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Research article

Brøndsted's fixed point theorem in LUR spaces

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Abstract: In this paper, we establish novel characterizations of locally uniformly rotund (LUR) Banach spaces. Building on these results, we prove that Zubelevich's relaxed version of Brøndsted's fixed point theorem for uniformly rotund spaces holds in the strictly broader class of LUR spaces. As an illustration, we derive existence results for fixed points of directional operators on Orlicz spaces under conditions where neither Brøndsted's nor Zubelevich's theorems apply.

Keywords: Brøndsted fixed point theorem; directional operator; locally uniformly rotund space; Orlicz space; renorming; uniformly rotund space
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1. Introduction

Throughout this paper, $(X, \|\cdot\|)$ is assumed to be a Banach space with closed unit ball B.

Let $M \subset X$ be a closed set such that $\|x\| > 1$ for all $x \in M$, and let $T : M \to M$ map each $x \in M$ in the direction of B, i.e., if $Tx \neq x$, then $x + t(Tx - x) \in B$ for some $t > 1$. Define a partial order on M as follows: for $x, y \in M$, write $x \preceq y$ if either $x = y$ or there exists $t > 1$ such that $x + t(y - x) \in B$. With this definition, the above condition on T can be written as $x \preceq Tx$ for all $x \in M$.

The following fixed point theorem is due to Brøndsted [1]:

Theorem 1.1. *Let M be a closed subset of a Banach space X, such that*

$$\inf\{\|y\| : y \in M\} > 1. \tag{1.1}$$

If the mapping $T : M \to M$ satisfies $x \preceq Tx$ for all $x \in M$, then T has a fixed point.

Quite recently, Zubelevich [2] demonstrated that, in uniformly rotund spaces, the hypotheses of Brønsted's theorem can be relaxed. For convenience, let us define those spaces through their characterization in [3, Proposition 5.2.8].

Definition 1.2. *The Banach space* $(X, \|\cdot\|)$ *is uniformly convex or uniformly rotund (UR) provided that whenever* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are sequences with* $\|x_n\| = \|y_n\| = 1$ $(n \in \mathbb{N})$ *and* $\lim_{n\to\infty} \|x_n + y_n\| = 2$*, it follows that* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Examples of UR spaces are Hilbert spaces and the Lebesgue spaces L^p (in particular, the sequence spaces ℓ^p), for $1 < p < \infty$; see, e.g., [4, p. 189 and Proposition 3.II.8]. More information on UR Banach spaces in connection with fixed point theory can be found, e.g., in [5].

Theorem 1.3. *([2, Theorem 1.4]) Assume* X *is a UR Banach space, and let* M *be a closed subset of* X *such that* $M \cap B = \emptyset$*. If the mapping* $T : M \to M$ *satisfies* $x \preceq Tx$ *for all* $x \in M$*, then* T *has a fixed point.*

Note that the hypothesis (1.1) in Theorem 1.1 is stronger than the hypothesis $M \cap B = \emptyset$ in Theorem 1.3 only in some infinite-dimensional spaces. In finite-dimensional spaces, the two conditions are equivalent due to the compactness of the closed unit ball B, which ensures a positive distance between the closed set M and B. If M is assumed to be weakly (hence, strongly) closed, they are also equivalent in infinite-dimensional reflexive spaces. In fact, in a reflexive Banach space, B is weakly compact. If $M \cap B = \emptyset$, then the distance δ between the weakly compact set B and the weakly closed set M is strictly positive; this implies (1.1), as for any $y \in M$ and $x = y/\|y\| \in B$, we have $\|y\| - 1 = \|y - x\| \geq \delta > 0$. Since UR spaces are reflexive [3, Theorem 5.2.15], under the hypothesis that M is weakly closed, Zubelevich's Theorem 1.3 becomes a special case of Brønsted's Theorem 1.1.

Motivated by Zubelevich's work [2], in Section 2 of this paper we introduce and investigate the class of Banach spaces termed locally uniformly rotund by segments (LURS), where Zubelevich's fixed point theorem holds. In Section 3, we show that local uniform rotundity (LUR) implies LURS, thereby establishing LUR, rather than UR, as a natural framework for Zubelevich's theorem and extending the applicability of Zubelevich's result beyond UR spaces. The case studied here is more general, since LUR norms are less restrictive than UR ones; in particular, LUR spaces need not be reflexive. Finally, in Section 4, we illustrate our advances by proving the existence of fixed points for directional operators on Orlicz spaces in settings where Brøndsted's and Zubelevich's theorems are inapplicable.

2. LURS spaces

Definition 2.1. *A Banach space* X *is said to be locally uniformly rotund for segments (LURS) provided that for every* $a \in X$*, with* $\|a\| = 1$*, and every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that whenever* $x \in X$ *satisfies*

(i) $1 < \|x\| \leq 1 + \delta$ *and*
(ii) $\|ta + (1-t)x\| > 1$ *for all* $t \in (0, 1)$*,*

we have $\|x - a\| \leq \varepsilon$*.*

This definition is inspired by [2, Proposition 2.1], where it is shown that UR spaces are LURS. Although Zubelevich did not use such a denomination, he gave this property a nice physical interpretation. Imagine a light source at a position x, just outside a spherical screen (the unit ball B), casting light onto a point a on the screen's surface ($\|a\| = 1$). If the light source is close to the screen ($1 < \|x\| \leq 1+\delta$) and the beam's path from x to a ($ta + (1-t)x$, $t \in (0,1)$) is beyond the screen's surface (norm greater than 1), then the light source must be nearly directly above the illuminated point a, meaning the distance between x and a is small ($\|x - a\| \leq \varepsilon$). Thus, LURS Banach spaces are characterized by the geometric constraint that the beam being beyond the screen forces the light source to align closely with the point it illuminates.

