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Research article

Generalized α-ψ-Geraghty contraction type fixed point theorems in strong b-metric spaces

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Abstract: This paper aims to present the new concept of generalized α-ψ-Geraghty contraction type for both single-valued and multi-valued mappings in a strong b-metric space. Our approach extends the concepts introduced by Karapınar for single-valued mappings and by Afshari et al. for multi-valued mappings. Several results on the fixed point theory of this contraction type are provided both globally (in the entire space) and locally (in the closed ball). Our findings expand and enhance several fixed point theorems, including the famous Geraghty's theorem. Examples and an application of integral equations are included to illustrate these results.

Keywords: strong b-metric space (SbMS); α-admissible; α-ψ-Geraghty contraction type
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1. Introduction

In nonlinear analysis, the fixed point theory is an important tool of mathematics that helps to solve different problems. This theory has been extensively applied across various scientific fields, including computer science, economics, engineering, chemistry, biology, and medical sciences. Recent research demonstrates the effectiveness of fixed point techniques in establishing the existence and uniqueness of solutions for a variety of complex mathematical models. These methods have been successfully applied to generalized coupled systems of finite-delay fractional differential equations, nonexpansive mappings in Banach spaces, and chaotic dynamical systems such as Chua's attractor, employing contraction principles, Schauder's fixed point theorem, and other iterative schemes. By providing a rigorous analytical framework, fixed point theory not only ensures theoretical solvability but also supports practical applications in areas including control systems, predator–prey dynamics, nonlinear integral equations, and the analysis of complex chaotic behaviors (see e.g., [8, 18, 27]).

Among such applications, Feinstein and Rudloff [15] investigated the relationship between Nash equilibria and Pareto optimality in multi-criteria noncooperative games. By formulating a vector optimization problem, they characterized Nash equilibria as Pareto-efficient solutions, where fixed point ideas play a central role in linking best-response functions to equilibrium strategies. Their work provides new mathematical insights for game theory and economics.

In the medical field, El Mamounia et al. [13] proposed a fractional-order model to describe the spread of COVID-19. By applying Krasnoselskii's fixed point theorem and Banach's contraction principle, they established the existence and uniqueness of solutions to their model, proving the stability of both disease-free and endemic equilibria. Their results illustrate how fixed point theory can provide rigorous mathematical foundations for understanding epidemic dynamics.

In computer science, Enescu and Sahbi [14] developed a multi-modal normalizing flow approach for image generation and classification. A key feature of their method is that the model parameters are obtained as an interpretable fixed point solution of their optimization criterion. This approach improves performance in learning multimodal distributions and avoids the difficulties of gradient descent tuning, showing how fixed point methods can enhance modern machine learning algorithms.

From a more abstract perspective, Younis and Öztürk [28] studied best proximity points for proximal contractions in extended b-metric spaces. Their results extend and generalize existing findings in optimal proximity theory, offering new insights into coincidence points in multivalued mappings. This work highlights how generalizations of fixed point results continue to enrich nonlinear analysis and provide useful benchmarks for applications.

Overall, these recent contributions—from economics and medicine to computer science and pure mathematics—demonstrate the wide applicability and significance of fixed point theory in addressing real-world problems.

The Banach contraction principle is one of the fundamental results in fixed point theory. It states that a contraction self-map f defined on a complete metric space has a unique fixed point. Since its introduction, many researchers have proposed extensions and generalizations of this principle in various directions. One notable result is due to Geraghty [17], who in 1973 introduced a new contractive condition extending Banach's theorem. Research on Geraghty's theorem remains active, with numerous authors working to broaden the contractive condition—such as by incorporating admissible functions α and replacing the auxiliary function β with more general alternatives [5, 6, 19, 23, 25]—or by modifying the underlying metric structure, including generalized metrics and b-metrics [1–3, 6, 9, 24].

Recent work also considers generalized contraction mappings—such as Banach, Kannan, Chatterjea, and Geraghty types—to address more complex problems, including those in strong b-metric spaces where standard conditions fail. For unifying the terminology of mappings in metric space settings, see [10].

The notion of a strong b-metric space was introduced by Kirk and Shahzad in 2014. They modified the relaxed triangle inequality of b-metric spaces, strengthening it in a way that grants the space several topological properties that standard b-metric spaces lack.

Definition 1.1. *[22] Let $\Xi \neq \emptyset$ be a set and $s \geq 1$. Let $ł : \Xi \times \Xi \to [0, +\infty)$ be a distance function satisfying for each ϖ_1, ϖ_2, and $\varpi_3 \in \Xi$:*
SbM1. $ł(\varpi_1, \varpi_2) \geq 0$,

SbM2. $\ell(\varpi_1, \varpi_2) = 0 \Leftrightarrow \varpi_1 = \varpi_2$,
SbM3. $\ell(\varpi_1, \varpi_2) = \ell(\varpi_2, \varpi_1)$,
SbM4. $\ell(\varpi_1, \varpi_3) \leq \ell(\varpi_1, \varpi_2) + s\,\ell(\varpi_2, \varpi_3)$.
Then the pair (Ξ, ℓ) *is called a strong b-metric space* $(SbMS)$.

By replacing condition $(SbM4)$ in Definition 1.1 with the inequality $\ell(\varpi_1, \varpi_3) \leq s[\ell(\varpi_1, \varpi_2) + \ell(\varpi_2, \varpi_3)]$, the pair (Ξ, ℓ) becomes a b-metric space.

Example 1.1. *[7] Let* $\Xi = \{1, 2, 3\}$, $\ell\colon \Xi \times \Xi \to [0, +\infty)$ *be defined by* $\ell(1, 1) = \ell(2, 2) = \ell(3, 3) = 0$, $\ell(1, 2) = \ell(2, 1) = 2$, $\ell(2, 3) = \ell(3, 2) = 1$, $\ell(1, 3) = \ell(3, 1) = 6$. *Then* (Ξ,ℓ) *is a strong b-metric space with* $s = 4$.

Example 1.2. *[4] Let* $\Xi = \mathbb{R}$, *and* $\ell(\varpi_1, \varpi_2) = \max\{|\varpi_1 - \varpi_2|, 2|\varpi_1 - \varpi_2| - 1\}$ *for all* $\varpi_1, \varpi_2 \in \mathbb{R}$, *then* (Ξ,ℓ) *is a strong b-metric with* $s = 2$ *but is not a metric space.*

Example 1.3. *[4, 12] Let* $\Xi = [1, +\infty), \ell(\varpi_1, \varpi_2) = (\varpi_1 - \varpi_2)^4$ *for all* $\varpi_1, \varpi_2 \in \Xi$. *Then,* ℓ *is a continuous b-metric on* Ξ *with* $s = 8$ *but is not a strong b-metric.*

The class of strong b-metric spaces is intermediate between the class of b-metric spaces and the class of metric spaces. This demonstrates the significance of strong b-metric spaces over b-metric spaces, as many well-known fixed point results that are valid in strong b-metric space do not fully hold in b-metric space; see, for example, [4].

In [17], Geraghty extended Banach's fixed point theorem in complete metric spaces and established a fixed point result for single-valued mappings that motivated many scholars to extend and generalize Geraghty's result to obtain new interesting fixed point results.

Later, Karapınar extended Geraghty's contraction to a more general form known as generalized α-ψ-Geraghty contractions, investigating the existence and uniqueness of fixed points for mappings satisfying this condition. [20, Theorems 3.3 and 3.4] were partially extended to the context of b-metric spaces as [3, Theorems 2.3 and 2.4] for single-valued mappings. These results were further generalized to multi-valued mappings as [2, Theorems 2.1 and 2.2].

Based on this, throughout this paper, we fully extend Karapınar's results in metric spaces [20] to the context of strong b-metric spaces. Furthermore, we enhance the results of [2, 20] by refining the generalized α-ψ contraction condition. Additionally, we provide examples to illustrate the validity and applicability of our findings. Throughout this manuscript, we denote by $\mathcal{CB}(\Xi)$ the family of all nonempty closed and bounded subsets of Ξ, and by $\mathcal{H}$ the Pompeiu–Hausdorff metric, that is,

$$\mathcal{H}(A, B) = \max\left\{\sup_{a\in A} D(a, B),\ \sup_{b\in B} D(b, A)\right\},$$

where, for $x_0 \in \Xi$, we define $D(x_0, A) = \inf_{a\in A} \ell(x_0, a)$.

First, let us review some fundamental concepts.

