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Research article

Stability and properties of Cauchy–Stieltjes Kernel families under generalized t-transformation

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Abstract: The study of the generalized t-deformation of free convolution from the lens of Cauchy–Stieltjes kernel (CSK) families provides an excellent mathematical framework for understanding noncommutative probability distributions. In this paper, we use the concept of generalized t-deformation to demonstrate different elements of the Marchenko–Pastur, free Gamma, and inverse semicircle measures in the CSK families setting. These findings advance our knowledge of generalized t-deformation in the non-commutative probability framework.

Keywords: Marchenko–Pastur distribution; free Gamma distribution; variance function; inverse semicircle distribution; Cauchy–Stieltjes transform
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1. Introduction

The study of families of measures and their stability under various operations, such as convolution or measure transformations, plays a crucial role in understanding the behavior and properties of random variables across different statistical models. In particular, the stability of a family of measures under these operations can provide deep insights into the underlying structure of the distributions and their applications in fields like statistical inference, signal processing, and stochastic modeling [13, 20, 30]. Convolution, a fundamental operation in probability theory, often arises in contexts such as the summation of independent random variables or in systems characterized by random noise. Investigating how a family of probability measures behaves under convolution helps in understanding the aggregation of uncertainty and the resulting distributions. Similarly, transformations of measures, which may include scaling, shifting, or other functional mappings, are essential tools for exploring how changes in the underlying model affect the stability and dispersion of the distributions [24–26]. This article is a continuation of the work presented in [1] for the investigation of generalized t-deformation introduced in [23]. We explore the stability of Cauchy–Stieltjes kernel (CSK) families of measures under such operations, focusing on the theoretical foundations. By examining the conditions under which stability is preserved, our investigation reveals new insights into generalized t-deformation. The objective of this paper will be given in more details after presenting the concept of generalized t-deformation and providing basic concepts on CSK families.

The subject of transforming measures and convolutions has been thoroughly examined and developed in several papers, see [10, 22]. The notion of generalized t-deformation (also called ($\mathbf{t} = (a, b)$)-deformation) of measures, denoted by $\widetilde{U}^{\mathbf{t}}(\cdot)$, is introduced in [23] for $a \in \mathbb{R}$ and $b > 0$, so that the t-deformation of Bożejko and Wysoczański [11, 12] is a particular case . Let ϱ be a real probability measure with a finite first moment. For $a \in \mathbb{R}$ and $b > 0$, consider $\mathbf{t} = (a, b)$ and introduce the ($\mathbf{t} = (a, b)$)-deformation as follows

$$1/\mathcal{G}_{\widetilde{U}^{\mathbf{t}}(\varrho)}(w) = (b - a)m_0^{\varrho} + (1 - b)w + b/\mathcal{G}_{\varrho}(w), \tag{1.1}$$

where m_0^{ϱ} denote the first moment of ϱ and

$$\mathcal{G}_{\varrho}(w) = \int \frac{\varrho(dx)}{w - x}, \quad w \in \mathbb{C} \setminus \operatorname{supp}(\varrho) \tag{1.2}$$

is the Cauchy transform of ϱ. If $t = a = b > 0$, $\widetilde{U}^{\mathbf{t}}(\cdot)$ is reduced to the original t-transformation [11,12]. The ($\mathbf{t} = (a, b)$)-deformation of ϱ (with all moments being finite) can be seen in terms of continued fractions. For

$$\mathcal{G}_{\varrho}(w) = \cfrac{1}{-\alpha_1 + w - \cfrac{\beta_1}{-\alpha_2 + w - \cfrac{\beta_2}{-\alpha_3 + w - \cfrac{\beta_3}{-\alpha_4 + w - ...}}}},$$

we have,

$$\mathcal{G}_{\widetilde{U}^{\mathbf{t}}(\varrho)}(w) = \cfrac{1}{-a\alpha_1 + w - \cfrac{b\beta_1}{-\alpha_2 + w - \cfrac{\beta_2}{-\alpha_3 + w - \cfrac{\beta_3}{-\alpha_4 + w - ...}}}}.$$

For $a \neq 0$ and $b > 0$, we write $\mathbf{t}^{-1} = (1/a, 1/b)$, then $\widetilde{U}^{\mathbf{t}^{-1}}$ and $\widetilde{U}^{\mathbf{t}}$ are the inverses of each other [23]. The $\mathbf{t}$-transformed free convolution $\boxplus_{(\mathbf{t})}$ is defined by

$$\mu \boxplus_{(\mathbf{t})} \varrho = \widetilde{U}^{\mathbf{t}^{-1}}(\widetilde{U}^{\mathbf{t}}(\mu) \boxplus \widetilde{U}^{\mathbf{t}}(\varrho)), \tag{1.3}$$

where μ and ϱ are real probabilities with finite first moments. The $\boxplus_{(\mathbf{t})}$-Poisson limit depends on two parameters and is provided directly using the orthogonal polynomials, but the $\boxplus_{(\mathbf{t})}$-central limit measure is the same as the one for the original t-deformation.

We now introduce the notion of CSK families. By substituting the CSK $\frac{1}{1-\gamma s}$ for the exponential kernel $\exp(\gamma s)$, the CSK families of probabilities are defined in free probability in a similar way to that of natural exponential families (NEFs) in classical probability. For measures with compact support, the CSK families have been examined in [6, 8]. Additional characteristics are demonstrated in [7] by utilizing measures with a one-sided support border, say from above. The CSK families serve as essential tools in free probability, providing a structured approach to studying noncommutative distributions and their convolutions. Characterized by the corresponding variance functions (VFs), these families facilitate the analysis of free infinite divisibility, asymptotic spectral distributions, and moment problems. Their utility extends to solving integral equations, characterizing fixed points of transformations, and modeling free convolution processes. Moreover, they play a crucial role in random matrix theory, where they describe the limiting behavior of eigenvalue distributions. Overall, CSK families offer a robust analytical framework for exploring fundamental problems in free probability and operator algebra. Denote by $\mathcal{P}_{ba}$ (respectively $\mathcal{P}_c$) the set of (non-degenerate) probabilities with a one–sided support boundary from above (respectively with compact support). Let $\varrho \in \mathcal{P}_{ba}$. Then

$$\mathcal{M}_\varrho(\gamma) = \int \frac{\varrho(ds)}{1-\gamma s}$$

converges for $\gamma \in [0, \gamma_+^\varrho)$ with $1/\gamma_+^\varrho = \max\{0, \sup \mathrm{supp}(\varrho)\}$. The CSK family induced by ϱ is the set

$$\mathcal{K}_+(\varrho) = \left\{ P_\gamma^\varrho(ds) = \frac{\varrho(ds)}{\mathcal{M}_\varrho(\gamma)(1-\gamma s)} : \ \ \gamma \in (0, \gamma_+^\varrho) \right\}.$$

