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Research article

The chromatic numbers of prime graphs of polynomials and power series over rings

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Abstract: A prime graph of a ring R, denoted by $PG^*(R)$, is a graph whose vertex set is the set of the strong zero divisors $S(R)$ of R, and its edge set is either $E(PG^*(R)) = \{(x, y) : xRy = 0$ or $yRx = 0, x \neq y$ and $x, y \in S(R)\}$. This graph is a subgraph of the prime graph $PG(R)$. In this paper, we investigate the chromatic numbers of the prime graphs of Artinian rings that satisfy certain conditions. In particular, if R is an Artinian ring with a unique prime ideal, then we prove that $\chi(PG(R)) \leq n + 1$, where n is the order of the prime ideal. Moreover, we explore the chromatic number of the prime graph of $M_2(\mathbb{Z}_n)$.

Keywords: prime graphs; polynomial rings; power series rings; chromatic numbers
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1. Introduction

In algebraic graph theory, the association of graphs with algebraic structures is an active area of research that continues to attract significant interest. This field explores various graph invariants using abstract algebraic methods, while also analyzing algebraic properties through graph-theoretic techniques. In 1988, Beck [1] pioneered the study of graphs associated with commutative rings by introducing and investigating the graph of a commutative ring R, making him the first to establish a connection between graphs and rings. This concept gained popularity when Anderson and Livingston [2] later renamed it the zero-divisor graph. Many studies focused on graph invariants, including those of the zero-divisor graphs [3–5]. Moreover, several studies examined invariants of polynomial rings and power series rings [6,7]. In 2010, a new development by Satyanarayana et al. [8] was introduced, where the prime graph $PG(R)$ was associated with a commutative ring R. Kalita and Patra [9] were the first to investigate the prime graphs of non-commutative rings. Furthermore, Pawar and Joshi [10] studied some invariants of the prime graph of $\mathbb{Z}_n$, such as the girth. In 2023, F. Maulidya et al. explored prime graphs of the Cartesian product of the rings $\mathbb{Z}_m \times \mathbb{Z}_n$ and their complements [11].

Additionally, Kalita and Patra also examined the relationship between the chromatic number of the prime graph of a finite product of commutative rings and that of the prime graphs of individual rings [12]. Moreover, the chromatic number of the prime graph of $\mathbb{Z}_n[x]$ was studied in [13]. Despite these developments, research on prime graphs over rings remains relatively limited. Therefore, we aim to extend this study by considering prime graphs in the context of polynomial and power series rings. Specifically, this paper focuses on determining the chromatic numbers of the prime graphs of R, $R[x]$, and $R[[x]]$ over Artinian rings, and provide some examples.

In Section 1, we provide definitions and recall results related to the prime graphs $PG(R)$ and $PG^*(R)$. Then, in Section 2, we study the chromatic number of the prime graphs of $R[x]$ and $R[[x]]$ over commutative rings. Finally, in the last section, we investigate the chromatic number of the prime graphs of R, $R[x]$ and $R[[x]]$ over Artinian rings that satisfy special cases. Moreover, we investigate the chromatic numbers of $PG(M_2(\mathbb{Z}_n))$.

2. Preliminaries

In this study, let R be a ring with identity. We denote the polynomial ring with one indeterminate by $R[x]$, and the power series with one indeterminate by $R[[x]]$. An element $a \in R$ is called a strong zero-divisor in R if either $\langle a\rangle\langle b\rangle = 0$ or $\langle b\rangle\langle a\rangle = 0$ for some nonzero element $b \in R$ [14]. We denote $S_l(R)$, $S_r(R)$, and $S(R)$ for the set of all left strong zero-divisors, right strong zero-divisors, and strong zero-divisors of R, respectively. Note that $S(R) \subseteq Z(R)$, where $Z(R)$ is the set of zero-divisors in R. If R is commutative, then $S(R) = Z(R)$. An ideal P in a ring R is said to be *prime* if $P \neq R$ and for any ideals A and B in R such that either $AB \subset P$, $A \subset P$, or $B \subset P$. An ideal M of a ring R is called a *maximal* ideal if $M \neq R$ and for every ideal N such that $M \subset N \subset R$, either $N = M$ or $N = R$. Let X be either an element or a subset of a ring R. The right annihilator of X in R is the set of all elements $r \in R$ such that $Xr = 0$. Similarly, the left annihilator of X in R is the set of all elements $r \in R$ such that $rX = 0$. A ring R is left [resp. right] *Noetherian* if R satisfies the ascending chain condition on left [resp. right] ideals. R is said to be *Noetherian* if it is both left and right *Noetherian*. A ring R is left [resp. right] Artinian if R satisfies the descending chain condition on left [resp. right] ideals. R is said to be Artinian if it is both left and right Artinian.

In [14], Behboodi et al. proved the following results:

Proposition 2.1. *Let R be a left Artinian ring with identity. Then, $S(R)$ is a union of maximal ideals.*

Theorem 2.2. *Let R be a Noetherian ring with $S(R) \neq R$. Then, there exist prime ideals $P_1, P_2, ..., P_n$ of R such that each P_i $(1 \leq i \leq n)$ is a left (resp. right) annihilator of a cyclic ideal and $S_l(R) = \bigcup_{i=1}^{n} P_i$ (resp. $S_r(R) = \bigcup_{i=1}^{n} P_i$). Consequently, $S(R) = \bigcup_{i=1}^{n} P'_i$ for some $n \in \mathbb{N}$ and prime ideals P'_i of R.*

A *graph* $G = (V, E)$ is a nonempty set V of objects called vertices together with a set E of unordered pairs of distinct vertices of G called edges. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. If the edge $e = \{u, v\}$, then the vertices, u and v, are called the *end vertices* or *ends* of the edge e. Each edge is said to join its ends; in this case, we say that e is *incident* with each one of its ends. Two vertices u and v are *adjacent* to each other in G if and only if there is an edge of G with u and v as its ends; otherwise, they are *nonadjacent*. If there exists a path between any two distinct vertices, then the graph is *connected*, and a graph is *complete* if every vertex is connected to every other vertex. A *path* in a graph G is a walk in which all vertices and edges are distinct. The *distance* between two distinct vertices a and b in a graph G, denoted by $d(a, b)$, is equal to the length of the shortest path if a path exists between them; otherwise, $d(a, b) = \infty$. The *diameter* of the graph G is defined as $max\{d(u, v) : u$ and v are vertices of G such that $u \neq v\}$, which is denoted by $diam(G)$. A *clique* of G is a maximal complete subgraph of G, which is denoted by $cl(G)$. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors needed to color the vertices of G such that no two adjacent vertices share the same color. In this paper all concepts in graph theory are taken from [15].