Definition 1.2. *Let* $f : \Xi \to \Xi$ *be a single-valued mapping and* $\alpha : \Xi \times \Xi \to [0, +\infty)$ *be a function. Then for* ϖ_1, ϖ_2, *and* $\varpi_3 \in \Xi$, f *is claimed to be:*

i. [26] α-admissible if $\alpha(\varpi_1, \varpi_2) \geq 1$ *implies* $\alpha(f\varpi_1, f\varpi_2) \geq 1$.
ii. [21] Triangular α-admissible if

1. *f is α-admissible;*
2. *$\alpha(\varpi_1, \varpi_3) \geq 1$ and $\alpha(\varpi_3, \varpi_2) \geq 1$, then $\alpha(\varpi_1, \varpi_2) \geq 1$.*

iii. [25] α-orbital admissible if $\alpha(\varpi_1, f\varpi_1) \geq 1$ implies $\alpha(f\varpi_1, f^2\varpi_1) \geq 1$.

iv. [25] Triangular α-orbital admissible if

1. *f is α-orbital admissible;*
2. *$\alpha(\varpi_1, \varpi_2) \geq 1$ and $\alpha(\varpi_2, f\varpi_2) \geq 1$, then $\alpha(\varpi_1, f\varpi_2) \geq 1$.*

Remark 1.1. *[25] Every α-admissible mapping is an α-orbital admissible mapping and every triangular α-admissible mapping is a triangular α-orbital admissible mapping.*

In [23], Mohammadi and others expanded the notion of α-admissible for multi-valued mappings as follows.

Definition 1.3. *The multi-valued mapping T is α-admissible whenever for each $\varpi_1 \in \Xi$ and $\varpi_2 \in T\varpi_1$ with $\alpha(\varpi_1, \varpi_2) \geq 1$, we have $\alpha(\varpi_2, \varpi_3) \geq 1$ for all $\varpi_3 \in T\varpi_2$.*

In the following, we present several known results: some will be used as auxiliary lemmas in the proofs of our main results, while others will serve as benchmarks for comparison with our findings.

Proposition 1.1. *[22] Let $(\varpi_n)_{n\in\mathbb{N}}$ be a sequence in an SbMS, and suppose $\sum_{n=1}^{+\infty} ł(\varpi_n, \varpi_{n+1}) < +\infty$. Then $(\varpi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Lemma 1.1. *[11] Let $(\Xi, ł)$ be a b-metric space. For any $\Gamma, \Omega \in \mathcal{CB}(\Xi)$ and any $\varpi, \omega \in \Xi$, we have the following:*

- *$D(\varpi, \Omega) \leq ł(\varpi, \omega)$ for any $\omega \in \Omega$;*
- *$D(\varpi, \Omega) \leq \mathcal{H}(\Gamma, \Omega)$ where $\varpi \in \Gamma$;*
- *$D(\varpi, \Gamma) \leq s[ł(\varpi, \omega) + D(\omega, \Gamma)]$.*

Lemma 1.2. *[11] Let Γ and Ω be nonempty closed and bounded subsets of a b-metric space $(\Xi, ł)$ and $\sigma > 1$. Then, for all $\gamma \in \Gamma$, there exists $\omega \in \Omega$ such that $ł(\gamma, \omega) \leq \sigma\mathcal{H}(\Gamma, \Omega)$.*

Lemma 1.3. *[25] Let $T : \Xi \to \Xi$ be a triangular α-orbital admissible mapping. Assume that there exists $\varpi_1 \in \Xi$ such that $\alpha(\varpi_1, T\varpi_1) \geq 1$. Define a sequence $(\varpi_n)_{n\in\mathbb{N}}$ by $\varpi_{n+1} = T\varpi_n$. Then we have $\alpha(\varpi_n, \varpi_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Theorem 1.1. *[20, Theorem 3.3] Let (Ξ, d) be a complete metric space and $T : \Xi \to \Xi$ be a generalized α-ϕ-Geraghty contraction type such that*

i. T is triangular α-admissible;
ii. there exists $\varpi_0 \in \Xi$ such that $\alpha(\varpi_0, T\varpi_0) \geq 1$;
iii. T is continuous, or Ξ satisfies the (C_α) condition; that is, for all $n \in \mathbb{N}$, $(\varpi_n)_{n\in\mathbb{N}} \subseteq \Xi$ converges to ϖ^* in Ξ and $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$, then $\alpha(\varpi_n, \varpi^*) \geq 1, \forall n$.

Then, T has a fixed point in Ξ.

Theorem 1.2. *[3, Theorems 2.3 and 2.4] Let (Ξ, d) be a complete b-metric space with $s \geq 1$, and $T : \Xi \to \Xi$ be a generalized α-ψ-Geraghty contraction type such that*

i. T is triangular α-admissible;
ii. there exists $\varpi_0 \in \Xi$ such that $\alpha(\varpi_0, T\varpi_0) \geq 1$;
iii. T is continuous or Ξ satisfies (C_α) condition.

Then, T has a fixed point in Ξ.

2. Main results

In this section, we define the concept of a generalized α-ψ-Geraghty contraction type in an SbMS for both single-valued mappings and multi-valued mappings. Then, we show and investigate the existence and uniqueness of a fixed point in these functions.

2.1. For single-valued mappings

The generalization of α-ψ-Geraghty contraction type for single-valued mappings in the context of SbMS is defined in this section. Following this, some results show the existence and uniqueness of a fixed point for this mapping.
Denote with $\mathcal{F}$ the class of all functions $\zeta : [0, +\infty) \to [0, 1)$ that satisfy the condition: If $\lim_{n\to+\infty} \zeta(t_n) = 1$, then $\lim_{n\to+\infty} t_n = 0$.

Definition 2.1. *Let (Ξ, d) be an SbMS with $s \geq 1$; then a single-valued mapping $T : \Xi \to \Xi$ is a generalized α-ψ- Geraghty contraction type whenever there exist $\alpha : \Xi\times\Xi \to [0, +\infty)$ and some $L \geq 0$ such that for*

$$\begin{aligned} M(\varpi_1, \varpi_2) &= \max\{d(\varpi_1, \varpi_2), d(\varpi_1, T\varpi_1), d(\varpi_2, T\varpi_2), \\ &\quad \frac{d(\varpi_1, T\varpi_2) + d(\varpi_2, T\varpi_1)}{2s}\}, \\ N(\varpi_1, \varpi_2) &= \min\{d(\varpi_1, T\varpi_1), d(\varpi_2, T\varpi_1)\}, \end{aligned}$$

we have

$$\begin{aligned} \alpha(\varpi_1, \varpi_2)\psi(d(T\varpi_1, T\varpi_2)) \leq\ & \zeta(\psi(M(\varpi_1, \varpi_2)))\psi(M(\varpi_1, \varpi_2)) + \\ & L\phi(N(\varpi_1, \varpi_2)), \end{aligned} \tag{2.1}$$

for all $\varpi_1, \varpi_2 \in \Xi, \zeta \in \mathcal{F}$, and $\psi, \phi \in \Psi$.
Where Ψ is the class of all non-decreasing and continuous functions $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi^{-1}(0) = 0$.

Remark 2.1. *The generalized α-ψ- Geraghty condition given in Definition 2.1 is considered an extension of the conditions presented in the context of metric spaces [20, Definition 3.1] and b-metric spaces [3, Definition 2.1].*

Theorem 2.1. *Let (Ξ, d) be a complete SbMS with $s \geq 1$, and $T : \Xi \to \Xi$ be a generalized α-ψ-Geraghty contraction type such that*

i. T is triangular α-orbital admissible;
ii. there exists $\varpi_0 \in \Xi$ such that $\alpha(\varpi_0, T\varpi_0) \geq 1$;
iii. T is continuous.

Then, T has a fixed point in Ξ.

Proof. Let $\varpi_0 \in \Xi$ be such that $\alpha(\varpi_0, T\varpi_0) \geq 1$. Define an iterative sequence $(\varpi_n)_{n\in\mathbb{N}} \subseteq \Xi$ by $\varpi_n = T\varpi_{n-1}$ for all $n \in \mathbb{N}$. If $\varpi_{n-1} = \varpi_n$ for some $n \in \mathbb{N}$, then the result is proved since ϖ_n is a fixed point of T. So, we assume that $\varpi_n \neq \varpi_{n+1}$ for all $n \in \mathbb{N}$. Since T is an α-admissible

and $\alpha(\varpi_0, T\varpi_0) \geq 1$, we have $\alpha(T\varpi_0, T^2\varpi_0) = \alpha(\varpi_1, \varpi_2) \geq 1$. Continuing this process, we deduce that for all $n \in \mathbb{N}, \alpha(\varpi_n, \varpi_{n+1}) \geq 1$.
Since T is a generalized α-ψ- Geraghty contraction type and ψ is non-decreasing, we have for all $n \in \mathbb{N}$

$$\begin{aligned}\psi(d(\varpi_{n+1}, \varpi_{n+2})) &= \psi(d(T\varpi_n, T\varpi_{n+1}))\\ &\leq \alpha(\varpi_n, \varpi_{n+1})\psi(d(T\varpi_n, T\varpi_{n+1}))\\ &\leq \zeta(\psi(M(\varpi_n, \varpi_{n+1})))\psi(M(\varpi_n, \varpi_{n+1})) + L\phi(N(\varpi_n, \varpi_{n+1}))\\ &< \psi(M(\varpi_n, \varpi_{n+1})) + L\phi(N(\varpi_n, \varpi_{n+1})), \qquad (2.2)\end{aligned}$$

where

$$\begin{aligned}M(\varpi_n, \varpi_{n+1}) &= \max\{d(\varpi_n, \varpi_{n+1}), d(\varpi_n, T\varpi_n), d(\varpi_{n+1}, T\varpi_{n+1}),\\ &\qquad \frac{d(\varpi_n, T\varpi_{n+1}) + d(\varpi_{n+1}, T\varpi_n)}{2s}\}\\ &= \max\{d(\varpi_n, \varpi_{n+1}), d(\varpi_{n+1}, \varpi_{n+2}), \frac{d(\varpi_n, \varpi_{n+2})}{2s}\},\end{aligned}$$

and

$$\begin{aligned}N(\varpi_n, \varpi_{n+1}) &= \min\{d(\varpi_n, T\varpi_n), d(\varpi_{n+1}, T\varpi_n)\}\\ &= \min\{d(\varpi_n, \varpi_{n+1}), d(\varpi_{n+1}, \varpi_{n+1})\} = 0.\end{aligned}$$