Zubelevich asserts that uniform rotundity is essential for the validity of his Proposition 2.1. However, this claim is inaccurate. LUR spaces are also LURS (Corollary 3.2 below), yet not all LUR spaces are UR (see [6] or Section 4 of this paper). Let us recall the definition of the LUR class (cf. [3, Definition 5.3.2 and Proposition 5.3.5]):

Definition 2.2. *The Banach space $(X, \|\cdot\|)$ is locally uniformly convex or locally uniformly rotund (LUR), if for each $a \in X$, with $\|a\| = 1$, and every sequence $\{x_n\}_{n=1}^{\infty} \subset X$, with $\|x_n\| = 1$ ($n \in \mathbb{N}$) and $\lim_{n\to\infty} \|x_n + a\| = 2$, we have $\lim_{n\to\infty} \|x_n - a\| = 0$.*

It turns out that the proof of Zubelevich's relaxed version (Theorem 1.3) of Brønsted's theorem (Theorem 1.1) depends only on the space being LURS, which leads to the following:

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a LURS Banach space, and let $M \subset X$ be a nonempty, closed set such that $M \cap B = \emptyset$. Let $T : M \to M$ be a map with the property that for any $x \in M$ such that $Tx \neq x$, there exists $t > 1$ with $x + t(Tx - x) \in B$. Then, T has a fixed point.*

The proof of Theorem 2.3, which we give below, adheres to the scheme in [2], using Zorn's lemma to find a maximal element in M, which will be a fixed point of T. Nevertheless, since some of the arguments in [2] are not explicit, we include full proofs for completeness.

The following two lemmas help clarify the partial order defined on M.

Lemma 2.4. *Let $N \subset X$ be such that $\|x\| > 1$ for all $x \in N$. Then, the binary relation $\preceq$ is a partial order in N.*

Proof. It is easily seen that the definition of $\preceq$ can be restated as follows: Given $x, y \in N$, we have that $x \preceq y$ if either $x = y$ or there exist $a \in B$ and $t \in (0,1)$ such that $y = ta + (1-t)x$.

To check reflexivity, antisymmetry, and transitivity, let $x, y, z \in N$.

(i) It is apparent that $x \preceq x$. Therefore, $\preceq$ is reflexive.

(ii) Assume $x \preceq y$, $y \preceq x$, $x \neq y$, so:

$$y = \beta a + (1-\beta)x, \quad x = \gamma b + (1-\gamma)y, \quad \|a\| \leq 1,\ \|b\| \leq 1, \quad \beta, \gamma \in (0,1).$$

Then,

$$x = \gamma b + (1-\gamma)[\beta a + (1-\beta)x],$$

$$x = \frac{(1-\gamma)\beta a + \gamma b}{(1-\gamma)\beta + \gamma},$$

$$\|x\| \leq \frac{(1-\gamma)\beta \|a\| + \gamma \|b\|}{(1-\gamma)\beta + \gamma} \leq \frac{(1-\gamma)\beta + \gamma}{(1-\gamma)\beta + \gamma} = 1,$$

contradicting $\|x\| > 1$. Thus, $x = y$ and $\preceq$ is antisymmetric.

(iii) Suppose $x \preceq y$, $y \preceq z$. If $x = y$ and $y = z$, there is nothing to prove. If $x = y$ but $y \neq z$, then $x \preceq z$ follows easily:

$$x = y = \beta a + (1-\beta)z, \quad \|a\| \leq 1, \quad \beta \in (0,1),$$

and similarly if $x \neq y$ but $y = z$. Thus, assume $x \neq y$, $y \neq z$, so that

$$y = \beta a + (1-\beta)x, \quad z = \gamma b + (1-\gamma)y, \quad \|a\| \leq 1,\ \|b\| \leq 1, \quad \beta, \gamma \in (0,1).$$

Then,

$$z = \gamma b + (1-\gamma)\beta a + (1-\beta)(1-\gamma)x.$$

Letting

$$d = \frac{\gamma b + (1-\gamma)\beta a}{\gamma + (1-\gamma)\beta}, \quad \alpha = \gamma + (1-\gamma)\beta \in (0,1),$$

we find

$$\|d\| = \left\|\frac{\gamma b + (1-\gamma)\beta a}{\gamma + (1-\gamma)\beta}\right\| \leq \frac{\gamma\,\|b\| + (1-\gamma)\beta\,\|a\|}{\gamma + (1-\gamma)\beta} \leq \frac{\gamma + (1-\gamma)\beta}{\gamma + (1-\gamma)\beta} = 1$$

and

$$1 - \alpha = (1-\beta)(1-\gamma).$$

Hence,

$$z = \alpha d + (1-\alpha)x$$

shows that $x \preceq z$. So, $\preceq$ is transitive and the proof is complete. □

Lemma 2.5. *Let $M \subset X$ be a closed set such that $M \cap B = \emptyset$. For $x, y \in M$, $x \neq y$, the following conditions are equivalent:*

(i) $x \preceq y$.
(ii) *There exists $t > 1$ such that $x + t(y-x) \in B$.*
(iii) *There exist $\tilde{t} > 1$ and $a \in X$, with $\|a\| = 1$, such that $x + \tilde{t}(y-x) = a$ and $x + t(y-x) \notin B$ for all $t < \tilde{t}$.*
(iv) *There exist $a \in X$, with $\|a\| = 1$, and $\tau \in (0,1)$, such that $y = \tau a + (1-\tau)x$ and $\|sa + (1-s)x\| > 1$ for all $s \in (0,1)$.*

Proof. (i) $\Leftrightarrow$ (ii) This is just the definition of the partial order $\preceq$.

(iii) $\Rightarrow$ (ii) If (iii) holds, then $x + \tilde{t}(y-x) = a \in B$, so ($ii$) holds with $t = \tilde{t}$.