The map $\gamma \mapsto k_\varrho(\gamma) = \int s P_\gamma^\varrho(ds)$ is bijective from $(0, \gamma_+^\varrho)$ into $k_\varrho((0, \gamma_+^\varrho)) = (m_0^\varrho, m_+^\varrho)$ which is said to be the (one–sided) mean domain of $\mathcal{K}_+(\varrho)$. For $x \in (m_0^\varrho, m_+^\varrho)$, write $Q_x^\varrho(ds) = P_{\psi_\varrho(x)}^\varrho(ds)$, where $\psi_\varrho(\cdot)$ is the inverse of $k_\varrho(\cdot)$. We obtain the mean parametrization $\mathcal{K}_+(\varrho) = \{Q_x^\varrho(ds) : \ \ x \in (m_0^\varrho, m_+^\varrho)\}$. We have $m_0^\varrho = \lim_{\gamma\to 0^+} k_\varrho(\gamma)$ and $m_+^\varrho = B_\varrho - \lim_{w\to B_\varrho^+} 1/\mathcal{G}_\varrho(w)$, where $B_\varrho = 1/\gamma_+^\varrho$. When the support of ϱ is bounded from below, we have $\gamma \in (\gamma_-^\varrho, 0)$, where γ_-^ϱ can be either $1/A_\varrho$ or $-\infty$ with $A_\varrho = \min\{0, \inf supp(\varrho)\}$. The CSK family is denoted $\mathcal{K}_-(\varrho)$ with the mean domain $(m_-^\varrho, m_0^\varrho)$, where $m_-^\varrho = A_\varrho - 1/\mathcal{G}_\varrho(A_\varrho)$. If $\varrho \in \mathcal{P}_c$, then $\gamma \in (\gamma_-^\varrho, \gamma_+^\varrho)$ and $\mathcal{K}(\varrho) = \mathcal{K}_-(\varrho) \cup \mathcal{K}_+(\varrho) \cup \{\varrho\}$ is the complete (two-sided) CSK family.

Let $\varrho \in \mathcal{P}_{ba}$. The VF [6]

$$x \mapsto \mathcal{V}_\varrho(x) = \int (s-x)^2 Q_x^\varrho(ds) \tag{1.4}$$

is a crucial concept in CSK families. If ϱ has not got a first moment, then in $\mathcal{K}_+(\varrho)$, all probabilities have infinite variance. A notion of pseudo-variance function (PVF) $\mathbb{V}_\mu(\cdot)$ is introduced in [7] as $\mathbb{V}_\varrho(x) = x\left(1/\psi_\mu(x) - x\right)$. If $m_0^\varrho = \int s\varrho(ds)$ is finite, the VF exists [7] and

$$\mathbb{V}_\varrho(x) = \frac{x\mathcal{V}_\varrho(x)}{x - m_0^\varrho}. \tag{1.5}$$

The generalized t-transformation is studied in [1] from the viewpoint of the VFs of CSK families. An expression is proved for the VF under the powers of $\boxplus_{(\mathbf{t})}$-convolution. Furthermore, an estimation is presented for the members of $\mathbf{t}$-transformed free Poisson and free Gaussian CSK families. Moreover, related to $\boxplus_{(\mathbf{t})}$-convolution, a new limiting theorem is shown by incorporating free multiplicative convolution and basing on the VF. This work is a continuation of the investigation of generalized t-transformation. We present certain features regarding the Marchenko–Pastur (MP), the free Gamma (FG), and the inverse semicircle (ISC) measures in the setting of CSK families, based on generalized t-transformation. More precisely, for $\varrho \in \mathcal{P}_{ba}$ we consider the $(\mathbf{t})$-deformation of measures in Section 2. For $a^2 = b$, we show that if $\widetilde{U}^{\mathbf{t}}(Q_x^\varrho) \in \mathcal{K}_+(\varrho)$, then, up to scaling, ϱ is either an ISC or a FG measure. In Section 3, we consider the notion of $\boxplus_{(\mathbf{t})}$-convolution. For $0 < \alpha \neq 1$ such that $(Q_m^\mu)^{\boxplus_{(\mathbf{t})}\alpha}$ is defined, we show that if $(Q_x^\varrho)^{\boxplus_{(\mathbf{t})}\alpha} \in \mathcal{K}_+(\varrho)$, then ϱ is a MP measure up to scaling. In addition, we demonstrate that if the $\boxplus_{(\mathbf{t})}$-convolution product of two given CSK families remains a CSK family, then we have MP CSK families. That is, for ϱ_1 and $\varrho_2 \in \mathcal{P}_c$, by introducing the family of measures

$$\mathcal{T} = \mathcal{K}(\varrho_1) \boxplus_{(\mathbf{t})} \mathcal{K}(\varrho_2) = \{Q_{r_1}^{\varrho_1} \boxplus_{(\mathbf{t})} Q_{r_2}^{\varrho_2}, \quad r_1 \in (m_-^{\varrho_1}, m_+^{\varrho_1}) \text{ and } r_2 \in (m_-^{\varrho_2}, m_+^{\varrho_2})\}, \tag{1.6}$$

we demonstrate that if $\mathcal{T}$ remains a CSK family (i.e $\mathcal{T} = \mathcal{K}(\tau)$ for $\tau \in \mathcal{P}_c$) then, up to affinity, τ, ϱ_1, and ϱ_2 are of the MP type law.

This section concludes with a remark that provides supporting evidence for the article's key results.

Remark 1.1. Let $\varrho \in \mathcal{P}_{ba}$.

(i) The law $Q_x^\varrho(ds)$ can be written as $Q_x^\varrho(ds) = l_\varrho(s,x)\varrho(ds)$ with

$$l_\varrho(s,x) := \begin{cases} \frac{\mathbb{V}_\varrho(x)}{\mathbb{V}_\varrho(x)+x(x-s)}, & x \neq 0; \\ 1, & m = 0, \ \mathbb{V}_\varrho(0) \neq 0; \\ \frac{\mathbb{V}'_\varrho(0)}{\mathbb{V}'_\varrho(0)-s}, & x = 0, \ \mathbb{V}_\varrho(0) = 0. \end{cases} \tag{1.7}$$