Hence

$$\psi(d(\varpi_{n+1}, \varpi_{n+2})) < \psi(M(\varpi_n, \varpi_{n+1})). \qquad (2.3)$$

Since

$$\begin{aligned}\frac{d(\varpi_n, \varpi_{n+2})}{2s} &\leq \frac{sd(\varpi_n, \varpi_{n+1}) + d(\varpi_{n+1}, \varpi_{n+2})}{2s}\\ &\leq \max\{d(\varpi_n, \varpi_{n+1}), d(\varpi_{n+1}, \varpi_{n+2})\},\end{aligned}$$

then we have

$$M(\varpi_n, \varpi_{n+1}) \leq \max\{d(\varpi_n, \varpi_{n+1}), d(\varpi_{n+1}, \varpi_{n+2})\}.$$

If $\max\{d(\varpi_n, \varpi_{n+1}), d(\varpi_{n+1}, \varpi_{n+2})\} = d(\varpi_{n+1}, \varpi_{n+2})$, then by (2.3) we have

$$\begin{aligned}\psi(d(\varpi_{n+1}, \varpi_{n+2})) &< \psi(M(\varpi_n, \varpi_{n+1}))\\ &\leq \psi(d(\varpi_{n+1}, \varpi_{n+2})),\end{aligned}$$

that is a contradiction. Hence, for all $n \in \mathbb{N}, d(\varpi_{n+1}, \varpi_{n+2}) < d(\varpi_n, \varpi_{n+1})$. Therefore, (2.2) becomes

$$\begin{aligned}\psi(d(\varpi_{n+1}, \varpi_{n+2})) &\leq \zeta(\psi(M(\varpi_n, \varpi_{n+1})))\psi(d(\varpi_n, \varpi_{n+1}))\\ &< \psi(d(\varpi_n, \varpi_{n+1})). \qquad (2.4)\end{aligned}$$

Since for all $n \in \mathbb{N}, d(\varpi_{n+1}, \varpi_{n+2}) < d(\varpi_n, \varpi_{n+1})$, then $(d(\varpi_n, \varpi_{n+1}))_{n\in\mathbb{N}}$ is a non-negative and decreasing sequence. As a consequence, there exists $c \geq 0$ such that $\lim_{n\to+\infty} d(\varpi_n, \varpi_{n+1}) = c$. Suppose $c > 0$; by continuity of ψ and Inequality (2.4), we have

$$1 = \lim_{n\to+\infty} \frac{\psi(d(\varpi_{n+1}, \varpi_{n+2}))}{\psi(d(\varpi_n, \varpi_{n+1}))} \leq \lim_{n\to+\infty} \zeta(\psi(d(\varpi_n, \varpi_{n+1}))) \leq 1.$$

It yields that $\lim_{n\to+\infty}\zeta(\psi(d(\varpi_n,\varpi_{n+1})))=1$, and since $\zeta\in\mathcal{F}$, then $\lim_{n\to+\infty}\psi(d(\varpi_n,\varpi_{n+1}))=0$. Considering the above information, $\lim_{n\to+\infty}d(\varpi_n,\varpi_{n+1})=c$, and since $\psi(t)=0 \iff t=0$, then $c=0$, which is a contradiction. Hence, we have

$$\lim_{n\to+\infty} d(\varpi_n,\varpi_{n+1})=0. \tag{2.5}$$

To show that $(\varpi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, suppose $\lim_{n,m\to+\infty}d(\varpi_n,\varpi_m)=\epsilon>0$. For $n<m$, we have

$$d(\varpi_n,\varpi_m)\le sd(\varpi_n,\varpi_{n+1})+d(\varpi_{n+1},\varpi_{m+1})+sd(\varpi_m,\varpi_{m+1}). \tag{2.6}$$

By Lemma 1.3 and by (2.1) we obtain

$$\begin{aligned}\psi(d(\varpi_{n+1},\varpi_{m+1})) &= \psi(d(T\varpi_n,T\varpi_m))\\ &\le \alpha(\varpi_n,\varpi_m)\psi(d(T\varpi_n,T\varpi_m))\\ &\le \zeta(\psi(M(\varpi_n,\varpi_m)))\psi(M(\varpi_n,\varpi_m))+L\phi(N(\varpi_n,\varpi_m)),\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}M(\varpi_n,\varpi_m) &= \max\{d(\varpi_n,\varpi_m),d(\varpi_n,T\varpi_n),d(\varpi_m,T\varpi_m),\\ &\quad \frac{d(\varpi_n,T\varpi_m)+d(\varpi_m,T\varpi_n)}{2s}\}\\ &= \max\{d(\varpi_n,\varpi_m),d(\varpi_n,\varpi_{n+1}),d(\varpi_m,\varpi_{m+1}),\\ &\quad \frac{d(\varpi_n,\varpi_{m+1})+d(\varpi_m,\varpi_{n+1})}{2s}\},\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}N(\varpi_n,\varpi_m) &= \min\{d(\varpi_n,T\varpi_n),d(\varpi_m,T\varpi_n)\}\\ &= \min\{d(\varpi_n,\varpi_{n+1}),d(\varpi_m,\varpi_{n+1})\}.\end{aligned} \tag{2.9}$$

Since

$$\begin{aligned}\frac{d(\varpi_n,\varpi_{m+1})+d(\varpi_m,\varpi_{n+1})}{2s} &\le \frac{sd(\varpi_n,\varpi_m)+d(\varpi_m,\varpi_{m+1})+sd(\varpi_n,\varpi_m)}{2s}+\\ &\quad \frac{d(\varpi_n,\varpi_{n+1})}{2s}\\ &= d(\varpi_n,\varpi_m)+\frac{d(\varpi_m,\varpi_{m+1})+d(\varpi_n,\varpi_{n+1})}{2s}.\end{aligned} \tag{2.10}$$

By (2.5), (2.8)–(2.10), we have

$$\lim_{n\to+\infty} M(\varpi_n,\varpi_m)=\lim_{n\to+\infty} d(\varpi_n,\varpi_m), \tag{2.11}$$

and

$$\lim_{n\to+\infty} N(\varpi_n,\varpi_m)=0. \tag{2.12}$$

By continuity of ψ and by (2.5) and (2.6), we have

$$\lim_{n\to+\infty}\psi(d(\varpi_n,\varpi_m))\le\lim_{n\to+\infty}\psi(d(\varpi_{n+1},\varpi_{m+1})). \tag{2.13}$$

Hence together with (2.7) and (2.11)–(2.13), we deduce

$$\begin{aligned}
\lim_{n\to+\infty}\psi(d(\varpi_n,\varpi_m)) &\leq \lim_{n\to+\infty}\psi(d(\varpi_{n+1},\varpi_{m+1}))\\
&= \lim_{n\to+\infty}\psi(d(T\varpi_n,T\varpi_m))\\
&\leq \lim_{n\to+\infty}(\zeta(\psi(M(\varpi_n,\varpi_m)))\psi(M(\varpi_n,\varpi_m)) + L\phi(N(\varpi_n,\varpi_m)))\\
&\leq \lim_{n\to+\infty}\zeta(\psi(M(\varpi_n,\varpi_m)))\psi(d(\varpi_n,\varpi_m))\\
&\leq \lim_{n\to+\infty}\psi(d(\varpi_n,\varpi_m)).
\end{aligned}$$

Hence by the assumption $\lim_{n,m\to+\infty} d(\varpi_n,\varpi_m) = \epsilon$, we have

$1 \leq \lim_{n\to+\infty}\zeta(\psi(M(\varpi_n,\varpi_m))) \leq 1$. Which implies $\lim_{n\to+\infty}\zeta(\psi(M(\varpi_n,\varpi_m))) = 1$. Thus, $\lim_{n\to+\infty}\psi(M(\varpi_n,\varpi_m)) = \lim_{n\to+\infty}\psi(d(\varpi_n,\varpi_m)) = 0$. It is a contradiction with the assumption $\lim_{n,m\to+\infty} d(\varpi_n,\varpi_m) = \epsilon > 0$. Thus, $(\varpi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and by completeness of Ξ, $(\varpi_n)_{n\in\mathbb{N}}$ converges to some $\varpi^* \in \Xi$.