(ii) $\Rightarrow$ (iii) Define the function $f : [0,\infty) \to [0,\infty)$ by

$$f(t) = \|x + t(y-x)\| \quad (t \geq 0).$$

Since $t \mapsto x + t(y-x)$ is a linear function in t, and the norm $\|\cdot\|$ is continuous, $f(t)$ is continuous. Thus, the set

$$E = \{t \geq 1 : \|x + t(y-x)\| \leq 1\} = f^{-1}([0,1]) \cap [1,\infty)$$

is nonempty (by (*ii*)), bounded below by 1, and closed. Therefore, there exists $\tilde{t} = \min E$, with $\tilde{t} \geq 1$ and

$$f(\tilde{t}) = \left\| x + \tilde{t}(y - x) \right\| \leq 1. \tag{2.1}$$

Actually, $\tilde{t} > 1$, as otherwise we would have $f(1) = \|y\| \leq 1$, contradicting the assumption that $y \in M$ with $M \cap B = \emptyset$.

Let $a = x + \tilde{t}(y - x)$. From (2.1), $\|a\| \leq 1$. If $\|a\| < 1$, since f is continuous at $\tilde{t}$, there exists a neighborhood of $\tilde{t} > 1$ where $f(t) < 1$. Thus, there exists t' with $1 \leq t' < \tilde{t}$ and $f(t') < 1$, which contradicts $\tilde{t} = \min E$. Therefore, $\|a\| = 1$. Moreover, for all $t < \tilde{t}$ we have $t \notin E$, so that $\|x + t(y - x)\| > 1$ and $x + t(y - x) \notin B$. This completes the proof that (*ii*) implies (*iii*).

(*iii*) $\Rightarrow$ (*iv*) Assume there exist $\tilde{t} > 1$ and $a \in X$, $\|a\| = 1$, such that $x + \tilde{t}(y - x) = a$ and $x + t(y - x) \notin B$ for all $t < \tilde{t}$. In particular,

$$y - x = \frac{a - x}{\tilde{t}} \quad \text{and} \quad y = \frac{a + (\tilde{t} - 1)x}{\tilde{t}} = \frac{a}{\tilde{t}} + \left(1 - \frac{1}{\tilde{t}}\right)x,$$

that is, $y = \tau a + (1 - \tau)x$, with

$$\tau = \frac{1}{\tilde{t}} \in (0, 1).$$

Now, given any $s \in (0, 1)$, set $t = s\tilde{t} < \tilde{t}$. As $\|x + t(y - x)\| > 1$, we find that

$$sa + (1 - s)x = \frac{t}{\tilde{t}}a + \left(1 - \frac{t}{\tilde{t}}\right)x = \frac{ta + (\tilde{t} - t)x}{\tilde{t}} = x + t\,\frac{a - x}{\tilde{t}} = x + t(y - x)$$

satisfies $\|sa + (1 - s)x\| > 1$, which establishes (*iv*).

(*iv*) $\Rightarrow$ (*iii*) Assume (*iv*) holds. Let $y = \tau a + (1 - \tau)x$, with $a \in X$, $\|a\| = 1$, $\tau \in (0, 1)$, and $\|sa + (1 - s)x\| > 1$ for all $s \in (0, 1)$. Let $\tilde{t} = 1/\tau > 1$. Then,

$$x + \tilde{t}(y - x) = x + \frac{1}{\tau}[\tau a + (1 - \tau)x - x] = a.$$

Moreover, for $t < \tilde{t}$, $x + t(y - x) \notin B$, because

$$x + t(y - x) = x + t[\tau a + (1 - \tau)x - x] = \tau t a + (1 - \tau t)x = sa + (1 - s)x,$$

with $0 < s = \tau t < \tau\tilde{t} = 1$, implies $x + t(y - x) > 1$.

The proof is thus complete. □

From now on, M will denote a closed subset of X such that $M \cap B = \emptyset$, endowed with the partial order $\preceq$. Let $C \subset M$ be a chain with respect to $\preceq$ (that is, for every $x, y \in C$, either $x \preceq y$ or $y \preceq x$), and put

$$\rho = \inf\{\|u\| : u \in C\},$$

so that $\rho \geq 1$. Then, $x \in C$ implies $\|x\| > 1$ whenever $\rho = 1$, and $\|x\| \geq \rho > 1$ whenever $\rho > 1$.

For any $x \in C$, define

$$K_x(\rho) = \{y \in M : x \preceq y,\ \|y\| \geq \rho\}.$$

The sets $K_x(\rho)$ are nonempty, because $x \in K_x(\rho)$. Furthermore, $x_1, x_2 \in C$ and $x_1 \preceq x_2$ implies $K_{x_2}(\rho) \subset K_{x_1}(\rho)$.

Lemma 2.6. *Let C be a chain in M and $\rho = \inf\{\|u\| : u \in C\}$. For any $x \in C$, the sets $K_x(\rho)$ are closed.*

Proof. Let $\{y_n\}_{n=1}^{\infty} \subset K_x(\rho)$ be such that $\lim_{n\to\infty} y_n = y \in M$. The continuity of the norm ensures that $\|y\| = \lim_{n\to\infty} \|y_n\| \geq \rho$. Since $x \leq y_n$, Lemma 2.5 yields $a_n \in X$, with $\|a_n\| = 1$, and $\tau_n \in (0,1)$, such that

$$y_n = \tau_n a_n + (1-\tau_n)x \quad (n \in \mathbb{N}). \tag{2.2}$$

By the Bolzano-Weierstrass theorem, passing to a subsequence if necessary, we may assume that $\{\tau_n\}_{n=1}^{\infty}$ converges to some $\tau \in [0,1]$. If $\tau = 0$, then the estimate

$$\|y_n - x\| = \tau_n \|a_n - x\| \leq \tau_n(1 + \|x\|) \quad (n \in \mathbb{N})$$

entails $y = x \in K_x(\rho)$. If $\tau \in (0,1)$, define

$$a_n = \frac{1}{\tau_n} y_n + \left(1 - \frac{1}{\tau_n}\right)x, \quad a = \frac{1}{\tau} y + \left(1 - \frac{1}{\tau}\right)x,$$

so that $\lim_{n\to\infty} a_n = a$. From $\|a_n\| = 1$ $(n \in \mathbb{N})$, we obtain $\|a\| = 1$ as well. In particular, $x + t(y - x) = a \in B$, with $t = 1/\tau > 1$; this shows that $x \leq y$ and proves that $y \in K_x(\rho)$. Finally, the case $\tau = 1$ must be excluded, as $y \in M$ forces $\|y\| > 1$, while assuming $\tau = 1$ and taking limits in (2.2) would imply $y = a$, with $\|a\| = 1$. □