A prime graph of a ring R, denoted by $PG(R)$, is defined as a graph whose vertex set is the whole ring R; for two distinct $x, y \in R$, there is an edge connecting x and y if and only if either $xRy = 0$ or $yRx = 0$. A prime graph of a ring R, denoted by $PG^*(R)$, is a graph whose vertex set is the set of the strong zero divisors $S(R)$ of R, and its edge set is either $E(PG(R)) = \{(x, y) : xRy = 0$ or $yRx = 0, x \neq y$, and $x, y \in S(R)\}$ [16]. This graph is a subgraph of the prime graph $PG(R)$. Moreover, Satyanarayana et al. [8] proved that any ring R, not necessarily commutative, is a prime ring if and only if $PG(R)$ is a star graph. Additionally, in [17], Satyanarayana et al. explored the relationship between the prime and the zero-divisor graphs. They proved that the prime graph $PG(R)$ is a subgraph of the zero divisor graph $\Gamma(R)$ an associative ring R. $PG^*(R)$ is actually the induced subgraph of $\Gamma(R)$ generated by $S(R)$, which is usually denoted as $\langle S(R)\rangle \subseteq \Gamma(R)$. In particular, if R is commutative, then $PG(R) \cong \Gamma(R)$. However, they also demonstrated that the converse of this theorem is false (see Example 4.1 in [17]).

Note that, if $S(R) = R$, then the prime graph $PG^*(R)$ coincides with the prime graph $PG(R)$.

The following propositions in [16] are used to prove the Propositions 4.2 and 4.3.

Proposition 2.3. *Let R be an Artinian ring with identity such that $diam(PG^*(R)) = 2$ and $S(R) = I_1 \cup I_2$, where I_1 and I_2 are distinct maximal prime ideals in $S(R)$. Then, $I_1 \cap I_2 = 0$ (in particular, for all $a_1 \in I_1$ and $a_2 \in I_2$, $a_1Ra_2 = \{0\}$).*

Proposition 2.4. *Let R be an Artinian ring with identity such that $diam(PG^*(R)) = 2$. If $S(R) = I_1 \cup I_2$ is the union of precisely two maximal prime ideals in $S(R)$, then $diam(PG^*(R[x])) = 2$.*

3. Chromatic number of prime graphs of $R[x]$ and $R[[x]]$ over commutative rings

In this section, let R be a commutative ring. Then, $PG(R) \cong \Gamma(R)$ when $0 \in Z(R)$. In [1], the chromatic number of the zero-divisor graph of certain commutative rings R was determined. Consequently, every result established about the chromatic number of the zero-divisor graph can be applied to the prime graph of some commutative rings. Furthermore, in [7], M. J. Park et al. determined the chromatic number of the zero-divisor graph of polynomials over $\mathbb{Z}_n$. Thus, in every result from [7], the zero-divisor graph can be replaced with the prime graph.

In the following results, we replace the zero-divisor graphs in [1] by the prime graphs.

Proposition 3.1. $\chi(PG(R)) = 2$ *if and only if R is an integral domain, $R \cong \mathbb{Z}_4$, or $R \cong \mathbb{Z}_2[x]/\langle x^2\rangle$.*

Corollary 3.2. $\chi(PG(R[x])) = 2$ *if and only if R is an integral domain.*

Note that the zero element is not considered as a vertex in the zero-divisor graph of $\mathbb{Z}_n[x]$ as it was stated in [7]. Therefore, the chromatic number of $PG^*(\mathbb{Z}_n[x])$ is greater than the chromatic number of the zero-divisor graph of $\mathbb{Z}_n[x]$ by 1. Accordingly, Propositions 3.3 and 3.4 can be rewritten as follows:

Proposition 3.3. *If $r \geq 2$ is an integer and $n = p_1 \ldots p_r$, where $p_1, \ldots, p_r$ are distinct primes, then $\chi(PG^*((\mathbb{Z}_n[X])) = r + 1$.*

Proposition 3.4. *If n is a multiple of the square of a prime, then $\chi(PG^*((\mathbb{Z}_n[X])) = \infty$.*

The proof of the above proposition is similar to the proof of Proposition 3.4 in [7]. In the following proposition, we now consider the remaining case, where n is a multiple of non-square of a prime.

Proposition 3.5. *If $R = \mathbb{Z}_n[x]$, with $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, where $p_1, \ldots, p_r$ are prime numbers and $\alpha_1, \ldots, \alpha_r$ are odd integers, then $\chi(PG^*((\mathbb{Z}_n[X])) = \infty$.*

Proof. Suppose that $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, where $\alpha_1, \ldots, \alpha_r$ are odd integers and $r \geq 1$.
Case(1): $\alpha_j > 1$ and $\alpha_i = 1$ when $i \neq j$ and $1 \leq i, j \leq r$. Let $k = p_1^{\alpha_1 - 1} p_2 \ldots p_r$ and let $C = \{kx^m : m \in \mathbb{N}\}$. Then, the product of any two elements of C is zero in $\mathbb{Z}_n[x]$ (i.e., if $p_1^{\alpha_j - 1} p_2 \ldots p_r x^{m_1}$ and $p_1^{\alpha_j - 1} p_2 \ldots p_r x^{m_2}$ in C, then $p_1^{2\alpha_1 - 2} p_2^2 \ldots p_r^2 x^{m_1 + m_2} = (p_1^{\alpha_1} p_2 \ldots p_r) p_1^{\alpha_1 - 2} p_2 \ldots p_r x^{m_1 + m_2} = n(p_1^{\alpha_1 - 2} p_2 \ldots p_r x^{m_1 + m_2}) = 0(p_1^{\alpha_1 - 2} p_2 \ldots p_r x^{m_1 + m_2}) = 0)$. Therefore, C is a maximal clique in $PG(\mathbb{Z}_n[x]))$ and the order of C is ∞. Thus, $cl(PG(\mathbb{Z}_n[x])) = \infty$. Since R is a commutative ring, $\chi(PG(\mathbb{Z}_n[x])) = cl(PG(\mathbb{Z}_n[x])) = \infty$.
Case(2): $\alpha_1, \ldots, \alpha_r > 1$. Let $k = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \ldots p_r^{\alpha_r - 1}$ and let $C = \{kx^m : m \in \mathbb{N}\}$. Then, the product of any two elements of C is zero in $\mathbb{Z}_n[x]$ (i.e., if $p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \ldots p_r^{\alpha_r - 1} x^{m_1}$ and $p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \ldots p_r^{\alpha_r - 1} x^{m_2}$ in C, then $p_1^{2\alpha_1 - 2} p_2^{2\alpha_2 - 2} \ldots p_r^{2\alpha_r - 2} x^{m_1 + m_2} = (p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}) p_1^{\alpha_1 - 2} p_2^{\alpha_2 - 2} \ldots p_r^{\alpha_r - 2} x^{m_1 + m_2} = n(p_1^{\alpha_1 - 2} p_2^{\alpha_2 - 2} \ldots p_r^{\alpha_r - 2} x^{m_1 + m_2}) = 0(p_1^{\alpha_1 - 2} p_2^{\alpha_2 - 2} \ldots p_r^{\alpha_r - 2} x^{m_1 + m_2}) = 0)$. Therefore, C is a maximal clique in $PG(\mathbb{Z}_n[x])$ and the order of C is ∞. Thus, $cl(PG(\mathbb{Z}_n[x])) = \infty$. Since R is a commutative ring, $\chi(PG(\mathbb{Z}_n[x])) = cl(PG(\mathbb{Z}_n[x])) = \infty$. □

Note that, $\chi(PG(R))$ is the same as $\chi(PG^*(R))$, since all elements in $R \backslash S(R)$ are just adjacent to the zero element. Therefore, these elements can be assigned to any color used in $PG^*(R)$, except the color assigned to the zero element.