By continuity of T we obtain that

$$d(\varpi^*,T\varpi^*) = d(\lim_{n\to+\infty}\varpi_{n+1},T\varpi^*) = d(\lim_{n\to+\infty}T\varpi_n,T\varpi^*) = 0.$$

Hence $\varpi^* = T\varpi^*$. □

Replacing the continuity of the mapping T in the above theorem by a suitable condition on Ξ.

Theorem 2.2. *Let (Ξ,d) be a complete SbMS with $s \geq 1$, and $T:\Xi\to\Xi$ be a generalized α-ψ-Geraghty contraction type such that*

i. T is triangular α-orbital admissible;
ii. there exists $\varpi_0 \in \Xi$ such that $\alpha(\varpi_0,T\varpi_0) \geq 1$;
iii. for all $n\in\mathbb{N}$, $(\varpi_n)_{n\in\mathbb{N}} \subseteq \Xi$ converges to ϖ^* in Ξ and $\alpha(\varpi_n,\varpi_{n+1}) \geq 1$, then $\alpha(\varpi_n,\varpi^*) \geq 1, \forall n$;

Then, T has a fixed point in Ξ.

Proof. As stated in the proof of Theorem 2.1, we have that $(\varpi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence that converges to some $\varpi^*\in\Xi$. Since $\alpha(\varpi_n,\varpi_{n+1}) \geq 1$, then condition (iii) leads to

$$\alpha(\varpi_n,\varpi^*) \geq 1, \forall n,$$

and since (Ξ,d) is an SbMS we have

$$d(\varpi^*,T\varpi^*) \leq sd(\varpi^*,\varpi_{n+1}) + d(\varpi_{n+1},T\varpi^*).$$

Let $n\to+\infty$

$$d(\varpi^*,T\varpi^*) \leq \lim_{n\to+\infty} d(T\varpi_n,T\varpi^*).$$

Since $\psi\in\Psi$, we have

$$\psi(d(\varpi^*,T\varpi^*)) \leq \lim_{n\to+\infty}\alpha(\varpi_n,\varpi^*)\psi(d(T\varpi_n,T\varpi^*))$$

$$\leq \lim_{n\to+\infty} (\zeta(\psi(M(\varpi_n,\varpi^*)))\psi(M(\varpi_n,\varpi^*)) + L\phi(N(\varpi_n,\varpi^*))), \tag{2.14}$$

where

$$\begin{aligned} M(\varpi_n,\varpi^*) &= \max\{d(\varpi_n,\varpi^*), d(\varpi_n,T\varpi_n), d(\varpi^*,T\varpi^*), \\ &\quad \frac{d(\varpi_n,T\varpi^*)+d(\varpi^*,T\varpi_n)}{2s}\} \\ &= \max\{d(\varpi_n,\varpi^*), d(\varpi_n,\varpi_{n+1}), d(\varpi^*,T\varpi^*), \\ &\quad \frac{d(\varpi_n,T\varpi^*)+d(\varpi^*,\varpi_{n+1})}{2s}\}, \end{aligned}$$

and

$$\begin{aligned} N(\varpi_n,\varpi^*) &= \min\{d(\varpi_n,T\varpi_n), d(\varpi^*,T\varpi_n)\} \\ &= \min\{d(\varpi_n,\varpi_{n+1}), d(\varpi^*,\varpi_{n+1})\}. \end{aligned}$$

Since

$$\frac{d(\varpi_n,T\varpi^*)+d(\varpi^*,\varpi_{n+1})}{2s} \leq \frac{sd(\varpi_n,\varpi^*)+d(\varpi^*,T\varpi^*)+d(\varpi^*,\varpi_{n+1})}{2s}.$$

Therefore

$$\lim_{n\to+\infty} \frac{d(\varpi_n,T\varpi^*)+d(\varpi^*,\varpi_{n+1})}{2s} \leq \frac{d(\varpi^*,T\varpi^*)}{2s}.$$

That implies

$$\lim_{n\to+\infty} M(\varpi_n,\varpi^*) = d(\varpi^*,T\varpi^*), \tag{2.15}$$

and

$$\lim_{n\to+\infty} N(\varpi_n,\varpi^*) = 0. \tag{2.16}$$

To show that ϖ^* is a fixed point of T, let us assume that $T\varpi^* \neq \varpi^*$. Then, by (2.14)–(2.16)

$$\begin{aligned} 1 = \lim_{n\to+\infty} \frac{\psi(d(\varpi^*,T\varpi^*))}{\psi(M(\varpi_n,\varpi^*))} &\leq \lim_{n\to+\infty} (\zeta(\psi(M(\varpi_n,\varpi^*))) \\ &\leq 1. \end{aligned} \tag{2.17}$$

Hence, $\lim_{n\to+\infty} \zeta(\psi(M(\varpi_n,\varpi^*))) = 1$, then $\lim_{n\to+\infty} \psi(M(\varpi_n,\varpi^*)) = \psi(d(\varpi^*,T\varpi^*)) = 0$. That is a contradiction. Hence, ϖ^* is a fixed point of T. □

Consider the following condition to obtain a unique fixed point of a generalized α-ψ-Geraghty contraction type mapping.
(U) For all ϖ_1, ϖ_2 belonging to the set of fixed points of T, either $\alpha(\varpi_1,\varpi_2) \geq 1$ or $\alpha(\varpi_2,\varpi_1) \geq 1$.

Theorem 2.3. *Adding condition (U) to the hypotheses of the previous theorems, Theorems 2.1 and 2.2, we obtain uniqueness of the fixed point of T.*

Proof. Suppose that ϖ^* and w^* are two distinct fixed points of T. Thus, $M(\varpi^*, w^*) = d(\varpi^*, w^*)$ and $N(\varpi^*, w^*) = 0$. Without loss of generality, let $\alpha(\varpi^*, w^*) \geq 1$; then we have

$$\begin{aligned} \psi(d(\varpi^*, w^*)) &\leq \alpha(\varpi^*, w^*)\psi(d(T\varpi^*, Tw^*)) \\ &\leq \zeta(\psi(d(\varpi^*, w^*)))\psi(d(\varpi^*, w^*)) \\ &< \psi(d(\varpi^*, w^*)), \end{aligned}$$

that is a contradiction. □

Corollary 2.1. *Let (Ξ, d) be a complete SbMS with $s \geq 1$, and $T : \Xi \to \Xi$ be a mapping on Ξ such that for all $\varpi_1, \varpi_2 \in \Xi$*

$$d(T\varpi_1, T\varpi_2) \leq \zeta(M(\varpi_1, \varpi_2))M(\varpi_1, \varpi_2),$$

where $\zeta \in \mathcal{F}$ and

$$M(\varpi_1, \varpi_2) = \max\{d(\varpi_1, \varpi_2), d(\varpi_1, T\varpi_1), d(\varpi_2, T\varpi_2), \frac{d(\varpi_1, T\varpi_2) + d(\varpi_2, T\varpi_1)}{2s}\},$$

then T has a unique fixed point.

Proof. Put $\alpha(\varpi_1, \varpi_2) = 1$ for all $\varpi_1, \varpi_2 \in \Xi$, $L = 0$ and $\psi(t) = t$ in Theorem 2.3. □

Definition 2.2. *Let (Ξ, d) be an SbMS with $s \geq 1$, and let $\alpha : \Xi \times \Xi \to [0, +\infty)$ be a function. Then a single-valued mapping $T : \Xi \to \Xi$ is an α-ψ- Geraghty contraction type if*

$$\alpha(\varpi_1, \varpi_2)\psi(d(T\varpi_1, T\varpi_2)) \leq \zeta(\psi(d(\varpi_1, \varpi_2)))\psi(d(\varpi_1, \varpi_2)), \tag{2.18}$$

for all $\varpi_1, \varpi_2 \in \Xi$, $\zeta \in \mathcal{F}$, and $\psi \in \Psi$.

Remark 2.2.
- *T is called ψ- Geraghty contraction type if $\alpha(\varpi_1, \varpi_2) = 1$ for all $\varpi_1, \varpi_2 \in \Xi$ in Definition 2.2.*
- *T is called α- Geraghty contraction type if $\psi(t) = t$ in Definition 2.2.*

Based on the proof of Theorems 2.1–2.3, proving the following theorem is straightforward.

Theorem 2.4. *Let (Ξ, d) be a complete SbMS with $s \geq 1$, and $T : \Xi \to \Xi$ be an α-ψ- Geraghty contraction type such that*

i. T is triangular α-orbital admissible;
ii. there exists $\varpi_0 \in \Xi$ such that $\alpha(\varpi_0, T\varpi_0) \geq 1$;
iii. T is continuous or Ξ satisfies (C_α) condition.

Then T has a fixed point in Ξ. Moreover, if (U) is satisfied, then the obtained fixed point is unique.