(ii) ϱ is determined by $\mathbb{V}_\varrho(\cdot)$: If we set $\varpi = \varpi(x) = x + \frac{\mathbb{V}_\varrho(x)}{x}$, then

$$\mathcal{G}_\varrho(\varpi) = \frac{x}{\mathbb{V}_\varrho(x)}. \tag{1.8}$$

(iii) Let $f : s \mapsto \vartheta s + \iota$ where $\vartheta \neq 0$ and $\iota \in \mathbb{R}$. Then, for x close enough to $m_0^{f(\varrho)} = f(m_0^\varrho) = \vartheta m_0^\varrho + \iota$, one has

$$\mathbb{V}_{f(\varrho)}(x) = \frac{\vartheta^2 x}{x-\iota}\mathbb{V}_\varrho\left(\frac{x-\iota}{\vartheta}\right). \tag{1.9}$$

If $\mathcal{V}_\varrho(\cdot)$ exists, then

$$\mathcal{V}_{f(\varrho)}(m) = \vartheta^2\mathcal{V}_\varrho\left(\frac{x-\iota}{\vartheta}\right). \tag{1.10}$$

(iv) Let $\varrho \in \mathcal{P}_{ba}$ with a finite m_0^ϱ. The free cumulant transform $\mathcal{R}_\varrho$ of ϱ is defined by

$$\mathcal{R}_\varrho(\mathcal{G}_\varrho(\upsilon)) = \upsilon - 1/\mathcal{G}_\varrho(\upsilon), \quad \upsilon \text{ close to } \infty. \tag{1.11}$$

The $\mathbf{t}$-deformed free cumulant transform, denoted by $\mathcal{R}^{(\mathbf{t})}(\cdot)$, is defined in [23] as

$$\mathcal{R}_\varrho^{(\mathbf{t})}(\varpi) := \frac{1}{b}\left(\mathcal{R}_{\widetilde{U}^{\mathbf{t}}(\varrho)}(\varpi) + (b-a)m_0^\varrho\right). \tag{1.12}$$

For $\alpha > 0$, for which $\varrho^{\boxplus_{(\mathbf{t})}\alpha}$ is defined,

$$\mathcal{R}^{(\mathbf{t})}_{\varrho^{\boxplus_{(\mathbf{t})}\alpha}}(\varpi) = \alpha\mathcal{R}_\varrho^{(\mathbf{t})}(\varpi).$$

For any w close enough to $m_0^{\widetilde{U}^{\mathbf{t}}(\varrho)} = am_0^{\varrho}$, we have [1]

$$\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(\varrho)}(w) = \frac{bw}{w+(b-a)m_0^{\varrho}}\mathbb{V}_{\mu}\left(\frac{w+(b-a)m_0^{\varrho}}{b}\right) + w\left(\frac{w+(b-a)m_0^{\varrho}}{b} - w\right). \tag{1.13}$$

In addition, for any w close enough to $m_0^{\varrho^{\boxplus(\mathbf{t})\alpha}} = \alpha m_0^{\varrho}$, one has

$$\mathbb{V}_{\varrho^{\boxplus(\mathbf{t})\alpha}}(w) = \alpha\mathbb{V}_{\varrho}\left(\frac{w}{\alpha}\right) + \left(\frac{1}{\alpha} - 1\right)w(w(1-b) + \alpha(b-a)m_0^{\varrho}). \tag{1.14}$$

2. A property of FG and ISC laws

In [17], the authors investigated CSK families and examined their behavior under V_a-transformation [21]. They established that a CSK family remains invariant under this transformation for all nonzero real parameter a if and only if its generating measure is the semicircle distribution, up to an affine transformation. This characterization emphasizes the noted role of the semicircle law in free probability, identifying it as the unique distribution with such invariance and further reinforcing its position as the free analog of the classical Gaussian distribution. In a related contribution, CSK families were studied in [16] under the t-transformation of measures, and it was proved that a CSK family with a one–sided support boundary remains invariant across such transformation if and only if the generating measure is of the Marchenko–Pastur type law. This provides a sharp and elegant characterization of the Marchenko–Pastur distribution via invariance of CSK families. Motivated by these results, we turn in this section to analogous properties for other important distributions in free probability. In particular, we investigate the inverse semicircle (ISC) type law and the free Gamma (FG) type law, employing the framework of $(\mathbf{t} = (a, b))$-transformation.

The ISC law is [7, p. 590]

$$\mathbf{ISC}(ds) = \frac{\sqrt{-1-4s}}{2\pi s^2}\mathbf{1}_{(-\infty,-\frac{1}{4})}(s)ds, \tag{2.1}$$

with $m_0^{\mathbf{ISC}} = -\infty$. We have $\mathbb{V}_{\mathbf{ISC}}(x) = x^3$, for all $x \in (m_0^{\mathbf{ISC}}, m_+^{\mathbf{ISC}}) = (-\infty, -1)$.

The FG law is given, for $a_1 \neq 0$, by

$$\mathbf{FG}_{a_1}(ds) = \frac{\sqrt{((a_1+\sqrt{a_1^2+1})^2 - s)(s - (-a_1+\sqrt{a_1^2+1})^2)}}{2\pi a_1^2 s^2}\mathbf{1}_{\left((-|a_1|+\sqrt{a_1^2+1})^2,(|a_1|+\sqrt{a_1^2+1})^2\right)}(s)ds, \tag{2.2}$$

with $m_0^{\mathbf{FG}_{a_1}} = 1$. We have $\mathcal{V}_{\mathbf{FG}_{a_1}}(x) = a_1^2x^2$.

Assume that $a_1 > 0$, (the same logic applies to the case where $a_1 < 0$). As shown in [15, Example 3.6], $m_-^{\mathbf{FG}_{a_1}} = \frac{1}{1+a_1^2}$ and

$$\mathcal{G}_{\mathbf{FG}_{a_1}}(z) = \frac{\frac{z-1}{a_1^2} + 2z - \sqrt{\left(2a_1 - \frac{z-1}{a_1}\right)^2 - 4(1+a_1^2)}}{2z^2}. \tag{2.3}$$

With $B_{\mathbf{FG}_{a_1}} = (\sqrt{a_1^2+1} + a_1)^2$, we obtain $m_+^{\mathbf{FG}_{a_1}} = B_{\mathbf{FG}_{a_1}} - \frac{1}{\mathcal{G}_{\mathbf{FG}_{a_1}}(B_{\mathbf{FG}_{a_1}})} = 1 + \frac{a_1}{\sqrt{a_1^2+1}}$. Thus,

$$\left(m_-^{\mathbf{FG}_{a_1}}, m_+^{\mathbf{FG}_{a_1}}\right) = \left(\frac{1}{1+a_1^2}, 1 + \frac{a_1}{\sqrt{a_1^2+1}}\right). \tag{2.4}$$

We now state and prove this section's main result.

