}{1-\omega}(\varkappa_2-\varkappa_1)})\Big) + \mathcal{Z}_7\mathcal{A}\frac{\mathcal{M}(\omega)}{(1-\omega)}|e^{\frac{-\omega}{1-\omega}\varkappa_2} \\
&- e^{\frac{-\omega}{1-\omega}\varkappa_1}|\Big(\frac{1-\omega}{\omega}(e^{\frac{\omega}{1-\omega}\varkappa_1}-1)-\varkappa_1\Big) + \frac{\mathcal{Z}_7\mathcal{A}\mathcal{M}(\omega)}{(1-\omega)}e^{\frac{-\omega}{1-\omega}\varkappa_2}\Big(\frac{1-\omega}{\omega}|e^{\frac{\omega}{1-\omega}\varkappa_2} - e^{\frac{\omega}{1-\omega}\varkappa_1}| - |\varkappa_2-\varkappa_1|\Big) \\
&+ \frac{(\mathcal{Z}_2+\mathcal{Z}_5)\mathcal{A}\mathcal{M}(\omega)}{\omega}|e^{\frac{-\omega}{1-\omega}\varkappa_2} - e^{\frac{-\omega}{1-\omega}\varkappa_1}|(e^{\frac{\omega}{1-\omega}\varkappa_1}-1) + \frac{(\mathcal{Z}_2+\mathcal{Z}_5)\mathcal{A}\mathcal{M}(\omega)}{\omega}e^{\frac{-\omega}{1-\omega}\varkappa_2}|e^{\frac{\omega}{1-\omega}\varkappa_2} - e^{\frac{\omega}{1-\omega}\varkappa_1}| \\
&+ \mathcal{Z}_3\frac{\mathcal{M}(\omega)}{1-\omega}\,|e^{\frac{-\omega}{1-\omega}\varkappa_2} - e^{\frac{-\omega}{1-\omega}\varkappa_1}|\Big[Q(\varkappa_1)e^{\frac{\omega}{1-\omega}\varkappa_1} + m_0 + d_0(e^{\frac{\omega}{1-\omega}\varkappa_1}-1)\Big] \\
&+ \mathcal{Z}_3\frac{\mathcal{M}(\omega)}{1-\omega}\,e^{\frac{-\omega}{1-\omega}\varkappa_2}\Big[|Q(\varkappa_2)e^{\frac{\omega}{1-\omega}\varkappa_2} - Q(\varkappa_1)e^{\frac{\omega}{1-\omega}\varkappa_1}| + d_0|e^{\frac{\omega}{1-\omega}\varkappa_1} - ^{\frac{\omega}{1-\omega}\varkappa_2}\,|\Big] \\
&+ \mathcal{Z}_5\frac{\mathcal{M}(\omega)}{\omega}d_0|e^{\frac{-\omega}{1-\omega}\varkappa_2} - e^{\frac{-\omega}{1-\omega}\varkappa_1}|(e^{\frac{\omega}{1-\omega}\varkappa_1}-1) + \mathcal{Z}_5\frac{\mathcal{M}(\omega)}{\omega}(d_0e^{\frac{-\omega}{1-\omega}\varkappa_2}+m_0)|e^{\frac{\omega}{1-\omega}\varkappa_2} - e^{\frac{\omega}{1-\omega}\varkappa_1}|.
\end{aligned} \tag{5.14}
$$

Upon merging Eqs (5.12)–(5.14), one obtains

$$
\begin{aligned}
\|\mathcal{G}\Phi(\varkappa_2) - \mathcal{G}\Phi(\varkappa_1)\|_{\mathfrak{x}} = &\sup_{\varkappa_1,\varkappa_2\in\mathfrak{I}}|\mathcal{G}\Phi(\varkappa_2)-\mathcal{G}\Phi(\varkappa_1)| + \sup_{\varkappa_1,\varkappa_2\in\mathfrak{I}}|\mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\rho}\Phi(\varkappa_2) - \mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\rho}\Phi(\varkappa_1)| \\
&+ \sup_{\varkappa_1,\varkappa_2\in\mathfrak{I}}|\mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\omega}\Phi(\varkappa_2) - \mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\omega}\Phi(\varkappa_1)|.
\end{aligned}
$$

Hence $\|\Phi(\varkappa_2)-\Phi(\varkappa_1)\|_{\mathfrak{x}} \to 0$ as $\varkappa_1 \to \varkappa_2$, and thus the operator $\mathcal{G}$ is equicontinuous.

Step 4. We start by showing that the operator $\mathcal{G}$ is a F-contraction on $\mathfrak{B}_{\mathcal{A}}$. For every bounded $\hat{W} \subset \mathfrak{B}_{\mathcal{A}}$ and for any $\nabla > 0$, by virtue of Lemma 3.7 and the properties of F, a sequence $\{\Phi_i\}_{i=1}^{\infty} \subset F$ exists satisfying

$$
\begin{aligned}
F(\mathcal{G}\hat{W}(\varkappa)) \leq 2F\Big\{&\widehat{\lambda_1}\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\{\Phi_i(\nu)\}_{i=1}^{\infty}e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu + \frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\{\Phi_i(\nu)\}_{i=1}^{\infty}d\nu \\
&+ \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\{\Phi_i(\varepsilon)\}_{i=1}^{\infty}e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon + \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\{\Phi_i(\varepsilon)\}_{i=1}^{\infty}e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon \\
&+ \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\mathbf{Q}(\nu,\{\Phi_i(\nu)\}_{i=1}^{\infty}, {}^{CF}\mathcal{D}^{\rho}\{\Phi_i(\nu)\}_{i=1}^{\infty}, {}^{CF}\mathcal{D}^{\omega}\{\Phi_i(\nu)\}_{i=1}^{\infty})d\nu\Big) \\
&+ \mathcal{Z}_7\int_0^{\varkappa}\{\Phi_i(\nu)\}_{i=1}^{\infty}e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu + \mathcal{Z}_2\int_0^{\varkappa}\{\Phi_i(\nu)\}_{i=1}^{\infty}d\nu + \mathcal{Z}_2\int_0^{\nu}\{\Phi_i(\nu)\}_{i=1}^{\infty}e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon \\
&+ \mathcal{Z}_2\int_0^{\nu}\{\Phi_i(\varepsilon)\}_{i=1}^{\infty}e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon + \mathcal{Z}_3\mathbf{Q}(\varkappa,\{\Phi_i(\varkappa)\}_{i=1}^{\infty}, {}^{CF}\mathcal{D}^{\rho}\{\Phi_i(\varkappa)\}_{i=1}^{\infty}, {}^{CF}\mathcal{D}^{\omega}\{\Phi_i(\varkappa)\}_{i=1}^{\infty}) \\
&+ \mathcal{Z}_5\int_0^{\varkappa}\mathbf{Q}(\nu,\{\Phi_i(\nu)\}_{i=1}^{\infty}, {}^{CF}\mathcal{D}^{\rho}\{\Phi_i(\nu)\}_{i=1}^{\infty}, {}^{CF}\mathcal{D}^{\omega}\{\Phi_i(\nu)\}_{i=1}^{\infty})d\nu\Big\} + \nabla.
\end{aligned}
$$

By Lemma 3.6, it follows that

$$\begin{aligned}
&\leq 4\Bigg\{\widehat{\lambda_1}\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon} F(\{\Phi_i(\nu)\}_{i=1}^{\infty})e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu+\frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon} F(\{\Phi_i(\nu)\}_{i=1}^{\infty})d\nu\\
&\quad+\frac{\kappa_2}{\kappa_4}\int_0^{\nu} F(\{\Phi_i(\varepsilon)\}_{i=1}^{\infty})e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon+\frac{\kappa_2}{\kappa_4}\int_0^{\nu} F(\{\Phi_i(\varepsilon)\}_{i=1}^{\infty})e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon\\
&\quad+\frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon} F\Big(\mathbf{Q}(\nu,\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\rho}\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\omega}\{\Phi_i(\nu)\}_{i=1}^{\infty})\Big)d\nu\Big)\\
&\quad+\mathcal{Z}_7\int_0^{\varkappa} F(\{\Phi_i(\nu)\}_{i=1}^{\infty})e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu+\mathcal{Z}_2\int_0^{\varkappa} F(\{\Phi_i(\nu)\}_{i=1}^{\infty})d\nu\\
&\quad+\mathcal{Z}_2\int_0^{\nu} F(\{\Phi_i(\nu)\}_{i=1}^{\infty})e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon+\mathcal{Z}_2\int_0^{\nu} F(\{\Phi_i(\varepsilon)\}_{i=1}^{\infty})e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon\\
&\quad+\mathcal{Z}_3F\Big(\mathbf{Q}(\varkappa,\{\Phi_i(\varkappa)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\rho}\{\Phi_i(\varkappa)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\omega}\{\Phi_i(\varkappa)\}_{i=1}^{\infty})\Big)\\
&\quad+\mathcal{Z}_5\int_0^{\varkappa} F\Big(\mathbf{Q}(\nu,\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\rho}\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\omega}\{\Phi_i(\nu)\}_{i=1}^{\infty})\Big)d\nu\Bigg\}+\nabla.
\end{aligned}\tag{5.15}$$

Applying the condition ($\aleph_4$), Eq (5.15) becomes

$$F(\mathcal{G}\hat{W}(\varkappa))\ \leq 4\Big(\mathfrak{W}_1+\mathfrak{W}_2\mathfrak{K}\Big)F(\{\Phi_i\}_{i=1}^{\infty})+\nabla,\quad \forall\ \nabla>0.$$

Then,

$$F(\mathcal{G}\hat{W})=\sup F(\mathcal{G}\hat{W}(\varkappa))\leq 4\Big(\mathfrak{W}_1+\mathfrak{W}_2\mathfrak{K}\Big)F(\mathfrak{B}_{\mathcal{A}}).\tag{5.16}$$

