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Research article

On the existence and stability of bounded solutions for a class of three-point boundary value problems of fractional type

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Abstract: This article investigates the existence and stability of solutions for a class of three-point boundary value problems involving Riemann-Liouville fractional derivatives of order less than one. We considered nonlinear fractional differential equations subject to nonlocal boundary conditions that include a singular condition at the origin and a global fractional condition at interior points. Our approach generalizes previous work, allowing for singular behavior near zero, and employs the Leray-Schauder fixed point theorem to establish the existence of bounded solutions without requiring contractive conditions on the nonlinear term. Instead, we imposed a local integrability condition known as the L^p-Carathéodory condition. Furthermore, we studied the Ulam-Hyers-Rassias stability of approximate solutions by means of a Gronwall-type inequality adapted to weakly singular kernels. Two concrete examples were also included to illustrate the theory.

Keywords: boundary value problem; fractional differential equation; stability; nonlocal boundary condition; L^p-Carathéodory condition; weakly singular kernel; Gronwall-type inequality; Riemann-Liouville fractional derivative
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1. Motivation

Fractional differential problems have attracted considerable attention over the last three decades, due both to the intense development of fractional calculus and to its wide range of applications in various fields of science and engineering. Since fractional derivatives offer a more nuanced representation of certain physical phenomena, modeling with fractional-order equations is often preferable to the classical integer-order approach. These models have found applications in diverse areas such as electromagnetism, porous media, control theory, diffusion, viscoelasticity, signal processing, biology, fluid dynamics, engineering, image processing, fractal theory, potential theory, chemistry, and more (see, for example, the in-depth works [10, 14, 15, 17, 20], as well as the references cited therein).

Much of the research in this area focuses on the qualitative properties of fractional nonlinear dynamics, including the existence, controllability, and stability of solutions. For more information, we recommend the monographs and papers [3, 5, 9, 16, 18, 19, 30–32] and the references contained therein.

On the other hand, three-point boundary value problems (BVPs) belong to the broader class of so-called nonlocal, multi-point, or m-point BVPs. These problems arise naturally in various domains of applied mathematics and physics, where they model numerous real-world phenomena. For example, the vibration of a guy-wire with uniform cross-section, composed of N segments with varying densities, can be modeled as a multi-point boundary value problem (BVP). Consequently, the fractional analogue of such problems has also gained considerable attention in recent research.

In [29], the authors investigated the existence and stability of the following three-point fractional BVP:

$$\begin{cases} \lambda D^{\alpha}u(t) + D^{\beta}u(t) = f(t, u(t)), \\ u(0) = 0, \quad \mu D^{\gamma_1}u(T) + I^{\gamma_2}u(\eta) = \gamma_3, \end{cases}$$

where D^{α} and D^{β} denote Riemann-Liouville (R-L) fractional derivatives, with the orders satisfying $1 < \alpha \leq 2$, $1 \leq \beta < \alpha$, $0 \leq \gamma_1 \leq \alpha - \beta$, and $\gamma_2 \geq 0$. The constants λ and μ are assumed to satisfy $0 < \lambda \leq 1$ and $0 \leq \mu \leq 1$.

The method used by these authors to prove the results of existence was based on fixed point theory, more specifically, on Banach's contraction principle and Krasnosel'skii's fixed point theorem, applied to an integral equation representing the problem. As a result, it was necessary to impose certain contractive-type conditions on the nonlinear function f, which, in a certain sense, can be considered strong.

In this article, in addition to several other distinct aspects related to the conditions to be imposed on the problem functions, we deepen the study initiated in [29], considering R-L derivatives of order in the interval $(0, 1)$. More precisely, we study the existence of bounded solutions to the following three-point BVP:

$$\lambda D^{\alpha}u(t) + D^{\beta}u(t) = f(t, u(t)), \tag{1.1}$$

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = u^0, \quad \mu D^{\gamma_1}u(T) + I^{\gamma_2}u(\eta) = \gamma_3, \tag{1.2}$$

where $0 \leq \gamma_1 < \alpha < 2/3$, $1 \leq \gamma_2 + \alpha$, and $\beta < \alpha$. Here, the constants λ and μ are required to satisfy certain conditions related to the bounded region in which the solutions are sought; see Theorem 4.3.

To apply a fixed point argument in this setting, special care must be taken when interpreting the notion of solution and establishing the relationship between a solution of the BVP (1.1)-(1.2) and the associated Volterra integral equation, especially due to the singular behavior near the origin. Since we employ the Leray-Schauder fixed point theorem, no contractive-type assumptions are imposed on the nonlinearity f; instead, we assume a local condition known as the L^q-Carathéodory condition; see Definition 4.1.

On the other hand, stability theory in mathematics began in the 17th century with the study of mechanical systems, such as the stability of a floating body, particularly through the studies of Euler and Lagrange. The modern foundation was laid by Aleksandr Lyapunov in the late 19th century. His work, now known as Lyapunov stability, provides a rigorous framework for determining whether the solutions of differential equations and the trajectories of dynamical systems remain close to a given solution or equilibrium point under small perturbations.

Similarly, the stability of functional equations concerns the question of whether an exact solution lies near a given approximate solution, i.e., a solution to a perturbed or inexact equation. This line of research was initiated in 1925 by Pólya and Szegö [21]. Subsequent foundational work by S. M. Ulam [24], D. H. Hyers [12], T. Aoki [1], and T. M. Rassias [23] led to what is now known as Ulam-Hyers-Rassias (U-H-R) stability (see Definition 5.1). It is well known that the U-H-R type of stability originated from a very concrete problem posed by Stanisław Ulam in 1940. Basically, he raised a fundamental question about the "closeness" of certain functions, in the sense of whether a function approximately satisfies a functional equation, whether it is (or is not) necessarily close to a function that satisfies the equation exactly. This concept has been widely applied (and also expanded in its definition [2, 6–8, 11, 22]) to various types of equations, including differential, integro-differential, and fractional differential equations.

In the context of the BVP (1.1)-(1.2), an approximate solution is a function y satisfying the inequality

$$|\lambda D^{\alpha} y(t) + D^{\beta} y(t) - f(t, y(t))| < \epsilon,$$

along with boundary conditions analogous to those in (1.2).

More precisely, we consider functions that share the same nonlocal structure as the solutions of (1.1)-(1.2), but whose behavior near the origin is governed by the global boundary conditions. Accordingly, the nonlocal condition remains the same, involving the parameters μ and γ_i $(i = 1, 2, 3)$ evaluated at the points T and η, while the local condition at the origin is adjusted to ensure consistency with the nonlocal component. That is,

$$\lim_{t\to 0^+} t^{1-\alpha} y(t) = y^0, \quad \mu D^{\gamma_1} y(T) + I^{\gamma_2} y(\eta) = \gamma_3. \tag{1.3}$$

To study the U-H-R stability of problem (1.1)-(1.2), we make use of a weakly singular Gronwall-type inequality established in [26]; see Lemma 5.1.

With regard to the organization of the article in the following sections, we will first have a preliminary section whose main objective is to briefly present the essential material that we will use in the later parts, whether in terms of definitions, basic known results, or consequent functional spaces. In Section 3, we will identify integral equations associated with part of our problem and with the complete problem. Specifically, we will first establish an equivalence between an initial value problem and a Volterra integral equation, and then we will identify our global problem with another integral equation. In Section 4, the main objective is to obtain sufficient conditions that are not too strong to guarantee the existence of a solution to the problem under study. This is achieved through a combination of arguments that we consider atypical in certain components, but effective. Section 5 deals with obtaining conditions that guarantee the Ulam-Hyers-Rassias stability of the problem under study, which is mainly supported by a Gronwall inequality for weakly singular kernels. In Section 6, we present two distinct examples of solutions that satisfy the class of BVP analyzed here and that illustrate the theory developed, namely in the steps to be verified in relation to the construction carried out, as well as in the exemplification of the conditions to be guaranteed and that such conditions do not represent the empty set. Finally, in the last section, we conclude the article with a brief summary of what has been done and some perspectives for the future.

2. Preliminaries

To correctly formulate the BVP (1.1)-(1.2), we first introduce the function spaces that will be used in the sequel. Let $L^p[0,T]$, for $1 \leq p < \infty$, denote the classical Lebesgue space of functions whose p-th power is Lebesgue integrable, endowed with the norm

$$\|f\|_p = \left(\int_0^T |f(s)|^p ds \right)^{1/p}.$$

The space $L^\infty[0,T]$ consists of essentially bounded functions, equipped with the norm

$$\|u\|_\infty = \text{ess} \sup_{t \in [0,T]} |u(t)|.$$

We will also make use of the $C[0,T]$ space of continuous functions on the interval $[0,T]$, endowed with the norm $\|f\|_\infty = \max_{t\in[0,T]} |f(t)|$. The Hölder space on $[0,T]$, denoted by $C^{0,\alpha}[0,T]$ for $0 < \alpha < 1$, will be also useful in our analysis, as it embeds continuously and compactly into $C^2[0,T]$; see Lemma 2.1 in [27].