Theorem 2.1. *Let $\mu \in \mathcal{P}_{ba}$. Assume that $b = a^2$, $a \neq 1$, and that for every $m \in (m_0^\mu, m_+^\mu)$, we have $\widetilde{U}^{\mathbf{t}}(Q_m^\mu) = Q_{h(m,a,b)}^\mu$ for some $h(m,a,b) \in (m_0^\mu, m_+^\mu)$. Then, $a > 0$, $h(m,a,b) = am$ and either*

(i) $m_0^\mu \in (0,+\infty)$ and μ is the image by $x \mapsto m_0^\mu x$ of the FG law (2.2) *with $\frac{m_0^\mu}{1+a_1^2} < am < m_0^\mu\left(1 + \frac{a_1}{\sqrt{a_1^2+1}}\right)$, or*

(ii) $m_0^\mu \in (-\infty, 0)$ and μ is the image by $x \mapsto m_0^\mu x$ of the FG law (2.2) *with $m_0^\mu\left(1 + \frac{a_1}{\sqrt{a_1^2+1}}\right) < am < \frac{m_0^\mu}{1+a_1^2}$, or*

(iii) $m_0^\mu = -\infty$ and there is a $\kappa > 0$ large enough such that μ is the image by $x \mapsto \frac{1}{\kappa}x$ of the ISC law (2.1).

Proof. Assume that $\widetilde{U}^{\mathbf{t}}(Q_m^\mu) = Q_{h(m,a,b)}^\mu$. For z that is large enough:

$$\mathcal{E}_{\widetilde{U}^{\mathbf{t}}(Q_m^\mu)}(z) = \mathcal{E}_{Q_{h(m,a,b)}^\mu}(z), \tag{2.5}$$

where

$$\mathcal{E}_\mu(z) = z - \frac{1}{\mathcal{G}_\mu(z)}. \tag{2.6}$$

Recall from Remark 1.1(v) that

$$m_0^{\widetilde{U}^{\mathbf{t}}(\mu)} = am_0^\mu. \tag{2.7}$$

Combining (2.7) with (2.5) and being aware, thanks to [16], that

$$\lim_{w\longrightarrow+\infty} \mathcal{E}_\mu(w) = m_0^\mu, \tag{2.8}$$

we get

$$h(m,a,b) = m_0^{Q_{h(m,a,b)}^\mu} = \lim_{z\longrightarrow+\infty} \mathcal{E}_{Q_{h(m,a,b)}^\mu}(z) = \lim_{z\longrightarrow+\infty} \mathcal{E}_{\widetilde{U}^{\mathbf{t}}(Q_m^\mu)}(z) = m_0^{\widetilde{U}^{\mathbf{t}}(Q_m^\mu)} = am_0^{Q_m^\mu} = am. \tag{2.9}$$

As long as $m \in (m_0^\mu, m_+^\mu)$, then $m = m_0^{Q_m^\mu} = \int x Q_m^\mu(dx)$ is finite. From the relations (1.1) and (2.6), we obtain

$$\mathcal{E}_{\widetilde{U}^{\mathbf{t}}(Q_m^\mu)}(z) = b\mathcal{E}_{Q_m^\mu}(z) + (a-b)m_0^{Q_m^\mu} = b\mathcal{E}_{Q_m^\mu}(z) + (a-b)m. \tag{2.10}$$

Combining (2.5), (2.9), and (2.10), we obtain

$$\mathcal{E}_{Q_{am}^\mu}(z) = b\mathcal{E}_{Q_m^\mu}(z) + (a-b)m. \tag{2.11}$$

From [16], for $w \neq \mathbb{V}_\mu(m)/m + m$, one has

$$\mathcal{E}_{Q_m^\mu}(w) = \frac{\left(\mathbb{V}_\mu(m)/m + m\right)\mathcal{E}_\mu(w) - mw}{\mathbb{V}_\mu(m)/m - (w - \mathcal{E}_\mu(w))}. \tag{2.12}$$

Using (2.12), Eq (2.11) is

$$\frac{\left(am+\frac{\mathbb{V}_\mu(am)}{am}\right)\mathcal{E}_\mu(z)-amz}{\frac{\mathbb{V}_\mu(am)}{am}-(z-\mathcal{E}_\mu(z))}=\frac{b\left(m+\frac{\mathbb{V}_\mu(m)}{m}\right)\mathcal{E}_\mu(z)-bmz}{\frac{\mathbb{V}_\mu(m)}{m}-(z-\mathcal{E}_\mu(z))}+(a-b)m. \tag{2.13}$$

After some calculations, Eq (2.13) becomes

$$\begin{aligned}&\left(am+\frac{\mathbb{V}_\mu(am)}{am}\right)\frac{\mathbb{V}_\mu(m)}{m}\mathcal{E}_\mu(z)-\left(am+\frac{\mathbb{V}_\mu(am)}{am}\right)\mathcal{E}_\mu(z)(z-\mathcal{E}_\mu(z))-az\mathbb{V}_\mu(m)\\&=\left(m+\frac{\mathbb{V}_\mu(m)}{m}\right)\frac{b\mathbb{V}_\mu(am)}{am}\mathcal{E}_\mu(z)-b\left(m+\frac{\mathbb{V}_\mu(m)}{m}\right)\mathcal{E}_\mu(z)(z-\mathcal{E}_\mu(z))-\frac{b\mathbb{V}_\mu(am)}{a}z-(a-b)m\mathcal{E}_\mu(z)(z-\mathcal{E}_\mu(z))\\&\quad+(a-b)\mathbb{V}_\mu(m)\frac{\mathbb{V}_\mu(am)}{am}-(a-b)\mathbb{V}_\mu(m)(z-\mathcal{E}_\mu(z))-(a-b)\frac{\mathbb{V}_\mu(am)}{a}(z-\mathcal{E}_\mu(z)).\end{aligned} \tag{2.14}$$

From [7, p. 580], we have

$$-\infty\le\int s\mu(ds)=m_0^\mu<m_+^\mu\le\sup\ supp(\mu)<+\infty. \tag{2.15}$$

We take into account the situations in which $-\infty<m_0^\mu<+\infty$ and $m_0^\mu=-\infty$ independently.