The following result is also derived using the same technique:

$$\begin{aligned}
&F(\mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\rho}\hat{W}(\varkappa))\leq 2F\Bigg\{\Big(\frac{\mathcal{M}(\rho)(1-\omega)}{(\rho-\omega)}(\mathcal{Z}_2+\mathcal{Z}_7)(e^{\frac{-\omega}{1-\omega}\varkappa}-e^{\frac{-\rho}{1-\rho}\varkappa})\\
&+\mathcal{Z}_0\sigma_2(\varkappa)\Big)\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\{\Phi_i(\nu)\}_{i=1}^{\infty}e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu+\frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\{\Phi_i(\nu)\}_{i=1}^{\infty}d\nu\\
&+\frac{\kappa_2}{\kappa_4}\int_0^{\nu}\{\Phi_i(\varepsilon)\}_{i=1}^{\infty}e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon+\frac{\kappa_2}{\kappa_4}\int_0^{\nu}\{\Phi_i(\varepsilon)\}_{i=1}^{\infty}e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon\\
&+\frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\mathbf{Q}(\nu,\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\rho}\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\omega}\{\Phi_i(\nu)\}_{i=1}^{\infty})d\nu\Big)\\
&+\mathcal{Z}_7\frac{\mathcal{M}(\rho)}{1-\rho}\int_0^{\varkappa}\{\Phi_i(\nu)\}_{i=1}^{\infty}e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu+\mathcal{Z}_7\frac{\omega\mathcal{M}(\rho)}{(1-\omega)(1-\rho)}\int_0^{\varkappa}\int_0^{\nu}e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}\{\Phi_i(\varepsilon)\}_{i=1}^{\infty}d\varepsilon d\nu\\
&+\mathcal{Z}_0\frac{(\kappa_2\mathcal{M}(\rho)+\rho)}{(1-\rho)}\int_0^{\varkappa}\{\Phi_i(\nu)\}_{i=1}^{\infty}e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu\\
&+\mathcal{Z}_3\frac{\mathcal{M}(\rho)}{(1-\rho)}\int_0^{\varkappa}\mathbf{Q}'(\nu,\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\rho}\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\omega}\{\Phi_i(\nu)\}_{i=1}^{\infty})e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu\\
&+\mathcal{Z}_0\frac{\rho}{(1-\rho)}\int_0^{\varkappa}\mathbf{Q}(\nu,\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\rho}\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\omega}\{\Phi_i(\nu)\}_{i=1}^{\infty})e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu\Bigg\}+\nabla,
\end{aligned}$$

so

$$F(\mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\rho}\hat{W}(\varkappa)) \leq 4(\mathfrak{W}_5 + \hat{\theta}_1\mathfrak{R})F(\{\Phi_i\}_{i=1}^{\infty}) + \nabla)F(\mathfrak{B}_{\mathcal{A}}).$$

$$F(\mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\rho}\hat{W}(\varkappa)) = \sup F(\mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\rho}\hat{W}(\varkappa)) \leq 4(\mathfrak{W}_5 + \hat{\theta_1\mathfrak{R}})F(\mathfrak{B}_{\mathcal{A}}). \tag{5.17}$$

By Lemma 3.6, along with ($\aleph_4$), the following estimate is obtained:

$$\begin{aligned}
F(\mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\omega}\Phi(\varkappa)) \leq{}& 2F\Big\{\widehat{\delta_1}\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\{\Phi_i(\nu)\}_{i=1}^{\infty}e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu + \frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\{\Phi_i(\nu)\}_{i=1}^{\infty}d\nu\\
&+ \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\{\Phi_i(\varepsilon)\}_{i=1}^{\infty}e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon + \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\{\Phi_i(\varepsilon)\}_{i=1}^{\infty}e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon\\
&+ \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\mathbf{Q}(\nu,\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\rho}\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\omega}\{\Phi_i(\nu)\}_{i=1}^{\infty})d\nu\Big)\\
&+ \mathcal{Z}_7\frac{\mathcal{M}(\omega)}{1-\omega}\int_0^{\varkappa}\{\Phi_i(\nu)\}_{i=1}^{\infty}e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu + \mathcal{Z}_7\frac{\omega\mathcal{M}(\omega)}{(1-\omega)^2}\int_0^{\varkappa}\int_0^{\nu}\{\Phi_i(\varepsilon)\}_{i=1}^{\infty}e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon d\nu\\
&+ (\mathcal{Z}_2+\mathcal{Z}_5)\frac{\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}\{\Phi_i(\nu)\}_{i=1}^{\infty}e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu\\
&+ \mathcal{Z}_3\frac{\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}\mathbf{Q}'(\nu,\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\rho}\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\omega}\{\Phi_i(\nu)\}_{i=1}^{\infty})e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu\\
&+ \mathcal{Z}_5\frac{\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}\mathbf{Q}(\nu,\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\rho}\{\Phi_i(\nu)\}_{i=1}^{\infty},{}^{CF}\mathcal{D}^{\omega}\{\Phi_i(\nu)\}_{i=1}^{\infty})e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu\Big\} + \nabla,
\end{aligned}$$

$$F(\mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\omega}\Phi(\varkappa)) \leq 4(\mathfrak{W}_3 + \hat{\theta}_2\mathfrak{R})F(\{\Phi_i\}_{i=1}^{\infty}) + \nabla. \tag{5.18}$$

It follows from Eqs (5.16)–(5.18) that

$$F(\mathcal{G}\hat{W}(\varkappa)) = \sup F(\mathcal{G}\hat{W}(\varkappa)) + \sup F(\mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\rho}\hat{W}(\varkappa)) + \sup F(\mathcal{G}^{C\mathcal{F}}\mathcal{D}^{\omega}\hat{W}(\varkappa)) \leq 4\hat{\theta}_3\, F(\mathfrak{B}_{\mathcal{A}}),$$

where

$$\hat{\theta}_3 = 4(\mathfrak{W}_1 + \mathfrak{W}_3 + \mathfrak{W}_5) + 4(\hat{\theta}_1 + \hat{\theta}_2 + \mathfrak{W}_2)\mathfrak{R}.$$

If $\hat{\theta}_3 < 1$, $\mathcal{G}$ is shown to be a F-contraction on $\mathfrak{B}_{\mathcal{A}}$. Therefore by Theorem 3.10, the boundary value problems (2.1) and (2.2) have at least one solution. This completes the proof. □

Theorem 5.2. *Suppose that ($\aleph_3$) holds. If*

$$\Big(\frac{(\mathfrak{W}_1 + \mathfrak{W}_3 + \mathfrak{W}_5)\mathfrak{W}_2}{1 - \mathfrak{W}_1} + \mathfrak{Y}\Big) < 1,$$

then the boundary value problems (2.1) *and* (2.2) *have a unique solution.*

Proof. Let Φ_1 and $\Phi_2 \in \mathfrak{X}$. Then for each $\varkappa \in \mathfrak{J}$, this implies

$$
\begin{aligned}
&|(\mathcal{G}\Phi_1)(\varkappa) - (\mathcal{G}\Phi_2)(\varkappa)| \leq \widehat{\lambda_1}\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon} |\Phi_1(\nu) - \Phi_2(\nu)| e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)} d\nu \\
&+ \frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon} |\Phi_1(\nu) - \Phi_2(\nu)| d\nu + \frac{\kappa_2}{\kappa_4}\int_0^{\nu} |\Phi_1(\varepsilon) - \Phi_2(\varepsilon)| \Big(e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)} + e^{\frac{\omega}{1-\omega}\varepsilon}\Big) d\varepsilon \\
&+ \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon} |\mathbf{Q}(\nu, \Phi_1(\nu), \mathcal{D}^{\rho}\Phi_1(\nu), \mathcal{D}^{\omega}\Phi_1(\nu)) - \mathbf{Q}(\nu, \Phi_2(\nu), \mathcal{D}^{\rho}\Phi_2(\nu), \mathcal{D}^{\omega}\Phi_2(\nu))| d\nu\Big) \\
&+ \mathcal{Z}_7\int_0^{\varkappa} |\Phi_1(\nu) - \Phi_2(\nu)| e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)} d\nu + \mathcal{Z}_2\int_0^{\varkappa} |\Phi_1(\nu) - \Phi_2(\nu)| d\nu + \mathcal{Z}_2\int_0^{\nu} |\Phi_1(\varepsilon) - \Phi_2(\varepsilon)| \Big(e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)} \\
&+ e^{\frac{\omega}{1-\omega}\varepsilon}\Big) d\varepsilon + \mathcal{Z}_3 |\mathbf{Q}(\varkappa, \Phi_1(\varkappa), \mathcal{D}^{\rho}\Phi_1(\varkappa), \mathcal{D}^{\omega}\Phi_1(\varkappa)) - \mathbf{Q}(\varkappa, \Phi_2(\varkappa), \mathcal{D}^{\rho}\Phi_2(\varkappa), \mathcal{D}^{\omega}\Phi_2(\varkappa))| \\
&+ \mathcal{Z}_5\int_0^{\varkappa} |\mathbf{Q}(\nu, \Phi_1(\nu), \mathcal{D}^{\rho}\Phi_1(\nu), \mathcal{D}^{\omega}\Phi_1(\nu)) - \mathbf{Q}(\nu, \Phi_2(\nu), \mathcal{D}^{\rho}\Phi_2(\nu), \mathcal{D}^{\omega}\Phi_2(\nu))| d\nu.
\end{aligned}
$$

From the Lipschitz condition $(\aleph_3)$, the result follows

$$
\begin{aligned}
\|(\mathcal{G}\Phi_1) - (\mathcal{G}\Phi_2)\| &\leq \mathfrak{W}_1\|\Phi_1 - \Phi_2\| \\
&+ \beth\, \mathfrak{W}_2\Big(\|\Phi_1 - \Phi_2\| + \|\mathcal{D}^{\rho}\Phi_1 - \mathcal{D}^{\rho}\Phi_2\| + \|\mathcal{D}^{\omega}\Phi_1 - \mathcal{D}^{\omega}\Phi_2\|\Big).
\end{aligned} \tag{5.19}
$$