Function spaces that allow for a singularity at zero are particularly appropriate when dealing with fractional derivatives. The space $C_\vartheta[0,T]$ is defined by

$$C_\vartheta[0,T] = \{f : [0,T] \longrightarrow \mathbb{R}, \quad f(t) = t^\vartheta g(t) \ a.e. \text{ for some } g \in C[0,T]\}.$$

This space becomes a Banach space when endowed with the norm $\|f\|_\vartheta := \|g\|_\infty$.

In particular, the spaces $C_{-\vartheta}[0,T]$, with $\vartheta > 0$, contain functions with singularities at zero. If $0 \leq \vartheta < 1$, then functions in $C_{-\vartheta}[0,T]$ are integrable and continuous on $(0,T]$ and the limit $\lim_{t\to 0^+} t^\vartheta f(t)$ exists.

Notice that for $0 \leq \vartheta \leq \omega < 1$ we have $C[0,T] = C_0[0,T] \subset C_{-\vartheta}[0,T] \subset C_{-\omega}[0,T]$.

We will use the well-known gamma function Γ, which for complex numbers z with a positive real part is defined by Euler's integral of the second kind:

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} \, dt$$

and is then defined in the complex plane as the analytic continuation of this integral function. The gamma function has several very useful and well-known properties, such as the fact that, for positive integers, it gives the same result as the factorial function shifted by one.

Definition 2.1. *The R-L fractional integral of order $\alpha > 0$ of a function $u \in L^1[0,T]$ is defined for t almost everywhere by*

$$I^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds.$$

If $\alpha = 1$, then I^α corresponds to the usual integration operator, which we denote simply by I.

$I^\alpha u$ is defined as an L^1 function; in particular, $I^\alpha u(t)$ is finite almost everywhere in t since it is given by a convolution of functions in $L^1[0,T]$. However, $I^\alpha u(0)$ is not necessarily defined for $u \in L^1[0,T]$. To see this, consider $0 < \alpha < \alpha + \epsilon < 1$ and the function $u(t) = t^{-\alpha-\epsilon}$. In this case we have

$$I^\alpha u(t) = \frac{\Gamma(1-\alpha-\epsilon)}{\Gamma(1-\epsilon)} t^{-\epsilon},$$

which diverges as $t \to 0^+$. Conditions for the existence of $I^\alpha u(0)$ can be found in [25].

On the other hand, the Newton-Leibniz formula in the fundamental theorem of calculus plays a crucial role in the interplay between integrals and derivatives. The space $AC[0,T]$ of absolutely continuous functions provides the appropriate framework for working with L^1 functions. Therefore, we use this space in the next definition.

Definition 2.2. *For $\alpha \in (0,1)$ and $u \in L^1[0,T]$ the R-L fractional derivative $D^\alpha u$ is defined when $I^{1-\alpha}u \in AC[0,T]$ by*

$$D^\alpha u(t) := DI^{1-\alpha}u(t) \quad a.e. \quad t \in [0,T].$$

Note that for $DI^{1-\alpha}u(t)$ to be defined almost everywhere in t it is necessary that $I^{1-\alpha}u$ be differentiable almost everywhere. Since we are transforming differential equations into integral equations and applying the Newton-Leibniz formula, we assume that $I^{1-\alpha}u \in AC[0,T]$.

In what follows, we will use the following well-known properties of R-L derivatives and integrals; see, e.g., [15, 25] and references therein.

Lemma 2.1. *Let $\alpha, \beta, \gamma > 0$.*

(1) If $u \in C_\gamma[0,T]$ and $\alpha + \beta \geq \gamma$, then $I^\alpha I^\beta u(t) = I^{\alpha+\beta}u(t)$ for all $t \in [0,T]$.
(2) If $u \in L^p[0,T]$ and $\alpha > \beta > 0$, then $D^\beta I^\alpha u(t) = I^{\alpha-\beta}u(t)$ for a.e. $t \in [0,T]$.
(3) If $0 < \alpha < 1$, $u \in L^1(0,T)$, and $u_{1-\alpha}(t) \in AC[0,T]$, then $I^\alpha D^\alpha u(t) = u(t) - \frac{u_{1-\alpha}(0)}{\Gamma(\alpha)}t^{\alpha-1}$, where $u_{1-\alpha}(t) = I^{1-\alpha}u(t)$.

The following result corresponds to Proposition 3.2 in [25] and Theorem 4.5 in [27].

Proposition 2.1. *Let $\alpha > 0$ and $0 \leq \vartheta < 1$.*

(1) I^α is a bounded operator from $L^p[0,T]$ into $L^p[0,T]$, $1 \leq p \leq \infty$.
(2) I^α is a bounded operator from $C_{-\vartheta}[0,T]$ into $C_{\alpha-\vartheta}[0,T]$.
(3) If $0 \leq \vartheta < \alpha < 1$, then I^α is a bounded operator from $C_{-\vartheta}[0,T]$ into $C[0,T]$.
(4) If $0 \leq \alpha < 1$, then I^α maps $L^\infty[0,T]$ into $C^{0,\alpha}[0,T]$.

3. BVP (1.1)-(1.2) and its complementary integral equation

To analyze the existence of solutions to the BVP (1.1)-(1.2) using fixed point methods for a related integral equation, we first establish an equivalence between an initial value problem (IVP) and a Volterra integral equation. This equivalence then allows us to link the BVP to an integral equation.

In [4, 27], this equivalence was proven for the fractional differential equation $D^\alpha u(t) = f(t, u(t))$, with initial condition $\lim_{t\to 0^+} t^{1-\alpha}u(t) = u^0$, $t \in (0,T]$. We adapt this result to our setting and include the proof to ensure the paper remains self-contained.

We recall that a function f is absolutely integrable on $(0,T]$ if f is Riemann integrable on every closed interval $[\eta, T]$, where $\eta \in (0,T]$, and $\lim_{\eta\to 0^+} \int_\eta^T |f(t)|dt$ exists and is finite, in which case $\int_0^T |f(t)|dt$ is defined to be

$$\int_0^T |f(t)|dt := \lim_{\eta\to 0^+} \int_\eta^T |f(t)|dt.$$

Furthermore, for continuous and absolutely integrable functions on $(0,T]$, we have the following result, which allows us to handle the initial condition near zero; see Theorem 6.1 in [4] and Lemma 5.5 in [27].

Lemma 3.1. *Let $u^0 \in \mathbb{R}$ and $\alpha \in (0,1)$. Suppose that a function u is continuous and absolutely integrable on $(0,T]$. Then*

$$\lim_{t\to0^+} u(t)t^{1-\alpha} = u^0 \quad \textit{if and only if} \quad \lim_{t\to0^+} I^{1-\alpha}u(t) = u^0\Gamma(\alpha).$$

On the other hand, in Proposition 2.1 of [4], it was proven that if f is a function defined on $(0,T]$ with a singularity at $t=0$, then $f \in L^1[0,T]$ whenever f is absolutely integrable on $(0,T]$. Conversely, if $f \in L^1[0,T]$ and is continuous on $(0,T]$, then f is absolutely integrable on $(0,T]$.

Next, we present a useful proposition which, although based on the methods used to construct the main result of [4], is specified here for the case in which the function f necessarily depends on the derivative of order β of the corresponding supposed solutions.

Proposition 3.1. *Let $0<\alpha<1$ and $u^0 \neq 0$. Let $f(t,u,v)$ be a continuous function on the set*

$$\mathcal{B} = \{(t,u,v) \in \mathbb{R}^3 \ : \ t \in (0,T], \ u \in J_1, \ v \in J_2\},$$

where $J_i \subset \mathbb{R}$, $i=1,2$ are unbounded intervals. If u and $f(t,u(t),D^\beta u(t))$ both are continuous and absolutely integrable functions on $(0,T]$ and $(t,u,D^\beta u) \mapsto f(t,u(t),D^\beta u(t))$ belongs to $L^1[0,T]$, then u satisfies the IVP

$$D^\alpha u(t) = f(t,u(t),D^\beta u(t)), \quad \lim_{t\to0^+} t^{1-\alpha}u(t) = u^0, \quad t \in (0,T] \tag{3.1}$$

if and only if it satisfies the Volterra integral equation

$$u(t) = u^0t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s),D^\beta u(s))ds, \quad t \in (0,T]. \tag{3.2}$$

Proof. Let $u : (0,T] \longrightarrow J_1$ be a continuous function satisfying IVP (3.1) and absolutely continuous on $(0,T]$. As f is continuous on the set $\mathcal{B}$, the function

$$\varphi(t) = f(t,u(t),D^\beta u(t))$$

is continuous on $(0,T]$. We have that

$$\varphi(t) = D^\alpha u(t) = DI^{1-\alpha}u(t),$$

and consequently, $I^{1-\alpha}u(t)$ is continuously differentiable on $(0,T]$. Then, integrating, we have

$$\int_\eta^t \varphi(s)ds = I^{1-\alpha}u(t) - I^{1-\alpha}u(\eta), \quad 0<\eta<t\le T.$$

Taking the limit $\eta \to 0^+$ and using Lemma 3.1, we obtain

$$I^{1-\alpha}u(t) = I\varphi(t) + u^0\Gamma(\alpha), \quad 0<t\le T.$$

Applying I^α to the equality above we get that

$$I^\alpha(I^{1-\alpha}u(t)) = Iu(t) = I^{\alpha+1}\varphi(t) + I^\alpha u^0\Gamma(\alpha).$$

We rewrite the expression as

$$\int_0^t u(s)ds = \int_0^t I^\alpha \varphi(s)ds + \frac{u^0\Gamma(\alpha)}{\Gamma(\alpha+1)}t^\alpha,$$

where $I^\alpha\varphi(s)$ is continuous on $(0, T]$ in virtue of Proposition 4.6 in [4].