4. Chromatic number of prime graphs of R, $R[x]$ and $R[[x]]$

In this section, let R be a finite non-commutative ring, unless otherwise stated. We establish some conditions on the rings R, $R[x]$, and $R[[x]]$ in the cases where the prime graphs of these rings have a chromatic number equal to 1,2, or 3. Furthermore, we divide this section into four subsections to discuss the chromatic number of prime graphs for certain non-commutative Artinian rings with some conditions. Finally, we use MATLAB to ensure that the results of each example are true.

Our next remark is about the prime graph with a chromatic number equal to one.

Remark 4.1. $\chi(PG(R[x])) = 1$ *if and only if* $R[x]$ *is zero. Additionally,* $\chi(PG(R[[x]])) = 1$ *if and only if* $R[[x]]$ *is zero.*

Proof. Assume that $R[x]$ is a zero polynomial. Thus, $\chi(PG(R[x])) = 1$. Conversely, let $\chi(PG(R[x])) = 1$. Then, 0 is adjacent to each element in $R[x]$; thus, $R[x]$ is a zero polynomial. A similar argument can be applied to $R[[x]]$. □

Proposition 4.2. *Let R be an Artinian ring such that $diam(PG^*(R)) = 2$, and let $S(R) = I_1 \cup I_2$ be a union of distinct maximal prime ideals I_1 and I_2. Then, $\chi(PG^*(R)) = 3$.*

Proof. Suppose that $S(R) = I_1 \cup I_2$ and $diam(PG^*(R)) = 2$. Let a and b be two elements in $S(R)$ such that a is adjacent to b. Since $S(R) = I_1 \cup I_2$, using Proposition 2.3, $a \in I_1$ and $b \in I_2$. Therefore, a is colored by one color and b is colored by another color. Hence, every element in $I_1 \backslash \{0\}$ is colored by one color, and every element of $I_2 \backslash \{0\}$ is colored by another color. Since 0 is adjacent to all vertices, it must be colored by a different color from $I_1 \backslash \{0\}$ and $I_2 \backslash \{0\}$. Therefore, we can conclude that $\chi(PG^*(R)) = 3$. □

Proposition 4.3. *Let R be an Artinian ring such that $diam(PG^*(R)) = 2$, and let $S(R) = I_1 \cup I_2$ be a union of distinct maximal prime ideals I_1 and I_2. Then, $\chi(PG^*(R[x])) = 3$.*

Proof. Using Proposition 2.4, $diam(PG^*(R[x])) = 2$. Let $f(x)$ and $g(x)$ be two elements of $S(R[x])$ such that $f(x)$ adjacent to $g(x)$. Using a similar technique to the proof of Proposition 2.4, $f(x) \in I_1[x]$ and $g(x) \in I_2[x]$. Therefore, $f(x)$ is colored by one color and $g(x)$ is colored by another color. Hence, every element in $I_1[x] \backslash \{0\}$ is colored by one color, and every element of $I_2[x] \backslash \{0\}$ is colored by another color. Since 0 is adjacent to all vertices, it is must colored by a different color from $I_1[x] \backslash \{0\}$ and $I_2[x] \backslash \{0\}$. Therefore, we can conclude that $\chi(PG^*(R[x])) = 3$. □

Proposition 4.4. *Let R be an Artinian ring such that $diam(PG^*(R)) = 2$, and let $S(R) = I_1 \cup I_2$ be a union of distinct maximal prime ideals I_1 and I_2. Then, $\chi(PG^*(R[[x]])) = 3$.*

Proof. This proof is similar to the proof of Proposition 4.3. □

Note that Propositions 4.2, 4.3, and 4.4 hold for $PG(R)$, $PG(R[x])$, and $PG(R[[x]])$. Indeed, as established earlier, all elements in $R \backslash S(R)$ are just adjacent to the zero element. Consequently, these elements can be assigned to any color used in $PG^*(R)$, except the color assigned to the zero element.

4.1. Chromatic number of prime graphs of simple rings

In this section, R will be a simple ring. To determine the chromatic number of the prime graph of a simple ring as in Proposition 4.6, we need to show that every simple ring is a prime ring. In [11], Nepal proved that every simple ring is a primitive ring (Lemma 3.2, [11]). Additionally, he proved that every primitive ring is a prime ring (Lemma 3.5, [11]). Therefore, every simple ring is indeed a prime ring. In the following lemma, we will prove it in an alternative way.

Lemma 4.5. *Every simple ring is a prime ring.*

Proof. Let R be a simple ring, and let $a, b \in R$ such that $aRb = 0$. Note that RaR is an ideal of R. Thus, either $RaR = R$ or $RaR = 0$, since R is a simple ring. If $RaR = 0$, then $Ra = 0$, since R is a ring with identity. Therefore, $a = 0$. Otherwise, if $RaR = R$, then $1_R aR = R$, which implies that $aRb = Rb$; however, $aRb = 0$, thus $Rb = 0$. Therefore, $b = 0$. Therefore, R is a prime ring. □

Proposition 4.6. *Let R be a finite simple ring. Then, $\chi(PG^*(R)) = 1$.*

Proof. Since R is a simple ring, it is a prime ring. Thus, $aRb = 0$, which implies that either $a = 0$ or $b = 0$. Therefore, $S(R) = \{0\}$. Hence, $\chi(PG^*(R)) = 1$. □

In this case, observe that the chromatic number of $PG(R)$ is not equal to the chromatic number of $PG^*(R)$. The following proposition proves this.

Proposition 4.7. *Let R be a finite simple ring. Then, $\chi(PG(R)) = 2$.*

Proof. Since R is a simple ring, it is a prime ring. Therefore, the prime graph of R is a star graph (Theorem 3.1, [8]). Hence, $\chi(PG(R)) = 2$. □

Corollary 4.8. *Let R be an $n \times n$ matrix over a division ring. Then, $PG(R)$ is a star graph, and its chromatic number is 2.*

Proof. Let R be $n \times n$ matrix over a division ring. Then, R is a simple ring, which implies that R is a prime ring. Hence, the prime graph of R is a star graph and $\chi(PG(R)) = 2$. □

Now, based on the above proposition, we provide the following example.

Example 4.9. *Let $R = M_{2\times2}(\mathbb{Z}_2)$. Then, the chromatic number of $PG(R)$ is equal to 2. Since $M_{2\times2}(\mathbb{Z}_2)$ is a simple ring, by Proposition 4.7, $PG(M_{2\times2}(\mathbb{Z}_2))$ is a star graph. Hence, $\chi(PG(M_{2\times2}(\mathbb{Z}_2))) = 2$.*

Now, we draw the prime graph of $M_{2\times2}(\mathbb{Z}_2)$, which confirms the validity of this result. In Figure 1, vertex 1 represents the zero matrix, which is connected to all other vertices. The other vertices represent the elements of $M_{2\times2}(\mathbb{Z}_2)$.