Similarly,

$$
\begin{aligned}
&|(\mathcal{G}\mathcal{D}^{\rho}\Phi_1)(\varkappa) - (\mathcal{G}\mathcal{D}^{\rho}\Phi_2)(\varkappa)| \leq \widehat{\delta_2}\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon} |\Phi_1(\nu) - \Phi_2(\nu)| e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)} d\nu \\
&+ \frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon} |\Phi_1(\nu) - \Phi_2(\nu)| d\nu + \frac{\kappa_2}{\kappa_4}\int_0^{\nu} |\Phi_1(\varepsilon) - \Phi_2(\varepsilon)| \Big(e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)} + e^{\frac{\omega}{1-\omega}\varepsilon}\Big) d\varepsilon \\
&+ \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon} |\mathbf{Q}(\nu, \Phi_1(\nu), \mathcal{D}^{\rho}\Phi_1(\nu), \mathcal{D}^{\omega}\Phi_1(\nu)) - \mathbf{Q}(\nu, \Phi_2(\nu), \mathcal{D}^{\rho}\Phi_2(\nu), \mathcal{D}^{\omega}\Phi_2(\nu))| d\nu\Big) \\
&+ (\mathcal{Z}_7\frac{\mathcal{M}(\rho)}{1-\rho} + \mathcal{Z}_0\frac{(\kappa_2\mathcal{M}(\rho)+\rho)}{(1-\rho)})\int_0^{\varkappa} |\Phi_1(\nu) - \Phi_2(\nu)| e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)} d\nu \\
&+ \mathcal{Z}_7\frac{\omega\mathcal{M}(\rho)}{(1-\omega)(1-\rho)}\int_0^{\varkappa}\int_0^{\nu} e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)} e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)} |\Phi_1(\varepsilon) - \Phi_2(\varepsilon)| d\varepsilon d\nu \\
&+ \mathcal{Z}_3\frac{\mathcal{M}(\rho)}{(1-\rho)} e^{\frac{-\rho}{1-\rho}\varkappa}\int_0^{\varkappa} |\mathbf{Q}'(\nu, \Phi_1(\nu), \mathcal{D}^{\rho}\Phi_1(\nu), \mathcal{D}^{\omega}\Phi_1(\nu)) - \mathbf{Q}'(\nu, \Phi_2(\nu), \mathcal{D}^{\rho}\Phi_2(\nu), \mathcal{D}^{\omega}\Phi_2(\nu))| e^{\frac{\rho}{1-\rho}\nu} d\nu \\
&+ \mathcal{Z}_0\frac{\rho}{(1-\rho)} e^{\frac{-\rho}{1-\rho}\varkappa}\int_0^{\varkappa} |\mathbf{Q}(\nu, \Phi_1(\nu), \mathcal{D}^{\rho}\Phi_1(\nu), \mathcal{D}^{\omega}\Phi_1(\nu)) - \mathbf{Q}(\nu, \Phi_2(\nu), \mathcal{D}^{\rho}\Phi_2(\nu), \mathcal{D}^{\omega}\Phi_2(\nu))| e^{\frac{\rho}{1-\rho}\nu} d\nu.
\end{aligned} \tag{5.20}
$$

By using the integration by part and applying the condition $(\aleph_3)$, Eq (5.20) becomes

$$
\begin{aligned}
\|(\mathcal{G}\mathcal{D}^{\rho}\Phi_1) - (\mathcal{G}\mathcal{D}^{\rho}\Phi_2)\| &\leq \mathfrak{W}_5\|\Phi_1 - \Phi_2\| \\
&+ \beth\, \hat{\theta}_1\Big(\|\Phi_1 - \Phi_2\| + \|\mathcal{D}^{\rho}\Phi_1 - \mathcal{D}^{\rho}\Phi_2\| + \|\mathcal{D}^{\omega}\Phi_1 - \mathcal{D}^{\omega}\Phi_2\|\Big).
\end{aligned} \tag{5.21}
$$

Using the same technique, it is found that

$$\begin{aligned}\|\mathcal{GD}^{\omega}\Phi_1 - \mathcal{GD}^{\omega}\Phi_2\| \leq \mathfrak{W}_3\|\Phi_1 - \Phi_2\| \\ + \beth\,\hat{\theta}_2\Big(\|\Phi_1 - \Phi_2\| + \|\mathcal{D}^{\rho}\Phi_1 - \mathcal{D}^{\rho}\Phi_2\| + \|\mathcal{D}^{\omega}\Phi_1 - \mathcal{D}^{\omega}\Phi_2\|\Big).\end{aligned} \tag{5.22}$$

It follows from Eqs (5.19), (5.21), and (5.22) that

$$\begin{aligned}\|\mathcal{G}\Phi_1 - \mathcal{G}\Phi_2\|_{\mathfrak{x}} \leq &(\mathfrak{W}_1 + \mathfrak{W}_3 + \mathfrak{W}_5)\|\Phi_1 - \Phi_2\| \\ &+ \beth\,\mathfrak{Y}\left(\|\Phi_1 - \Phi_2\| + \|\mathcal{D}^{\rho}\Phi_1 - \mathcal{D}^{\rho}\Phi_2\| + \|\mathcal{D}^{\omega}\Phi_1 - \mathcal{D}^{\omega}\Phi_2\|\right).\end{aligned} \tag{5.23}$$

At this point, we need to compute the term $\|\Phi_1 - \Phi_2\|$, and it is determined as follows:

$$\|\Phi_1 - \Phi_2\| \leq \frac{\beth\,\mathfrak{W}_2}{1 - \mathfrak{W}1}\Big(\|\Phi_1 - \Phi_2\| + \|\mathcal{D}^{\rho}\Phi_1 - \mathcal{D}^{\rho}\Phi_2\| + \|\mathcal{D}^{\omega}\Phi_1 - \mathcal{D}^{\omega}\Phi_2\|\Big). \tag{5.24}$$

By means of Eq (5.24), Eq (5.23) takes the form

$$\|\mathcal{G}\Phi_1 - \mathcal{G}\Phi_2\|_{\mathfrak{x}} \leq \hat{\beta}\,\|\Phi_1 - \Phi_2\|_{\mathfrak{x}}.$$

$$\text{where} \qquad \hat{\beta} = \beth\left(\frac{(\mathfrak{W}_1 + \mathfrak{W}_3 + \mathfrak{W}_5)\mathfrak{W}_2}{1 - \mathfrak{W}_1} + \mathfrak{Y}\right).$$

If $\hat{\beta} < 1$. Then, by the Banach contraction principle, the boundary value problems (2.1) and (2.2) have a unique solution. □

6. Stability results

Stability is a key qualitative aspect of fractional-order systems, where the analysis is more intricate than in their integer-order counterparts. This importance arises from the sensitivity of physical models to perturbations and measurement errors. In this section, the Ulam–Hyers and Ulam–Hyers–Rassias stability analyses of the Caputo–Fabrizio $\mathcal{FDE}$ (2.1), subject to the boundary condition (2.2), are presented.

Definition 6.1. *The Eq* (2.1) *is Ulam–Hyers stable if a real number* $\hat{\eta}_f > 0$ *exists such that for each* $\hat{\xi} > 0$ *and for each solution* $\mathfrak{T} \in C^1(\mathfrak{J}, \mathfrak{R})$ *of the inequality, we have*

$$|{}^{\mathcal{CF}}\mathcal{D}^{\rho}\mathfrak{T}(\varkappa) + {}^{\mathcal{CF}}\mathcal{D}^{\omega}\mathfrak{T}(\varkappa) + \mathfrak{T}(\varkappa) - \mathbf{Q}\left(\varkappa, \mathfrak{T}(\varkappa), {}^{\mathcal{CF}}\mathcal{D}^{\rho}\mathfrak{T}(\varkappa), {}^{\mathcal{CF}}\mathcal{D}^{\omega}\mathfrak{T}(\varkappa)\right)| \leq \hat{\xi}, \quad \varkappa \in \mathfrak{J}. \tag{6.1}$$

A solution $\Phi \in C^1(\mathfrak{J}, \mathfrak{R})$ *of Eq* (2.1) *exists with*

$$|\mathfrak{T}(\varkappa) - \Phi(\varkappa)| \leq \hat{\eta}_f\hat{\xi}, \quad \varkappa \in \mathfrak{J}.$$

Remark 6.2. *A function* $\mathfrak{T}(\varkappa) \in C(\mathfrak{J}, \mathfrak{R})$ *is a solution of the inequality* (6.1) *if and only if a function* $\mathfrak{V}$ *exists such that*

$$\begin{aligned}&(i)|\mathfrak{V}(t)| \leq \hat{\xi}, \textit{for all } \varkappa \in \mathfrak{J}, \\ &(ii){}^{\mathcal{CF}}\mathcal{D}^{\rho}\mathfrak{T}(\varkappa) + {}^{\mathcal{CF}}\mathcal{D}^{\omega}\mathfrak{T}(\varkappa) + \mathfrak{T}(\varkappa) = Q(\varkappa) + \mathfrak{V}(\varkappa), \quad \forall \varkappa \in \mathfrak{J}.\end{aligned} \tag{6.2}$$

Definition 6.3. *Eq* (2.1) *is Ulam–Hyers–Rassias stable with respect to* $\hat{\mho} \in C(\Im, \Re_+)$*, if a real number* $\hat{\eta}_c > 0$ *exists such that for each* $\hat{\xi} > 0$ *and for each solution* $\mathfrak{T} \in C^1(\Im, \Re)$ *of the inequality, we have*

$$|{}^{C\mathcal{F}}\mathcal{D}^{\rho}\mathfrak{T}(\varkappa) + {}^{C\mathcal{F}}\mathcal{D}^{\omega}\mathfrak{T}(\varkappa) + \mathfrak{T}(\varkappa) - \mathbf{Q}\left(\varkappa, \mathfrak{T}(\varkappa), {}^{C\mathcal{F}}\mathcal{D}^{\rho}\mathfrak{T}(\varkappa), {}^{C\mathcal{F}}\mathcal{D}^{\omega}\mathfrak{T}(\varkappa)\right)| \le \hat{\xi}\hat{\mho}(\varkappa), \quad \varkappa \in \Im. \tag{6.3}$$