Example 2.1. *Let $\Xi = [0, +\infty)$ endowed with the strong b-metric $d(\varpi_1, \varpi_2) = \max\{|\varpi_1 - \varpi_2|, 2|\varpi_1 - \varpi_2| - 1\}$ for all $\varpi_1, \varpi_2 \in \Xi$, with $s = 2$. Let $\psi(t) = t$ and $\zeta(t) = \frac{1}{2+t}$ for all $t \geq 0$. Then $\psi \in \Psi$ and $\zeta \in \mathcal{F}$. Define the mapping $T : \Xi \to \Xi$ by*

$$T\varpi = \begin{cases} \frac{\varpi}{4} & \text{if } \varpi \in [0, 1], \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

It is clear that T is continuous. Define the mapping $\alpha : \Xi \times \Xi \to [0, +\infty)$ by

$$\alpha(\varpi_1, \varpi_2) = \begin{cases} 1 & \text{if } \varpi_1, \varpi_2 \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Hence, T is a triangular α- orbital admissible mapping and $\alpha(1, T1) \geq 1$. Now we shall prove that T satisfies Inequality (2.18). For $\varpi_1, \varpi_2 \in [0,1]$

$$\begin{aligned} \zeta(\psi(d(\varpi_1,\varpi_2)))\psi(d(\varpi_1,\varpi_2)) - \alpha(\varpi_1,\varpi_2)\psi(d(T\varpi_1,T\varpi_2)) &= \frac{|\varpi_1-\varpi_2|}{2+|\varpi_1-\varpi_2|} - \frac{|\varpi_1-\varpi_2|}{4} \\ &= \frac{|\varpi_1-\varpi_2|(2-|\varpi_1-\varpi_2|^2)}{4(2+|\varpi_1-\varpi_2|)} \\ &\geq 0. \end{aligned}$$

Hence, we deduce that $\alpha(\varpi_1,\varpi_2)\psi(d(T\varpi_1,T\varpi_2)) \leq \zeta(\psi(d(\varpi_1,\varpi_2)))\psi(d(\varpi_1,\varpi_2))$ for $0 \leq \varpi_1, \varpi_2 \leq 1$. If $\varpi_1 > 1$ or $\varpi_2 > 1$, then $\alpha(\varpi_1,\varpi_2) = 0$; hence, Inequality (2.18) holds. Consequently, all assumptions of Theorem 2.4 are satisfied, and therefore T has a unique fixed point $\varpi^* = 0$.

The following corollary is a version of Geraghty's theorem in strong b-metric spaces.

Corollary 2.2. *Let (Ξ, d) be a complete SbMS with $s \geq 1$, and $T : \Xi \to \Xi$ be a mapping on Ξ such that for all $\varpi_1, \varpi_2 \in \Xi$, and $\zeta \in \mathcal{F}$,*

$$d(T\varpi_1, T\varpi_2) \leq \zeta(d(\varpi_1,\varpi_2))\, d(\varpi_1,\varpi_2),$$

then T has a unique fixed point.

Proof. Put $\alpha(\varpi_1,\varpi_2) = 1$ for all $\varpi_1, \varpi_2 \in \Xi$ and $\psi(t) = t$ in Theorem 2.4. □

Example 2.2. *Let $\Xi = \mathbb{R}$, and $d(\varpi_1,\varpi_2) = \max\{|\varpi_1-\varpi_2|, 2|\varpi_1-\varpi_2| - 1\}$ for all $\varpi_1, \varpi_2 \in \mathbb{R}$, then (Ξ, d) is a strong b-metric space with $s = 2$. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by*

$$T\varpi = \begin{cases} 3\varpi - \frac{9}{4} & \text{if } \varpi > 1, \\ \frac{3\varpi}{4} & \text{if } \varpi \in [0,1], \\ 0 & \text{if } \varpi < 0, \end{cases}$$

and

$$\alpha(\varpi_1, \varpi_2) = \begin{cases} 1 & \text{if } \varpi_1, \varpi_2 \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly that T is continuous and T is triangular α-admissible since

- $\alpha(\varpi_1, T\varpi_1) \geq 1 \Rightarrow \varpi_1, T\varpi_1 \in [0,1] \Rightarrow T\varpi_1, T^2\varpi_1 \in [0,1] \Rightarrow \alpha(T\varpi_1, T^2\varpi_1) \geq 1$.
- $\alpha(\varpi_1,\varpi_2) \geq 1$ *and* $\alpha(\varpi_2, T\varpi_2) \geq 1$, *then* $\alpha(\varpi_1, T\varpi_2) \geq 1$.

For $\varpi_1, \varpi_2 \in [0,1]$, assume $\zeta(t) = 3/4$ and $\psi(t) = \frac{t}{2}$ where $t \geq 0$, then we have

$$\alpha(\varpi_1,\varpi_2)\,\psi(d(T\varpi_1,T\varpi_2)) = \frac{1}{2}\cdot\frac{3}{4}|\varpi_1-\varpi_2|$$

$$\leq \frac{3}{4}\psi(d(\varpi_1,\varpi_2))$$
$$\leq \zeta(\psi(M(\varpi_1,\varpi_2)))\psi(M(\varpi_1,\varpi_2)).$$

For $\varpi_1, \varpi_2 \in \mathbb{R} \setminus [0,1], \alpha(\varpi_1,\varpi_2) = 0$, *then the mapping* T *is a generalized* α-ψ- *Geraghty contraction type. Also, when* $\varpi_0 = \frac{1}{3}$ *we have* $\alpha(\frac{1}{3}, T(\frac{1}{3})) = \alpha(\frac{1}{3}, \frac{1}{4}) \geq 1$.

All assumptions of Theorem 2.1 hold, so T *has a fixed point* 0.

Remark 2.3. *In the previous example, we note the following:*

- *Theorem 1.1 cannot be applied since* (Ξ, d) *is not a metric space.*
- *Theorem 1.2 cannot be applied since for all* $t \geq 0$, $\zeta(t) \geq \frac{1}{2}$.
- *Corollary 2.2 cannot be applied since*

$$d(T0,T2) = 2|0 - \frac{15}{4}| - 1 = \frac{13}{2} \nleq \zeta(d(0,2))\, d(0,2) = \frac{9}{4}.$$

2.2. For multi-valued mappings

This section treats the case $s > 1$. First, we consider $\mathcal{F}_{sb}$ the class of all functions $\zeta : [0,+\infty) \to [0,1/s)$, which satisfies the condition: If $\lim_{n\to+\infty} \zeta(t_n) = 1/s$, then $\lim_{n\to+\infty} t_n = 0$.

Define Ψ by the class of all non-decreasing and continuous functions $\psi : [0,+\infty) \to [0,+\infty)$ with $\psi^{-1}(0) = 0$ and such that for $c > 1, \psi(ct) \leq c\psi(t)$, and we introduce the notion of a generalized α-ψ-Geraghty contraction type for multi-valued mapping as follows:

Definition 2.3. *Let* (Ξ, d) *be an SbMS with* $s \geq 1$; *then a multi-valued mapping* $T : \Xi \to C\mathcal{B}(\Xi)$ *be a generalized* α-ψ-*Geraghty contraction type whenever there exist* $\alpha : \Xi \times \Xi \to [0,+\infty)$ *and some* $L \geq 0$ *such that for*

$$M(\varpi_1,\varpi_2) = \max\{d(\varpi_1,\varpi_2), D(\varpi_1,T\varpi_1), D(\varpi_2,T\varpi_2), \frac{D(\varpi_1,T\varpi_2)+D(\varpi_2,T\varpi_1)}{2s}\},$$
$$N(\varpi_1,\varpi_2) = \min\{D(\varpi_1,T\varpi_1), D(\varpi_2,T\varpi_1)\},$$

we have

$$\alpha(\varpi_1,\varpi_2)\psi(\mathcal{H}(T\varpi_1,T\varpi_2)) \leq \zeta(\psi(M(\varpi_1,\varpi_2)))\psi(M(\varpi_1,\varpi_2)) + L\phi(N(\varpi_1,\varpi_2)), \tag{2.19}$$

for all $\varpi_1,\varpi_2 \in \Xi, \zeta \in \mathcal{F}_{sb}$, *and* $\psi, \phi \in \Psi$.

Remark 2.4. *The generalized* α-ψ- *Geraghty condition given in Definition 2.3 is considered an extension of the conditions presented in the context of b-metric spaces [2, Definition 4].*

Theorem 2.5. *Let* (Ξ, d) *be a complete SbMS (with* $s > 1$*) and* $T : \Xi \to C\mathcal{B}(\Xi)$ *be a generalized* α-ψ-*Geraghty contraction type multi-valued mapping such that*

i. T *is* α-*admissible;*
ii. there exists $\varpi_0 \in \Xi$; *and* $\varpi_1 \in T\varpi_0$ *such that* $\alpha(\varpi_0,\varpi_1) \geq 1$;
iii. T *is continuous.*

Then, T has a fixed point in Ξ.