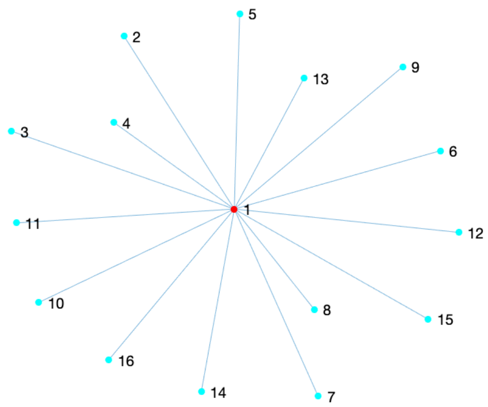

Figure 1. $\chi(PG(M_2(\mathbb{Z}_2))) = 2$.

After determining the chromatic number of the prime graph of R, we aim to determine the chromatic numbers of the prime graphs of $R[x]$ and $R[[x]]$ in some specific cases. Recall that, every division ring is a simple ring. Using this, we can prove the following propositions.

Lemma 4.10. *Let R be a division ring. Then, $R[x]$ and $R[[x]]$ are prime rings.*

Proof. Since R is a division ring, every element in R is a unit. Therefore, $R[x]$ has no zero divisors. Thus, if $f(x), g(x) \in R[x]$, and $f(x)R[x]g(x) = 0$, then either $f(x) = 0$ or $g(x) = 0$. Thus, $R[x]$ is a prime ring. A similar argument can be applied to $R[[x]]$. □

Proposition 4.11. *Let R be a division ring. Then, $\chi(PG^*(R[x])) = \chi(PG^*(R[[x]])) = 1$.*

Proof. Since R is a division ring, $R[x]$ and $R[[x]]$ are prime rings. Thus, $PG(R[x])$ and $PG(R[[x]])$ have only the zero element as a vertex. Hence, $\chi(PG(R[x])) = \chi(PG(R[[x]])) = 1$. □

Proposition 4.12. *Let R be a division ring. Then, $\chi(PG(R[x])) = \chi(PG(R[[x]])) = 2$.*

Proof. Since R is a division ring, $R[x]$ and $R[[x]]$ are prime rings. Thus, $PG(R[x])$ and $PG(R[[x]])$ are star graphs (Theorem 3.1, [8]). Hence, $\chi(PG(R[x])) = \chi(PG(R[[x]])) = 2$. □

4.2. Chromatic number of prime graphs of rings with one prime ideal

Lemma 4.13. *Let R be an Artinian ring with a unique prime ideal I. For every $a \in R$ and $aRb = 0$, $b \in I$.*

Proof. Since R is an Artinian ring with identity, I is a maximal ideal. Now, consider R/I. Since I is a maximal ideal, R/I is a division ring.

Suppose that, there exists $b \in R$ such that $aRb = 0$ for every $a \in R$, where both a and b non-zero elements. Since R has an identity, $ab = 0$. Then, $a + I, b + I \in R/I$ are units since R/I is a division ring. This implies that $(a + I)(b + I) = ab + I = I$ since $ab = 0$. However, I is prime, and either $a + I \subseteq I$ or $b + I \subseteq I$. If $b + I \subseteq I$, then $b \in I$; otherwise, $a + I \subseteq I \Rightarrow a \in R$, which is a contradiction. □

Proposition 4.14. *Let R be an Artinian ring with only a unique prime ideal I. Then, $\chi(PG^*(R)) \leq n+1$, where n is the order of I.*

Proof. By Lemma 4.13, every element in $R\backslash I$ can be colored with a single color, as there are no edges between them. Moreover, the elements of I may annihilate each other; thus, the color of elements of I will be maximum equal to the order of I. Therefore, $\chi(PG^*(R)) \leq n + 1$. □

If we consider that the zero element is adjacent to all elements outside the ideal, the chromatic number of $PG(R)$ still satisfies the same condition in the above proposition.

Corollary 4.15. *Let R be an Artinian ring with only a unique prime ideal I. Then, $\chi(PG(R)) \leq n + 1$, where n is the order of I.*

Proposition 4.16. *Let $R = M_2(\mathbb{Z}_p) = \left\{\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{0} \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_p, \text{ where } p \text{ is a prime number} \right\}$. Then, $\chi(PG(R)) = p + 1$.*

Proof. Let $R = M_2(\mathbb{Z}_p) = \left\{\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{0} \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_p\right\}$. Then, we have only one prime ideal of the form $I = \left\{\begin{pmatrix} \overline{0} & \overline{a} \\ \overline{0} & \overline{0} \end{pmatrix} : a \in \mathbb{Z}_p\right\}$. Since every element of I annihilates all elements in R, they must be colored with p distinct colors, as the order of I is p. Additionally, every element in $R\backslash I$ can be colored with a single color, as there are no edges between them. Thus, $\chi(PG(R)) = p + 1$. □

Example 4.17. *Consider the following ring:*

$$R = M_2(\mathbb{Z}_2) = \left\{\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{0} \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_2\right\}.$$

Now, we determine the chromatic number of $PG(R)$. First, we write all elements of R as follows:

$$R = \left\{ A_1 = \begin{pmatrix} \overline{0} & \overline{0} \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} \overline{1} & \overline{0} \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} \overline{0} & \overline{1} \\ 0 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} \overline{1} & \overline{1} \\ 0 & 0 \end{pmatrix} \right\}.$$

Then, we find the vertex set such that the verification is either $A_iRA_j = 0$ or $A_jRA_i = 0$, where $1 \leq i$ and $j \leq 4$ holds. Thus, $S(R) = \{A_1, A_2, A_3, A_4\}$, which means that all elements in R are strong zero divisors. This implies that $PG(R) = PG^*(R)$. Note that, there is only one prime ideal in R, that is, $I = \left\{ \begin{pmatrix} \overline{0} & \overline{a} \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Z}_2 \right\}$. Clearly, every element in I annihilates all elements of R. Therefore, the edge set $E(PG(R)) = \{A_1A_2, A_1A_3, A_1A_4, A_2A_3, A_3A_4\}$.

Since the vertices A_1 and A_3 annihilate all elements in R, they must be colored with two distinct colors. Next, we consider A_2 and A_4: since there is no edge between them, they can be colored with the same color, which is different from the colors of A_1 and A_3. Hence, $\chi(PG(M_2(\mathbb{Z}_2))) = 3$. Figure 2 shows the prime graph of R. Note that the vertex A_1 is renamed as 1, A_2 as 2, and so on.

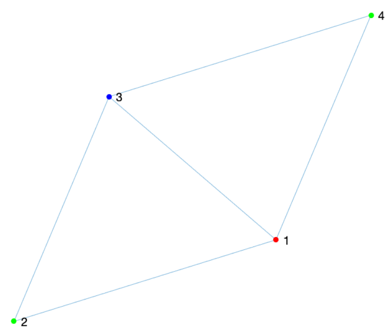

Figure 2. $\chi(PG(M_2(\mathbb{Z}_2))) = 3$.

Example 4.18. *Consider the following ring:*

$$R = M_2(\mathbb{Z}_7) = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ 0 & 0 \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_7 \right\}.$$

Now, we determine the chromatic number of $PG(R)$. Using MATLAB, we obtain Figure 3.