A solution $\Phi \in C^1(\Im, \Re)$ *of Eq* (2.1) *exists with*

$$|\mathfrak{T}(\varkappa) - \Phi(\varkappa)| \le \hat{\eta}_c\hat{\xi}\hat{\mho}(\varkappa), \quad \varkappa \in \Im.$$

Theorem 6.4. *If condition* ($\aleph_3$) *holds with* $\mathfrak{W}_7 < 1$*, then the boundary value problems* (2.1) *and* (2.2) *are Ulam–Hyers stable.*

Proof. Let $\mathfrak{T}(\varkappa) \in C(\Im, \Re)$ be any solution of the inequality (6.1). Then, by Remark (6.2), we have

$${}^{C\mathcal{F}}\mathcal{D}^{\rho}\mathfrak{T}(\varkappa) + {}^{C\mathcal{F}}\mathcal{D}^{\omega}\mathfrak{T}(\varkappa) + \mathfrak{T}(\varkappa) = \mathbf{Q}\left(\varkappa, \mathfrak{T}(\varkappa), {}^{C\mathcal{F}}\mathcal{D}^{\rho}\mathfrak{T}(\varkappa), {}^{C\mathcal{F}}\mathcal{D}^{\omega}\mathfrak{T}(\varkappa)\right) + \mathfrak{V}(\varkappa).$$

Let $\hat{\xi} > 0$ be arbitrary and $\mathfrak{T} \in C(\Im, \Re)$ be any solution satisfying inequality (6.1). Then

$$\begin{aligned}
&\Bigg|\mathfrak{T}(\varkappa) - \left(\mathcal{Z}_4\left(1 - e^{\frac{-\omega}{1-\omega}\varkappa}\right) + \mathcal{Z}_1 e^{\frac{-\omega}{1-\omega}\varkappa} + \mathcal{Z}_5\varkappa + \mathcal{Z}_3 + \mathcal{Z}_0\right)\Bigg(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\mathfrak{T}(\nu)e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu \\
&- \frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\mathfrak{T}(\nu)d\nu - \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon + \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon \\
&+ \frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(Q(\Upsilon) - Q(0)) + \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(Q(\nu) - Q(0))d\nu\Bigg) - \mathcal{Z}_7\int_0^{\varkappa}\mathfrak{T}(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu \\
&+ \mathcal{Z}_2\int_0^{\varkappa}\mathfrak{T}(\nu)d\nu + \mathcal{Z}_2\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon - \mathcal{Z}_2\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon - \mathcal{Z}_3(Q(\varkappa) - Q(0)) \\
&- \mathcal{Z}_5\int_0^{\varkappa}(Q(\nu) - Q(0))d\nu\Bigg| \le 2\hat{\xi}\,\frac{(1+\rho(\varkappa-1))}{\mathcal{M}(\rho)}.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\Bigg|\mathcal{D}^{\rho}\mathfrak{T}(\varkappa) - \Bigg(\frac{\mathcal{M}(\rho)(1-\omega)}{(\rho-\omega)}(\mathcal{Z}_2 - \mathcal{Z}_7)(e^{\frac{-\omega}{1-\omega}\varkappa} - e^{\frac{-\rho}{1-\rho}\varkappa}) - \mathcal{Z}_0\sigma_2(\varkappa)\Bigg)\Bigg(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\mathfrak{T}(\nu)e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu \\
&- \frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\mathfrak{T}(\nu)d\nu - \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon \\
&+ \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon + \frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(Q(\Upsilon - Q(0)) + \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(Q(\nu) - Q(0))d\nu\Bigg) \\
&- \mathcal{Z}_7\frac{\mathcal{M}(\rho)}{1-\rho}\int_0^{\varkappa}\mathfrak{T}(\nu)e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu + \frac{\mathcal{Z}_7\omega\mathcal{M}(\rho)}{(1-\omega)(1-\rho)}\int_0^{\varkappa}\int_0^{\nu}e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}\mathfrak{T}(\varepsilon)d\varepsilon d\nu \\
&+ \frac{\mathcal{Z}_0(\kappa_2\mathcal{M}(\rho)+\rho)}{(1-\rho)}\int_0^{\varkappa}\mathfrak{T}(\nu)e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu - \frac{\mathcal{Z}_2\,\omega}{1-\omega} - \mathcal{Z}_3\frac{\mathcal{M}(\rho)}{(1-\rho)}\int_0^{\varkappa}Q'(\nu)e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu \\
&- \mathcal{Z}_0\frac{\rho}{(1-\rho)}\int_0^{\varkappa}Q(\nu)e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu + \mathcal{Z}_0(1 - e^{\frac{-\rho}{1-\rho}\varkappa})Q(0)\Bigg| \le 2\hat{\xi}\,\sigma_2(\varkappa),
\end{aligned}$$

and

$$
\begin{aligned}
&\Big|\mathcal{D}^{\omega}\mathfrak{T}(\varkappa)-\Big((\mathcal{Z}_2-\mathcal{Z}_7)\frac{\mathcal{M}(\omega)}{1-\omega}e^{\frac{-\omega}{1-\omega}\varkappa}\Upsilon+\sigma_1(\varkappa)\mathcal{Z}_5\frac{\mathcal{M}(\omega)}{\omega}\Big)\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\mathfrak{T}(\nu)e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu\\
&-\frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\mathfrak{T}(\nu)d\nu-\frac{\kappa_2}{\kappa_4}\int_0^{\nu}\mathfrak{T}(\varepsilon)\,e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon\\
&+\frac{\kappa_2}{\kappa_4}\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon+\frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(Q(\Upsilon)-Q(0))+\frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(Q(\nu)-Q(0))d\nu\Big)\\
&-\frac{\mathcal{Z}_7\mathcal{M}(\omega)}{1-\omega}\int_0^{\varkappa}\mathfrak{T}(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu+\frac{\mathcal{Z}_7\omega\mathcal{M}(\omega)}{(1-\omega)^2}\int_0^{\varkappa}\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon d\nu\\
&+(\mathcal{Z}_2+\mathcal{Z}_5)\frac{\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}\mathfrak{T}(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu-\mathcal{Z}_2\,\frac{\omega}{1-\omega}-\frac{\mathcal{Z}_3\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}Q'(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu\\
&-\frac{\mathcal{Z}_5\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}Q(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu-\frac{\mathcal{Z}_5\mathcal{M}(\omega)}{\omega}(1-e^{\frac{-\omega}{1-\omega}\varkappa})Q(0)\Big|\\
&\le 2\hat{\xi}\,\frac{\rho\mathcal{M}(\omega)}{\omega\mathcal{M}(\rho)}\sigma_1(\varkappa).
\end{aligned}
$$

For $\mathfrak{T},\Phi\in C(\mathfrak{J},\mathfrak{R})$, it means that

$$
\begin{aligned}
&\Big|\mathfrak{T}(\varkappa)-\Phi(\varkappa)\Big|\le\Big|\mathfrak{T}(\varkappa)-\Big(\mathcal{Z}_4\Big(1-e^{\frac{-\omega}{1-\omega}\varkappa}\Big)+\mathcal{Z}_1e^{\frac{-\omega}{1-\omega}\varkappa}+\mathcal{Z}_5\varkappa+\mathcal{Z}_3+\mathcal{Z}_0\Big)\\
&\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\Phi(\nu)e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu-\frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\Phi(\nu)d\nu-\frac{\kappa_2}{\kappa_4}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon\\
&+\frac{\kappa_2}{\kappa_4}\int_0^{\nu}\Phi(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon+\frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(Q(\Upsilon)-Q(0))+\frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(Q(\nu)-Q(0))d\nu\Big)\\
&-\mathcal{Z}_7\int_0^{\varkappa}\Phi(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu+\mathcal{Z}_2\int_0^{\varkappa}\Phi(\nu)d\nu+\mathcal{Z}_2\int_0^{\nu}\Phi(\varepsilon)e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon-\mathcal{Z}_2\int_0^{\nu}\Phi(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon\\
&-\mathcal{Z}_3(Q(\varkappa)-Q(0))-\mathcal{Z}_5\int_0^{\varkappa}(Q(\nu)-Q(0))d\nu\Big|.
\end{aligned}
$$

It follows from ($\aleph_3$) that

$$
\begin{aligned}
\|\mathfrak{T}-\Phi\|\le 2\hat{\xi}\,\frac{(1+\rho(\varkappa-1))}{\mathcal{M}(\rho)(1-\mathfrak{W}1)}+\daleth\frac{\mathfrak{W}_2}{(1-\mathfrak{W}1)}\Big(&\|\mathfrak{T}-\Phi\|\\
&+\|\mathcal{D}^{\rho}\mathfrak{T}-\mathcal{D}^{\rho}\Phi\|+\|\mathcal{D}^{\omega}\mathfrak{T}-\mathcal{D}^{\omega}\Phi\|\Big).
\end{aligned}
\tag{6.4}
$$

In the same manner, one obtains

$$
\begin{aligned}
\|\mathcal{D}^{\rho}\mathfrak{T}-\mathcal{D}^{\rho}\Phi\|&\le 2\hat{\xi}\left(\sigma_2(\varkappa)+\frac{(1+\rho(\varkappa-1))\mathfrak{W}_5}{\mathcal{M}(\rho)(1-\mathfrak{W}_1)}\right)\\
&+\daleth\left(\frac{\mathfrak{W}_5\mathfrak{W}_2}{1-\mathfrak{W}_1}+\hat{\theta}_1\right)\Big(\|\mathfrak{T}-\Phi\|+\|\mathcal{D}^{\rho}\mathfrak{T}-\mathcal{D}^{\rho}\Phi\|+\|\mathcal{D}^{\omega}\mathfrak{T}-\mathcal{D}^{\omega}\Phi\|\Big).
\end{aligned}
\tag{6.5}
$$