Lemma 2.7. *Let C be a chain in M and $\rho = \inf\{\|u\| : u \in C\}$. Fix $x \in C$ and $z \in K_x(\rho)$.*

(i) If $\rho > 1$, then:

$$\|z - x\| \leq (\|x\| - \rho)\,\frac{\|x\| + 1}{\rho - 1}.$$

(ii) If X is LURS and $\rho = 1$, then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x\| \leq 1 + \delta$ implies $\|z - x\| \leq \varepsilon$.

Proof. Assume $\rho > 1$. If $z = x$, the estimate is obvious. Otherwise, Lemma 2.5 yields $t > 1$ and $a \in X$, $\|a\| = 1$, such that $x + t(z - x) = a$. Then,

$$z = \frac{a + (t-1)x}{t},$$

$$\rho \leq \|z\| \leq \frac{1}{t} + \frac{t-1}{t}\,\|x\|,$$

$$\frac{1}{t} \leq \frac{\|x\| - \rho}{\|x\| - 1},$$

whence

$$\|z - x\| = \frac{1}{t}\,\|a - x\| \leq \frac{1}{t}\,(\|x\| + 1) \leq \frac{\|x\| - \rho}{\|x\| - 1}(\|x\| + 1) \leq (\|x\| - \rho)\frac{\|x\| + 1}{\rho - 1}.$$

Now, assume $\rho = 1$. If $z \neq x$ (otherwise, the estimate is, again, obvious), then, by Lemma 2.5, there exist $a \in X$, with $\|a\| = 1$, and $\tau \in (0,1)$ such that $z = \tau a + (1-\tau)x$ and $\|ta + (1-t)x\| > 1$ for all $t \in (0,1)$. Since X is LURS and $\|x\| > 1$, given $\varepsilon > 0$, Definition 2.1 guarantees the existence of $\delta > 0$ such that $\|x\| \leq 1 + \delta$ implies $\|x - a\| \leq \varepsilon$, so

$$\|z - x\| = \tau\,\|a - x\| \leq \varepsilon,$$

as asserted. □

Lemma 2.8. *Suppose X is LURS. Let C be a chain in M and $\rho = \inf\{\|u\| : u \in C\}$. Then, for any $\varepsilon > 0$, there exists $\tilde{x} \in C$ such that $x \in C$ and $\tilde{x} \preceq x$ implies* $\operatorname{diam} K_x(\rho) \leq \varepsilon$.

Proof. Assume $\rho > 1$. From the definition of infimum, given $\varepsilon > 0$, there exists $\tilde{x} \in C$ such that $\|\tilde{x}\| \leq \rho + \varepsilon$. Then, for any $z_1, z_2 \in K_{\tilde{x}}(\rho)$, Lemma 2.7 allows us to write

$$\|z_1 - z_2\| \leq \|z_1 - \tilde{x}\| + \|z_2 - \tilde{x}\| \leq 2\left(\|\tilde{x}\| - \rho\right) \frac{\|\tilde{x}\| + 1}{\rho - 1} \leq 2\varepsilon \frac{\rho + \varepsilon + 1}{\rho - 1}.$$

The right-hand side of the above estimate can be made arbitrarily small. To complete the proof of this case, it suffices to recall that $\tilde{x}, x \in C$ with $\tilde{x} \preceq x$ implies $K_x(\rho) \subset K_{\tilde{x}}(\rho)$, so $\operatorname{diam} K_x(\rho) \leq \operatorname{diam} K_{\tilde{x}}(\rho)$.

Now, assume $\rho = 1$. Given $\varepsilon > 0$, Lemma 2.7 yields $\delta > 0$ such that $\tilde{x} \in C$, $\|\tilde{x}\| \leq 1 + \delta$ and $z \in K_{\tilde{x}}(1)$ implies $\|z - \tilde{x}\| \leq \varepsilon$. The existence of such an $\tilde{x}$ is guaranteed by the definition of ρ as an infimum. At this point, the same argument as in the case $\rho > 1$ completes the proof. □

Proof of Theorem 2.3. Given any chain $C \subset M$, let $\rho = \inf\{\|u\| : u \in C\}$ and consider the nested family $\{K_x(\rho)\}_{x \in C}$, consisting of closed sets (Lemma 2.6) whose diameters tend to zero (Lemma 2.8). By the celebrated Cantor theorem, their intersection is a singleton:

$$\bigcap_{x \in C} K_x(\rho) = \{m\}.$$

The point $m \in M$ is an upper bound for C. Indeed, for any $x \in C$ we have $m \in K_x(\rho)$, and therefore $x \preceq m$.

Thus, every chain in the nonempty poset $(M, \preceq)$ has an upper bound. By Zorn's lemma, M contains, at least, one maximal element m^*. The hypothesis on T is that $x \preceq Tx$ for every $x \in M$. In particular, $m^* \preceq Tm^*$, so maximality forces $Tm^* = m^*$ and completes the proof. □

3. LUR spaces

In this section, we demonstrate that the class of LURS Banach spaces (Definition 2.1) is broader than the class of LUR Banach spaces (Definition 2.2); this is the content of Corollary 3.2. The proof of this result hinges on Proposition 3.1, which establishes two characterizations of LUR spaces. Although these characterizations might be known, we have been unable to identify direct references in the literature.