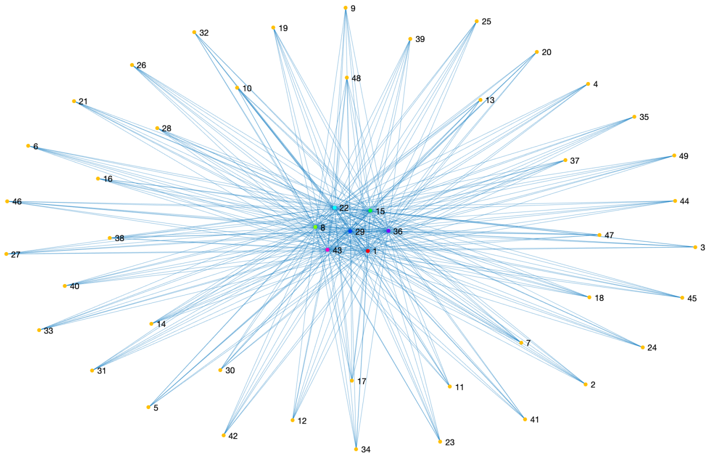

Figure 3. $\chi(PG(M_2(\mathbb{Z}_7))) = 8$.

We note that the vertices at the center of the prime graph only represent the prime ideal. Note that the center of the graph is actually the complete graph K_3 with verices from $I = \begin{pmatrix} \overline{0} & \mathbb{Z}_7 \\ \overline{0} & \overline{0} \end{pmatrix}$. *Each vertex of the prime ideal was assigned to a different color. However, the elements outside the ideal, which were not adjacent to each other but adjacent to the elements of the prime ideal, were assigned to the same color. This result is in agreement with a previously proven theorem.*

From the above figure, note that the chromatic number of $PG(R)$ is 8. Addtionally, by using Proposition 4.16, $\chi(PG(R)) = 7 + 1 = 8$.

It is well known that, if I is an ideal in R, then $I[x]$ is an ideal of $R[x]$ (see Ch.16 Ex.38 P.303 [18]). In the following lemma, we prove that I is a prime ideal of R if and only if $I[x]$ is a prime ideal of $R[x]$, where R is a commutative ring.

We know that if R is a commutative ring and I is a prime ideal of R, then $I[x]$ is a prime ideal of $R[x]$. Conversely, let $I[x]$ be a prime ideal of $R[x]$. We want to prove that I is a prime ideal of R. Assume that $a, b \in I$ such that $ab \in I$. Therefore, $abx \in I[x]$. Let $f(x) = a$ and $g(x) = bx$; thus, $f(x)g(x) \in I[x]$. Since $I[x]$ is a prime ideal, either $a \in I[x]$ or $bx \in I[x]$, which implies that either $a \in I$ or $b \in I$, respectively. Thus, I is a prime ideal.

Proposition 4.19. *Let R be a finite commutative ring with a unique prime ideal I. Then, $\chi(PG^*(R[x])) \leq n + 1$, where n is the order of $I[x]$.*

Proof. Since R is a finite commutative ring, $R[x]$ is an Artinian ring. Additionally, since I is a prime ideal of R, $I[x]$ is a prime ideal of $R[x]$. Now, by Lemma 4.13, every element in $R[x] \backslash I[x]$ can be colored with a single color, as there are no edges between them. Since the elements of $I[x]$ may annihilate each other, the number of colors assigned to the elements of $I[x]$ will be at most equal to the order of $I[x]$. Thus, $\chi(PG^*(R[x])) \leq n + 1$. □

If we consider that the zero element is adjacent to all elements outside the ideal, the chromatic number of $PG(R[x])$ still satisfies the same condition in the above proposition.

4.3. Chromatic number of prime graphs of rings with at least two prime ideals

Proposition 4.20. *Let R be an Artinian ring and $S(R) = I_1 \cup I_2$, where I_1 and I_2 are two prime ideals. Then, $\chi(PG^*(R)) = 2 + k$, where k is the order of the intersection between I_1 and I_2.*

Proof. Suppose $S(R) = I_1 \cup I_2$. Then, we divide the elements of $S(R)$ into three sets, denoted by A, B, and C. The set A is the set of all elements in $I_1 \cap I_2$, the set B is the set of all elements in $I_1 \backslash (I_1 \cap I_2)$, and the set C is the set of all elements in $I_2 \backslash (I_1 \cap I_2)$. Since the elements in $I_1 \backslash (I_1 \cap I_2)$ are not adjacent, by Proposition 3.11 in [16], B is colored by one color. Similarly, C is colored by one color different from B, as each element in B is adjacent to every element in C. Now, let $a \in A$. Since a belongs to I_1, a is adjacent to all element in C. Then, a must be colored by a different color from C. On the other hand, a is also in I_2; therefore, it must be colored by a different color from B. If there is another element in A, say b, such that $b \in I_2$, then as $a \in I_1$, it will be adjacent to b. Therefore, b is colored by a different color from a. Thus, every element in A is colored by different colors. Hence, $\chi(PG^*(R)) = 2 + k$, where k is the order of the intersection between I_1 and I_2. □

Corollary 4.21. *Let R be an Artinian ring and $S(R) = \bigcup_{j=1}^{n} I_j$ for some natural number $n \geq 2$ and I_j are the prime ideals for all j. Then, $\chi(PG^*(R)) = n + k$, where k is the order of the intersection $\bigcap_{j=1}^{n} I_j$.*

Proposition 4.22. *Let $R[x]$ be a finite commutative ring and $S(R[x]) = \bigcup_{j=1}^{n} I_j[x]$, where $I_j[x]$ is the prime ideal for all j. Then, $\chi(PG^*(R[x])) = n + k$, where k is the order of the intersection $\bigcap_{j=1}^{n} I_j[x]$.*

Proof. Since $R[x]$ is a finite commutative ring, it follows that $R[x]$ is an Artinian ring. Then, the proof is similar to that of Proposition 4.20. □

In the following example, we applied Proposition 4.20 to determine the chromatic number.

Example 4.23. *Consider the following ring:*

$$R = M_2(\mathbb{Z}_3) = \{\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{c} \end{pmatrix} : \overline{a}, \overline{b}, \overline{c} \in \mathbb{Z}_3\}.$$

Note that R has 27 elements. First, we find the vertex set. Since the ring consists of upper triangular matrices, we have $S(R) = Z(R) = \left\{\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{c} \end{pmatrix} : \overline{a}, \overline{b}, \overline{c} \in \mathbb{Z}_3 \text{ and } ac = 0\right\}$ (see Example 3.1 [14]).

Thus, $S(R) = \left\{A_1 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix}, A_2 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}, A_3 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{2} \end{pmatrix}, A_4 = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{0} \end{pmatrix}, A_5 = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{1} \end{pmatrix}, A_6 = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{2} \end{pmatrix},\right.$ $A_7 = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{0} & \overline{0} \end{pmatrix}, A_8 = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{0} & \overline{1} \end{pmatrix}, A_9 = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{0} & \overline{2} \end{pmatrix}, A_{10} = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix}, A_{11} = \begin{pmatrix} \overline{1} & \overline{1} \\ \overline{0} & \overline{0} \end{pmatrix}, A_{12} = \begin{pmatrix} \overline{1} & \overline{2} \\ \overline{0} & \overline{0} \end{pmatrix}, A_{13} = \begin{pmatrix} \overline{2} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix},$ $\left. A_{14} = \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{0} & \overline{0} \end{pmatrix}, A_{15} = \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{0} & \overline{0} \end{pmatrix}\right\}$. Observe that this ring has two prime ideals:

$P_1 = \left\{\begin{pmatrix} \overline{0} & \overline{b} \\ \overline{0} & \overline{c} \end{pmatrix} : \overline{b}, \overline{c} \in \mathbb{Z}_3\right\} = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9\}$.