Moreover

$$|\mathcal{D}^{\omega}\mathfrak{T}-\mathcal{D}^{\omega}\Phi|\leq|\mathcal{D}^{\omega}\mathfrak{T}-\Big((\mathcal{Z}_2+\mathcal{Z}_7)\frac{\mathcal{M}(\omega)}{1-\omega}e^{\frac{-\omega}{1-\omega}\varkappa}\Upsilon+\sigma_1(\varkappa)\mathcal{Z}_5\frac{\mathcal{M}(\omega)}{\omega}\Big)$$
$$\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}|\Phi(\nu)|e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu+\frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}|\Phi(\nu)|d\nu+\frac{\kappa_2}{\kappa_4}\int_0^{\nu}|\Phi(\varepsilon)|e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon$$
$$+\frac{\kappa_2}{\kappa_4}\int_0^{\nu}|\Phi(\varepsilon)|e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon+\frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(|\mathcal{Q}(\Upsilon)|+|\mathcal{Q}(0)|)+\frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(|\mathcal{Q}(\nu)|+|\mathcal{Q}(0)|)d\nu\Big)$$
$$-\mathcal{Z}_7\frac{\mathcal{M}(\omega)}{1-\omega}\int_0^{\varkappa}|\Phi(\nu)|e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu-\mathcal{Z}_7\frac{\omega\mathcal{M}(\omega)}{(1-\omega)^2}\int_0^{\varkappa}\int_0^{\nu}|\Phi(\varepsilon)|e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon d\nu$$
$$-\mathcal{Z}_5\frac{\mathcal{M}(\omega)}{\omega}(1-e^{\frac{-\omega}{1-\omega}\varkappa})|\mathcal{Q}(0)|-\frac{(\mathcal{Z}_2+\mathcal{Z}_5)\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}|\Phi(\nu)|e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu-\frac{\mathcal{Z}_2\,\omega}{1-\omega}$$
$$-\frac{(\mathcal{Z}_3+\mathcal{Z}_5)\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}(|\mathcal{Q}'(\nu)|+|\mathcal{Q}(\nu)|)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu|.$$

$$\|\mathcal{D}^{\omega}\mathfrak{T}-\mathcal{D}^{\omega}\Phi\|\leq 2\hat{\xi}\frac{\mathcal{M}(\omega)\rho\sigma_1(\varkappa)}{\mathcal{M}(\rho)\omega}+\beth\,\hat{\theta}_2\Big(\|\mathfrak{T}-\Phi\|+\|\mathcal{D}^{\rho}\mathfrak{T}-\mathcal{D}^{\rho}\Phi\|+\|\mathcal{D}^{\omega}\mathfrak{T}-\mathcal{D}^{\omega}\Phi\|\Big)$$
$$+\mathfrak{W}_3\|\mathfrak{T}-\Phi\|.$$

From Eq (6.4), it becomes

$$\|\mathcal{D}^{\omega}\mathfrak{T}-\mathcal{D}^{\omega}\Phi\|\leq 2\hat{\xi}\frac{\mathcal{M}(\omega)\rho\sigma_1(\varkappa)}{\mathcal{M}(\rho)\omega}+\beth\,\hat{\theta}_2\Big(\|\mathfrak{T}-\Phi\|+\|\mathcal{D}^{\rho}\mathfrak{T}-\mathcal{D}^{\rho}\Phi\|+\|\mathcal{D}^{\omega}\mathfrak{T}-\mathcal{D}^{\omega}\Phi\|\Big)$$
$$+2\hat{\xi}\,\mathfrak{W}_3\,\frac{(1+\rho(\varkappa-1))}{\mathcal{M}(\rho)(1-\mathfrak{W}1)}+\beth\frac{\mathfrak{W}_2\mathfrak{W}_3}{(1-\mathfrak{W}1)}\Big(\|\mathfrak{T}-\Phi\|+\|\mathcal{D}^{\rho}\mathfrak{T}-\mathcal{D}^{\rho}\Phi\|+\|\mathcal{D}^{\omega}\mathfrak{T}-\mathcal{D}^{\omega}\Phi\|\Big),$$

$$\|\mathcal{D}^{\omega}\mathfrak{T}-\mathcal{D}^{\omega}\Phi\|\leq 2\hat{\xi}\left(\frac{(1-\mathfrak{W}_1)\mathcal{M}(\omega)\rho\sigma_1(\varkappa)+(1+\rho(x-1))\mathfrak{W}_3\omega}{\mathcal{M}(\rho)\omega(1-\mathfrak{W}_1)}\right)$$
$$+\beth\left(\frac{\mathfrak{W}_2\mathfrak{W}_3}{(1-\mathfrak{W}_1)}+\hat{\theta}_2\right)\Big(\|\mathfrak{T}-\Phi\|+\|\mathcal{D}^{\rho}\mathfrak{T}-\mathcal{D}^{\rho}\Phi\|+\|\mathcal{D}^{\omega}\mathfrak{T}-\mathcal{D}^{\omega}\Phi\|\Big).\tag{6.6}$$

From Eqs (6.4)–(6.6), the following result is obtained:

$$\|\Phi_1-\Phi_2\|_{\mathfrak{x}}\leq\hat{\eta}_f\hat{\xi},\quad\text{where}\quad\hat{\eta}_f=2\Big(\frac{T_1+T_2+T_3}{1-\mathfrak{W}_7}\Big).$$

Hence, the boundary value problems (2.1) and (2.2) are Ulam–Hyers stable. □

Theorem 6.5. *If the condition ($\aleph_3$) holds with $\mathfrak{W}_1\mathfrak{W}_7-(\mathfrak{W}_1+\mathfrak{W}_7)<1$, then the boundary value problems (2.1) and (2.2) are Ulam–Hyers–Rassias stable.*

Proof. Consider an arbitrary $\hat{\xi}>0$ and a solution $\mathfrak{T}\in C(\mathfrak{J},\mathfrak{R})$ satisfying inequality (6.3). Then, for every $\varkappa \in \mathfrak{J}$, the following estimates hold:

$$\begin{aligned}
&\Bigg|\mathfrak{T}(\varkappa) - \Big(\mathcal{Z}_4\Big(1 - e^{\frac{-\omega}{1-\omega}\varkappa}\Big) + \mathcal{Z}_1 e^{\frac{-\omega}{1-\omega}\varkappa} + \mathcal{Z}_5\varkappa + \mathcal{Z}_3 + \mathcal{Z}_0\Big)\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\mathfrak{T}(\nu)e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu \\
&- \frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\mathfrak{T}(\nu)d\nu - \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon + \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon \\
&+ \frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(Q(\Upsilon) - Q(0)) + \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(Q(\nu) - Q(0))d\nu\Big) - \mathcal{Z}_7\int_0^{\varkappa}\mathfrak{T}(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu \\
&+ \mathcal{Z}_2\int_0^{\varkappa}\mathfrak{T}(\nu)d\nu + \mathcal{Z}_2\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon - \mathcal{Z}_2\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon \\
&- \mathcal{Z}_3(Q(\varkappa) - Q(0)) - \mathcal{Z}_5\int_0^{\varkappa}(Q(\nu) - Q(0))d\nu\Bigg| \le \hat{\xi}\, V_{1\hat{\mho}}\,\hat{\mho}(\varkappa).
\end{aligned} \tag{6.7}$$

Moreover, a corresponding estimate for the fractional derivative of $\mathfrak{T}$ with respect to the order ρ is given by

$$\begin{aligned}
&\Bigg|\mathcal{D}^{\rho}\mathfrak{T}(\varkappa) - \Big(\frac{\mathcal{M}(\rho)(1-\omega)}{(\rho-\omega)}(\mathcal{Z}_2 - \mathcal{Z}_7)(e^{\frac{-\omega}{1-\omega}\varkappa} - e^{\frac{-\rho}{1-\rho}\varkappa}) - \mathcal{Z}_0\sigma_2(\varkappa)\Big) \\
&\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\mathfrak{T}(\nu)e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu - \frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\mathfrak{T}(\nu)d\nu - \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon \\
&+ \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon + \frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(Q(\Upsilon) + Q(0)) + \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(Q(\nu) + Q(0))d\nu\Big) \\
&- \mathcal{Z}_7\frac{\mathcal{M}(\rho)}{1-\rho}\int_0^{\varkappa}\mathfrak{T}(\nu)e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu + \mathcal{Z}_7\frac{\omega\mathcal{M}(\rho)}{(1-\omega)(1-\rho)}\int_0^{\varkappa}\int_0^{\nu}e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}\mathfrak{T}(\varepsilon)d\varepsilon d\nu \\
&+ \mathcal{Z}_0\frac{(\kappa_2\mathcal{M}(\rho)+\rho)}{(1-\rho)}\int_0^{\varkappa}\mathfrak{T}(\nu)e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu - \mathcal{Z}_2\,\frac{\omega}{1-\omega} - \mathcal{Z}_3\frac{\mathcal{M}(\rho)}{(1-\rho)}\int_0^{\varkappa}Q'(\nu)e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu \\
&- \mathcal{Z}_0\frac{\rho}{(1-\rho)}\int_0^{\varkappa}Q(\nu)e^{\frac{-\rho}{1-\rho}(\varkappa-\nu)}d\nu + \mathcal{Z}_0(1 - e^{\frac{-\rho}{1-\rho}\varkappa})Q(0)\Bigg| \le \hat{\xi}\, V_{2\hat{\mho}}\,\hat{\mho}(\varkappa).
\end{aligned}$$

