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Research article

Weak degeneracy of the square of the line graph of a subcubic graph

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Abstract: Weak degeneracy is a refined variation of degeneracy that retains many of the useful structural properties of degeneracy, such as facilitating efficient graph orientation, enabling compact representations, and supporting algorithmic applications in graph theory. We focus on the Erdős-Nešetřil Conjecture from the prespective of weak degeneracy. In this paper, we prove that for every subcubic graph G with a maximun average degree less than $\frac{33}{16}$, $\frac{27}{11}$, $\frac{13}{5}$, and $\frac{36}{13}$, the weak degeneracy of the corresponding graph $(L(G))^2$ is at most 5, 6, 7, and 8, respectively, where $(L(G))^2$ is the square of the line graph of G.

Keywords: weak degeneracy; strong chromatic index; subcubic graph
Mathematics Subject Classification: 05C15, 05C10

1. Introduction

All graphs in this paper are finite and simple. For a graph $G = (V, E)$ and $e_1, e_2 \in E$, the *distance* $d_G(e_1, e_2)$ between e_1 and e_2 is the length of the shortest path between the end vertices of e_1 and the end vertices of e_2 in G. For a graph G, $(L(G))^2$ denotes *square of the line graph*, where

$$V((L(G))^2) = E(G), E((L(G))^2) = \{e_ie_j : d_G(e_i, e_j) \leq 1\}.$$

A proper coloring of G is an assignment of colors to the vertices of G so that no two adjacent vertices receive the same color. A proper coloring of $(L(G))^2$ is called a *strong edge-coloring* of G. The *chromatic number* of $(L(G))^2$, denoted by $\chi'_s(G)$, is called the *strong chromatic index* of G, where the strong chromatic index is the minimum number of colors needed to color its edges so that each color class is an induced matching. The concept of strong chromatic index was introduced in [6, 7].

In 1985, Erdős and Nešetřil [5] presented the following conjecture.

Conjecture 1. Assume G is a graph with a maximum degree of Δ. Then,

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even;} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

This conjecture received a lot of attention. It was verified for graphs G with $\Delta(G) \leq 3$ [1, 11], and remains open for graphs G with $\Delta(G) \geq 4$. For graphs G with $\Delta(G) = 4$, the conjectured upper bound for $\chi'_s(G)$ is 20. The current known upper bound is 21 [12]. For graphs with a large maximum degree, Molly and Reed [15] proved that $\chi'_s(G) \leq 1.998\Delta^2$, which was subsequently improved by Bonamy, Perrett, and Postle [3] to $\chi'_s(G) \leq 1.835\Delta^2$, and by Hurley, Verclos, and Kang [13] to $\chi'_s(G) \leq 1.772\Delta^2$.

We usually use $\text{mad}(G) = \max_{H \subseteq G, |V(H)| \geq 1} \frac{2|E(H)|}{|V(H)|}$ to denote the maximum average degree of the graph G. Hocquard and Valicow [10] studied the strong chromatic index of subcubic graphs with limited mad(G), and provided the following theorem.

Theorem 1. *Let G be a subcubic graph:*

(a) If $mad(G) < \frac{15}{7}$, *then* $\chi'_s(G) \leq 6$;

(b) If $mad(G) < \frac{27}{11}$, *then* $\chi'_s(G) \leq 7$;

(c) If $mad(G) < \frac{13}{5}$, *then* $\chi'_s(G) \leq 8$;

(d) If $mad(G) < \frac{36}{13}$, *then* $\chi'_s(G) \leq 9$.

Later, Hocquard et al. [9] improved Theorem 1 and produced Theorem 2.

Theorem 2. *Let G be a subcubic graph:*

(a) If $mad(G) < \frac{7}{3}$, *then* $\chi'_s(G) \leq 6$;

(b) If $mad(G) < \frac{5}{2}$, *then* $\chi'_s(G) \leq 7$;

(c) If $mad(G) < \frac{8}{3}$, *then* $\chi'_s(G) \leq 8$;

(d) If $mad(G) < \frac{20}{7}$, *then* $\chi'_s(G) \leq 9$.