Proof. By assumption (ii), there exists $\varpi_0 \in \Xi$ and $\varpi_1 \in T\varpi_0$ such that $\alpha(\varpi_0, \varpi_1) \geq 1$. If $\varpi_1 = \varpi_0$, then we have nothing to prove. Let $\varpi_1 \neq \varpi_0$ and assume that $\varpi_1 \notin T\varpi_1$, since otherwise ϖ_1 is a fixed point of T. Let σ be a real number such that $\sigma \in (1, s)$. Then

$$0 < \psi(D(\varpi_1, T\varpi_1)) \quad < \quad \sigma\alpha(\varpi_0, \varpi_1)\psi(\mathcal{H}(T\varpi_0, T\varpi_1)).$$

Hence, there exists $\varpi_2 \in T\varpi_1$, such that

$$\begin{aligned} \psi(d(\varpi_1, \varpi_2)) &\leq \sigma\alpha(\varpi_0, \varpi_1)\psi(\mathcal{H}(T\varpi_0, T\varpi_1)) \\ &\leq \sigma\zeta(\psi(M(\varpi_0, \varpi_1)))\psi(M(\varpi_0, \varpi_1)) + \sigma L\phi(N(\varpi_0, \varpi_1)) \\ &< \frac{\sigma}{s}\psi(M(\varpi_0, \varpi_1)) + \sigma L\phi(N(\varpi_0, \varpi_1)), \end{aligned}$$

where

$$\begin{aligned} M(\varpi_0, \varpi_1) &= \max\{d(\varpi_0, \varpi_1), D(\varpi_0, T\varpi_0), D(\varpi_1, T\varpi_1), \\ &\quad \frac{D(\varpi_0, T\varpi_1) + D(\varpi_1, T\varpi_0)}{2s}\} \\ &\leq \max\{d(\varpi_0, \varpi_1), D(\varpi_1, T\varpi_1), \frac{D(\varpi_0, T\varpi_1)}{2s}\}, \end{aligned}$$

and

$$\begin{aligned} N(\varpi_0, \varpi_1) &= \min\{D(\varpi_0, T\varpi_0), D(\varpi_1, T\varpi_0)\} \\ &\leq \min\{d(\varpi_0, \varpi_1), d(\varpi_1, \varpi_1)\} = 0. \end{aligned}$$

Since

$$\frac{D(\varpi_0, T\varpi_1)}{2s} \leq \frac{sd(\varpi_0, \varpi_1) + D(\varpi_1, T\varpi_1)}{2s} \leq \max\{d(\varpi_0, \varpi_1), D(\varpi_1, T\varpi_1)\},$$

then we have

$$M(\varpi_0, \varpi_1) \leq \max\{d(\varpi_0, \varpi_1), D(\varpi_1, T\varpi_1)\}.$$

If $\max\{d(\varpi_0, \varpi_1), D(\varpi_1, T\varpi_1)\} = D(\varpi_1, T\varpi_1)$, then we get

$$\psi(D(\varpi_1, T\varpi_1)) \leq \psi(d(\varpi_1, \varpi_2)) < \frac{\sigma}{s}\psi(D(\varpi_1, T\varpi_1)),$$

that is a contradiction. Hence, we obtain $\max\{d(\varpi_0, \varpi_1), D(\varpi_1, T\varpi_1)\} = d(\varpi_0, \varpi_1)$, so

$$\psi(d(\varpi_1, \varpi_2)) < \frac{\sigma}{s}\psi(d(\varpi_0, \varpi_1)).$$

Since $\frac{s}{\sigma} > 1$ and ψ is non-decreasing, we have

$$\psi(\frac{s}{\sigma}d(\varpi_1, \varpi_2)) \leq \frac{s}{\sigma}\psi(d(\varpi_1, \varpi_2)) < \psi(d(\varpi_0, \varpi_1)).$$

Hence

$$d(\varpi_1, \varpi_2) < \frac{\sigma}{s}d(\varpi_0, \varpi_1).$$

Since $\varpi_2 \in T\varpi_1$ and $\varpi_1 \notin T\varpi_1$, so it is clear that $\varpi_1 \neq \varpi_2$. Put

$$\sigma_1 = \frac{\sigma}{s}\frac{\psi(d(\varpi_0, \varpi_1))}{\psi(d(\varpi_1, \varpi_2))} > 1.$$

Now if $\varpi_2 \in T\varpi_2$, then ϖ_2 is a fixed point of T. Assume that $\varpi_2 \notin T\varpi_2$, and since $\alpha(\varpi_1, \varpi_2) \geq 1$, then we have

$$0 < \psi(D(\varpi_2, T\varpi_2)) \quad < \quad \sigma_1\alpha(\varpi_1, \varpi_2)\psi(\mathcal{H}(T\varpi_1, T\varpi_2)).$$

Therefore, there exists $\varpi_3 \in T\varpi_2$, such that

$$\begin{aligned} 0 < \psi(D(\varpi_2, T\varpi_2)) \leq \psi(d(\varpi_2, \varpi_3)) &\leq \sigma_1\alpha(\varpi_1, \varpi_2)\psi(\mathcal{H}(T\varpi_1, T\varpi_2)) \\ &\leq \sigma_1\zeta(\psi(M(\varpi_1, \varpi_2)))\psi(M(\varpi_1, \varpi_2)) + \\ &\quad \sigma_1 L\phi(N(\varpi_1, \varpi_2)) \\ &< \frac{\sigma_1}{s}\psi(M(\varpi_1, \varpi_2)) + \sigma_1 L\phi(N(\varpi_1, \varpi_2)). \end{aligned}$$

Similarly, $M(\varpi_1, \varpi_2) \leq d(\varpi_1, \varpi_2)$ and $N(\varpi_1, \varpi_2) = 0$. Hence, we get

$$\psi(d(\varpi_2, \varpi_3)) \leq \frac{\sigma_1}{s}\psi(d(\varpi_1, \varpi_2)) < \frac{\sigma}{s^2}\psi(d(\varpi_0, \varpi_1)).$$

Also, since $\frac{s^2}{\sigma} > 1$ we have

$$\psi(\frac{s^2}{\sigma}d(\varpi_2, \varpi_3)) \leq \frac{s^2}{\sigma}\psi(d(\varpi_2, \varpi_3)) < \psi(d(\varpi_0, \varpi_1)).$$

Hence

$$d(\varpi_2, \varpi_3) \leq \frac{\sigma}{s^2}d(\varpi_0, \varpi_1).$$

Since $\varpi_3 \in T\varpi_2$ and $\varpi_2 \notin T\varpi_2$, so it is clear that $\varpi_2 \neq \varpi_3$. Put

$$\sigma_2 = \frac{\sigma}{s^2}\frac{\psi(d(\varpi_0, \varpi_1))}{\psi(d(\varpi_2, \varpi_3))} > 1.$$

Continuing in this manner, we obtain a sequence $(\varpi_n)_{n\in\mathbb{N}} \subseteq \Xi$ such that for all $n \in \mathbb{N}, \varpi_n \in T\varpi_{n-1}, \varpi_{n-1} \neq \varpi_n, \alpha(\varpi_{n-1}, \varpi_n) \geq 1$, and $d(\varpi_n, \varpi_{n+1}) \leq \sigma(\frac{1}{s})^n d(\varpi_0, \varpi_1)$. By completeness of Ξ and by Proposition 1.1, since $\sum_{i=1}^{+\infty} d(\varpi_n, \varpi_{n+1}) < +\infty$ we have $(\varpi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence that converges to some $\varpi^* \in \Xi$.

By continuity of T we obtain

$$D(\varpi^*, T\varpi^*) = D(\lim_{n\to+\infty} \varpi_{n+1}, T\varpi^*) \leq \mathcal{H}(\lim_{n\to+\infty} T\varpi_n, T\varpi^*) = 0.$$

Hence $\varpi^* \in T\varpi^*$. □

Theorem 2.6. *Let* (Ξ, d) *be a complete SbMS (with* $s > 1$*) and* $T : \Xi \to C\mathcal{B}(\Xi)$ *be a generalized* α*-*ψ*-Geraghty contraction type multi-valued mapping such that*

i. T *is* α*-admissible;*

ii. there exists $\varpi_0 \in \Xi$ *and* $\varpi_1 \in T\varpi_0$ *such that* $\alpha(\varpi_0, \varpi_1) \geq 1$;
iii. for all $n \in \mathbb{N}$, *if* $(\varpi_n)_{n\in\mathbb{N}} \subseteq \Xi$ *converges to* ϖ^* *in* Ξ, *and* $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$, *then* $\alpha(\varpi_n, \varpi^*) \geq 1, \forall n$;

then, T *has a fixed point in* Ξ.