Finally, the bound related to the fractional derivative of $\mathfrak{T}$ of order ω is obtained in the form

$$\begin{aligned}
&\Bigg|\mathcal{D}^{\omega}\mathfrak{T}(\varkappa) - \Big((\mathcal{Z}_2 - \mathcal{Z}_7)\frac{\mathcal{M}(\omega)}{1-\omega}e^{\frac{-\omega}{1-\omega}\varkappa}\Upsilon + \sigma_1(\varkappa)\mathcal{Z}_5\frac{\mathcal{M}(\omega)}{\omega}\Big)\Big(\frac{\omega\kappa_1}{\kappa_4(1-\omega)}\int_0^{\Upsilon}\mathfrak{T}(\nu)e^{\frac{-\omega}{1-\omega}(\Upsilon-\nu)}d\nu \\
&+ \frac{\kappa_2\mathcal{M}(\rho)+\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}\mathfrak{T}(\nu)d\nu + \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\mathfrak{T}(\varepsilon)\, e^{\frac{-\omega}{1-\omega}(\Upsilon-\varepsilon)}d\varepsilon + \frac{\kappa_2}{\kappa_4}\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{\omega}{1-\omega}\varepsilon}d\varepsilon \\
&+ \frac{1-\rho}{\kappa_4\mathcal{M}(\rho)}(Q(\Upsilon) + Q(0)) + \frac{\rho}{\kappa_4\mathcal{M}(\rho)}\int_0^{\Upsilon}(Q(\nu) + Q(0))d\nu\Big) - \mathcal{Z}_7\frac{\mathcal{M}(\omega)}{1-\omega}\int_0^{\varkappa}\mathfrak{T}(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu \\
&+ \mathcal{Z}_7\frac{\omega\mathcal{M}(\omega)}{(1-\omega)^2}\int_0^{\varkappa}\int_0^{\nu}\mathfrak{T}(\varepsilon)e^{\frac{-\omega}{1-\omega}(\varkappa-\varepsilon)}d\varepsilon d\nu + (\mathcal{Z}_2 + \mathcal{Z}_5)\frac{\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}\mathfrak{T}(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu \\
&- \mathcal{Z}_2\,\frac{\omega}{1-\omega} - \mathcal{Z}_3\frac{\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}Q'(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu - \mathcal{Z}_5\frac{\mathcal{M}(\omega)}{(1-\omega)}\int_0^{\varkappa}Q(\nu)e^{\frac{-\omega}{1-\omega}(\varkappa-\nu)}d\nu \\
&- \mathcal{Z}_5\frac{\mathcal{M}(\omega)}{\omega}(1 - e^{\frac{-\omega}{1-\omega}\varkappa})Q(0)\Bigg| \le \hat{\xi}\, V_{3\hat{\mho}}\,\hat{\mho}(\varkappa).
\end{aligned}$$

Using the condition ($\aleph_3$), it is easily rewritten as

$$\|\mathfrak{T} - \Phi\| \leq \hat{\xi}\, \frac{V_{1\hat{\mho}}\, \hat{\mho}(\varkappa)}{1 - \mathfrak{W}_1} + \frac{\beth\, \mathfrak{W}_2}{1 - \mathfrak{W}_1}\Big(\|\mathfrak{T} - \Phi\| + \|\mathcal{D}^{\rho}\mathfrak{T} - \mathcal{D}^{\rho}\Phi\| + \|\mathcal{D}^{\omega}\mathfrak{T} - \mathcal{D}^{\omega}\Phi\|\Big). \tag{6.8}$$

By the same technique, it holds that

$$\begin{aligned} \|\mathcal{D}^{\rho}\mathfrak{T} - \mathcal{D}^{\rho}\Phi\| \leq \hat{\xi}\, & \frac{\Big(V_{2\hat{\mho}}\, \hat{\mho}(\varkappa)(1 - \mathfrak{W}_1) + \mathfrak{W}_5\, V_{1\hat{\mho}}\, \hat{\mho}(\varkappa)\Big)}{1 - \mathfrak{W}_1} \\ & + \beth\left(\frac{\mathfrak{W}_5\, \mathfrak{W}_2}{1 - \mathfrak{W}_1} + \theta_1\right)\Big(\|\mathfrak{T} - \Phi\| + \|\mathcal{D}^{\rho}\mathfrak{T} - \mathcal{D}^{\rho}\Phi\| + \|\mathcal{D}^{\omega}\mathfrak{T} - \mathcal{D}^{\omega}\Phi\|\Big), \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} \|\mathcal{D}^{\omega}\mathfrak{T} - \mathcal{D}^{\omega}\Phi\| \leq \hat{\xi}\, & \frac{\Big(V_{3\hat{\mho}}\, \hat{\mho}(\varkappa)(1 - \mathfrak{W}_1) + \mathfrak{W}_3\, V_{1\hat{\mho}}\, \hat{\mho}(\varkappa)\Big)}{1 - \mathfrak{W}_1} \\ & + \beth\left(\frac{\mathfrak{W}_2\, \mathfrak{W}_3}{1 - \mathfrak{W}_1} + \theta_2\right)\Big(\|\mathfrak{T} - \Phi\| + \|\mathcal{D}^{\rho}\mathfrak{T} - \mathcal{D}^{\rho}\Phi\| + \|\mathcal{D}^{\omega}\mathfrak{T} - \mathcal{D}^{\omega}\Phi\|\Big). \end{aligned} \tag{6.10}$$

Combining inequalities (6.8)–(6.10) yields $\|\mathfrak{T} - \Phi\|_{\mathfrak{x}} \leq \hat{\xi}\hat{\eta}_c\hat{\mho}(\varkappa)$, where

$$\hat{\eta}_c = \left(\frac{(1 + \mathfrak{W}_3 + \mathfrak{W}_5)V_{1\hat{\mho}} + (V_{2\hat{\mho}} + V_{3\hat{\mho}})(1 - \mathfrak{W}_1)}{(1 + \mathfrak{W}_1\mathfrak{W}_7 - \mathfrak{W}_1 - \mathfrak{W}_7)}\right).$$

Hence, the boundary value problems (2.1) and (2.2) are Ulam–Hyers–Rassias stable. □

7. Example

An example is given in this section to illustrate the usefulness of the main results.

Example 7.1. *Consider the following boundary value problem:*

$$\begin{aligned} & {}^{C\mathcal{F}}\mathcal{D}^{\rho}\Phi(\varkappa) + {}^{C\mathcal{F}}\mathcal{D}^{\omega}\Phi(\varkappa) + \Phi(\varkappa) = \frac{1}{b + \varkappa} + \frac{\Phi(\varkappa)}{b + \Phi(\varkappa)} + \frac{{}^{C\mathcal{F}}\mathcal{D}^{\rho}\Phi}{b\, e^{({}^{C\mathcal{F}}\mathcal{D}^{\rho}\Phi)} + \varkappa} + \frac{|{}^{C\mathcal{F}}\mathcal{D}^{\omega}\Phi|e^{-\varkappa}}{b + |{}^{C\mathcal{F}}\mathcal{D}^{\omega}\Phi|^2}, \\ & \Phi(0) = \Phi(1), \quad \varkappa \in [0, 1]. \end{aligned} \tag{7.1}$$

Given $a = 0$, $\Upsilon = 1$, *and* $\rho = 0.2$, $\omega = 0.6$, $b = 10$, *applying the Lipschitz condition leads to* $\beth = 0.1$, *and*

$$\hat{\beta} = \left(\frac{(\mathfrak{W}_1 + \mathfrak{W}_3 + \mathfrak{W}_5)\mathfrak{W}_2}{1 - \mathfrak{W}_1} + \mathfrak{Y}\right) \iff 0.1246852 < 1.$$

Then, the boundary value problem (7.1) *has a unique solution.*

To estimate at least one solution, from the condition

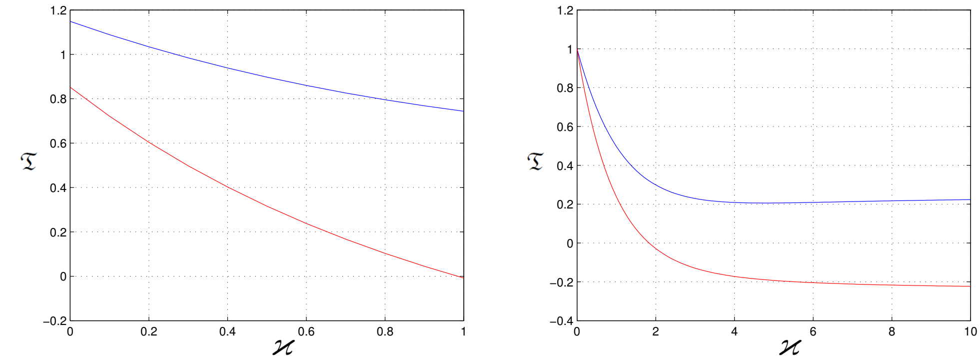
$$F(\mathbf{Q}(W)) \leq \frac{1}{b + \varkappa} F(W) \to \mathfrak{R} = \frac{1}{b + \varkappa},$$

$\hat{\theta}_3 = 4(\mathfrak{W}_1 + \mathfrak{W}_3 + \mathfrak{W}_5) + 4(\hat{\theta}_1 + \hat{\theta}_2 + \mathfrak{W}_2)\mathfrak{R} = 0.6480568 < 1$. *The requirement of Theorem 5.1 is satisfied. Hence, the boundary value problem* (7.1) *has at least one solution on* $[0, 1]$.