Moreover, for (a), (b), and (d) of Theorem 2, Hocquard et al. [9] provided some graphs that satisfied with $\text{mad}(G) = \frac{7}{3}$ $(\frac{5}{2}, \frac{20}{7})$ and $\chi'_s(G) > 6$ (7 *and* 9, respectively). These examples showed that the corresponding bounds were sharp.

An edge list L of a graph G is a mapping that assigns a finite set to each edge of G. We say that L is a k-edge list if $|L(e)| \geq k$ for every edge in G. The graph G is strongly L-edge colorable if there exists a strong edge-coloring c of G such that $c(e) \in L(e)$ for every edge e of G. For a positive integer k, a graph G is strongly k-edge choosable if for every k-edge list L, G is strongly L-edge colorable. The list strong chromatic index $\chi'_{ls}(G)$ is the minimum positive integer k for which G is strongly k-edge choosable. There are many works on the list strong chromatic index. For example, $\chi'_{ls}(G)$ for subcubic graphs was investigated by Ma et al. [14], which presented the following theorem.

Theorem 3. *Let G be a subcubic graph:*

(a) If $mad(G) < \frac{15}{7}$, *then* $\chi'_{ls}(G) \leq 6$;

(b) If mad(G) $< \frac{27}{11}$, *then* $\chi'_{ls}(G) \leq 7$;

(c) If mad(G) $< \frac{13}{5}$, *then* $\chi'_{ls}(G) \leq 8$;

(d) If mad(G) $< \frac{36}{13}$, *then* $\chi'_{ls}(G) \leq 9$.

For (b), (c) and (d) of Theorem 3, Zhu and Miao [18] provided a better upper bound with mad(G) using the following Theorem 4.

Theorem 4. *Let G be a subcubic graph:*

(a) If mad(G) $< \frac{5}{2}$, *then* $\chi'_{ls}(G) \leq 7$;

(b) If mad(G) $< \frac{8}{3}$, *then* $\chi'_{ls}(G) \leq 8$;

(c) If mad(G) $< \frac{14}{5}$, *then* $\chi'_{ls}(G) \leq 9$.

For a graph G, the k-assignment L assigns a list $L(v)$ (a set of colors) with $|L(v)| = k$ to each vertex v. G is defined as L-colorable if there is a proper coloring f where $f(v) \in L(v)$. If G is L-colorable for any k-assignment L, then G is k-choosable. The *choice number of G, denoted* $\chi_l(G)$*, is the smallest integer k such that G is k-choosable.* Based on the definition of a choice number, it is easy to observe that the choice number $\chi_l((L(G))^2)$ of $(L(G))^2$ is equal to $\chi'_{ls}(G)$ for every graph G. The choice number $\chi_l((L(G))^2)$ of $(L(G))^2$ has been previously studied in a few papers. In [17], it was proved that $\chi_l((L(G))^2) \leq 22$ for G with $\Delta \leq 4$.

The *degeneracy* of graphs comes from the greedy algorithms for graph coloring. The basic greedy algorithm considers the vertices of G one at a time. Considering how to color one vertex u in the proper coloring, we assign to it an arbitrary color, say α, from $L(u)$ (which is the set of available colors of u). At this point, to ensure that the coloring is proper, we have to remove the colors which correspond to α from the lists of colors available to the neighbors of u. Thus, the list size for every neighbor u may decrease by 1, while all the other lists remain unchanged. If no list size reduces to 0 throughout this process (i.e., if every uncolored vertex always has at least one available color), then we successfully obtain a proper coloring. This idea is formally captured in the notion of the *degeneracy* of the graph.

To motivate the definition of *weak degeneracy* of graphs, consider a vertex $u \in V(G)$ and let A be a subset of its all neighbors. In general, if we assign a color to u, then every vertex of A may lose one of its colors. However, suppose that $|L(u)| > \Sigma_{v \in A}|L(v)|$ (i.e., u has strictly more available colors than the set of all available colors of vertices in A). In this case, there must be a color in $L(u)$ that does not correspond to any color in $\bigcup_{v \in A} L(v)$, and assigning such a color to u does not affect the number of available colors of any vertex contained in A (of course, the other neighbors of u may still lose a color). In this way, we "save" an extra color for every vertex in A. This idea naturally leads to the notion that we call *weak degeneracy*.