$P_2 = \left\{\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{0} \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_3\right\} = \{A_1, A_4, A_7, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}\}$.

If we examine closely, we see that the vertex set $S(R)$ is the union of these prime ideals. Additionally, there is an intersection between them, $P_1 \cap P_2 = \{A_1, A_4, A_7\}$. Therefore, every element in this intersection annihilates all elements in $S(R)$; therefore, they must be colored with three distinct colors.

Now, observe that each element in $P_1 \backslash (P_1 \cap P_2)$ does not annihilate any other element, which allows us to color $P_1 \backslash (P_1 \cap P_2)$ with a single color, distinct from the colors used for the intersection. Similarly, each element in $P_2 \backslash (P_1 \cap P_2)$ does not annihilate each others; therefore, we can color $P_2 \backslash (P_1 \cap P_2)$ with a single color, different from those used for the intersection and $P_1 \backslash (P_1 \cap P_2)$. Thus, the chromatic number is 5*.*

Finally, in Figure 4, we verify this result using MATLAB and draw the prime graph of R, noting that the vertex A_1 is renamed as 1*, A_2 as* 2*, and so on.*

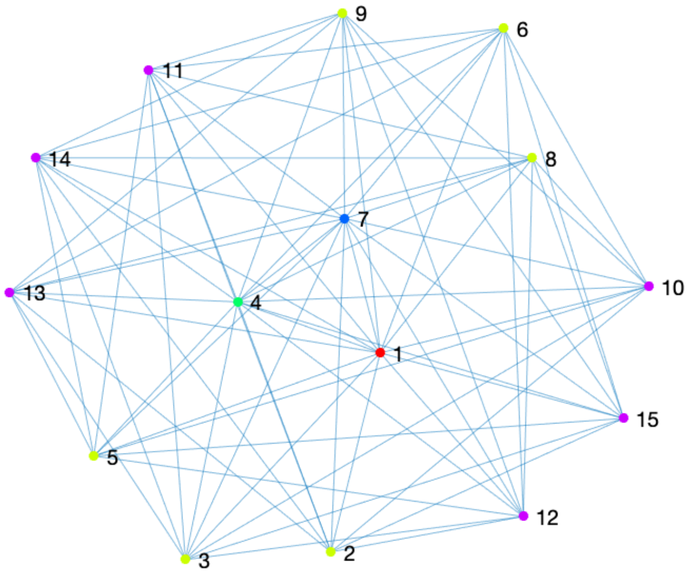

Figure 4. $\chi(PG^*(M_2(\mathbb{Z}_3))) = 5$.

Note that the chromatic number of $PG(R)$ in Propositions 4.21 and 4.23 is the same as that of $PG^*(R)$, since the all elements in $R \backslash S(R)$ are adjacent to the zero element. Therefore, these elements can be assigned to any color used in $PG^*(R)$, except the color assigned to the zero element.

4.4. Examples of the chromatic number in prime graphs of rings without prime ideals

When we have prime ideals, we could easily investigate the chromatic numbers of prime graphs as in the previous sections. In the next example, there are proper ideals, but there is no prime ideal. Moreover, we cannot find a rule governing the number of colors in this graph.

Example 4.24. *Consider the following ring:*

$$R = M_2(\mathbb{Z}_4) = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ 0 & 0 \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_4 \right\}.$$

Now, we determine the chromatic number of the prime graph of R.

Proof. First, we write all elements of R as follows:

$$R=\left\{A_1=\begin{pmatrix}\overline{0}&\overline{0}\\ 0&0\end{pmatrix}, A_2=\begin{pmatrix}\overline{0}&\overline{1}\\ 0&0\end{pmatrix}, A_3=\begin{pmatrix}\overline{0}&\overline{2}\\ 0&0\end{pmatrix}, A_4=\begin{pmatrix}\overline{0}&\overline{3}\\ 0&0\end{pmatrix}, A_5=\begin{pmatrix}\overline{1}&\overline{0}\\ 0&0\end{pmatrix}, A_6=\begin{pmatrix}\overline{1}&\overline{1}\\ 0&0\end{pmatrix}, A_7=\begin{pmatrix}\overline{1}&\overline{2}\\ 0&0\end{pmatrix},\right.$$
$$A_8=\begin{pmatrix}\overline{1}&\overline{3}\\ 0&0\end{pmatrix}, A_9=\begin{pmatrix}\overline{2}&\overline{0}\\ 0&0\end{pmatrix}, A_{10}=\begin{pmatrix}\overline{2}&\overline{1}\\ 0&0\end{pmatrix}, A_{11}=\begin{pmatrix}\overline{2}&\overline{2}\\ 0&0\end{pmatrix}, A_{12}=\begin{pmatrix}\overline{2}&\overline{3}\\ 0&0\end{pmatrix}, A_{13}=\begin{pmatrix}\overline{3}&\overline{0}\\ 0&0\end{pmatrix}, A_{14}=\begin{pmatrix}\overline{3}&\overline{1}\\ 0&0\end{pmatrix},$$
$$\left. A_{15}=\begin{pmatrix}\overline{3}&\overline{2}\\ 0&0\end{pmatrix} A_{16}=\begin{pmatrix}\overline{3}&\overline{3}\\ 0&0\end{pmatrix}\right\}.$$

Next, we find the vertex set such that the condition is either $A_iRA_j = 0$ or $A_jRA_i = 0$, where $1 \leq i, j \leq 16$ holds. Thus, $S(R) = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}\}$, which means that all elements in R are strong zero divisors. This implies that $PG(R) = PG^*(R)$. Note that there are two proper ideals:

$$I_1=\left\{\begin{pmatrix}\overline{0}&\overline{0}\\ 0&0\end{pmatrix},\begin{pmatrix}\overline{2}&\overline{0}\\ 0&0\end{pmatrix},\begin{pmatrix}\overline{0}&\overline{2}\\ 0&0\end{pmatrix},\begin{pmatrix}\overline{2}&\overline{2}\\ 0&0\end{pmatrix}\right\}.$$

$$I_2=\left\{\begin{pmatrix}\overline{0}&\overline{0}\\ 0&0\end{pmatrix},\begin{pmatrix}\overline{0}&\overline{1}\\ 0&0\end{pmatrix},\begin{pmatrix}\overline{0}&\overline{2}\\ 0&0\end{pmatrix},\begin{pmatrix}\overline{0}&\overline{3}\\ 0&0\end{pmatrix}\right\}.$$

Observe that I_1 is not a prime ideal, since $\begin{pmatrix}\overline{2}&\overline{1}\\ 0&0\end{pmatrix}\begin{pmatrix}\overline{2}&\overline{3}\\ 0&0\end{pmatrix}=\begin{pmatrix}\overline{0}&\overline{2}\\ 0&0\end{pmatrix}\in I_1$; however, neither $\begin{pmatrix}\overline{2}&\overline{1}\\ 0&0\end{pmatrix}$ nor $\begin{pmatrix}\overline{2}&\overline{3}\\ 0&0\end{pmatrix}$ belong to I_1. Similarly, I_2 is not a prime ideal, since $\begin{pmatrix}\overline{2}&\overline{0}\\ 0&0\end{pmatrix}\begin{pmatrix}\overline{2}&\overline{2}\\ 0&0\end{pmatrix}=\begin{pmatrix}\overline{0}&\overline{0}\\ 0&0\end{pmatrix}\in I_2$; however, neither $\begin{pmatrix}\overline{2}&\overline{0}\\ 0&0\end{pmatrix}$ nor $\begin{pmatrix}\overline{2}&\overline{2}\\ 0&0\end{pmatrix}$ belong to I_2.