The Ulam–Hyers stability result for Eq (7.1) *is now considered. Set* $\Phi(\varkappa) = e^{-\varkappa}$. *Then according to Definition 3.1, one obtains*

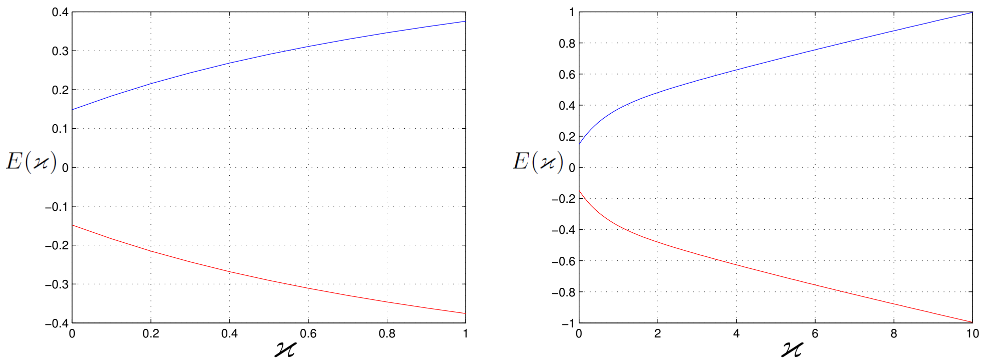
$$ {}^{C\mathcal{F}}_{a}D^{\rho}_{\varkappa}(e^{-\varkappa}) = \frac{\mathcal{M}(\rho)}{2\rho - 1}\left(e^{\frac{-\rho}{1-\rho}\varkappa} - e^{-\varkappa}\right), $$

$\Phi(\varkappa)$ *satisfies Eq* (6.1) *with* $\hat{\xi} = 0.2097746$. *Then, the Eq* (7.1) *is Ulam–Hyers stable with* $\|\mathfrak{T} - \Phi\|_{\mathfrak{X}} \leq 0.2097746\ \hat{\eta_f}$. *Next, to estimate the Ulam–Hyers–Rassias stability for Eq* (7.1) *, let* $\Phi = e^{-\varkappa}$ *and* $\hat{\mho}(\varkappa) = e^{-\varkappa}$, *by Theorem 6.5, the results is* $\|\mathfrak{T} - \Phi\|_{\mathfrak{X}} \leq \hat{\xi}\hat{\eta_c}\hat{\mho}(\varkappa) \iff (0.2097746) * (0.7065934) = 0.14822534 > 0$. *To investigate the behavior of the functions* $\mathfrak{T}$ *and* $\|E(\varkappa)\| = \|\mathfrak{T} - \Phi\|_{\mathfrak{X}}$, *we used Matlab, and the results are depicted in Figures 1 and 2.*



(a) **(b)**

Figure 1. Behavior of $\mathfrak{T}$ when (a) $\varkappa \in [0, 1]$, and (b) $\varkappa \in [0, 10]$.



(a) **(b)**

Figure 2. Behavior of $E(\varkappa)$ when (a) $\varkappa \in [0, 1]$, and (b) $\varkappa \in [0, 10]$.

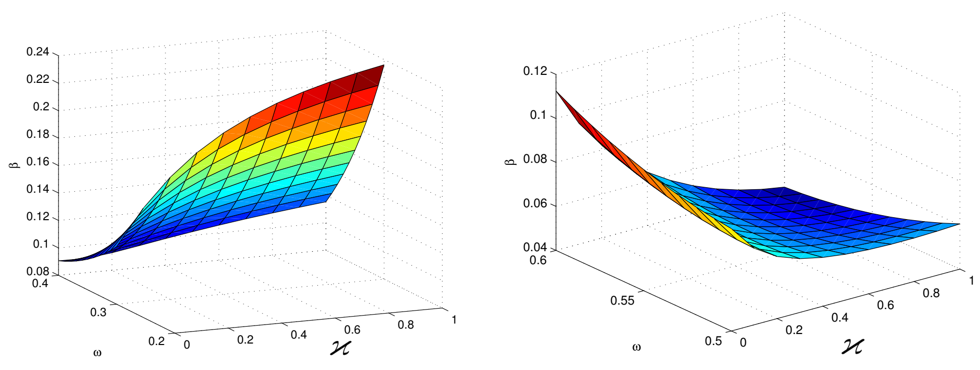
8. Discussion

To provide further insight into the qualitative behavior of the solution, graphical data based on the illustrative example in the manuscript were generated by varying the parameters ρ and ω. This approach facilitates the investigation of the solution's existence and potential stability under different scenarios, demonstrating its behavior within the framework of Banach's fixed-point theorem and verifying the fulfillment of Ulam-type stability conditions. The resulting graphical evidence thereby supports and validates the theoretical findings presented throughout the paper.

Our main objective is to assess the theoretical result of the Banach contraction principle and to evaluate the effectiveness of the $\mathcal{UH}$ and $\mathcal{UHR}$ stability concepts for the problem (7.1) when $\varkappa \in [0, 1]$. First, Table 1 and Figures 3–5 demonstrate the values and graphical representation of $\hat{\beta}$ at various fractional orders of $(\rho, \omega) = \{(0.3, 0.85), (0.5, 0.9), (0.65, 0.75), (0.5, 0.35), (0.8, 0.55), (0.9, 0.6)\}$. Second, to demonstrate the sufficiency of the $\mathcal{UH}$ stability for the solution, the stability is analyzed by examining the behavior of $\hat{\eta}_f \hat{\xi}$, as illustrated in Figures 6 and 7 for selected fractional orders. Specifically, the analysis considers $\rho = 0.3$ with $\omega \in [0.1, 0.4]$ and $\omega \in [0.6, 0.9]$, as well as $\rho = [0.1, 0.4]$ with $\omega = 0.3$ and $\omega = 0.9$. For further details, refer to Table 2. Moreover, to evaluate the efficiency of the $\mathcal{UHR}$ stability, the value of $\hat{\xi}\hat{\eta}_c\hat{\mho}(\varkappa)$ is calculated for $\hat{\xi} = 0.2097746$ and $\hat{\mho}(\varkappa) = e^{-\varkappa}$ at several orders corresponding to $(\rho, \omega) = \{(0.45, 0.8), (0.55, 0.9), (0.6, 0.3), (0.8, 0.45), (0.9, 0.55)\}$, as shown in Table 3. Additionally, the behavior of $\hat{\xi}\hat{\eta}_c\hat{\mho}(\varkappa)$ at selected orders is illustrated in Figures 8 and 9.

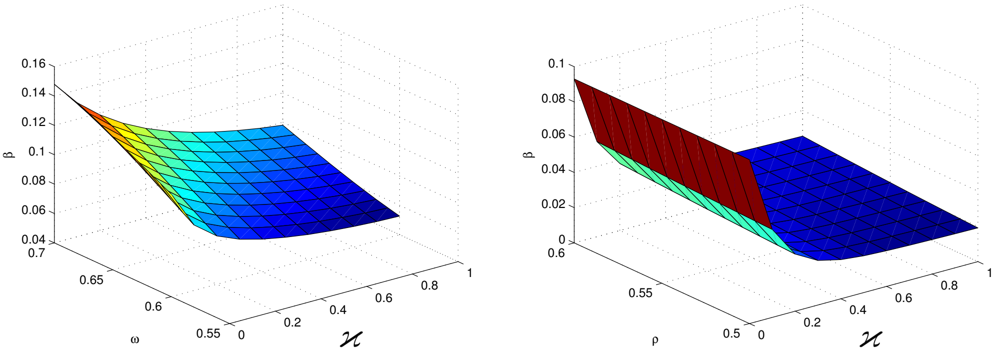
Table 1. Values of $\hat{\beta}$ when $0 < \rho < 1$, and $0 < \omega < 1$, for Example 7.1.

$\varkappa$	$\rho = 0.3$, $\omega = 0.85$	$\rho = 0.5$, $\omega = 0.9$	$\rho = 0.65$, $\omega = 0.75$	$\rho = 0.8$, $\omega = 0.35$	$\rho = 0.8$, $\omega = 0.55$	$\rho = 0.9$, $\omega = 0.6$
0	0.115223	0.093032	0.146157	0.102221	0.095086	0.109877
0.1	0.067309	0.049904	0.113061	0.112551	0.080196	0.084621
0.2	0.040362	0.032404	0.087813	0.122229	0.070466	0.072901
0.3	0.025034	0.025227	0.068426	0.130695	0.063691	0.066461
0.4	0.016205	0.022209	0.053459	0.137886	0.058733	0.062302
0.5	0.011033	0.020871	0.041856	0.143943	0.054991	0.059307
0.6	0.007928	0.020211	0.032831	0.149059	0.052142	0.057041
0.7	0.005998	0.019827	0.025794	0.153425	0.050007	0.055318
0.8	0.004740	0.019558	0.020301	0.157203	0.048477	0.054054
0.9	0.003871	0.019337	0.016014	0.160525	0.047479	0.053198
1	0.003228	0.019142	0.012679	0.163489	0.046961	0.052717



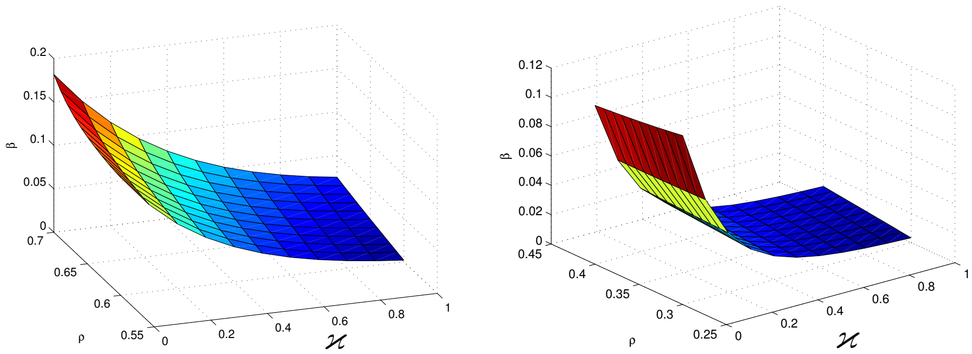
(a) **(b)**

Figure 3. The values of $\hat{\beta}$ when (a) $\rho = 0.8$, $\omega \in [0.3, 0.4]$, and (b) $\rho = 0.8$, $\omega \in [0.5, 0.6]$.



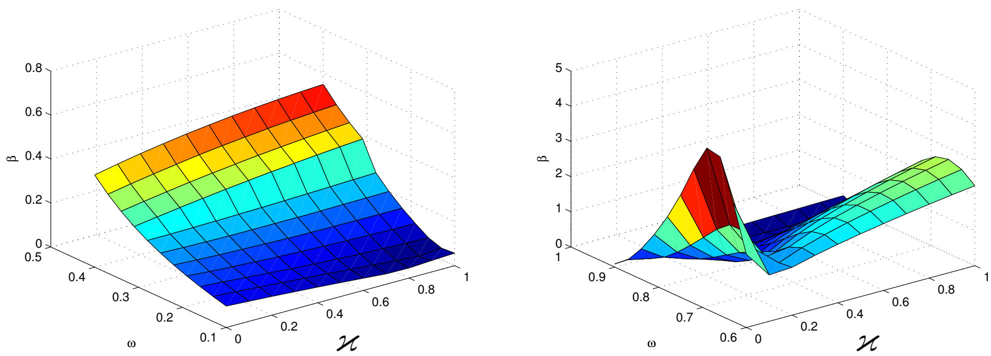
(a) **(b)**

Figure 4. The values of $\hat{\beta}$ when (a) $\rho = 0.9,\ \omega \in [0.6, 0.7]$, and (b) $\rho \in [0.5, 0.6],\ \omega = 0.9$.