In this paper, we are interested in the weak degeneracy of $(L(G))^2$. The weak degeneracy of a graph is a variation of its degeneracy, which was recently introduced recently by Bernshteyn and Lee [2]. Let $G = (V, E)$ be a graph, $\mathbb{Z}$ be the set of integers, and $\mathbb{Z}^G$ be the set of mappings $f : V(G) \to \mathbb{Z}$. An integer $k \in \mathbb{Z}$ is viewed as a constant mapping in $\mathbb{Z}^G$ defined as $k(v) = k$ for all $v \in V(G)$. For $f, g \in \mathbb{Z}^G$, $f \pm g \in \mathbb{Z}^G$ is defined as $(f \pm g)(v) = f(v) \pm g(v)$ for all $v \in V(G)$. For $f \in \mathbb{Z}^G$ and a subset U of $V(G)$, let $f|_U$ be the restriction of f to U, and let $f_{-U} : V(G) - U \to \mathbb{Z}$ be defined as $f_{-U}(x) = f(x) - |N_G(x) \cap U|$ for $x \in V(G) - U$. For convenience, we may use f for $f|_U$, and write f_{-v} for $f_{-\{v\}}$. We denote the set of edges incident to u for any $u \in V(G)$ by $E(u)$, and let $E(U) = \cup_{u \in U} E(u)$. For a subset X of $V(G)$, δ_X is the characteristic function of X, which is defined as $\delta_X(v) = 1$ if $v \in X$ and $\delta_X(v) = 0$ otherwise. Let $\mathcal{L}$ be the set of pairs (G, f), where G is a graph and $f \in \mathbb{Z}^G$.

When comparing the pair (G, f) with the algorithm for graph coloring, it is clear that G is a graph and f is mapping that initially assigns a positive integer for every vertex in G. While we intend to find a proper coloring for G, you could view the positive integer as the number of available colors for every vertex. Based on this idea and the above improved algorithm for graph coloring, we introduce the delete-save operation for (G, f).

Definition 1. For $u \in V(G)$ and $A \subseteq V(G)$, the *deletion-save operation* $\mathsf{DeleteSave}(u, A) : \mathcal{L} \to \mathcal{L}$ is defined as follows:

$$\mathsf{DeleteSave}(u, A)(G, f) = (G - u, f_{-u} + \delta_A).$$

We say $\mathsf{DeleteSave}(u, A)$ is *legal* for (G, f) if $A \subseteq N_G(u)$, $f(u) \geq \sum_{w \in A} f(w) + |A|$, and both f and $f_{-u} + \delta_A$ are non-negative.

It is evident that the delete-save operation has a strong connection with the improved algorithm for graph coloring. After we use legal $\mathsf{DeleteSave}$(u,A) for (G, f), the new pair $(G - u, f_{-u} + \delta_A)$ can be viewed as follows:

(1) u is deleted (in the graph coloring, we usually say that u is colored);

(2) If $v \in V(G) - u$ and $v \notin N_G(u)$, then $f_{new}(v) = f(v)$;

(3) If $v \in V(G) - u$ and $v \in A$, then $f_{new}(v) = f(v)$ (comparing with the improved algorithm for graph coloring, the number of availbale colors for these vertices which are contained in A does not change, because we select one color α for u and α does not appera in the set of available colors of any vertex in A);

(4) $f_{new}(v) = f(v) - 1$(after u is colored, the number of available colors of these vertices will decrease at most 1);

where $f_{new} = f_{-u} + \delta_A$.

The above definition is slightly different from the one in [2], where $\mathsf{DeleteSave}(u, A)$ is restricted to the case that $|A| \leq 1$. Actually, we only consider $|A| \leq 1$ in this paper; $\mathsf{DeleteSave}(u, \emptyset)$ is called a *deletion move* and is denoted by $\mathsf{Delete}(u)$.