Proof. As stated in the proof of Theorem 2.5, we have that $(\varpi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence that converges to some $\varpi^* \in \Xi$. And $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$. Then, condition (iii) leads to

$$\alpha(\varpi_n, \varpi^*) \geq 1, \forall n,$$

and since (Ξ, d) is an SbMS we have

$$\begin{aligned} D(\varpi^*, T\varpi^*) &\leq sd(\varpi^*, \varpi_{n+1}) + d(\varpi_{n+1}, z), \quad z \in T\varpi^* \\ &\leq sd(\varpi^*, \varpi_{n+1}) + \sigma\mathcal{H}(T\varpi_n, T\varpi^*), \quad 1 < \sigma < s. \end{aligned}$$

Let $n \to +\infty$

$$D(\varpi^*, T\varpi^*) \leq \lim_{n\to+\infty} \sigma\mathcal{H}(T\varpi_n, T\varpi^*).$$

Since $\psi \in \Psi$, we have

$$\begin{aligned} \psi(D(\varpi^*, T\varpi^*)) &\leq \sigma \lim_{n\to+\infty} \psi(\mathcal{H}(T\varpi_n, T\varpi^*)) \\ &\leq \sigma \lim_{n\to+\infty} \alpha(\varpi_n, \varpi^*)\psi(\mathcal{H}(T\varpi_n, T\varpi^*)) \\ &\leq \sigma \lim_{n\to+\infty} (\zeta(\psi(M(\varpi_n, \varpi^*)))\psi(M(\varpi_n, \varpi^*)) + L\phi(N(\varpi_n, \varpi^*))), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} M(\varpi_n, \varpi^*) &= \max\{d(\varpi_n, \varpi^*), D(\varpi_n, T\varpi_n), D(\varpi^*, T\varpi^*), \\ &\quad \frac{D(\varpi_n, T\varpi^*) + D(\varpi^*, T\varpi_n)}{2s}\} \\ &\leq \max\{d(\varpi_n, \varpi^*), d(\varpi_n, \varpi_{n+1}), D(\varpi^*, T\varpi^*), \\ &\quad \frac{D(\varpi_n, T\varpi^*) + d(\varpi^*, \varpi_{n+1})}{2s}\}, \end{aligned}$$

and

$$\begin{aligned} N(\varpi_n, \varpi^*) &= \min\{D(\varpi_n, T\varpi_n), D(\varpi^*, T\varpi_n)\} \\ &\leq \min\{d(\varpi_n, \varpi_{n+1}), d(\varpi^*, \varpi_{n+1})\}. \end{aligned}$$

Since

$$\frac{D(\varpi_n, T\varpi^*) + d(\varpi^*, \varpi_{n+1})}{2s} \leq \frac{sd(\varpi_n, \varpi^*) + D(\varpi^*, T\varpi^*) + d(\varpi^*, \varpi_{n+1})}{2s}.$$

Therefore

$$\lim_{n\to+\infty} \frac{D(\varpi_n, T\varpi^*) + d(\varpi^*, \varpi_{n+1})}{2s} \leq \frac{D(\varpi^*, T\varpi^*)}{2s}.$$

This implies that

$$\lim_{n\to+\infty} M(\varpi_n, \varpi^*) \leq D(\varpi^*, T\varpi^*), \tag{2.21}$$

and

$$\lim_{n\to+\infty} N(\varpi_n, \varpi^*) = 0. \tag{2.22}$$

Hence, by (2.20)–(2.22), we have

$$\psi(D(\varpi^*, T\varpi^*)) \leq \frac{\sigma}{s}\psi(D(\varpi^*, T\varpi^*)),$$

that means $\psi(D(\varpi^*, T\varpi^*)) = 0$,then $D(\varpi^*, T\varpi^*) = 0$; that gives $\varpi^* \in T\varpi^*$ and ϖ^* is a fixed point of T. □

The following theorem studies the existence of a fixed point on the closed ball that is a subset of Ξ.

Theorem 2.7. *Let (Ξ, d) be a complete SbMS with $s > 1$, and $T : \Xi \to C\mathcal{B}(\Xi)$ be a generalized α-ψ-Geraghty contraction type multi-valued mapping such that*

i. *T is α-admissible;*
ii. *there exists $\varpi_0 \in B[\varpi_0, r]$ and $\varpi_1 \in T\varpi_0$ such that $\alpha(\varpi_0, \varpi_1) \geq 1$;*
iii. *T is continuous or the closed ball ($B[\varpi_0, r]$) satisfies that for all $n \in \mathbb{N}, (\varpi_n)_{n\in\mathbb{N}} \subseteq B[\varpi_0, r]$ converges to ϖ^* in $B[\varpi_0, r]$ and $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$, then $\alpha(\varpi_n, \varpi^*) \geq 1, \forall n$;*
iv. *for $x_0 \in B[\varpi_0, r]$, there exists $\varpi_1 \in T\varpi_0$ such that $\forall n \in \mathbb{N}$ and $r > 0$, $\sum_{i=0}^{n}(\frac{1}{s})^{i+1} < \frac{r}{\sigma s d(\varpi_0, \varpi_1)}$, where $\sigma \in (1, s)$.*

Then, T has a fixed point in $B[\varpi_0, r]$.

Proof. Similarly to the proof of Theorems 2.5 and 2.6, we have a Cauchy sequence $(\varpi_n)_{n\in\mathbb{N}} \subseteq \Xi$ such that for all $n \in \mathbb{N}, \varpi_n \in T\varpi_{n-1}, \varpi_{n-1} \neq \varpi_n, \alpha(\varpi_{n-1}, \varpi_n) \geq 1$, and $d(\varpi_n, \varpi_{n+1}) \leq \sigma(\frac{1}{s})^n d(\varpi_0, \varpi_1)$. Since by (iv) we have that

$$\begin{aligned} d(\varpi_{n+1}, \varpi_0) &\leq sd(\varpi_0, \varpi_1) + sd(\varpi_1, \varpi_2) + sd(\varpi_2, \varpi_3) + \cdots + sd(\varpi_n, \varpi_{n+1}) \\ &\leq \sigma sd(\varpi_0, \varpi_1)[\frac{1}{s} + (\frac{1}{s})^2 + (\frac{1}{s})^3 + \cdots + (\frac{1}{s})^{n+1}] \\ &\leq \sigma sd(\varpi_0, \varpi_1)\sum_{i=0}^{n}(\frac{1}{s})^{i+1} \\ &< r. \end{aligned}$$

Hence, we have $(\varpi_n)_{n\in\mathbb{N}} \subseteq B[\varpi_0, r]$. By completeness of $B[\varpi_0, r]$ and by Proposition 1.1, we have $(\varpi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence that converges to some $\varpi^* \in B[\varpi_0, r]$. Then, imitating the rest of proof of Theorem 2.5 and proof of Theorem 2.6, we obtain the desired result. □

The following theorem provides the condition under which a fixed point is unique for a generalized α-ψ-Geraghty contraction type multi-valued mapping.

Theorem 2.8. *Adding condition (U) to the hypotheses of the previous theorems, Theorems 2.5–2.7, we obtain uniqueness of the fixed point of T.*

Proof. Suppose that ϖ^* and w^* are two fixed points of T. Thus, $M(\varpi^*, w^*) = d(\varpi^*, w^*)$ and $N(\varpi^*, w^*) = 0$. Without loss of generality, let $\alpha(\varpi^*, w^*) \geq 1$; then for $1 < c < s$ we have,

$$\psi(d(\varpi^*, w^*)) \leq c\alpha(\varpi^*, w^*)\psi(\mathcal{H}(T\varpi^*, Tw^*))$$

$$\begin{aligned} &\leq c\zeta(\psi(d(\varpi^*, w^*)))\psi(d(\varpi^*, w^*)) \\ &\leq \frac{c}{s}\psi(d(\varpi^*, w^*)) \\ &< \psi(d(\varpi^*, w^*)), \end{aligned}$$

that is a contradiction. □

Example 2.3. *Let $\Xi = \mathbb{R}$, and $d(\varpi, w) = \max\{|\varpi - w|, 2|\varpi - w| - 1\}$ for all $\varpi, w \in \mathbb{R}$; then (Ξ, d) is an SbMS with $s = 2$ (see [4]). Let $T : \mathbb{R} \to \mathcal{CB}(\mathbb{R})$ be defined by*

$$T\varpi = \begin{cases} [0, \frac{\varpi}{9}] & \text{if } \varpi \in [0, 1], \\ [\frac{1}{9}, \frac{3}{9}] & \text{otherwise,} \end{cases} \quad \text{and} \quad \alpha(\varpi, w) = \begin{cases} 1 & \text{if } \varpi, w \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

For $\psi(t) = t$ and $\zeta = \frac{1}{9}$, then for $\varpi, w \in [0, 1]$ with $\varpi \leq w$, we have

$$\begin{aligned} \alpha(\varpi, w)\psi(\mathcal{H}(T\varpi, Tw)) &\leq \frac{1}{9}d(\varpi, w) \\ &\leq \zeta(\psi(M(\varpi, w)))\psi(M(\varpi, w)) + L\phi(N(\varpi, w)), \end{aligned}$$

where $M(\varpi, w) = \max\{d(\varpi, w), D(\varpi, T\varpi), D(w, Tw), \frac{D(\varpi,Tw)+D(w,T\varpi)}{2s}\}$ and $N(\varpi, w) = D(\varpi, T\varpi)$. For $\varpi, w \in \mathbb{R} \setminus [0, 1], \alpha(\varpi, w) = 0$, then Inequality (2.19) holds. By the definitions of T and α, we have

- *T is α-admissible;*
- *for $\varpi_0 = 1$ and $\varpi_1 = \frac{1}{9}$ we have $\alpha(1, \frac{1}{9}) = 1$;*
- *If $(\varpi_n)_{n\in\mathbb{N}} \subseteq \Xi$ converges to ϖ^* in Ξ and $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $(\varpi_n)_{n\in\mathbb{N}} \subseteq [0, 1]$, so $\varpi^* \in [0, 1]$, hence $\alpha(\varpi_n, \varpi^*) \geq 1, \forall n$.*

All assumptions of Theorem 2.6 hold, so T has a fixed point 0.