Proposition 3.1. *Let X be a Banach space. The following are equivalent:*

(i) X is LUR.

(ii) For every $a \in X$, with $\|a\| = 1$, and every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$, with $1 < \|x\| \leq 1 + \delta$ and

$$\left\|\frac{x + a}{2}\right\| > 1 - \delta,$$

implies $\|x - a\| \leq \varepsilon$.

(iii) For every $a \in X$, with $\|a\| = 1$, and every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$, with $1 < \|x\| \leq 1 + \delta$ and

$$\left\|\frac{x + a}{2}\right\| > 1, \tag{3.1}$$

implies $\|x - a\| \leq \varepsilon$.

Proof. (i) $\Rightarrow$ (ii) Assume X is LUR. Then, for every $a \in X$, with $\|a\| = 1$, and every sequence $\{b_n\}_{n=1}^{\infty}$ with $\|b_n\| = 1$ ($n \in \mathbb{N}$), such that $\lim_{n\to\infty} \|b_n + a\| = 2$, we must have $\lim_{n\to\infty} \|b_n - a\| = 0$.

Given any $a \in X$, with $\|a\| = 1$, and any $\varepsilon > 0$, we need to find $\delta > 0$ such that for all $x \in X$ with $1 < \|x\| \le 1 + \delta$ and $\|x + a\| > 2(1 - \delta)$, there holds $\|x - a\| \le \varepsilon$. Arguing by contradiction, suppose there exist $a \in X$, with $\|a\| = 1$, and $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, some $x_n \in X$ satisfies $1 < \|x_n\| \le 1 + 1/n$ and

$$\left\|\frac{x_n + a}{2}\right\| > 1 - \frac{1}{n}, \tag{3.2}$$

but

$$\|x_n - a\| > 2\varepsilon. \tag{3.3}$$

For each $n \in \mathbb{N}$, let $b_n = x_n/\|x_n\|$. Then, $\|b_n\| = 1$ and $x_n = (1 + r_n)\, b_n$, where $0 < r_n = \|x_n\| - 1 \le 1/n$, so that $\lim_{n\to\infty} r_n = 0$. From (3.2) and $b_n = x_n - r_n b_n$, we thus have:

$$1 \ge \left\|\frac{b_n + a}{2}\right\| = \left\|\frac{x_n + a - r_n b_n}{2}\right\| \ge \left\|\frac{x_n + a}{2}\right\| - \frac{\|r_n b_n\|}{2} > 1 - \frac{1}{n} - \frac{r_n}{2} \to 1 \quad \text{as } n \to \infty.$$

By the LUR property, this implies

$$\lim_{n\to\infty} \|b_n - a\| = 0. \tag{3.4}$$

However, from (3.3) and $b_n = x_n - r_n b_n$, it follows that

$$\|b_n - a\| \ge \|x_n - a\| - \|r_n b_n\| > 2\varepsilon - r_n \quad (n \in \mathbb{N}).$$

For large n, $r_n < \varepsilon$, so $\|b_n - a\| > \varepsilon$, contradicting (3.4).

(ii) $\Rightarrow$ (iii) This is apparent.

(iii) $\Rightarrow$ (i) Assume condition (iii) holds. To show X is LUR, let $a \in X$ with $\|a\| = 1$, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $\|x_n\| = 1$ ($n \in \mathbb{N}$) and $\lim_{n\to\infty} \|x_n + a\| = 2$. Suppose, for contradiction, that $\lim_{n\to\infty} \|x_n - a\| > 0$. Then, passing to a subsequence if necessary, some $\varepsilon_0 > 0$ is such that $\|x_n - a\| \ge \varepsilon_0$ ($n \in \mathbb{N}$).

Let $0 < \varepsilon' < \min\{1, \varepsilon_0/2\}$. By condition ($iii$), there exists $\delta' > 0$ such that if $x \in X$ satisfies $1 < \|x\| \le 1 + \delta'$ and $\|x + a\| > 2$, then $\|x - a\| \le \varepsilon'$.

Choose $\delta > 0$ such that $\delta \le \delta'$ and $\eta = \delta/(2 + \delta) < \varepsilon'$. Since $\lim_{n\to\infty} \|x_n + a\| = 2$, for sufficiently large n we must have

$$\|x_n + a\| > 2 - \frac{\eta}{2}.$$

Define $y_n = (1 + \eta)x_n$ ($n \in \mathbb{N}$). We verify that, for large enough n, y_n satisfies the estimates of (iii), with respect to δ' and ε'. In the first place,

$$1 < \|y_n\| = (1 + \eta)\,\|x_n\| = 1 + \eta = 1 + \frac{\delta}{2 + \delta} \le 1 + \delta \le 1 + \delta'.$$

Moreover, for large n,

$$\|y_n + a\| = \|(1 + \eta)x_n + a\| \ge (1 + \eta)\,\|x_n + a\| - \eta\,\|a\|$$

$$> (1+\eta)\left(2-\frac{\eta}{2}\right)-\eta = 2+2\eta-\frac{\eta}{2}-\frac{\eta^2}{2}-\eta$$
$$= 2+\frac{\eta}{2}-\frac{\eta^2}{2} = 2+\frac{\eta}{2}(1-\eta) > 2,$$

because $\eta < 1$. Therefore, provided n is large enough, (*iii*) ensures that $\|y_n - a\| \le \varepsilon'$. Hence,

$$\|x_n - a\| = \|y_n - a - \eta x_n\| \le \|y_n - a\| + \eta\,\|x_n\|$$
$$\le \varepsilon' + \eta < 2\varepsilon' < \varepsilon_0.$$

Thus, $\|x_n - a\| < \varepsilon_0$ for sufficiently large n, contradicting $\|x_n - a\| \ge \varepsilon_0$ for all $n \in \mathbb{N}$. This contradiction shows that $\lim_{n\to\infty} \|x_n - a\| = 0$, so X is LUR and the proof is complete. □