Note that every element in I_2 annihilates all elements of R; thus, they must be colored with four distinct colors: red, yellow, green, and turquoise, since the order of I_2 is 4. Additionally, all remaining elements are colored with a single color, say blue, which is different from the colors of the elements in I_2. Until now, we have five colors for coloring.

Since 2 is a zero divisor in Z_4, we take all matrices where the a_{11} entry is 2. This gives us four matrices: A_9, A_{10}, A_{11}, and A_{12}. Note that, the vertex A_{11} is adjacent to A_9, A_{10}, and A_{12}; therefore, it is colored with another color, pink. Vertex A_9 is adjacent to A_{10}, A_{11}, and A_{12}; therefore, if we remain the vertex color A_9 by blue,then the vertices A_{12} and A_{10} are colored with the same color, say purple, as there is no edge between them. Thus, the chromatic number is 7.

Finally, in Figure 5, we draw the prime graph of R, noting that the vertex A_1 is renamed as 1, A_2 as 2, and so on. □

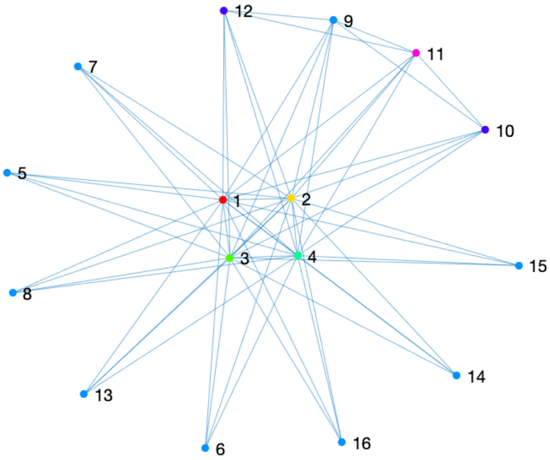

Figure 5. $\chi(PG(M_2(\mathbb{Z}_4))) = 7$.

Example 4.25. *Consider the following ring:*

$$R = M_2(\mathbb{Z}_6) = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ 0 & 0 \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_6 \right\}.$$

We determine the chromatic number of $PG(R)$.

Proof. In Figure 6, using MATLAB, we obtain the prime graph $PG(R)$ as follows:

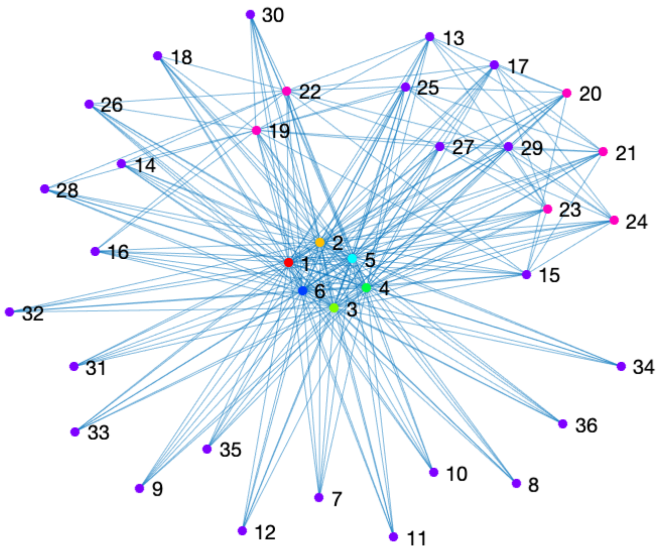

Figure 6. $\chi(PG(M_2(\mathbb{Z}_6))) = 8$.

Note that this ring has a proper ideal of the form $I = \left\{\begin{pmatrix}\overline{0} & \overline{b} \\ \overline{0} & \overline{0}\end{pmatrix} : \overline{b} \in \mathbb{Z}_6\right\}$, which is not a prime ideal. It is easy to see that every element in I annihilates every element in R. Therefore, each element in I must be colored with a unique color; thus, we require 6 colors. The center of the graph is exactly I. Additionally, all remaining elements are colored with a new color. This brings the total to 7 colors.

However, since $\mathbb{Z}_6$ contains zero divisors, there exist matrices outside the ideal that are adjacent to each other. In the figure above, one additional color is required because every matrix with $a_{11} = \overline{3}$ annihilates all matrices with $a_{11} = \overline{2}$ and all matrices with $a_{11} = \overline{4}$. Consequently, the vertices which have $a_{11} = \overline{3}$ will be colored with a new color, say pink. In Figure 6, the matrices with a pink color are $\left\{A_{19} = \begin{pmatrix}\overline{3} & \overline{0} \\ \overline{0} & \overline{0}\end{pmatrix}, A_{20} = \begin{pmatrix}\overline{3} & \overline{1} \\ \overline{0} & \overline{0}\end{pmatrix}, A_{21} = \begin{pmatrix}\overline{3} & \overline{2} \\ \overline{0} & \overline{0}\end{pmatrix}, A_{22} = \begin{pmatrix}\overline{3} & \overline{3} \\ \overline{0} & \overline{0}\end{pmatrix}, A_{23} = \begin{pmatrix}\overline{3} & \overline{4} \\ \overline{0} & \overline{0}\end{pmatrix}, A_{24} = \begin{pmatrix}\overline{3} & \overline{5} \\ \overline{0} & \overline{0}\end{pmatrix}\right\}$. Note that, the vertex A_{19} is renamed as 19, A_{20} as 20, and so on. Hence, $\chi(PG(M_2(\mathbb{Z}_6))) = 6 + 2 = 8$. □

In the above example, the chromatic number of $PG(R)$ is the same as that of $PG^*(R)$, because the vertex set is the whole ring R.

Corollary 4.26. *Let $R = M_2(\mathbb{Z}_n)$ denote the matrix ring consisting of 2×2 upper triangular matrices over $\mathbb{Z}_n$ in the following form:*

$$R = \left\{\begin{pmatrix}\overline{a} & \overline{b} \\ \overline{0} & \overline{0}\end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_n\right\}.$$

If n is a not prime number, then the chromatic number of the prime graph $PG(M_2(\mathbb{Z}_n))$ is greater than $n + 1$.