Suppose that $0\le m_0^\mu<+\infty$ (the case $-\infty<m_0^\mu\le 0$ can be deduced easily by considering the reflection $\varphi: x\mapsto -x$). In (2.14), on both sides, divide by z and let $z\to+\infty$. Knowing from [16] that

$$\lim_{w\to+\infty}\frac{\mathcal{E}_\mu(w)}{w}=0 \qquad\text{and}\qquad \lim_{w\to+\infty}\frac{(w-\mathcal{E}_\mu(w))\mathcal{E}_\mu(w)}{w}=m_0^\mu, \tag{2.16}$$

we get

$$\left(\frac{am-m_0^\mu}{am}\right)\mathbb{V}_\mu(am)=b\mathbb{V}_\mu(m)\left(\frac{m-m_0^\mu}{m}\right). \tag{2.17}$$

Combining (2.17) and (1.5), we get

$$\mathcal{V}_\mu(am)=b\mathcal{V}_\mu(m),\quad\forall\ m\in(m_0^\mu,m_0^\mu),\quad\forall\,a\in\mathbb{R}\backslash\{0\},\quad\text{and}\quad\forall\,b>0. \tag{2.18}$$

The non–degeneracy of μ gives $\mathcal{V}_\mu(\cdot)\ne 0$. With $b=a^2$, Eq (2.18) implies $\mathcal{V}_\nu(m)=\lambda m^2$ for $\lambda>0$.

- A VF of the form $\mathcal{V}(m)=\lambda m^2$, where $\lambda>0$, cannot exist if $m_0^\mu=0$; see [2].
- If $m_0^\mu\ne 0$, then μ is the image by $s\mapsto m_0^\mu s$ of FG_{a_1} presented by (2.2).

When $m_0^\mu=-\infty$, on both sides of (2.14), divide by $z\mathcal{E}_\mu(z)$, and let $z\to+\infty$, yielding

$$\mathbb{V}_\mu(ma)=a^3\mathbb{V}_\mu(m),\quad\ \forall\,m\in(m_0^\mu,m_+^\mu)\ \text{and}\ \forall\,a\in\mathbb{R}\backslash\{0\}. \tag{2.19}$$

Thus, $\mathbb{V}_\mu(m)=\kappa m^3$, with $\kappa>0$, and μ is the image by $s\mapsto\frac{1}{\kappa}s$ of ISC given by (2.1).

Remark 2.2. We show that $h(m,a,b)=ma$ belongs to the mean domain.
Case 1. If $0<m_0^\mu<+\infty$ and μ is the image by $s\mapsto m_0^\mu s$ of FG_{a_1} given by (2.2), one has

$$(m_-^\mu,m_+^\mu)=\left(\frac{m_0^\mu}{1+a_1^2},m_0^\mu\left(1+\frac{a_1}{\sqrt{a_1^2+1}}\right)\right).$$

The parameter a must be strictly positive and the condition in Theorem 2.1(i) is required.
Case 2. If $m_0^\mu = -\infty$ and μ is the image by $s \mapsto \frac{1}{\kappa}s$ of ISC (2.1). We have, $(m_0^\mu, m_+^\mu) = (-\infty, -\frac{1}{\kappa})$. The parameter a must be strictly positive and the parameter κ should be large enough so that if $m \in (-\infty, -\frac{1}{\kappa})$, then $h(m,a,b) = am \in (-\infty, -\frac{1}{\kappa})$.

We now demonstrate that the opposite implication in Theorem 2.1 is false.
Case 1. Suppose that $m_0^\mu \in (0, +\infty)$, $h(m,a,b) = am$ for $a > 0$, and μ is the image by $s \mapsto m_0^\mu s$ of FG_{a_1} (2.2) so that $\frac{m_0^\mu}{1+a_1^2} < am < m_0^\mu(1 + \frac{a_1}{\sqrt{a_1^2+1}})$. We aim to show that

$$\widetilde{U}^{\mathbf{t}}(Q_m^\mu) \neq Q_{am}^\mu. \tag{2.20}$$

We have that $m_0^{\widetilde{U}^{\mathbf{t}}(Q_m^\mu)} = am = m_0^{Q_{am}^\mu}$. Then there is an $\varepsilon > 0$ so that $x \mapsto \mathbb{V}_{\widetilde{U}^{\mathbf{t}}(Q_m^\mu)}(x)$ and $x \mapsto \mathbb{V}_{Q_{am}^\mu}(x)$ are defined on $(ma, ma + \varepsilon)$. We know from [2, Eq 47] that

$$\mathbb{V}_{Q_m^\mu}(y) = \frac{y^3(a_1^2 m^2 - (y-m)(m - m_0^\mu))}{(y-m)(y(m - m_0^\mu) + m m_0^\mu)}, \quad y \neq m. \tag{2.21}$$

Now, for $x \in (am, am + \varepsilon)$, one sees from (1.13) (with $b = a^2$) and (2.21) that

$$\begin{aligned}
\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(Q_m^\mu)}(x) &= \frac{a^2 x}{x + (a^2 - a) m_0^{Q_m^\mu}} \mathbb{V}_{Q_m^\mu}\left(\frac{x + (a^2-a) m_0^{Q_m^\mu}}{a^2}\right) + x\left(\frac{x + (a^2-a) m_0^{Q_m^\mu}}{a^2} - x\right)\\
&= \frac{a^2 x}{x + (a^2 - a) m} \mathbb{V}_{Q_m^\mu}\left(\frac{x + (a^2-a) m}{a^2}\right) + x\left(\frac{x + (a^2-a) m}{a^2} - x\right)\\
&= x\left(\frac{x + (a^2-a)m}{a^2}\right)^2 \left[\frac{a_1^2 a^2 m^2 - (x - am)(-m_0^\mu + m)}{(x - am)\left(\frac{x + (a^2-a)m}{a^2}(-m_0^\mu + m) + m m_0^\mu\right)}\right] + x\left(\frac{x + (a^2-a)m}{a^2} - x\right)\\
&\neq \mathbb{V}_{Q_{am}^\mu}(x).
\end{aligned}$$

This concludes the proof of (2.20) by the use of (1.8).
Case 1. Suppose that $m_0^\mu = -\infty$, $h(m,a,b) = am$ for $a > 0$, and μ is the image by $s \mapsto \frac{1}{\kappa}s$ of the ISC given by (2.1). We have to show that

$$\widetilde{U}^{\mathbf{t}}(Q_m^\mu) \neq Q_{am}^\mu. \tag{2.22}$$

We have $m_0^{\widetilde{U}^{\mathbf{t}}(Q_m^\mu)} = am = m_0^{Q_{am}^\mu}$. Then there is an $\varepsilon > 0$ such that the functions $x \mapsto \mathbb{V}_{\widetilde{U}^{\mathbf{t}}(Q_m^\mu)}(x)$ and $x \mapsto \mathbb{V}_{Q_{am}^\mu}(x)$ are well defined on $(am, am + \varepsilon)$. For $u \in \left(-\infty, -\frac{1}{\kappa}\right)$, we have

$$\mathbb{V}_\mu(u) = \kappa u^3. \tag{2.23}$$

From [9, Eq (2.9)], we get

$$y = \frac{u^2\left(\kappa m^3\right) - m^2\left(\kappa u^3\right)}{u\left(\kappa m^3\right) - m\left(\kappa u^3\right)} = \frac{um}{u+m}. \tag{2.24}$$