Corollary 3.2. *UR Banach spaces are LUR, and LUR Banach spaces are LURS.*

Proof. The first assertion is well known [3, Proposition 5.3.3]. The second assertion follows from the fact that the segment condition (*ii*) in Definition 2.1 implies the midpoint condition (3.1), which is obtained by particularizing $t = 1/2$. □

Corollary 3.3. *Let $(X, \|\cdot\|)$ be a LUR Banach space, let $M \subset X$ be a nonempty, closed set, and let $T : M \to M$ be a map such that, for each $x \in M$, $\|x\| > 1$ and $x \preceq Tx$, that is, either $Tx = x$ or there exists $t > 1$ with $\|tTx + (1-t)x\| \le 1$. Then, T has a fixed point.*

Proof. Combine Theorem 2.3 and Corollary 3.2. □

Remark 3.4. *Corollary 3.2 implies [2, Proposition 2.1].*

Remark 3.5. *We do not know whether LURS Banach spaces are LUR. However, it is not difficult to prove that LURS implies rotundity [3, Definition 5.1.1] and the Kadec–Klee property [7, Definition II.1.1]. Consequently, by Troyanksi's theorem [7, Corollary IV.3.6], every LURS Banach space admits a LUR renorming.*

4. An example in Orlicz space

In this section, we construct an example where our results apply, but Brøndsted's Theorem 1.1 and Zubelevich's Theorem 1.3 do not. Our ambient space will be an Orlicz space $X = L_\Phi[0,1]$ (see, e.g., [8]), which is LUR but not UR.

4.1. The Orlicz space $L_\Phi[0,1]$

Recall that an Orlicz function $\Phi : \mathbb{R} \to [0,\infty)$ is a convex, even function, continuous at 0, satisfying $\Phi(0) = 0$, $\Phi(t) > 0$ for $t \ne 0$, and $\lim_{|t|\to\infty} \Phi(t)/|t| = \infty$. Consider the Orlicz function

$$\Phi(t) = (1+|t|)\ln(1+|t|) - |t| \quad (|t| \in \mathbb{R}).$$

Consider also the finite and atomless measure space obtained by restricting the Lebesgue measure to the interval $[0,1]$. The Orlicz space $L_\Phi[0,1]$ consists of all those measurable functions $f : [0,1] \to \mathbb{R}$ such that the modular

$$I_\Phi\left(\frac{f}{\lambda}\right) = \int_0^1 \Phi\left(\frac{|f(x)|}{\lambda}\right) dx$$

is finite for some $\lambda > 0$, endowed with the Luxemburg norm

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : \int_0^1 \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\} \quad (f \in L_\Phi[0,1]).$$

We claim that $L_\Phi[0,1]$ is LUR but not UR.

On the one hand, according to [9, Theorem 1], $L_\Phi[0,1]$ is LUR provided that Φ is strictly convex on $\mathbb{R}$ and satisfies the Δ_2-condition, i.e., there exist $k, t_0 > 0$ such that $\Phi(2t) \le k\Phi(t)$ for $|t| \ge t_0$, with $\Phi(t_0) > 0$. Indeed, Φ is strictly convex on $\mathbb{R}$ because

$$\Phi''(t) = \frac{1}{1+|t|} > 0 \quad (t \in \mathbb{R}).$$

Furthermore, Φ satisfies the Δ_2-condition: as $\Phi(t) \sim |t| \ln |t|$ asymptotically as $|t| \to \infty$, we obtain

$$\lim_{|t|\to\infty} \frac{\Phi(2t)}{\Phi(t)} = \lim_{|t|\to\infty} \frac{|2t| \ln |2t|}{|t| \ln |t|} = 2.$$

Consequently, $L_\Phi[0,1]$ is LUR.

On the other hand, according to [10, Theorem 3], $L_\Phi[0,1]$ is UR provided that Φ is strictly convex on $\mathbb{R}$, meets the Δ_2-condition, and is uniformly convex for large arguments, meaning that there exists $t_0 > 0$ such that for all $x \in (0,1)$ and some $\delta_x \in (0,1)$,

$$\Phi\left(\frac{(1+x)t}{2}\right) \le (1-\delta_x)\,\frac{\Phi(t)+\Phi(xt)}{2} \tag{4.1}$$

whenever $|t| \ge t_0$. We assert that Φ fails the latter requirement. In fact, fix $x \in (0,1)$. Given the asymptotic behavior

$$\Phi(t) \sim |t| \ln |t|, \quad \Phi(xt) \sim x|t| \ln(x|t|) = x|t| \ln x + x|t| \ln |t|,$$
$$\Phi\left(\frac{(1+x)t}{2}\right) \sim \frac{(1+x)|t|}{2} \ln\left(\frac{(1+x)|t|}{2}\right),$$

as $|t| \to \infty$, the ratio of the left-hand side to the right-hand side of (4.1) is found to be

$$\frac{\Phi\left(\frac{(1+x)t}{2}\right)}{\frac{\Phi(t)+\Phi(xt)}{2}} \sim \frac{\ln |t| + \ln\left(\frac{1+x}{2}\right)}{\ln |t| + \frac{x \ln x}{1+x}} \quad \text{as } |t| \to \infty.$$

Hence,

$$\lim_{|t|\to\infty} \frac{\Phi\left(\frac{(1+x)t}{2}\right)}{\frac{\Phi(t)+\Phi(xt)}{2}} = 1.$$

This implies that for any $\delta \in (0,1)$, there exists a sufficiently large t_δ such that

$$\Phi\left(\frac{(1+x)t}{2}\right) > (1-\delta)\,\frac{\Phi(t)+\Phi(xt)}{2} \quad \text{for all } |t| > t_\delta.$$

Therefore, there does not exist $t_0 > 0$ and $\delta_x \in (0,1)$ satisfying (4.1) for all $|t| \ge t_0$, which prevents $L_\Phi[0,1]$ from being UR and completely proves our claim.