Differentiating both sides of the last equality, we obtain

$$u(t) = u^0 t^{\alpha-1} + I^\alpha\varphi(t), \quad 0 < t \leq T,$$

which corresponds to Eq (3.2).

Now, assume that u satisfies the Volterra integral equation (3.2). Then, applying D^α to both sides of (3.2), we have

$$\begin{aligned} D^\alpha u(t) &= D^\alpha u^0 t^{\alpha-1} + D^\alpha I^\alpha \varphi(t) \\ &= DI^{1-\alpha} u^0 t^{\alpha-1} + D^\alpha I^\alpha \varphi(t) \\ &= Du^0\Gamma(\alpha) + DI^{1-\alpha} I^\alpha \varphi(t) \\ &= f(t, u(t), D^\beta u(t)). \end{aligned}$$

Finally, we prove that u satisfies the initial condition. Applying $I^{1-\alpha}$ to the integral equation (3.2), we have

$$I^{1-\alpha}u(t) = I^{1-\alpha}u^0 t^{\alpha-1} + I^{1-\alpha} I^\alpha \varphi(t) = u^0\Gamma(\alpha) + I\varphi(t).$$

Therefore,

$$\lim_{t\to 0^+} I^{1-\alpha}u(t) = u^0\Gamma(\alpha) + \lim_{t\to 0^+} \int_0^t f(s, u(s), D^\beta u(s))ds = u^0\Gamma(\alpha),$$

which finishes the proof. □

In the last result, we can consider $u \in C_{\alpha-1}[0, T]$ and $t \mapsto f(t, u, D^\alpha u) \in C_{-\vartheta}[0, T]$ for some $0 < \vartheta < 1$, since in this case $I^\alpha f \in C_{\alpha-\vartheta}[0, T] \subset C_{\alpha-1}[0, T]$ by Proposition 2.1, thus all terms belong to the same space.

To establish the equivalence between BVP (1.1)-(1.2) and a nonlinear integral equation, we assume in the following result that all variables satisfy $0 < s \leq t \leq \eta < T$. Furthermore, to apply the semigroup and commutativity properties of R-L integrals and derivatives, we impose certain conditions on the orders α, γ_1, and γ_2.

Proposition 3.2. *Let* $0 < \gamma_1 < \alpha < 1$*,* $\gamma_2 + \alpha \geq 1$*, and* $u^0 \neq 0$*. Let* $f(t, u, v)$ *be a continuous function on the set*

$$\mathcal{B} = \{(t, u, v) \in \mathbb{R}^3 \ : \ t \in (0, T], \ u \in J_1, \ v \in J_2\},$$

where $J_i \subset \mathbb{R}$*,* $i = 1, 2$ *are unbounded intervals. If* u *and* $f(t, u(t), D^\beta u(t))$ *both are continuous and absolutely integrable functions on* $(0, T]$ *and* $(t, u, D^\beta u) \mapsto f(t, u(t), D^\beta u(t))$ *belongs to* $L^1[0, T]$*, then* u *satisfies the BVP* (1.1)-(1.2) *if and only if it satisfies the integral equation*

$$u(t) = \frac{\Omega}{\Theta}t^{\alpha-1} - \frac{1}{\lambda}\left[I^\alpha D^\beta u(t) - I^\alpha f(t, u(t))\right], \quad 0 < t \leq T, \tag{3.3}$$

where Ω and Θ are real constants (with Ω depending on u and f) given by

$$\Omega := \gamma_3 + \frac{\mu}{\lambda}\left(I^{\alpha-\gamma_1}D^{\beta}u(T) - I^{\alpha-\gamma_1}f(T,u(T))\right) + \frac{1}{\lambda}\left(I^{\alpha+\gamma_2}D^{\beta}u(\eta) - I^{\alpha+\gamma_2}f(\eta,u(\eta))\right),$$
$$\Theta := \mu\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}T^{\alpha-\gamma_1-1} + \frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma_2)}\eta^{\alpha+\gamma_2-1}.$$

Proof. Let us rewrite the differential equation as

$$D^{\alpha}u(t) = -\frac{1}{\lambda}[D^{\beta}u(t) - f(t,u(t))],$$

where the right-hand side of the equality is a continuous and absolutely integrable function on $(0,T]$. From Proposition 3.1, we have that u is a solution of the IVP

$$D^{\alpha}u(t) = -\frac{1}{\lambda}[D^{\beta}u(t) - f(t,u(t))], \quad \lim_{t\to 0^+} t^{1-\alpha}u(t) = u^0, \quad t \in (0,T]$$

if and only if it is a solution of the Volterra integral equation

$$\begin{aligned} u(t) &= u^0 t^{\alpha-1} - \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[D^{\beta}u(s) - f(s,u(s))]ds \\ &= u^0 t^{\alpha-1} - \frac{1}{\lambda}[I^{\alpha}D^{\beta}u(t) - I^{\alpha}f(t,u(t))]. \end{aligned} \tag{3.4}$$

Now, we apply D^{γ_1} to Eq (3.4). Then,

$$D^{\gamma_1}u(t) = u^0 D^{\gamma_1}t^{\alpha-1} - \frac{1}{\lambda}[D^{\gamma_1}I^{\alpha}D^{\beta}u(t) - D^{\gamma_1}I^{\alpha}f(t,u(t))].$$

Using the properties of the R-L derivatives and integrals, we obtain the following expression for $\alpha > \gamma_1$:

$$D^{\gamma_1}u(t) = u^0\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1} - \frac{1}{\lambda}[I^{\alpha-\gamma_1}D^{\beta}u(t) - I^{\alpha-\gamma_1}f(t,u(t))]. \tag{3.5}$$

Applying I^{γ_2} in (3.4), from the condition $\gamma_2 + \alpha \geq 1$, we obtain

$$\begin{aligned} I^{\gamma_2}u(t) &= u^0 I^{\gamma_2}t^{\alpha-1} - \frac{1}{\lambda}[I^{\gamma_2}I^{\alpha}D^{\beta}u(t) - I^{\gamma_2}I^{\alpha}f(t,u(t))] \\ &= u^0\frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma_2)}t^{\alpha+\gamma_2-1} - \frac{1}{\lambda}[I^{\gamma_2+\alpha}D^{\beta}u(t) - I^{\gamma_2+\alpha}f(t,u(t))]. \end{aligned} \tag{3.6}$$

Using the boundary condition $\mu D^{\gamma_1}u(T) + I^{\gamma_2}u(\eta) = \gamma_3$, from (3.5) and (3.6) we have

$$\begin{aligned} \gamma_3 &= u^0\frac{\mu\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}T^{\alpha-\gamma_1-1} - \frac{\mu}{\lambda}[I^{\alpha-\gamma_1}D^{\beta}u(T) - I^{\alpha-\gamma_1}f(T,u(T))] \\ &\quad + u^0\frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma_2)}\eta^{\alpha+\gamma_2-1} - \frac{1}{\lambda}[I^{\gamma_2+\alpha}D^{\beta}u(\eta) - I^{\gamma_2+\alpha}f(\eta,u(\eta))]. \end{aligned} \tag{3.7}$$

Letting

$$\Theta = \mu\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}T^{\alpha-\gamma_1-1} + \frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma_2)}\eta^{\alpha+\gamma_2-1},$$

from (3.7) we obtain

$$u^0 = \frac{1}{\Theta}\left[\gamma_3 + \frac{\mu}{\lambda}(I^{\alpha-\gamma_1}D^{\beta}u(T) - I^{\alpha-\gamma_1}f(T,u(T))) + \frac{1}{\lambda}(I^{\gamma_2+\alpha}D^{\beta}u(\eta) - I^{\gamma_2+\alpha}f(\eta,u(\eta)))\right] = \frac{\Omega}{\Theta}.$$

Substituting it into Eq (3.4), we conclude that any solution of the BVP (1.1)-(1.2) is also a solution of the integral equation (3.3), and vice versa. □

Remark 3.1. *Note that some integrals in formula* (3.3) *may vanish depending on the values of the variables* t, s, η. *For instance, if* $0 < \eta \le s \le T$, *then*

$$I^{\alpha+\gamma_2}D^{\beta}u(\eta) = I^{\alpha+\gamma_2}f(\eta,u(\eta)) = 0,$$

and for $0 < t \le s$,

$$I^{\alpha}D^{\beta}u(t) = I^{\alpha}f(t,u(t)) = 0.$$

4. On the existence of solutions for the BVP (1.1)-(1.2)

The existence of a solution to the BVP (1.1)-(1.2) will be established by proving the existence of a fixed point of an integral operator **T**, defined by the right-hand side of equality (3.3). To achieve this, we must properly define the operator **T** in an appropriate Banach space. Let

$$T_1u(t) := I^{\alpha}D^{\beta}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}D^{\beta}u(s)ds,$$
$$T_2u(t) := I^{\alpha}f(t,u(t)) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))ds.$$

Thus, the right-hand side of equality (3.3) can be rewritten as

$$\mathbf{T}u(t) = \frac{\Omega}{\Theta}t^{\alpha-1} - \frac{1}{\lambda}\left[T_1u(t) - T_2u(t)\right].$$

Our next task is to define **T** as an operator in an appropriate Banach space. Since the term $t^{\alpha-1}$ appears in the definition of **T**, we choose the space $C_{\alpha-1}[0,T]$ as the input space.