Definition 2. A *removal scheme* $\Omega = \mathsf{DeleteSave}(\theta_1, \theta_2, \ldots, \theta_k) : \mathcal{L} \to \mathcal{L}$, where for each i, $\theta_i = (u_i, A_i)$, is recursively defined as follows:

$$\mathsf{DeleteSave}(\theta_1, \theta_2, \ldots, \theta_k)(G, f) = \mathsf{DeleteSave}(\theta_k)(\mathsf{DeleteSave}(\theta_1, \theta_2, \ldots, \theta_{k-1})(G, f)).$$

We say $\mathsf{DeleteSave}(\theta_1, \theta_2, \ldots, \theta_k)$ is legal for (G, f) if $\mathsf{DeleteSave}(\theta_1, \theta_2, \ldots, \theta_{k-1})$ is legal for (G, f) and $\mathsf{DeleteSave}(\theta_k)$ is legal for $\mathsf{DeleteSave}(\theta_1, \theta_2, \ldots, \theta_{k-1})(G, f)$. A move $\theta_i = (u_i, A_i)$ *removes* u_i from G. A graph G is *weakly f-degenerate* if there is a removal scheme $\Omega = \mathsf{DeleteSave}(\theta_1, \theta_2, \ldots, \theta_n)$ that is legal for (G, f) and removes all vertices of G. The *weak degeneracy* of G, denoted by $\mathsf{wd}(G)$, is the minimum integer d such that G is weakly d-degenerate.

Note that if all vertices of G can be removed by a sequence of legal deletion operations (i.e., operation of the form $(u, \emptyset)$) with respect to (G, f), then G is *f-degenerate*, and the *degeneracy* of G, denoted by $\mathsf{d}(G)$, is the minimum d such that G is d-degenerate. For every graph, its weak degeneracy is less than or equal to the degeneracy.

For simplicity, we shall denote $\theta_i = (u_i, \emptyset)$ by $\theta_i = (u_i)$ and $\theta_i = (u_i, \{w_i\})$ by $\theta_i = (u_i, w_i)$.

Let us show the difference between the weak degeneracy and the degeneracy. For the bipartite complete graph $K_{3,3}$ with two independent sets $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$, it is clear that its degeneracy is 3. However, its weak degeneracy is at most 2 because the following removal scheme $\Omega = \mathsf{DeleteSave}((u_1), (u_2, v_1), (u_3, \{v_2, v_3\}), (v_1), (v_2), (v_3))$ is legal for $(K_{3,3}, 2)$ such that all six vertices could be removed. Here, $f(u) = 2$ for every vertex in $K_{3,3}$ at first (see Figure 1(a)). In step one, $u = u_1$ and $A = \emptyset$, and the resulting pair is shown in Figure 1(b). In step two, $u = u_2, A = \{v_1\}$, and the resulting pair is shown by Figure 1(c). In step three, $u = u_3, A = \{v_2, v_3\}$, and the resulting pair is shown in Figure 1(d). In step four, $u = v_1$ and $A = \emptyset$. In step five, $u = v_2$ and $A = \emptyset$. In step six, $u = v_3$ and $A = \emptyset$. By iteratively deleting vertices, all vertices are removed by legal delete-save operations, which implies that $\mathsf{wd}(K_{3,3}) \leq 2$.