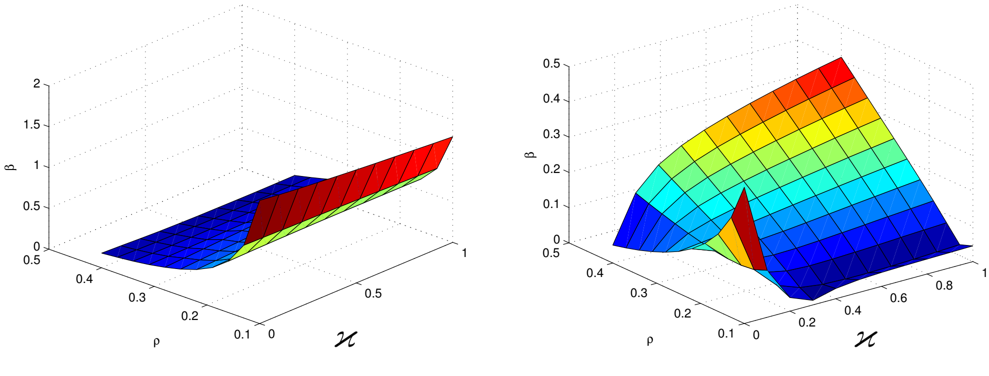
(a) **(b)**

Figure 5. The values of $\hat{\beta}$ when (a) $\rho \in [0.6, 0.7], \omega = 0.75$, and (b) $\rho \in [0.3, 0.4], \omega = 0.85$.



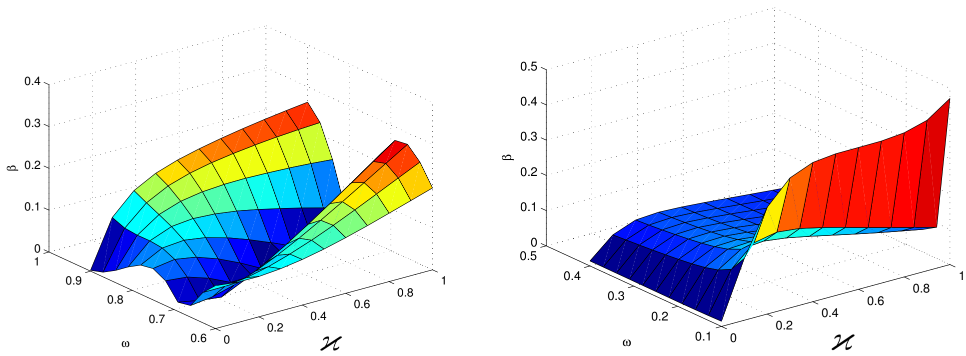
(a) **(b)**

Figure 6. The values of $\hat{\eta}_f \hat{\xi}$ when (a) $\rho = 0.3, \omega \in [0.1, 0.4]$, and (b) $\rho = 0.3, \omega \in [0.6, 0.9]$.



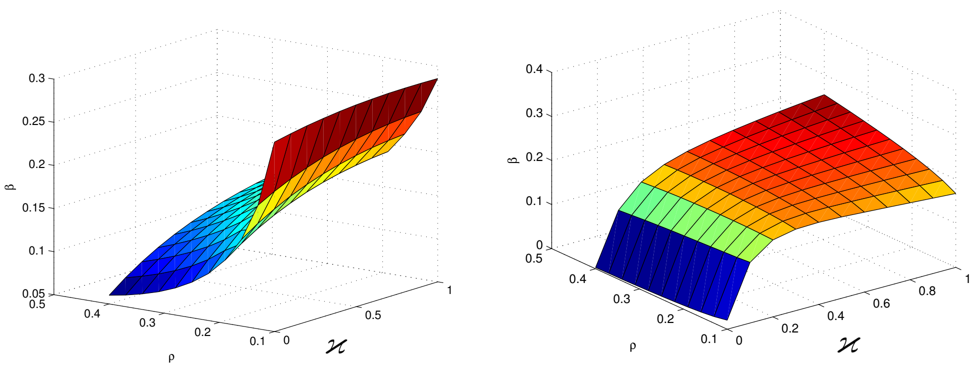
(a) **(b)**

Figure 7. The values of $\hat{\eta}_f\hat{\xi}$ when (a) $\rho \in [0.1, 0.4]$, $\omega = 0.3$, and (b) $\rho \in [0.1, 0.4]$, $\omega = 0.9$.



(a) **(b)**

Figure 8. The values of $\hat{\xi}\hat{\eta}_c\hat{\mho}(\varkappa)$ when (a) $\rho = 0.45$, $\omega \in [0.6, 0.9]$, and (b) $\rho = 0.9$, $\omega \in [0.1, 0.4]$.



(a) **(b)**

Figure 9. The values of $\hat{\xi}\hat{\eta}_c\hat{\mho}(\varkappa)$ when (a) $\rho \in [0.1, 0.4]$, $\omega = 0.2$, and (b) $\rho \in [0.1, 0.4]$, $\omega = 0.9$.

Table 2. Values of $\hat{\eta}_f\hat{\xi}$ of $\mathcal{UH}$ stability when $0 < \rho < 1$ and $0 < \omega < 1$ for Example 7.1.

$\varkappa$	$\rho = 0.3$ $\omega = 0.5$	$\rho = 0.6$ $\omega = 0.8$	$\rho = 0.8$ $\omega = 0.9$	$\rho = 0.5$ $\omega = 0.3$	$\rho = 0.7$ $\omega = 0.5$	$\rho = 0.9$ $\omega = 0.6$
0	0.721926	0.207777	0.002551	0.146942	0.180329	0.037618
0.1	0.759346	0.287687	0.408271	0.145205	0.201126	0.268935
0.2	0.793342	0.329523	0.606063	0.139622	0.203481	0.381079
0.3	0.828302	0.348975	0.710957	0.130882	0.194436	0.426924
0.4	0.862619	0.354757	0.770417	0.119599	0.178844	0.445203
0.5	0.896056	0.351888	0.805878	0.106321	0.159997	0.450355
0.6	0.928322	0.343334	0.827798	0.091531	0.140081	0.448079
0.7	0.959096	0.330881	0.841656	0.075661	0.126251	0.440846
0.8	0.988041	0.315631	0.850521	0.059085	0.131958	0.429909
0.9	1.014811	0.298284	0.856201	0.058509	0.134925	0.416045
1	1.039073	0.2793045	0.859815	0.062368	0.135826	0.399823

Table 3. Values of $\hat{\xi}\hat{\eta}_c\hat{\mho}(\varkappa)$ of Ulam–Hyers–Rassias stability for Example 7.1.

$\varkappa$	$\rho = 0.45$ $\omega = 0.8$	$\rho = 0.55$ $\omega = 0.9$	$\rho = 0.6$ $\omega = 0.3$	$\rho = 0.8$ $\omega = 0.45$	$\rho = 0.9$ $\omega = 0.55$
0	0.058198	0.001113	0.014388	0.032992	0.012241
0.1	0.099223	0.103129	0.008245	0.009509	0.066011
0.2	0.113944	0.153368	0.005425	0.032744	0.094014
0.3	0.114752	0.183472	0.003922	0.044839	0.103992
0.4	0.107861	0.204768	0.003061	0.050391	0.106871
0.5	0.096573	0.221655	0.002557	0.052065	0.106583
0.6	0.082701	0.235941	0.002267	0.051432	0.104594
0.7	0.067246	0.248422	0.002113	0.049421	0.101496
0.8	0.050765	0.259491	0.002049	0.046583	0.09756
0.9	0.033565	0.269369	0.002047	0.043245	0.092971
1	0.015822	0.278209	0.002089	0.039601	0.087837

9. Conclusions

This paper has addressed the solution of a class of Caputo–Fabrizio $\mathcal{FDE}$s with boundary conditions, particularly those associated with Basset-type models that arise in physical systems exhibiting memory effects, such as fluid dynamics and viscoelastic materials.

The existence and uniqueness of solutions were established through the application of Darbo's fixed-point theorem and the Banach contraction mapping principle. The Ulam–Hyers and Ulam–Hyers–Rassias stabilities of the boundary value problems (2.1) and (2.2) were investigated, confirming the robustness of the solution under small perturbations. An illustrative example, supported by tables and graphical representations, was presented to validate the theoretical findings and to demonstrate the effectiveness of the proposed model in addressing Basset-type problems.

The originality of this study lies in the dual-framework methodology that combines the Caputo–Fabrizio derivative in the theoretical formulation with the modified Caputo–Fabrizio operator in the solution process. To the best of our knowledge, this combination has not been previously explored, providing new theoretical insights and practical significance.

Future research may focus on extending the current model to incorporate more general boundary conditions, variable-order fractional operators, or systems influenced by external forces and distributed delays. Such extensions would further enhance the applicability of the model to complex phenomena in engineering and applied sciences.

Authors contributions

Shayma Adil Murad: Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Resources, Writing-original draft, Writing-review & editing, Supervision; Ava Sh. Rafeeq: Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Writing-original draft,Resources, Supervision; Alan M. Omar: Formal analysis, Investigation, Methodology, Software, Validation, Writing-original draft, Resources; Mohammed O. Mohammed: Formal analysis, Investigation, Methodology, Software, Validation, Writing-original draft, Resources; Thabet Abdeljawad: Administration, Validation, Visualization, Review and editing; Manar A. Alqudah: Validation, Resources, Formal analysis. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Thabet Abdeljawad would like to thank Prince Sultan University for the support through TAS research lab. Manar A. Alqudah would like to thank Princess Nourah bint Abdulrahman University researchers supporting project number (PNURSP2025R14), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of interest

The authors declare no competing interests.

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