Notice from Proposition 2.1 that

$$T_1, T_2 : C_{\alpha-1}[0,T] \longrightarrow C_{2\alpha-1}[0,T].$$

Moreover, if we assume that $0 \le \gamma_1 < \alpha < \frac{2}{3}$, we observe that $C_{2\alpha-1}[0,T] \subset C_{\alpha-1}[0,T]$. This means that

$$\mathbf{T} : C_{\alpha-1}[0,T] \longrightarrow C_{\alpha-1}[0,T]$$

is well-defined and bounded.

Since $u(t) = t^{\alpha-1}v(t)$ with $v \in C[0,T]$, then u is a solution of the integral equation (3.3) in $C_{\alpha-1}[0,T]$ if and only if $v(t) = t^{1-\alpha}u(t)$ is a solution in $C[0,T]$ of the equation

$$v(t) = \frac{\Omega}{\Theta} - \frac{t^{1-\alpha}}{\lambda}\left[I^{\alpha}D^{\beta}t^{\alpha-1}v(t) - I^{\alpha}f(t,s^{\alpha-1}v(t))\right], \quad 0 < t \le T.$$

We define

$$\widehat{T}_1 v(t) := I^{\alpha} D^{\beta} t^{\alpha-1} v(t), \quad \widehat{T}_2 v(t) := I^{\alpha} f(t, t^{\alpha-1} v(t)),$$

where $\widehat{T}_1$ and $\widehat{T}_2$ are bounded operators from $C[0,T]$ into itself. Meaning that the operator

$$\widehat{\mathbf{T}} v(t) := \frac{\Omega}{\Theta} - \frac{t^{1-\alpha}}{\lambda}\left[\widehat{T}_1 v(t) - \widehat{T}_2 v(t)\right] \tag{4.1}$$

is well-defined and bounded from $C[0,T]$ into itself.

To prove the existence of a fixed point for the operator $\widehat{\mathbf{T}}$, we will apply the Leray-Schauder fixed point theorem.

Theorem 4.1. *(Leray-Schauder) Let F be a closed convex subset in a Banach space X and assume that $S : F \longrightarrow F$ is a continuous mapping such that $S(F)$ is a relatively compact subset of F. Then S has a fixed point.*

Since the Leray-Schauder theorem is a localization result for fixed points, we follow the approach in [27] and use the following definition to prove the compactness of $\widehat{\mathbf{T}}$.

Definition 4.1. *A function $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to be L^q_{loc} $(1 \le q \le \infty)$ if f is measurable and, for each $R > 0$, there exists a function $f_R \in L^q[0,T]$ such that $|f(t,u)| \le f_R(t)$ for all u with $|u| \le R$.*

In addition to other less significant differences, we would like to point out that the main distinction between Carathéodory's classical condition and the L^q-Carathéodory condition we are using lies in the different integrability required for the function f_R (which acts as an upper bound of the function that has the property in question), and also in the circumstance that this upper bound needs to occur, in our case, only locally (and not necessarily throughout the domain, as in the classical case).

Conditions for the complete continuity of the operator $\widehat{T}_2$ are given in the following result.

Theorem 4.2. *[27, Theorem 7.2] Let $f : [0,T] \times \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be continuous and suppose there exist $0 \le \nu < \alpha < 1$ and an $L^{\infty}_{\mathrm{loc}}$ function $g : [0,T] \times \mathbb{R} \longrightarrow [0,\infty)$ such that $|f(t, t^{\alpha-1} v)| \le t^{-\nu} g(t,v)$ for $t > 0$. Then the operator*

$$\widehat{T}_2 v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, s^{\alpha-1} v(s)) ds$$

is completely continuous in $C[0,T]$.

To prove the complete continuity of the operator $\widehat{T}_1$, note that

$$\widehat{T}_1 v(t) = \frac{d}{dt} I^{1-\beta} t^{\alpha-1} v(t).$$

Since $t^{\alpha-1} v(t) \in L^{\infty}[0,T]$, then from Proposition 2.1, $I^{1-\beta} t^{\alpha-1} v(t) \in C^{0,1-\beta}[0,T]$, meaning that the function $D I^{1-\beta} t^{\alpha-1} v(t)$ is a bounded function for each $v \in C[0,T]$ and since the Hölder space $C^{0,1-\beta}[0,T]$ embeds continuously and compactly in $C[0,T]$ (Lemma 2.1 on [27]), we conclude that $\widehat{T}_1$ is a completely continuous operator on $C[0,T]$.

Finally, since the terms Θ, Ω, and $t^{1-\alpha}$ do not affect compactness, we conclude that $\widehat{\mathbf{T}}$ is a completely continuous operator on $C[0,T]$.

Now, let $B_R = \{u \in C[0,T] \ : \ \|u\|_\infty \le R\}$ be the closed ball of radius $R > 0$ in $C[0,T]$ and let

$$t^{\alpha-1}B_R = \{t^{\alpha-1}v(t) \ : \ v \in B_R\}, \quad t \in (0,T].$$

Since for any $0 \le \sigma, \varsigma < 1$, we have that $I^\sigma D^\varsigma$ is a compact operator on $C[0,T]$, then $I^\sigma D^\varsigma t^{\alpha-1}B_R$ is a bounded set in $C[0,T]$. That is, there exists $M_R > 0$ such that

$$\|I^\sigma D^\varsigma w\|_\infty \le M_R, \quad \text{for all } w \in t^{\alpha-1}B_R. \tag{4.2}$$

Moreover, for $\varsigma < \sigma$ and function f satisfying the hypothesis of Theorem 4.2, we have

$$\begin{aligned} I^\sigma f(t, t^{\alpha-1}v(t)) &= \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1} f(s, s^{\alpha-1}v(s))ds \\ &\le \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1} s^{-\varsigma} g(s, v(s))ds \\ &\le \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1} s^{-\varsigma} ds\|g\|_\infty \\ &= \frac{\Gamma(1-\varsigma)}{\Gamma(\sigma-\varsigma+1)}\|g\|_\infty. \end{aligned} \tag{4.3}$$

Hence, $\|I^\sigma f\|_\infty \le \frac{\Gamma(1-\varsigma)}{\Gamma(\sigma-\varsigma+1)}\|g\|_\infty$.

We are now ready to state and prove our main existence result.

Theorem 4.3. *(Existence of solutions) Let us assume that the function $f : [0,T] \times \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is continuous and suppose that there exist $0 \le \vartheta < \alpha$ and an L^∞_{loc} function $g : [0,T] \times \mathbb{R} \longrightarrow [0,\infty)$ such that $|f(t, t^{\alpha-1}v)| \le t^{-\vartheta} g(t,v)$, for $t > 0$. Then, the BVP*

$$\begin{cases} \lambda D^\alpha u(t) + D^\beta u(t) = f(t, u(t)), \\ \lim_{t\to 0^+} t^{1-\alpha}u(t) = u^0, \quad \mu D^{\gamma_1}u(T) + I^{\gamma_2}u(\eta) = \gamma_3, \end{cases} \tag{4.4}$$

where the orders on the R-L derivatives and integrals satisfy

$$0 \le \gamma_1 < \alpha < 2/3, \quad 1 \le \gamma_2 + \alpha, \quad \beta < \alpha, \tag{4.5}$$

has at least one solution in the closed ball B_R of radius R on $C_{\alpha-1}[0,T]$, provided $R \ge \|g\|_\infty$, and the constants λ and μ are chosen such that the real numbers Ω and Θ, given in Proposition 3.2, satisfy

$$\left|\frac{\Omega}{\Theta}\right| \le \frac{R}{2}.$$

In particular, λ should be such that

$$|\lambda| \ge \max\left\{\frac{4\Gamma(1-\vartheta)T^{1-\alpha}}{\Gamma(1+\alpha-\vartheta)}, \frac{4M_R T^{1-\alpha}}{R}\right\},$$

where M_R is given by (4.2).

Proof. Let $u \in B_R$. We have shown that, under conditions (4.5), any solution of the fixed-point equation $\widehat{\mathbf{T}}u(t) = u(t)$, where the operator $\widehat{\mathbf{T}}$ is defined in (4.1), is also a solution of the BVP (4.4). Moreover, we have established that $\widehat{\mathbf{T}}$ is a completely continuous operator on $C[0,T]$.

Therefore, by the Leray-Schauder fixed point theorem, it remains to prove that $\widehat{\mathbf{T}}$ maps the closed ball B_R into itself.