From (2.24), we get

$$u = \frac{ym}{m-y}. \tag{2.25}$$

Using (2.23), (2.25), and [9, Eq (2.10)], one gets

$$\mathbb{V}_{Q^\mu_m}(y) = y\left(\frac{\mathbb{V}_\mu(u)}{u} + u - y\right) = y^3\left(\frac{\kappa m^2 + (m-y)}{(y-m)^2}\right), \quad y \neq m. \tag{2.26}$$

Now, for $x \in (\alpha m, \alpha m + \varepsilon)$, one sees from (1.13) (with $b = a^2$) and (2.26) that

$$\begin{aligned}\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(Q^\mu_m)}(x) &= \frac{a^2x}{x+(a^2-a)m}\mathbb{V}_{Q^\mu_m}\left(\frac{x+(a^2-a)m}{a^2}\right) + x\left(\frac{x+(a^2-a)m}{a^2} - x\right)\\ &= x(x+(a^2-a)m)^2\left(\frac{\kappa a^2m^2 + (am-x)}{(x-am)^2}\right)\\ &\neq \mathbb{V}_{Q^\mu_{am}}(x).\end{aligned}$$

This concludes the proof of (2.22). □

3. Notes on the MP law based on $\boxplus_{(\mathbf{t})}$-convolution

The MP law constitutes the limiting spectral distribution for a class of large random covariance matrices, occupying a position in free probability analogous to the Poisson distribution in classical probability. In the framework of CSK families, the MP measure is uniquely characterized by the fact that the family it generates remains invariant under t-transformation of measures (or Boolean convolution); see [16]. This invariance property underscores the distinguished position of the MP law as one of the few free probability distributions singled out by convolution stability, thereby reinforcing its structural importance in both theoretical developments and applications such as signal processing and wireless communications.

The MP law is given, for $a_1 \neq 0$, by

$$\mathbf{MP}_{a_1}(dy) = \frac{\sqrt{((a_1+1)^2-y)\,(y-(a_1-1)^2)}}{2\pi a_1^2 y}\mathbf{1}_{((a_1-1)^2,(a_1+1)^2)}(y)dy + (1-1/a_1^2)^+\delta_0 \tag{3.1}$$

with $m_0^{\mathbf{MP}_{a_1}} = 1$. We have

$$\mathbb{V}_{\mathbf{MP}_{a_1}}(m) = \frac{a_1^2m^2}{m-1} \quad \text{and} \quad (m_-^{\mathbf{MP}_{a_1}}, m_+^{\mathbf{MP}_{a_1}}) = \begin{cases} (1-|a_1|, 1+|a_1|), & \text{if } \ a_1^2 \leq 1;\\ (0, 1+|a_1|), & \text{if } \ a_1^2 > 1.\end{cases} \tag{3.2}$$

See [16, Section 3]. We have the following property of the MP law based on $\boxplus_{(\mathbf{t})}$-convolution.

Theorem 3.1. *Let $\mu \in \mathcal{P}_{ba}$ with a finite m_0^μ. Assume that $a \neq 0$, $b > 0$, and $0 < \alpha \neq 1$ is such that for every $m \in (m_0^\mu, m_+^\mu)$, the power $(Q^\mu_m)^{\boxplus_{(t)}\alpha}$ is well defined and we have $(Q^\mu_m)^{\boxplus_{(t)}\alpha} = Q^\mu_{g(m,a,b,\alpha)}$ for some $g(m,a,b,\alpha) \in (m_-^\mu, m_+^\mu)$. Then $m_0^\mu \neq 0$, $g(m,a,b,\alpha) = \alpha m$, and μ is the image of MP_{a_1} by $x \mapsto m_0^\mu x$, where $|a_1|$ is sufficiently large.*

Proof. Assume that $(Q^\mu_m)^{\boxplus_{(\mathbf{t})}\alpha} = Q^\mu_{g(m,a,b,\alpha)}$. That is,

$$\mathcal{R}^{(\mathbf{t})}_{(Q^\mu_m)^{\boxplus_{(\mathbf{t})}\alpha}}(\varpi) = \mathcal{R}^{(\mathbf{t})}_{Q^\mu_{g(m,a,b,\alpha)}}(\varpi), \qquad \varpi \text{ close to } 0. \tag{3.3}$$

On the basis of [19], we may write

$$\mathcal{R}_{Q^\mu_m}(\varpi) = m_0^{Q^\mu_m} + \mathrm{Var}(Q^\mu_m)\varpi + \varpi\varepsilon(\varpi), \tag{3.4}$$

where $m_0^{Q^\mu_m} = m$, $\mathrm{Var}(Q^\mu_m) = \mathcal{V}_\mu(m)$, and $\lim_{\varpi\to 0}\varepsilon(\varpi) = 0$. That is

$$\mathcal{R}_{Q_m^\mu}(\varpi) = m + \mathcal{V}_\mu(m)\varpi + \varpi\varepsilon(\varpi). \tag{3.5}$$

Using (3.5) and (1.12), one gets

$$\mathcal{R}^{(\mathbf{t})}_{Q_m^\mu}(\varpi) = m + \mathcal{V}_\mu(m)\varpi + \varpi\varepsilon(\varpi). \tag{3.6}$$

Equation (3.6) implies

$$\mathcal{R}^{(\mathbf{t})}_{(Q_m^\mu)^{\boxplus(\mathbf{t})\alpha}}(\varpi) = \alpha\mathcal{R}^{(\mathbf{t})}_{Q_m^\mu}(\varpi) = \alpha m + \alpha\mathcal{V}_\mu(m)\varpi + \alpha\varpi\varepsilon(\varpi), \tag{3.7}$$

and

$$\mathcal{R}^{(\mathbf{t})}_{Q^\mu_{g(m,a,b,\alpha)}}(\varpi) = g(m,a,b,\alpha) + \mathcal{V}_\mu(g(m,a,b,\alpha))\varpi + \varpi\varepsilon(\varpi). \tag{3.8}$$

Equations (3.3), (3.7), and (3.8) give $g(m,a,b,\alpha) = \alpha m$ and then

$$\mathcal{V}_\mu\,(m\alpha) = \alpha\mathcal{V}_\mu(m), \quad \forall\ m \in (m_0^\mu, m_+^\mu) \text{ and } \forall\ 0 < \alpha \neq 1. \tag{3.9}$$

The non–degeneracy of μ implies that $\mathcal{V}_\mu(\cdot) \neq 0$. Then $\mathcal{V}_\mu(m) = \beta m$ for $\beta \neq 0$.