Proof. Since there exists a proper ideal of the form $I = \left\{\begin{pmatrix}\overline{0} & \overline{b} \\ \overline{0} & \overline{0}\end{pmatrix} : \overline{b} \in \mathbb{Z}_n\right\}$, which is not a prime ideal, it is easy to see that every element in I annihilates every element in R. This implies that $PG(R) = PG^*(R)$, since each element in R is a vertex. Therefore, each element in I must be colored with a unique color, and that requires n colors. Moreover, all remaining elements are at least colored with one color, which brings the total to $n + 1$ colors.

Since $\mathbb{Z}_n$ is not a field, it contains zero divisors. We select all matrices where the a_{11} entry is a zero divisor. Then, we carefully study the adjacency relationships to color these matrices so that no two adjacent vertices share the same color. Thus, the chromatic number of this graph is greater than $n + 1$. □

Table 1 presents some results related to the chromatic numbers of the prime graphs $PG(M_2(\mathbb{Z}_n))$, where n is not a prime number. In particular, in Figure 7, using MATLAB, we obtain the chromatic number of the prime graph $PG(M_2(\mathbb{Z}_8))$.

Table 1. Chromatic numbers of prime graphs PG.

Rings	Number of Zero Divisors of $\mathbb{Z}_n$	Number of Nilpotent Elements $\mathbb{Z}_n$	Chromatic Numbers of PG
$M_2(Z_4) = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{0} \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_4 \right\}$	1	1	7
$M_2(Z_6) = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{0} \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_6 \right\}$	3	0	8
$M_2(Z_8) = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{0} \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_8 \right\}$	3	3	14
$M_2(Z_9) = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{0} \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_9 \right\}$	2	2	16
$M_2(Z_{10}) = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{0} \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_{10} \right\}$	5	0	12
$M_2(Z_{12}) = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{0} \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_{12} \right\}$	7	1	21
$M_2(Z_{14}) = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{0} \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_{14} \right\}$	7	1	16
$M_2(Z_{15}) = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{0} \end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_{15} \right\}$	6	0	17

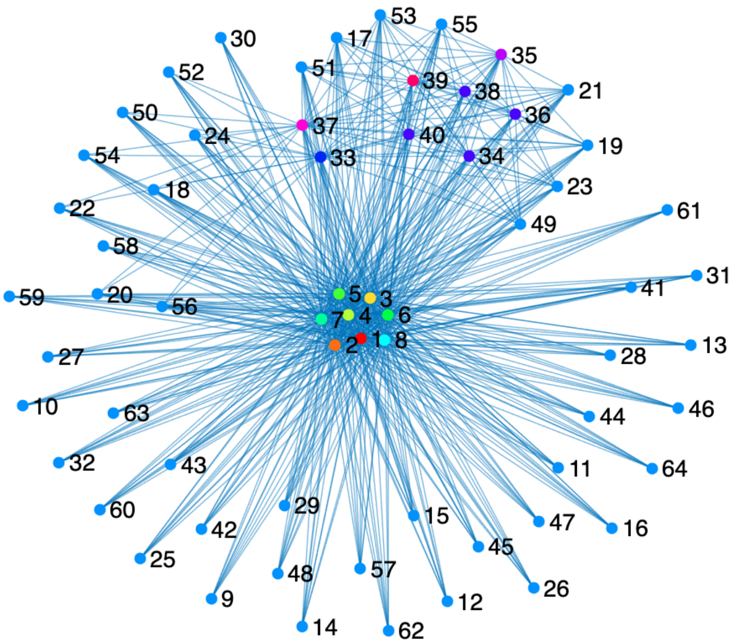

Figure 7. $\chi(PG(M_2(\mathbb{Z}_8))) = 14$.

5. Conclusions

In this paper, we investigated the chromatic number of prime graphs for certain rings, and proved that when R is a non-commutative division ring, the chromatic number of the prime graph for $R[x]$ and $R[[x]]$ is 2. Additionally, we explored other cases for the chromatic number of these graphs, including examples that involved matrix rings.

Author contributions

Walaa Alqarafi: Investigation, methodology, writing - original draft, writing - review and editing, software; Alaa Altassan: Writing - original draft, writing - review and editing, software, supervision; Wafaa Fakieh: Investigation, methodology, writing - original draft, writing - review and editing, supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest regarding the publishing of this paper.

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Appendix

We create MATLAB code to implement the following algorithm, which verifies the validity of our results. Let $M_2(\mathbb{Z}_n) = \left\{\begin{pmatrix}\overline{a} & \overline{b}\\ \overline{0} & \overline{0}\end{pmatrix} : \overline{a}, \overline{b} \in \mathbb{Z}_n\right\}$, where n is a positive integer. In this case, if A and B are two vertices of $PG(M_2(\mathbb{Z}_n))$, there is an edge between them whenever $AB = 0$ or $BA = 0$, as described in Example 3.1 of [14].

Algorithm: Chromatic Numbers of $PG(M_2(\mathbb{Z}_n))$

Input:

- **n**: A positive integer greater than 1, representing the modulus of $\mathbb{Z}_n$.

Output:

- **Vertices:** A list of 2×2 zero-divisor matrices of the form $\left(\begin{smallmatrix}\overline{a} & \overline{b}\\ \overline{0} & \overline{0}\end{smallmatrix}\right)$ over $\mathbb{Z}_n$.
- **Edges:** Pairs of indices representing connections between vertices where $AB = 0$ or $BA = 0$ modulo n.
- **Chromatic Number:** The minimum number of colors needed to color the graph such that no two adjacent vertices share the same color.
- **Independent Sets:** A collection of maximal sets of vertices where no two vertices are connected by an edge.

Steps:

1. **Input Validation:**
   - Ensure $n > 1$. If $n \leq 1$, display an error message.
2. **Generate Vertices:**
   - Initialize an empty list of vertices.
   - For all combinations of $a, b \in \{0, 1, ..., n-1\}$, construct 2×2 matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$.
   - Append each matrix to the vertex list in row-vector form $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$.
3. **Find Edges:**
   - Initialize an empty list of edges.
   - For every pair of vertices (A, B), reshape them into 2×2 matrices.
   - Check if $AB = 0$ or $BA = 0$ modulo n. If true, add an edge (i, j) between the corresponding indices i and j.
4. **Display Vertices and Edges:**
   - Display the list of vertices as 2×2 matrices.
   - Display the list of edges as pairs of connected vertex indices.
5. **Construct the Graph:**
   - Create a graph object using the edges.
6. **Greedy Coloring Algorithm:**
   - Initialize a color array for all vertices, setting the first vertex to color 1.
   - For each subsequent vertex:
     - Find the colors of adjacent vertices.
     - Assign the smallest unused color to the current vertex.
   - Determine the chromatic number as the maximum color used.
7. **Find Independent Sets:**
   - Initialize an empty list of independent sets and mark all vertices as unvisited.
   - While unvisited vertices exist:
     - Start a new independent set with an unvisited vertex.
     - For each other unvisited vertex, check if it is not adjacent to any vertex in the current set.
     - Add non-adjacent vertices to the current set and mark them as visited.
     - Append the current set to the list of independent sets.
8. **Display Results:**
   - Display the chromatic number.

- Display the independent sets.

9. **Plot the Graph:**

- Define a color-map with distinct colors based on the chromatic number.
- Plot the graph with vertices colored according to the greedy coloring.
- Add a color bar and title for visualization.

End Algorithm