Observe that the condition $0 \leq \vartheta < \alpha$ ensures that the bound (4.3) holds for the operators $\widehat{T}_2$. That is,

$$\|\widehat{T}_2 v\|_\infty \leq \frac{\Gamma(1-\vartheta)}{\Gamma(1+\alpha-\vartheta)}\|g\|_\infty.$$

Here, $\|g\|_\infty$ is taken on $[0,T]\times\mathbb{R}$. Also, from (4.2) we have

$$\|\widehat{T}_1 v\|_\infty \leq M_R.$$

Then, we obtain

$$\|\widehat{\mathbf{T}}v\|_\infty \leq \left|\frac{\Omega}{\Theta}\right| + \frac{T^{1-\alpha}}{|\lambda|}\left[\|\widehat{T}_1 v\|_\infty + \|\widehat{T}_2 v\|_\infty\right] \leq \left|\frac{\Omega}{\Theta}\right| + \frac{T^{1-\alpha}}{|\lambda|}\left[M_R + \frac{\Gamma(1-\vartheta)}{\Gamma(1+\alpha-\vartheta)}\|g\|_\infty\right].$$

Now, since the value of the real number Ω depends on R, we are going to estimate an upper bound for this number:

$$\begin{aligned}
|\Omega| &\leq |\gamma_3| + \left|\frac{\mu}{\lambda}\right|\left(|I^{\alpha-\gamma_1}D^\beta u(T)| + |I^{\alpha-\gamma_1}f(T,u(T))|\right) + \frac{1}{|\lambda|}\left(|I^{\alpha+\gamma_2}D^\beta u(\eta)| + |I^{\alpha+\gamma_2}f(\eta,u(\eta))|\right)\\
&= |\gamma_3| + \left|\frac{\mu}{\lambda}\right|\left(|u(T)|I^{\alpha-\gamma_1}D^\beta \mathbf{1}(T) + |f(T,u(T))|I^{\alpha-\gamma_1}\mathbf{1}(T)\right)\\
&\quad + \frac{1}{|\lambda|}\left(|u(\eta)|I^{\alpha+\gamma_2}D^\beta\mathbf{1}(\eta) + |f(\eta,u(\eta))|I^{\alpha+\gamma_2}\mathbf{1}(\eta)\right)\\
&\leq |\gamma_3| + \left|\frac{\mu}{\lambda}\right|\left(RI^{\alpha-\gamma_1}D^\beta\mathbf{1}(T) + \|f\|_\infty I^{\alpha-\gamma_1}\mathbf{1}(T)\right) + \frac{1}{|\lambda|}\left(RI^{\alpha+\gamma_2}D^\beta\mathbf{1}(\eta) + \|f\|_\infty I^{\alpha+\gamma_2}\mathbf{1}(\eta)\right).
\end{aligned}$$

In the estimates above, by $\mathbf{1}(t)$ we denote the constant function equals to 1 for each $t \in [0,T]$.

Evaluating the R-L derivatives and integrals above, we obtain the following upper bound for $|\gamma_3|$:

$$\begin{aligned}
|\gamma_3| &\leq |\mu||D^{\gamma_1}u(T)| + |I^{\gamma_2}u(\eta)| \leq R(|\mu|D^{\gamma_1}\mathbf{1}(T) + I^{\gamma_2}\mathbf{1}(\eta))\\
&= R\left(\frac{|\mu|T^{-\gamma_1}}{\Gamma(1-\gamma_1)} + \frac{\eta^{\gamma_2}}{\Gamma(\gamma_2+1)}\right).
\end{aligned}$$

Therefore, using this and the fact that $R \geq \|g\|_\infty$, we obtain the estimate

$$\begin{aligned}
|\Omega| &\leq R\left(\frac{|\mu|T^{-\gamma_1}}{\Gamma(1-\gamma_1)} + \frac{\eta^{\gamma_2}}{\Gamma(\gamma_2+1)}\right) + \left|\frac{\mu}{\lambda}\right|\left(R\frac{T^{-\beta}t^{\alpha-\gamma_1}}{\Gamma(1-\beta)\Gamma(\alpha-\gamma_1+1)} + T^{-\vartheta}\|g\|_\infty\frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)}\right)\\
&\quad + \frac{1}{|\lambda|}\left(R\frac{\eta^{-\beta}}{\Gamma(1-\beta)}\frac{t^{\alpha+\gamma_2}}{\Gamma(\alpha+\gamma_2+1)} + \eta^{-\vartheta}\|g\|_\infty\frac{\eta^{\alpha+\gamma_2}}{\Gamma(\alpha+\gamma_2+1)}\right)\\
&\leq R\left[\left(\frac{|\mu|T^{-\gamma_1}}{\Gamma(1-\gamma_1)} + \frac{\eta^{\gamma_2}}{\Gamma(\gamma_2+1)}\right) + \left|\frac{\mu}{\lambda}\right|\left(\frac{T^{\alpha-\beta-\gamma_1}}{\Gamma(1-\beta)\Gamma(\alpha-\gamma_1+1)} + \frac{T^{\alpha-\gamma_1-\vartheta}}{\Gamma(\alpha-\gamma_1+1)}\right)\right.
\end{aligned}$$

$$+\frac{1}{|\lambda|}\left(\frac{\eta^{-\beta}}{\Gamma(1-\beta)}\frac{T^{\alpha+\gamma_2}}{\Gamma(\alpha+\gamma_2+1)}+\frac{\eta^{\alpha+\gamma_2-\vartheta}}{\Gamma(\alpha+\gamma_2+1)}\right)\bigg].$$

On the other hand, since

$$|\Theta| = \left|\mu\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}T^{\alpha-\gamma_1-1}+\frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma_2)}\eta^{\alpha+\gamma_2-1}\right|,$$

we can choose $|\mu|$ sufficiently small and $|\lambda|$ large enough such that $\left|\frac{\Omega}{\Theta}\right| \leq \frac{R}{2}$.

Finally, from the estimate above and the fact that

$$|\lambda| \geq \max\left\{\frac{4\Gamma(1-\vartheta)T^{1-\alpha}}{\Gamma(1+\alpha-\vartheta)}, \frac{4M_R T^{1-\alpha}}{R}\right\},$$

we obtain

$$\begin{aligned}\|Tv\|_\infty &\leq \frac{R}{2}+\frac{T^{1-\alpha}}{|\lambda|}\left[M_R+\frac{\Gamma(1-\vartheta)R}{\Gamma(1+\alpha-\vartheta)}\right]\\ &\leq \frac{R}{2}+\frac{R}{4}+\frac{R}{4}=R.\end{aligned}$$

This proves the result. □

5. Ulam-Hyers-Rassias stability

In this section, we study the U-H-R stability of the BVP (1.1)-(1.2). From this point onward, we assume that the nonlinear function f, along with all the derivative and integral orders, α, β, γ_1, and γ_2 and the parameters λ, μ, and γ_3 in the boundary conditions, satisfy the assumptions of Theorem 4.3. As a result, the problem admits at least one solution within a certain closed ball in $C_{\alpha-1}[0,T]$.

To correctly apply the semigroup property of the R-L integral and use Gronwall's inequality, we must impose additional constraints on the parameters α and β. Specifically, we require

$$0 \leq \gamma_1 < \alpha, \quad 1/2 \leq \alpha < 2/3, \quad 1 \leq \gamma_2+\alpha, \quad \beta < \alpha, \quad 2\alpha-\beta > 1 \tag{5.1}$$

(see also the conditions in (4.5)).

The stability conclusion relay in the following Gronwall inequality for weakly singular kernels (Theorem 3.2 on [26]):

Lemma 5.1. *Let $a \geq 0$ and $b > 0$ be constants and suppose that $\beta > 0$, $\gamma \geq 0$, and $\beta+\gamma < 1$. Suppose that u is a non-negative function in $L^\infty[0,T]$ satisfying that*

$$u(t) \leq a + b\int_0^t (t-s)^{-\beta}s^{-\gamma}u(s)ds, \quad \textit{for a.e. } t \in [0,T].$$

We write $B_0 := B(1-\beta, 1-\gamma)$, for the Beta function. For $r > 0$ let $t_r = \left(\frac{r}{bB_0}\right)^{\frac{1}{1-\beta-\gamma}}$, and let $r_0 := bB_0T^{1-\beta-\gamma}$ so that $t_r \leq T$ for $r \leq r_0$. Then, if $r \leq r_0$ and also $r < 1$, we have

$$u(t) \leq \frac{a}{1-r}\exp\left(\frac{bt_r^{-\beta}}{(1-r)(1-\gamma)}t^{1-\gamma}\right) \quad \textit{for a.e. } t \in [0,T].$$

Moreover, for $r_1 = \beta/(1-\gamma)$*, we always have*

$$u(t) \leq \frac{a(1-\gamma)}{1-\beta-\gamma} \exp\left(\frac{bt_{r_1}^{-\beta}}{1-\beta-\gamma} t^{1-\gamma}\right) \quad \textit{for a.e. } t \in [0,T].$$

In particular, there is an explicit constant $C(b,\beta,\gamma,T)$ *such that* $u(t) \leq C(b,\beta,\gamma,T)$ *for a.e.* $t \in [0,T]$*. Moreover, the exponent constant* $\frac{bt_{r_1}^{-\beta}}{1-\beta-\gamma)}$ *is optimal in the sense that it is the smallest possible for admissible choices of r.*

Definition 5.1. *The BVP* (1.1)-(1.2) *is U-H-R stable if for given* $\epsilon > 0$*, and a function* $y \in C_{\alpha-1}[0,T]$ *satisfying the boundary conditions on* (1.3)*, and the inequality*

$$|\lambda D^{\alpha} y(t) + D^{\beta} y(t) - f(t, y(t))| < \epsilon \Upsilon(t),$$

there exists a solution $u \in C_{\alpha-1}[0,T]$ *of BVP* (1.1)-(1.2) *such that*

$$|y(t) - u(t)| \leq \epsilon \Psi(t) \Upsilon(t), \quad t \in (0,T],$$

where Ψ *and* Υ *are some functions that do not depend on u and y.*

Now, we prove the U-H-R stability of BVP (1.1)-(1.2).