• A VF $\mathcal{V}(m) = \beta m$, where $\beta \neq 0$, cannot exist if $m_0^\mu = 0$; see [16].

• If $m_0^\mu \neq 0$, then μ is the image of the MP law (3.1) via $s \mapsto m_0^\mu s$ and $\beta = a_1^2 m_0^\mu$.

Remark 3.2. This comment justifies the selection of the parameter $a_1 \neq 0$ of MP_{a_1}. We need $\alpha m \in (m_0^\mu, m_+^\mu)$ for $m \in (m_0^\mu, m_+^\mu)$. The image of MP_{a_1} by $\phi : s \mapsto m_0^\mu s$ is the law μ. We obtain $m_+^\mu = \phi(m_+^{\mathbf{MP}_{a_1}}) = m_0^\mu(1 + |a_1|)$ if $m_0^\mu > 0$. With $m_-^\mu = m_0^\mu(1 + |a_1|)$, the mean domain is (m_-^μ, m_0^μ) if $m_0^\mu < 0$. We should always have $a_1^2 > 1$ such that $|a_1|$ is sufficiently big that αm belongs to the mean domain.

We justify that the reverse implication of Theorem 3.1 is not valid. In this case, $m_0^\mu > 0$, $g(m,a,b,\alpha) = \alpha m$ is what we know where μ is the image (via $s \mapsto m_0^\mu s$) of $\mathbf{MP}_{a_1}$ for $a_1^2 > 1$, such that $|a_1|$ is sufficiently big. Here $(m_0^\mu, m_0^\mu(1 + |a_1|))$ is the mean domain for $\mathcal{K}_+(\mu)$. One has $\alpha m \in (m_0^\mu, m_0^\mu(1 + |a_1|)$ for any $|a_1|$ large enough. In order to refute the inverse implication, we must establish that

$$(Q_m^\mu)^{\boxplus(\mathbf{t})\alpha} \neq Q_{\alpha m}^\mu. \tag{3.10}$$

Similarly, if $z > \alpha m$ is close enough to αm, one has

$$\mathbb{V}_{(Q_m^\mu)^{\boxplus(\mathbf{t})\alpha}}(z) \neq \mathbb{V}_{Q_{\alpha m}^\mu}(z). \tag{3.11}$$

Therefore (3.10) is achieved from Remark 1.1(ii). One has $m_0^{(Q_m^\mu)^{\boxplus(\mathbf{t})\alpha}} = \alpha m_0^{Q_m^\mu} = \alpha m = m_0^{Q_{\alpha m}^\mu}$. Thus $\tau > 0$ exists such that $\mathbb{V}_{(Q_m^\mu)^{\boxplus(\mathbf{t})\alpha}}(\cdot)$ and $\mathbb{V}_{Q_{\alpha m}^\mu}(\cdot)$ are defined on $(\alpha m, \alpha m + \tau)$. We know from [2, Eq 53] that

$$\mathbb{V}_{Q_m^\mu}(y) = y^2\left(\frac{a_1^2 m_0^\mu}{y - m} + \frac{m_0^\mu}{m} - 1\right), \quad y \neq m. \tag{3.12}$$

Using (1.14) and (3.12), we have the following, for $z \in (\alpha m, \alpha m + \tau)$ and $0 < \alpha \neq 1$:

$$\begin{aligned}
\mathbb{V}_{(Q_m^\mu)^{\boxplus(\mathbf{t})\alpha}}(z) &= \alpha\mathbb{V}_{Q_m^\mu}(z/\alpha) + (1/\alpha - 1)z(z(1-b) + \alpha(b-a)m) \\
&= z\left(\frac{a_1^2 z m_0^\mu}{z - \alpha m} + z\left[\frac{m_0^\mu}{\alpha m} - \frac{1}{\alpha}\right]\right) + (1/\alpha - 1)z(z(1-b) + \alpha(b-a)m) \\
&\neq z\left(\frac{a_1^2 z m_0^\mu}{z - \alpha m} + z\left[\frac{m_0^\mu}{\alpha m} - 1\right]\right) = \mathbb{V}_{Q_{\alpha m}^\mu}(z).
\end{aligned}$$

This concludes the demonstration of (3.10). □

Next, we show that the $\boxplus_{(\mathbf{t})}$-convolution product of two CSK families remains a CSK family only for MP CSK families.

Theorem 3.3. *Let $\varrho_1, \varrho_2 \in \mathcal{P}_c$. Consider the family of measures $\mathcal{T}$ introduced by* (1.6)*. If $\mathcal{T}$ remains a CSK family (i.e., $\mathcal{T} = \mathcal{K}(\tau)$ for $\tau \in \mathcal{P}_c$), then, up to affinity, τ, ϱ_1 and ϱ_2 are of the MP type law.*

Proof. Suppose that $\mathcal{T} = \mathcal{K}(\tau)$ with $\tau \in \mathcal{P}_c$. (We may assume, without loss of generality, that $0 \in (m_-^\tau, m_+^\tau)$). In this case, $\forall\, r_1 \in (m_-^{\varrho_1}, m_+^{\varrho_1})$ and $\forall\, r_2 \in (m_-^{\varrho_2}, m_+^{\varrho_2})$ there exists $r \in (m_-^\tau, m_+^\tau)$, such that

$$Q_r^\tau = Q_{r_1}^{\varrho_1} \boxplus_{(\mathbf{t})} Q_{r_2}^{\varrho_2}. \tag{3.13}$$

That is,

$$\mathcal{R}^{(\mathbf{t})}_{Q_r^\tau}(\varpi) = \mathcal{R}^{(\mathbf{t})}_{Q_{r_1}^{\varrho_1}}(\varpi) + \mathcal{R}^{(\mathbf{t})}_{Q_{r_2}^{\varrho_2}}(\varpi), \qquad \varpi \text{ close to } 0. \tag{3.14}$$

Using (3.4), one may write

$$\mathcal{R}^{(\mathbf{t})}_{Q_r^\tau}(\varpi) = r + \mathcal{V}_\tau(r)\varpi + \frac{\varpi}{t}\epsilon(\varpi), \qquad \varepsilon(\varpi) \xrightarrow{\varpi\to 0} 0. \tag{3.15}$$

We also have

$$\mathcal{R}^{(\mathbf{t})}_{Q_{r_1}^{\varrho_1}}(\varpi) = r_1 + \mathcal{V}_{\varrho_1}(r_1)\varpi + \varpi\varepsilon_1(\varpi), \qquad \varepsilon_1(\varpi) \xrightarrow{\varpi\to 0} 0, \tag{3.16}$$

and

$$\mathcal{R}^{(\mathbf{t})}_{Q_{r_2}^{\varrho_2}}(\varpi) = r_2 + \mathcal{V}_{\varrho_2}(r_2)\varpi + \varpi\varepsilon_2(\varpi), \qquad \varepsilon_2(\varpi) \xrightarrow{\varpi\to 0} 0. \tag{3.17}$$