Example 2.4. *Let $(\Xi, d), T, \alpha, \psi$ and ζ be defined as an Example 2.3. Let $r = \frac{8}{3}, \varpi_0 = \frac{1}{2}$ and $\varpi_1 = \frac{1}{18}$ then the closed ball centered at $\frac{1}{2}$ with radius $\frac{8}{3}$, $B[\frac{1}{2}, \frac{8}{3}]$, which is $[\frac{-4}{3}, \frac{7}{3}]$. For $\sigma = \frac{5}{4}$ then all conditions of Theorem 2.7 are satisfied. Hence T has a fixed point $0 \in B[\frac{1}{2}, \frac{8}{3}]$.*

3. Application

Finally, we support our results by introducing the following application. Integral equations, in which the unknown function appears under an integral sign, are fundamental tools in mathematical modeling. They arise in many applied sciences, such as potential theory, heat conduction, population dynamics, and engineering models, including signal processing and image restoration. Their ability to naturally incorporate boundary and initial conditions makes them especially powerful in describing real-world phenomena. Motivated by these advantages, we consider the following nonlinear integral equation:

$$\varpi(t) = h(t) + \int_0^1 k(t, \nu)f(\nu, \varpi(\nu))\,d\nu, \quad \forall t \in [0, 1]. \tag{3.1}$$

Equation (3.1) is a nonlinear Hammerstein-type integral equation, which arises in applications such as nonlocal heat conduction, population dynamics with memory effects, and engineering systems with feedback mechanisms. In our setting, the associated operator

$$(T\varpi)(t) = h(t) + \int_0^1 k(t, \nu)f(\nu, \varpi(\nu))\,d\nu$$

acts on a strong b-metric space (X, d). Within our generalized α-ψ-Geraghty contraction framework, T admits a unique fixed point ϖ^* in X provided that certain conditions are satisfied, and that the iterative process

$$\varpi_{n+1} = T\varpi_n, \quad n \geq 0,$$

converges to ϖ^* for any initial choice $\varpi_0 \in X$. This provides a robust theoretical basis for solving such equations in spaces beyond the classical metric or normed settings (see, for example, [16] for more information on strong b-normed spaces) and supports the development of computational schemes based on successive approximations.

Assume the following conditions:

i. $h : [0, 1] \to \mathbb{R}$ is a continuous function;
ii. $k : [0, 1] \times [0, 1] \to [0, +\infty)$ is a function such that $k(t, .) \in L^1([0, 1])$ for all $t \in [0, 1]$;
iii. $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(t, x) \geq 0$ and there exist the functions $\eta : X \times X \to [0, +\infty)$ and $\omega : [0, +\infty) \to [0, 1)$ such for all $\varpi, y \in X$, the following
   - $|f(\nu, \varpi(\nu)) - f(\nu, y(\nu))| \leq \eta(\varpi, y)|\varpi(\nu) - y(\nu)|$,
   - $\int_0^1 k(t, \nu)\eta(\varpi, y)d\nu \leq \frac{1}{2}\,\omega(d(\varpi, y))$ for all $t \in [0, 1]$, and $\omega(t_n) \to 1$ as $n \to +\infty$ implies that $\lim_{n\to+\infty} t_n = 0$.

Consider $X = C[0, 1]$, the space of continuous functions on $[0, 1]$, with the distance given for all $\varpi, y \in C[0, 1]$ by

$$d(\varpi, y) = \max_{t\in[0,1]}\{|\varpi(t) - y(t)|, 2|\varpi(t) - y(t)| - 1\}.$$

The space (X, d) is a complete SbMS with $s = 2$.

Theorem 3.1. *Under assumptions (i)–(iii), Eq (3.1) has a unique solution in $C[0, 1]$.*

Proof. Consider the operator $T : X \to X$ defined by

$$T\varpi(t) = h(t) + \int_0^1 k(t, \nu)f(\nu, \varpi(\nu))d\nu, \quad \forall t \in [0, 1].$$

Firstly, T is well defined since if $\varpi \in X$, then $T\varpi \in X$.
Now for $\varpi, y \in X$, we have

$$\begin{aligned}\left|T(\varpi)(t) - T(y)(t)\right| &= \left|\int_0^1 k(t, \nu)f(\nu, \varpi(\nu))d\nu - \int_0^1 k(t, \nu)f(\nu, y(\nu))d\nu\right| \\ &\leq \int_0^1 |k(t, \nu)|\left|f(\nu, \varpi(\nu)) - f(\nu, y(\nu))\right|d\nu \\ &\leq \int_0^1 |k(t, \nu)|\,\eta(\varpi, y)|\varpi(\nu) - y(\nu)|d\nu.\end{aligned}$$

Then we have,

$$\begin{aligned}\left|T(\varpi)(t) - T(y)(t)\right| &\leq \int_0^1 |k(t, \nu)|\,\eta(\varpi, y)d(\varpi, y)d\nu \qquad \text{(Since } |\varpi(\nu) - y(\nu)| \leq d(\varpi, y) \text{ for all } \nu\text{)} \\ &\leq d(\varpi, y)\int_0^1 |k(t, \nu)|\,\eta(\varpi, y)d\nu \qquad \text{(pulling out the constant } d(\varpi, y)\text{)}\end{aligned}$$

$$\leq \frac{1}{2}\,\omega(d(\varpi, y))\,d(\varpi, y).$$

Now we have two cases,

1. if $d(T\varpi, Ty) = 2|T\varpi(t) - Ty(t)| - 1$, then
$$d(T\varpi, Ty) \leq \omega(d(\varpi, y))\,d(\varpi, y) - 1.$$
2. If $d(T\varpi, Ty) = |T\varpi(t) - Ty(t)|$, then
$$d(T\varpi, Ty) \leq \frac{1}{2}\,\omega(d(\varpi, y))\,d(\varpi, y).$$

From two cases above we obtain that,

$$d(T\varpi, Ty) \leq \omega(d(\varpi, y))\, d(\varpi, y).$$

Since $\omega \in \mathcal{F}$, put $\omega(t) = \zeta(t)$ in Corollary 2.2; then the integral equation (3.1) has a unique solution in $C[0, 1]$ and the proof is completed. □

Example 3.1. *Consider the following nonlinear integral equation:*

$$\varpi(t) = t^2 + \int_0^1 \frac{\nu^3}{2e^t(1+t)} \frac{|\varpi(\nu)|}{1 + |\varpi(\nu)|}\, d\nu, \quad t \in [0, 1]. \tag{3.2}$$

It is observed that the above equation is a special case of (3.1) *with*

$$h(t) = t^2, \quad k(t, \nu) = \frac{\nu^3}{1+t} \quad \text{and} \quad f(t, \varpi) = \frac{|\varpi|}{2e^t(1 + |\varpi|)}.$$

Now, for arbitrary $\varpi \geq y$ and for $t \in [0, 1]$, we obtain

$$\begin{aligned} |f(t,\varpi) - f(t, y)| &= \left|\frac{|\varpi|}{2e^t(1+|\varpi|)} - \frac{|y|}{2e^t(1+|y|)}\right| \\ &= \frac{1}{2e^t}\left|\frac{|\varpi|}{1+|\varpi|} - \frac{|y|}{1+|y|}\right| \\ &\leq \frac{1}{2}|\varpi - y| = \eta(\varpi, y)|\varpi - y|. \end{aligned}$$

Also, we have

$$\begin{aligned} \int_0^1 k(t,\nu)\eta(\varpi, y)\, d\nu &= \frac{1}{2}\int_0^1 \frac{\nu^3}{1+t}\, d\nu \\ &= \frac{1}{8(1+t)} \quad \text{for all } t \in [0, 1] \\ &\leq \frac{1}{2}(\frac{1}{4}) = \frac{1}{2}\omega(d(\varpi, y)). \end{aligned}$$

Consequently, all the conditions of Theorem 3.1 are satisfied, and hence the integral equation (3.2) *has a unique solution in* $C[0, 1]$.

4. Conclusions

In this paper, we introduced the concept of generalized α-ψ-Geraghty contraction type for both single-valued and multi-valued mappings in strong b-metric spaces. Our approach extends and unifies earlier results by Karapınar and Afshari et al., providing new fixed point theorems that hold both globally and locally. Through examples and comparisons, we demonstrated that our results remain applicable in situations where several existing theorems, including well-known Geraghty-type results, fail to apply. In particular, Example 2.2 demonstrates a case where (X, d) is not a metric space; hence, the classical Geraghty theorem in metric spaces does not apply, while our result still guarantees the existence of a fixed point. The application to nonlinear integral equations further illustrates the practical value of our approach in modeling and solving real-world problems.

Author contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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