Theorem 5.1. *(U-H-R stability) Let* $\epsilon > 0$ *be given, and let* $B_{R(\epsilon)}$ *denote the closed ball of radius* $R(\epsilon) > 0$ *in* $C_{\alpha-1}[0,T]$*, where* $R(\epsilon)$ *depends on* ϵ *and satisfies the conditions established in Theorem 4.3. Let* $f_{R(\epsilon)} \in L^{\infty}[0,T]$ *be the function such that* $g(t,u) \leq f_{R(\epsilon)}(t)$ *for all* $u \in B_{R(\epsilon)}$*, and suppose that*

$$\|f_{R(\epsilon)}\|_{\infty} \leq \epsilon \frac{M_{R(\epsilon)}}{R(\epsilon)}.$$

Let $y \in B_{R(\epsilon)}$ *be a function satisfying the inequality*

$$|\lambda D^{\alpha} y(t) + D^{\beta} y(t) - f(t, y(t))| < \epsilon \Upsilon(t), \tag{5.2}$$

together with the boundary conditions

$$\lim_{t \to 0^+} t^{1-\alpha} y(t) = y^0, \quad \mu D^{\gamma_1} y(T) + I^{\gamma_2} y(\eta) = \gamma_3. \tag{5.3}$$

If $u \in B_{R(\epsilon)}$ *is a solution of the BVP* (1.1)-(1.2)*, then*

$$|y(t) - u(t)| \leq \epsilon \Psi(t) \Upsilon(t), \quad \textit{for all } t \in (0,T].$$

Proof. Note that if $y \in C_{\alpha-1}[0,T]$ is a solution of inequality (5.2), then there exists a function $m \in C_{\alpha-1}[0,T]$, with $|m(t)| \leq \epsilon \Upsilon(t)$, such that y satisfies the equation

$$\lambda D^{\alpha} y(t) + D^{\beta} y(t) = f(t, y(t)) + m(t).$$

Letting $F(t, y(t), D^{\alpha} y(t)) := f(t, y(t)) - D^{\alpha} y(t) + m(t)$ we find that F satisfies all the hypotheses of Proposition 3.1. Moreover, since y satisfies the boundary conditions (5.3), we can follow the proof of Proposition 3.2 to conclude that y can be written as

$$y(t) = \frac{\widetilde{\Omega}}{\Theta} t^{\alpha-1} - \frac{1}{\lambda} [I^{\alpha} D^{\beta} y(t) - I^{\alpha} f(t, y(t)) - I^{\alpha} m(t)], \quad t \in (0,T]$$

with

$$\widetilde{\Omega} := \Omega - \frac{1}{\lambda}[I^{\alpha-\gamma_1}m(T) + I^{\gamma_2+\alpha}m(\eta)],$$

where Ω and Θ are as in Proposition 3.2.

Let $u \in C_{\alpha-1}[0,T]$ be a solution of BVP (1.1). Then we write

$$u(t) = \frac{\Omega}{\Theta}t^{\alpha-1} - \frac{1}{\lambda}[I^{\alpha}D^{\beta}u(t) - I^{\alpha}f(t,u(t))], \quad t \in (0,T].$$

We have the following estimate:

$$\begin{aligned}|u(t) - y(t)| \le & \left|\frac{1}{\lambda\Theta}(I^{\alpha-\gamma_1}m(T) - I^{\alpha+\gamma_2}m(\eta))\right| t^{\alpha-1} \\ & + \frac{1}{|\lambda|}\left|I^{\alpha}D^{\beta}(u(t)-y(t)) - I^{\alpha}(f(t,u(t)) - f(t,y(t))) - I^{\alpha}m(t)\right| \\ \le & \frac{\|m\|_\infty}{|\lambda\Theta|}\left(\frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \frac{\eta^{\alpha+\gamma_2}}{\Gamma(\alpha+\gamma_2+1)}\right)t^{\alpha-1} \\ & + \frac{1}{|\lambda|}|I^{\alpha}D^{\beta}(u(t)-y(t))| + \frac{2\|f_{R(\epsilon)}\|_\infty}{|\lambda|}\frac{\Gamma(1-\vartheta)}{\Gamma(\alpha-\vartheta+1)}t^{\alpha-\vartheta} + \frac{\|m\|_\infty}{|\lambda|\Gamma(\alpha+1)}t^{\alpha}.\end{aligned}$$

Since $\alpha \ge 1/2$, from Lemmas 2.1 and 3.1, we have that

$$\begin{aligned}I^{\alpha}D^{\beta}(u(t)-y(t)) &= I^{\alpha-\beta}I^{\beta}D^{\beta}(u(t)-y(t)) \\ &= I^{\alpha-\beta}\left(u(t) - y(t) - \frac{t^{\beta-1}}{\Gamma(\beta)}(u^0\Gamma(\beta) - y^0\Gamma(\beta))\right) \\ &= I^{\alpha-\beta}(u(t)-y(t)) - (u^0 - y^0)I^{\alpha-\beta}t^{\beta-1} \\ &= I^{\alpha-\beta}(u(t)-y(t)) - \frac{1}{\lambda\Theta}[I^{\alpha-\gamma_1}m(T) + I^{\gamma_2+\alpha}m(\eta)]\frac{\Gamma(\beta)}{\Gamma(\alpha)}t^{\alpha-1}.\end{aligned}$$

In the above equality, we use the fact that

$$u^0 - y^0 = \frac{1}{\lambda\Theta}[I^{\alpha-\gamma_1}m(T) + I^{\gamma_2+\alpha}m(\eta)].$$

Hence, we obtain the estimate

$$\begin{aligned}|u(t) - y(t)| \le & \frac{\|m\|_\infty}{|\lambda\Theta|}\left(\frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \frac{\eta^{\alpha+\gamma_2}}{\Gamma(\alpha+\gamma_2+1)}\right)t^{\alpha-1} + \frac{1}{|\lambda|}|I^{\alpha-\beta}(u(t)-y(t))| \\ & + \frac{\|m\|_\infty}{\lambda^2|\Theta|}\left(\frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \frac{\eta^{\alpha+\gamma_2}}{\Gamma(\alpha+\gamma_2+1)}\right)\frac{\Gamma(\beta)}{\Gamma(\alpha)}t^{\alpha-1} \\ & + \frac{2\|f_{R(\epsilon)}\|_\infty}{|\lambda|}\frac{\Gamma(1-\vartheta)}{\Gamma(\alpha-\vartheta+1)}t^{\alpha-\vartheta} + \frac{\|m\|_\infty}{|\lambda|\Gamma(\alpha+1)}t^{\alpha} \\ = & \left(\frac{1}{|\lambda|} + \frac{\Gamma(\beta)}{\Gamma(\alpha)\lambda^2}\right)\left(\frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \frac{\eta^{\alpha+\gamma_2}}{\Gamma(\alpha+\gamma_2+1)}\right)\frac{\|m\|_\infty t^{\alpha-1}}{|\Theta|} \\ & + \frac{2\|f_{R(\epsilon)}\|_\infty}{|\lambda|}\frac{\Gamma(1-\vartheta)t^{\alpha-\vartheta}}{\Gamma(\alpha-\vartheta+1)} + \frac{\|m\|_\infty t^{\alpha}}{|\lambda|\Gamma(\alpha+1)} + \frac{1}{|\lambda|}I^{\alpha-\beta}|u(t)-y(t)|.\end{aligned}$$