Combining (3.14)–(3.17), we obtain $r = r_1 + r_2$, and then

$$\mathcal{V}_\tau(r_1 + r_2) = \mathcal{V}_{\varrho_1}(r_1) + \mathcal{V}_{\varrho_2}(r_2). \tag{3.18}$$

According to [14], the relation (3.18) gives

$$\mathcal{V}_\tau(m) = \mathcal{V}_{\varrho_1}(m) = \rho m + \theta \quad \text{and} \quad \mathcal{V}_{\varrho_2}(m) = \rho m \quad \text{for} \quad \rho \neq 0 \quad \text{and} \quad \theta > 0. \tag{3.19}$$

Note that $\theta = \mathcal{V}_\tau(0)$ must be positive strictly. Furthermore, $m_0^{\varrho_2} \neq 0$; otherwise, $m \mapsto \mathcal{V}_{\varrho_2}(m) = \rho m$ cannot be a VF (see [16]). It is clear from (3.2) and (3.19) that τ, ϱ_1 and ϱ_2 follow, up to affinity, the MP type law. □

4. Conclusions

The theory of CSK families has emerged as a significant extension of NEFs into the realm of non-commutative probability, where classical independence is replaced by notions such as freeness. These families are characterized by a kernel-type parametrization using the Cauchy–Stieltjes transform, with the VF playing a central role in encoding the analytic and probabilistic properties of the family. While foundational work [7] has established the core structure of CSK families, recent developments in free harmonic analysis and non-commutative statistics have expanded the context for these studies. Notably, recent advances in free cumulant techniques [3], operator-valued free probability [29], and the interplay between free convolution semigroups and random matrix models [5, 18] have provided new tools and frameworks in the context of free harmonic analysis and CSK families. In the context of random matrix theory, the evolution of spectral distributions under certain dynamics (e.g., Dyson Brownian motion) and their description through subordination functions and Stieltjes transforms [4] illustrate the relevance of such analytic tools, strengthening the link between CSK theory and contemporary models of large-dimensional behavior.

One of the key motivations in this area is understanding how such families behave under natural transformations, particularly the generalized t-transformations, which encompass dilations, convolutions, and more abstract deformations of measures. Understanding the structural robustness of CSK families in free probability requires their stability under generalized t-deformation. These families are crucial for describing noncommutative distributions because of their VFs. The preservation of fixed-point structures determines their stability in the presence of generalized t-transformation. This stability emphasizes the value of CSK families in modeling free convolution processes by ensuring that the important probabilistic and spectral properties are maintained during deformation. In this study, we have used the generalized t-transformation of measures to illustrate various features pertaining to the FG and ISC measures. In order to get different MP law properties, we have additionally employed the generalized t-transformation of free convolution. These studies contribute to our understanding of noncommutative probability, which has implications for operator algebras, random matrix theory, and free harmonic analysis.

The stability of CSK families under generalized t-transformations carries significant implications for applications in engineering and physics. In signal processing and communications, where non-Gaussian noise and correlated signals frequently arise, CSK distributions are stable under such transformations, providing accurate models while preserving the structural properties under operations like filtering or superposition, thereby simplifying both analysis and prediction. In physics, particularly in random matrix theory, CSK families are essential for describing the spectral behavior of large ensembles, and their invariance under generalized t-transformations ensures that these spectral laws remain predictable under perturbations, which is crucial for modeling complex quantum systems. Likewise, in control and reliability engineering, stable CSK distributions facilitate closed–form expressions for variances and higher-order moments after transformations, supporting robust system design and risk assessment. Beyond these general applications, stability under generalized t-transformations naturally highlights the importance of specific free probability measures like the MP, FG, and ISC laws in modeling complex stochastic systems. The MP law, for example, describes the limiting spectral law of random covariance matrices and is widely applied in multi–antenna communication systems to analyze channel capacity and interference. The FG distribution proves useful in stochastic models dominated by multiplicative or growth–type randomness, including queuing networks, reliability studies, and energy landscapes in physics. The ISC distribution, by contrast, captures the singular or heavy–tailed effects encountered in anomalous diffusion, turbulence, or inverse random operators, and its invariance under transformations makes it particularly relevant for strongly nonlinear systems. Taken together, these examples show that the invariance of CSK families is not only mathematically elegant but also provides a structurally robust foundation for modeling randomness across diverse domains, thereby bridging abstract free probability with practical challenges in physics and engineering. However, placing our findings within the context of the original t-deformation proposed by Bożejko and Wysoczański [11, 12] is critical for defining the breadth of our contribution. The generalized t-transformation we investigate broadens the traditional t-deformation in two main dimensions. First, whereas the original t-deformation was described in terms of the convolution structures associated with free and Boolean probability, the extended version is constructed using the Cauchy–Stieltjes kernel technique, which allows for a larger range of measures to be included. Second, the generalized setup allows for a more flexible definition of the VFs and stability qualities, making it possible to accommodate families of distributions that are not covered by the initial t-deformation. The relationship between the original t-deformation and the extended version in exceptional circumstances when they coincide, as well as the additional classes of distributions and transformations, is found exclusively in the generalized framework. This comparison demonstrates that our findings not only recover the Bożejko–Wysoczański structure as a specific example, but also apply it to a broader context.

On the other hand, VFs provide a unifying perspective in both classical and free probability, and our results can be naturally benchmarked against these two frameworks. In the classical setting, VFs play a central role in the theory of NEFs, where the classification of quadratic and cubic VFs [27,28] identifies a small number of distinguished distributions, such as the Gaussian, Poisson, and Gamma distributions, that remain stable under convolution or scaling. In free probability, an analogous role is played by CSK families, where the VF again serves as a key structural tool. In this broader context, our results extend the principle that invariance under transformations of measures is tightly linked to the functional form of the VF. Just as in the classical case where only a few NEFs admit simple variance structures, in the free setting, we show that the stability of CSK families under V_a and (a, b)-transformations singles out only specific laws, namely the semicircle, Marchenko–Pastur, inverse semicircle–type, and free Gamma–type distributions. This parallel reinforces the view that VFs provide a fundamental bridge between classical and free probability, while our findings enlarge the catalog of free distributions uniquely determined by their invariance properties.

Author contributions

Fatimah Alshahrani: Conceptualization, methodology, software, supervision, validation, resources, project administration; Raouf Fakhfakh: Formal analysis, investigation, writing-review and editing, visualization, writing-original draft preparation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest regarding this manuscript.

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