4.2. *Construction of the subset M*

Define a partition $\{A_n\}_{n=1}^{\infty}$ of $[0, 1]$ through the intervals

$$A_n = \left(\frac{1}{n+1}, \frac{1}{n}\right] \quad (n \in \mathbb{N})$$

so that $[0, 1] = \bigcup_{n=1}^{\infty} A_n \cup \{0\}$, where each A_n has positive Lebesgue measure

$$|A_n| = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \quad (n \in \mathbb{N})$$

and $\{0\}$ has measure zero.

Given $n \in \mathbb{N}$, let

$$M_n = \left\{ f \in L_\Phi[0, 1] : f\chi_{[0,1]\setminus A_n} = 0 \text{ a.e. and } \|f\|_\Phi \geq 1 + \frac{1}{n} \right\},$$

where χ_E is the characteristic function of the set E. We want to show that $M = \bigcup_{n=1}^{\infty} M_n \subset L_\Phi[0, 1]$ satisfies:

(i) $M \cap B = \emptyset$, where $B = \{f \in L_\Phi[0, 1] : \|f\|_\Phi \leq 1\}$;
(ii) $\inf\{\|f\|_\Phi : f \in M\} = 1$; and
(iii) M is closed in $L_\Phi[0, 1]$.

In fact, (i) follows immediately from the definition of M. To show (ii), it then suffices to construct a sequence $\{f_n\}_{n=1}^{\infty} \subset M$ such that $\lim_{n\to\infty} \|f_n\|_\Phi = 1$. For each $n \in \mathbb{N}$, define

$$f_n(t) = c_n\chi_{A_n}(t),$$

where $c_n > 0$ is chosen so that

$$\|f_n\|_\Phi = 1 + \frac{1}{n}.$$

Clearly, these functions belong to M, provided the existence of such coefficients is guaranteed. The Luxemburg norm of f_n is

$$\|f_n\|_\Phi = \inf\left\{\lambda > 0 : \int_0^1 \Phi\left(\frac{c_n\chi_{A_n}(t)}{\lambda}\right) dt \leq 1\right\} = \inf\left\{\lambda > 0 : \Phi\left(\frac{c_n}{\lambda}\right) \frac{1}{n(n+1)} \leq 1\right\}. \tag{4.2}$$

Set

$$\lambda_n = 1 + \frac{1}{n} = \frac{n+1}{n}$$

and choose $c_n > 0$ so that

$$\Phi\left(\frac{c_n}{\lambda_n}\right) \frac{1}{n(n+1)} = 1,$$

or, equivalently,

$$\Phi\left(\frac{c_n}{\lambda_n}\right) = n(n+1). \tag{4.3}$$

Since Φ is continuous and increasing with $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $|t| \to \infty$, there exists a unique $c_n > 0$ satisfying (4.3). For this c_n, the value λ_n is admissible on the right-hand side of (4.2). If $\lambda < \lambda_n$, then $c_n/\lambda > c_n/\lambda_n$, and, Φ being strictly increasing, we find

$$\Phi\left(\frac{c_n}{\lambda}\right) > \Phi\left(\frac{c_n}{\lambda_n}\right) = n(n+1),$$

whence

$$\Phi\left(\frac{c_n}{\lambda}\right)\frac{1}{n(n+1)} > 1.$$

This shows that such a λ is not an admissible value on the right-hand side of (4.2) and allows us to conclude that

$$\|f_n\|_\Phi = \lambda_n = 1 + \frac{1}{n} \to 1 \text{ as } n \to \infty,$$

which establishes (*ii*).

To prove that M is closed, consider some sequence $\{f_k\}_{k=1}^\infty \subset M$ such that $\lim_{k\to\infty} f_k = f$ in $L_\Phi[0,1]$. Thus, for every $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $f_k \in M_{n_k}$, which means that f_k vanishes a.e. outside A_{n_k} and $\|f_k\|_\Phi \geq 1 + 1/n_k$. We distinguish two cases based on the sequence $\{n_k\}_{k=1}^\infty$.

If $\{n_k\}_{k=1}^\infty$ is unbounded, then, passing to a subsequence (still denoted $\{n_k\}_{k=1}^\infty$) if necessary, we may assume $\lim_{k\to\infty} n_k = \infty$, so

$$\|f\|_\Phi = \lim_{k\to\infty} \|f_k\|_\Phi \geq 1.$$

Furthermore, for any fixed $m \in \mathbb{N}$, $A_{n_k} \cap A_m = \emptyset$ whenever $n_k > m$, so $f_k|_{A_m} = 0$. Since $\{f_k\}_{k=1}^\infty$ converges to f in $L_\Phi[0,1]$, it also converges in measure [11, proof of Theorem 2.4], and therefore some subsequence of $\{f_k\}_{k=1}^\infty$ converges to f a.e., implying $f|_{A_m} = 0$ a.e. As $m \in \mathbb{N}$ was arbitrary and $[0,1] = \bigcup_{m=1}^\infty A_m \cup \{0\}$, where $\{0\}$ has measure zero, we conclude that $f = 0$ a.e., which contradicts $\|f\|_\Phi \geq 1$. Hence, $\{n_k\}_{k=1}^\infty$ must be bounded, that is, there exists $N \in \mathbb{N}$ such that $n_k \leq N$ for every $k \in \mathbb{N}$. Consequently,

$$\{f_k\}_{k=1}^\infty \subset \bigcup_{n=1}^N M_n. \tag{4.4}$$

Each M_n ($n \in \mathbb{N}$) is closed. Indeed, fix $n \in \mathbb{N}$ and let $\{g_m\}_{m=1}^\infty \subset M_n$ converge to g in $L_\Phi[0,1]$. By definition of M_n, $g_m\chi_{[0,1]\setminus A_n} = 0$ a.e. and $\|g_m\|_\Phi \geq 1 + 1/n$ for all $m \in \mathbb{N}$. As above, some subsequence $\{g_{m_j}\}_{j=1}^\infty$ converges to g a.e. For a.e. $t \in [0,1] \setminus A_n$ and all $j \in \mathbb{N}$, we have $g_{m_j}(t) = 0$, so $g(t) = 0$. Thus, $g\chi_{[0,1]\setminus A_n} = 0$ a.e. Additionally, by continuity of the norm,

$$\|g\|_\Phi = \lim_{j\to\infty} \left\|g_{m_j}\right\|_\Phi \geq 1 + \frac{1}{n}.$$

This shows that $g \in M_n$, so M_n is closed. The finite union of closed sets $\bigcup_{n=1}^N M_n$ remains closed, and from (4.4) we conclude that $f \in \bigcup_{n=1}^N M_n \subset M$, establishing (*iii*).