Since $u, y \in C_{\alpha-1}[0,T]$, we write $u(t) = t^{\alpha-1}\widetilde{u}(t)$ and $y(t) = t^{\alpha-1}\widetilde{y}(t)$, where $\widetilde{u}, \widetilde{y} \in C[0,T]$. Thus, we rewrite the inequality above as

$$\begin{aligned}|\widetilde{u}(t) - \widetilde{y}(t)| \leq& \left(\frac{1}{|\lambda|} + \frac{\Gamma(\beta)}{\Gamma(\alpha)\lambda^2}\right)\left(\frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \frac{\eta^{\alpha+\gamma_2}}{\Gamma(\alpha+\gamma_2+1)}\right)\frac{\|m\|_\infty}{|\Theta|}\\ &+ \frac{2\|f_{R(\epsilon)}\|_\infty}{|\lambda|}\frac{\Gamma(1-\vartheta)t^{1-\vartheta}}{\Gamma(\alpha-\vartheta+1)} + \frac{\|m\|_\infty t}{|\lambda|\Gamma(\alpha+1)}\\ &+ \frac{t^{1-\alpha}}{\Gamma(\alpha-\beta)|\lambda|}\int_0^t (t-s)^{\alpha-\beta-1}s^{\alpha-1}|\widetilde{u}(s) - \widetilde{y}(s)|ds\\ \leq& \left(\frac{1}{|\lambda|} + \frac{\Gamma(\beta)}{\Gamma(\alpha)\lambda^2}\right)\left(\frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \frac{\eta^{\alpha+\gamma_2}}{\Gamma(\alpha+\gamma_2+1)}\right)\frac{\|m\|_\infty}{|\Theta|}\\ &+ \frac{2\|f_{R(\epsilon)}\|_\infty}{|\lambda|}\frac{\Gamma(1-\vartheta)T^{1-\vartheta}}{\Gamma(\alpha-\vartheta+1)} + \frac{\|m\|_\infty T}{|\lambda|\Gamma(\alpha+1)}\\ &+ \frac{T^{1-\alpha}}{\Gamma(\alpha-\beta)|\lambda|}\int_0^t (t-s)^{\alpha-\beta-1}s^{\alpha-1}|\widetilde{u}(s) - \widetilde{y}(s)|ds.\end{aligned}$$

Letting

$$\begin{aligned}a :=& \left(\frac{1}{|\lambda|} + \frac{\Gamma(\beta)}{\Gamma(\alpha)\lambda^2}\right)\left(\frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \frac{\eta^{\alpha+\gamma_2}}{\Gamma(\alpha+\gamma_2+1)}\right)\frac{\|m\|_\infty}{|\Theta|}\\ &+ \frac{2\|f_{R(\epsilon)}\|_\infty}{|\lambda|}\frac{\Gamma(1-\vartheta)T^{1-\vartheta}}{\Gamma(\alpha-\vartheta+1)} + \frac{\|m\|_\infty T}{|\lambda|\Gamma(\alpha+1)},\\ b :=& \frac{T^{1-\alpha}}{\Gamma(\alpha-\beta)|\lambda|},\end{aligned}$$

we have that the above inequality takes the form

$$|\widetilde{u}(t) - \widetilde{y}(t)| \leq a + b\int_0^t (t-s)^{\alpha-\beta-1}s^{\alpha-1}|\widetilde{u}(s) - \widetilde{y}(s)|ds.$$

Thus, since $2\alpha - \beta > 1$, from Lemma 5.1, we conclude that

$$|\widetilde{u}(t) - \widetilde{y}(t)| \leq a\Psi(t),$$

where Ψ is an exponential function. Finally, since $\|f_{R(\epsilon)}\|_\infty \leq \epsilon\frac{M_{R(\epsilon)}}{R(\epsilon)}$ and taking into account that $|\lambda| \geq \frac{4M_{R(\epsilon)}T^{1-\alpha}}{R(\epsilon)}$, we conclude that

$$\begin{aligned}|\widetilde{u}(t) - \widetilde{y}(t)| \leq \epsilon\Bigg[&\left(\frac{1}{|\lambda|} + \frac{\Gamma(\beta)}{\Gamma(\alpha)\lambda^2}\right)\left(\frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \frac{\eta^{\alpha+\gamma_2}}{\Gamma(\alpha+\gamma_2+1)}\right)\frac{1}{|\Theta|}\\ &+ \frac{\Gamma(1-\vartheta)T^{\alpha-\vartheta}}{\Gamma(\alpha-\vartheta+1)} + \frac{T}{|\lambda|\Gamma(\alpha+1)}\Bigg]\Upsilon(t)\Psi(t).\end{aligned}$$

That is, the BVP (1.1)-(1.2) is U-H-R stable. □

6. Examples

In this section, we illustrate the applicability of the existence and stability results established in the previous sections. Specifically, we analyze concrete fractional BVPs, verifying the conditions of Theorems 4.3 and 5.1. These examples highlight how our theoretical framework can be used to determine the admissible ranges of the parameters involved, ensure solvability, and, when possible, establish U-H-R stability.

Example 6.1. Consider the fractional BVP

$$\lambda D^{\frac{1}{2}}u(t) + D^{\frac{1}{4}}u(t) = \frac{\Gamma(1/2)}{\Gamma(1/4)}(u(t))^{3/2}, \quad \lambda \geq 40, \tag{6.1}$$

$$\lim_{t\to 0^+} t^{1/2}u(t) = 1, \quad \Gamma(1/4)D^{\frac{1}{4}}u(1) + I^{\frac{1}{2}}u(\eta) = 2\sqrt{\pi}, \quad \eta \in (0,1). \tag{6.2}$$

Then (6.1)-(6.2) has at least one solution $u \in C_{-1/2}[0,1]$ contained in a closed ball $B(R)$ with $2 \leq R \leq 4$.

Solution. **Step 1.** We first verify that $u(t) = t^{-1/2}$ solves the associated integral equation

$$u(t) = t^{-1/2} - \frac{1}{\lambda}\left[I^{\frac{1}{2}}D^{\frac{1}{4}}u(t) - I^{\frac{1}{2}}\frac{\Gamma(1/2)}{\Gamma(1/4)}|u(t)|^{3/2}\right]. \tag{6.3}$$

Indeed, for each $t \in (0,1]$,

$$D^{\frac{1}{4}}u(t) = D^{\frac{1}{4}}t^{-1/2} = \frac{\Gamma(1/2)}{\Gamma(1/4)}t^{-3/4} = \frac{\Gamma(1/2)}{\Gamma(1/4)}|u(t)|^{3/2},$$

and by linearity of the R-L integral, the second term in the right-hand side of (6.3) vanishes. Hence, $u(t) = t^{-1/2}$ solves (6.3).

Step 2. Next, we check that (6.3) matches the integral form (3.3) in Proposition 3.2. This reduces to verifying $\Omega = \Theta$. Indeed,

$$\Omega = 2\sqrt{\pi} + 0 + 0 = 2\sqrt{\pi},$$

and

$$\Theta = \Gamma(1/4)\frac{\Gamma(1/2)}{\Gamma(1/4)} + \frac{\Gamma(1/2)}{\Gamma(1/2+1/2)} = 2\Gamma(1/2) = 2\sqrt{\pi}.$$

Thus, $u(t) = t^{-1/2}$ satisfies (6.1)-(6.2).

Step 3. To apply Theorem 4.3, set $f(t,u) = \frac{\Gamma(1/2)}{\Gamma(1/4)}|u(t)|^{3/2}$. Then,

$$|f(t,t^{-1/2}v)| = \frac{\Gamma(1/2)}{\Gamma(1/4)}|t^{-1/2}v|^{3/2} = \frac{\Gamma(1/2)}{\Gamma(1/4)}t^{-3/8}t^{-3/8}|v|^{3/2}.$$

Setting $\vartheta = 3/8$ and defining $g(t,v) = \frac{\Gamma(1/2)}{\Gamma(1/4)}|t^{-1/4}v|^{3/2}$, we rewrite

$$|f(t,t^{-1/2}v)| = t^{-\vartheta}g(t,v),$$

where g is an L^∞_{loc} function, since for $|v| \leq R$,

$$|g(t,v)| \leq \frac{\Gamma(1/2)}{\Gamma(1/4)}R^{3/2}t^{-3/8} =: g_R(t),$$

with $g_R \in L^\infty[0,1]$.

Moreover, since $C_{-3/8}[0,1] \subset L^\infty[0,1]$, for $v \in B(R)$, we estimate

$$\|g\|_\infty \le \|g\|_{-3/8} \le \frac{\Gamma(1/2)}{\Gamma(1/4)}\|v\|_\infty^{3/2} \le \frac{\Gamma(1/2)}{\Gamma(1/4)}R^{3/2} \le \frac{R^{3/2}}{2}.$$

Thus, the condition $\|g\|_\infty \le R$ in Theorem 4.3 holds for $R \le 4$, whereas condition $\left|\frac{\Omega}{\Theta}\right| \le \frac{R}{2}$ holds for $R \ge 2$. Hence, combining both conditions we conclude that $2 \le R \le 4$.

Step 4. Verification of the lower bound for λ. Since $|D^{\frac{1}{4}}u| = |DI^{\frac{3}{4}}u|$ and $I^{\frac{3}{4}}u \in C^{0,3/4}[0,1]$, $I^{\frac{3}{4}}u$ has bounded variation. Therefore, for fixed $\frac{1}{2} < \delta < 1$,

$$\begin{aligned}
|DI^{\frac{3}{4}}u(t_0)| &= \left|\lim_{t\to t_0} \frac{I^{\frac{3}{4}}u(t) - I^{\frac{3}{4}}u(t_0)}{t - t_0}\right| \le \sup_{\frac{\delta}{2}<|t-t_0|<\delta} \left|\frac{I^{\frac{3}{4}}u(t) - I^{\frac{3}{4}}u(t_0)}{t - t_0}\right| \\
&\le \sup_{\frac{\delta}{2}<|t-t_0|<\delta} \frac{I^{\frac{3}{4}}|u(t)| + I^{\frac{3}{4}}|u(t_0)|}{|t - t_0|} \le \sup_{\frac{\delta}{2}<|t-t_0|<\delta} \frac{2\|u\|_\infty I^{\frac{3}{4}}\mathbf{1}(t)}{|t - t_0|} \\
&= 2\|u\|_\infty \sup_{\frac{\delta}{2}<|t-t_0|<\delta} \frac{t^{3/4}}{\Gamma(1+3/4)}\frac{1}{|t-t_0|} \\
&\le \frac{2\|u\|_\infty}{\Gamma(7/4)} \sup_{\frac{\delta}{2}<|t-t_0|<\delta} \frac{1}{|t-t_0|} \le \frac{2R}{\Gamma(7/4)}\frac{2}{\delta} < \frac{8R}{\Gamma(7/4)}.
\end{aligned}$$

Hence, for each $t \in (0,1]$ we obtain

$$|I^{\frac{1}{2}}D^{\frac{1}{4}}u(t)| \le I^{\frac{1}{2}}|D^{\frac{1}{4}}u(t)| \le \frac{8R}{\Gamma(7/4)}I^{\frac{1}{2}}\mathbf{1}(t) = \frac{8R}{\Gamma(7/4)\Gamma(3/2)}t^{1/2}.$$

Thus, we conclude that

$$\|I^{\frac{1}{2}}D^{\frac{1}{4}}u\|_\infty \le \frac{8R}{\Gamma(7/4)\Gamma(3/2)} =: M_R.$$

Therefore, for $\vartheta = 3/8$,

$$\max\left\{\frac{4\Gamma(1-\vartheta)}{\Gamma(1+\frac{1}{2}-\vartheta)}, \frac{4M_R}{R}\right\} = \max\left\{\frac{4\Gamma(5/8)}{\Gamma(9/8)}, \frac{32}{\Gamma(7/4)\Gamma(3/2)}\right\} = \frac{32}{\Gamma(7/4)\Gamma(3/2)} \le 40 \le \lambda,$$

and so the condition on λ is satisfied.

Thus, Theorem 4.3 guarantees at least one solution in $B(R)$ for $R \in [2,4]$ for BVP (6.1)-(6.2). Since $\|t^{-1/2}\|_{-1/2} = \|\mathbf{1}\|_\infty = 1$, the function $u(t) = t^{-1/2}$ belongs to this set.

Lastly, note that Theorem 5.1 cannot be applied here, since the condition $2\alpha - \beta > 1$ fails for $\alpha = 1/2, \beta = 1/4$. □

Remark 6.1. *The independence of the solution with respect to η in Example 6.1 is not due to the low order of the differential equation, but to the invariance of $u(t) = t^{-1/2}$ under the R-L integral of order $\frac{1}{2}$, as $I^{\frac{1}{2}}\eta^{-1/2} \equiv \Gamma(1/2)$ for all $\eta \in (0,1]$. For other orders, e.g., $\gamma_2 = 1$, the boundary condition introduces a dependence on η, leading to a unique admissible value of η: the function $u(t) = t^{-1/2}$ is a solution of the corresponding integral equation if $\Omega = \Theta$, which in this case are $\Omega = 2\Gamma(1/2)$ and $\Theta = \Gamma(1/2) + \frac{\Gamma(1/2)}{\Gamma(3/2)}\eta^{1/2}$. The equality is satisfied only for $\eta = (\Gamma(3/2))^2$. That is, the value of the point η is unique for this BVP.*

Example 6.2. Consider the fractional BVP

$$\lambda D^{\frac{5}{8}}u(t) + D^{\frac{1}{4}}u(t) = \frac{\Gamma(5/8)}{\Gamma(7/8)}u^{1/3}(t), \quad \lambda \geq 40, \tag{6.4}$$

$$\lim_{t\to 0^+} t^{3/8}u(t) = 1, \quad \Gamma(1/4)D^{\frac{1}{4}}u(1) + Iu(\eta) = \Gamma(1/2), \tag{6.5}$$

where

$$\eta = \left[\Gamma(13/8)\left(\frac{\Gamma(1/2)}{\Gamma(5/8)} - \frac{\Gamma(1/4)}{\Gamma(1/8)}\right)\right]^{8/5}.$$

Then (6.4)-(6.5) has at least one solution $u \in C_{-3/8}[0,1]$ lying in a closed ball $B(R)$ with

$$R \geq \left(\frac{\Gamma(5/8)}{\Gamma(13/8)}\right)^{3/2}.$$

Solution. Set $u(t) = t^{-3/8}$. As before,

$$D^{\frac{1}{4}}u(t) = \frac{\Gamma(5/8)}{\Gamma(7/8)}u^{1/3}(t), \quad \text{for each } t \in (0,1],$$

and consequently $\Omega = \Gamma(1/2)$. On the other hand,

$$\Theta = \Gamma(1/4)\frac{\Gamma(5/8)}{\Gamma(1/8)} + \frac{\Gamma(5/8)}{\Gamma(13/8)}\eta^{5/8},$$

and $\Omega = \Theta$ holds for the stated η. Therefore, $u(t) = t^{-3/8}$ solves (6.4)-(6.5).

For $f(t,v) = \frac{\Gamma(5/8)}{\Gamma(13/8)}v^{1/3}$,

$$|f(t, t^{-3/8}v)| \leq t^{-1/8}g(t,v), \quad g(t,v) = \frac{\Gamma(5/8)}{\Gamma(13/8)}v^{1/3}.$$

For $v \in B(R)$,

$$\|g\|_\infty \leq \frac{\Gamma(5/8)}{\Gamma(13/8)}R^{1/3} \leq R \implies R \geq \left(\frac{\Gamma(5/8)}{\Gamma(13/8)}\right)^{3/2}.$$

Also, $|\Omega/\Theta| \leq R/2$ requires $R \geq 2$. Thus, the lower bound is $\max\{2, (\Gamma(5/8)/\Gamma(13/8))^{3/2}\} \leq R$.

Finally, for $\vartheta = 1/8$,

$$\max\left\{\frac{4\Gamma(1-\vartheta)}{\Gamma(1+\frac{5}{8}-\vartheta)}, \frac{4M_R}{R}\right\} = \max\left\{\frac{4\Gamma(7/8)}{\Gamma(3/2)}, \frac{32}{\Gamma(7/4)\Gamma(3/2)}\right\} = \frac{32}{\Gamma(7/4)\Gamma(3/2)} \leq 40 \leq \lambda,$$

thereby satisfying the parameter condition.

Since the orders of derivatives and integrals in the present example satisfy the conditions (5.1), and for $\epsilon > 0$ such that

$$\left(\frac{\Gamma(5/8)}{\Gamma(13/8)}\right)^{3/2} \leq R \leq \left(\frac{8\Gamma(13/8)\epsilon}{\Gamma(5/8)\Gamma(7/4)\Gamma(3/2)}\right)^3,$$

we have $\|g\|_\infty \leq \epsilon\frac{M_R}{R}$, and then, by Theorem 5.1, we can conclude that the BVP is U-H-R stable. □

The two examples presented illustrate how our combination of arguments and fixed point structure effectively deal with non-trivial fractional BVPs, even in the presence of singular behavior and non-standard growth conditions. In particular, the examples also show how the theoretical bounds of the parameters (such as R and λ) can be explicitly computed and when the stability results apply.

7. Conclusions

In conclusion, we introduced a new combination of techniques that allow us to consider BVPs with a certain degree of complexity and without imposing very strong sufficient conditions in view of having the desired conclusions. The work extends existing literature by addressing singular behavior near the origin through the L^p-Carathéodory condition and employing the Leray-Schauder fixed point theorem without contractive assumptions. Moreover, we have opted for a different approach when comparing with most of the works related to BVPs having orders of differentiability not greater than one. Namely, in addition to the initial condition, which, considered only with the differential equation, allows us (without further ado) to "translate" the equation and condition equivalently through a single Volterra-type integral equation (cf. Proposition 3.1), we chose to add an extra condition, with a non-local character, which will naturally limit the existence of many different solutions. Furthermore, the motivation for choosing this specific additional condition (of several points) has to do with the mathematical richness it implies, which is also illustrated (and used) in several other works. Therefore, we can say that we are proposing a new way of considering BVPs with an order of differentiation less than one and where an additional condition is also included, which is not used to translate the problem in terms of an integral (Volterra) equation, but which restricts the possibility of having more solutions to a problem than the ones proposed here.

It is in this context that Section 4 presents, as its most significant result, a theorem that guarantees the existence of solutions to the problem in question. In addition, Section 5 presents different conditions under which U-H-R stability is guaranteed.

Moreover, two concrete examples of BVPs within the conditions considered here were also analyzed to illustrate the conditions in question and the possible solutions. We deliberately chose to show a case where it is possible to guarantee the stability of U-H-R according to our conditions and another where it is not, thus also illustrating the relevance of our sufficient conditions (which are therefore neither absolute nor represent the empty set).

In terms of possibilities for continuing this research, several directions could be pursued, among which we highlight the application of current methods to other classes of BVPs, to other types of stability (e.g., of exponential nature [13, 28]), and, additionally, the consideration of other classes of function spaces that are "weaker" than the continuity ones considered here.

Author contributions

L. P. Castro and E. M. Rojas: Conceptualization, Formal analysis, Investigation, Writing–original draft, Writing–review & editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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