4.3. Construction of a family of operators $T_n : M \to M$ ($n \in \mathbb{N}$)

Given $n \in \mathbb{N}$, let $h_n = a_n\chi_{A_n}$, where $a_n > 0$ is chosen such that $\|h_n\|_\Phi = 1$. For each $f \in M$, define

$$\lambda_{n,f} = \min\{\lambda \in [0,1] : \lambda f + (1-\lambda)h_n \in M\}, \quad T_n f = \lambda_{n,f} f + \left(1 - \lambda_{n,f}\right) h_n.$$

The map $\lambda \mapsto \lambda f + (1-\lambda)h_n$ is continuous, and M is closed. Therefore, the set

$$\{\lambda \in [0,1] : \lambda f + (1-\lambda)h_n \in M\}$$

is closed, nonempty (because it contains $\lambda = 1$, as $f \in M$), and bounded below, ensuring that $\lambda_{n,f}$ exists. As $h_n \in B$ and $M \cap B = \emptyset$, we have $h_n \notin M$, so $\lambda_{n,f} > 0$.

By definition of $\lambda_{n,f}$, $T_n f = \lambda_{n,f} f + \left(1 - \lambda_{n,f}\right) h_n \in M$. If $\lambda_{n,f} = 1$, then $T_n f = f$. If $0 < \lambda_{n,f} < 1$, let

$$t = t_{n,f} = \frac{1}{1-\lambda_{n,f}} > 1.$$

Then,

$$f + t\,(T_n f - f) = f + t\left(1 - \lambda_{n,f}\right)(h_n - f) = h_n.$$

Since $h_n \in B$, we obtain $\|f + t\,(T_n f - f)\|_\Phi = 1$. Thus, $f \preceq T_n f$.

It is worth noting that T_n is not the identity operator. Indeed, pick $f \in M_n$ with $\|f\|_\Phi > 1 + 1/n$ (any function of the form $f = c\chi_{A_n}$, with a suitably chosen $c > 0$, will do). The function $\varphi_n(\lambda) = \|\lambda f + (1-\lambda)h_n\|_\Phi$ ($\lambda \in [0,1]$) is continuous, with $\varphi_n(0) = \|h_n\|_\Phi = 1$ and $\varphi_n(1) = \|f\|_\Phi > 1 + 1/n$. The intermediate value theorem then yields $\tilde{\lambda} \in (0,1)$ such that $\varphi_n(\tilde{\lambda}) = 1 + 1/n$. Since $[\tilde{\lambda} f + (1-\tilde{\lambda})h_n]\chi_{[0,1]\setminus A_n}$ $=0$ a.e., we find that $\tilde{\lambda} f + (1-\tilde{\lambda})h_n \in M_n \subset M$. Therefore, $\lambda_{n,f} \le \tilde{\lambda} < 1$, so $T_n f \ne f$.

4.4. Existence of a fixed point for T_n ($n \in \mathbb{N}$)

The space $L_\Phi[0,1]$ being LUR, Corollary 3.3 guarantees a fixed point for each of the constructed T_n ($n \in \mathbb{N}$). Note that Brøndsted's Theorem 1.1 does not apply due to $\inf\{\|f\|_\Phi : f \in M\} = 1$, and Zubelevich's Theorem 1.3 does not apply as $L_\Phi[0,1]$ is not UR.

Remark 4.1. *The fact that, for each $n \in \mathbb{N}$, the operator T_n above is not the identity relies on the existence, in each M_n, of a function f such that a small segment $\lambda f + (1-\lambda)\, h_n$, with $\lambda < 1$, is inside M. Similar operators can be constructed on arbitrary LUR-not-UR spaces with this segment property, thus providing a wealth of new examples.*

5. Conclusions

Quite recently, Zubelevich established a relaxed version of Brøndsted's fixed point theorem in uniformly rotund (UR) Banach spaces. In the present paper, we introduce and investigate a class of Banach spaces, termed locally uniformly rotund by segments (LURS), where Zubelevich's fixed point theorem holds. We also provide new characterizations of locally uniformly rotund (LUR) spaces. These characterizations show that LUR spaces are LURS, revealing that LUR, rather than UR, is a fundamental geometric setting for Zubelevich's theorem and extending its scope from the class of UR spaces to the strictly broader LUR class. In order to illustrate this generalization, we prove fixed point existence results for directional operators on Orlicz spaces in cases where both Brøndsted's and Zubelevich's theorems fail.

The question of whether LURS and LUR classes coincide remains open. Moreover, the directional nature of the operators considered in Brøndsted's and Zubelevich's theorems suggests further research directions. Future work could explore the extension of Zubelevich's result to spaces with weaker rotundity, including weak local uniform rotundity (WLUR), midpoint local uniform rotundity (MLUR), or uniform convexity in every direction (UCED), and whether such geometric structures support fixed point theorems for significant classes of directional operators. However, the case of MLUR norms could be difficult: In descriptive Banach spaces, MLUR often implies the existence of a LUR renorming, so a meaningful investigation of MLUR should concentrate on the narrower class of non-descriptive spaces, where such an implication may fail [12, 13].

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no conflicts of interest to declare.

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