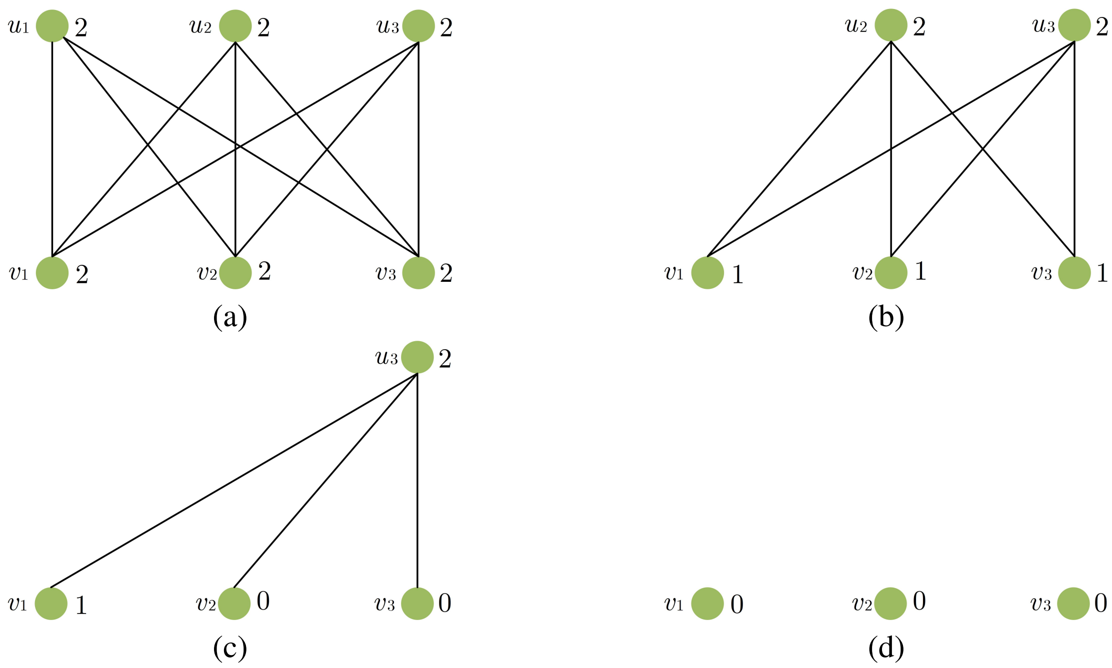

Figure 1. Three resulting pairs after delete-save operations.

The weak degeneracy of a graph gives an upper bound for many coloring parameters. In [2], it was proved that for any graph G,

$$\chi(G) \leq \chi_l(G) \leq \chi_{DP}(G) \leq \chi_{DPP}(G) \leq \mathsf{wd}(G) + 1,$$

and

$$\chi(G) \leq \chi_l(G) \leq \chi_P(G) \leq \chi_{DPP}(G) \leq \mathsf{wd}(G) + 1.$$

Here, $\chi_l(G), \chi_P(G), \chi_{DP}(G)$, and $\chi_{DPP}(G)$ are the choice number, paint number, DP-chromatic number, and DP-paint number of G, respectively. The reader can refer [4, 16, 19] for the definitions of these parameters.

We have to recall that $\chi'_{ls}(G) = \chi_l((L(G))^2)$ for any graph G. According to the definition of weak degeneracy and the fact that $\chi_l(G) \leq \mathsf{wd}(G) + 1$, a new prespective of weak degeneracy will help us to think about the Erdős-Nešetřil Conjecture. Herein, the paper will focus on subcubic graphs G with different mad(G), and the study of $\mathsf{wd}((L(G))^2)$. Moreover, as an extension, the upper bound of $\mathsf{wd}((L(G))^2)$ reflects some upper bound for other parameters. This paper proves Theorem 5 as follows.

Theorem 5. *Let G be a subcubic graph and $H = (L(G))^2$:*

(a) If mad(G) $< \frac{33}{16}$, then $\mathsf{wd}(H) \leq 5$;

(b) If mad(G) $< \frac{27}{11}$, then $\mathsf{wd}(H) \leq 6$;

(c) If mad(G) $< \frac{13}{5}$, then $\mathsf{wd}(H) \leq 7$;

(d) If mad(G) $< \frac{36}{13}$, then $\mathsf{wd}(H) \leq 8$.

The following sections are organized as follows: in Section 2, an important proposition is introduced, and the proof of Theorem 5(a) is completed; and in Sections 3–5, the proofs of Theorem 5(b)–(d) are given, respectively.

Before our next section, some definitions are introduced in advance in order to help us explain more clearly. The degree of vertex u is denoted by $d(u)$. A vertex of degree k (at least k, at most k) is referred to as $k(k^+, k^-)$-vertex. A $k(k^+, k^-)$-neighbor of u is a neighbor of u and the degree of this neighbor is k (at least k, at most k). A 2_k-vertex is a 2-vertex, and the number of its 3-neighbors is k exactly. A 3_k-vertex is a 3-vertex, and the number of its 2-neighbors is k exactly. A 2_k-neighbor of u is one neighbor of u so that the neighbor is 2_k-vertex. A 3_k-neighbor of u is one neighbor of u so that the neighbor is 3_k-vertex. A 3-vertex is an old 3-vertex if it is adjacent to one 1-vertex, otherwise it is young.

2. The proof of Theorem 5(a)

Before we introduce our proof for Theorem 5, because the next Proposition 1 plays an important role during the process, Proposition 1, which was proven in, [8] is shown here.

Proposition 1. If $G - U$ is weakly f_{-U}-degenerate, then G is weakly f-degenerate if and only if $G[U]$ is weakly f-degenerate. In particular, if $f(x) \geq \deg_G(x)$, then G is weakly f-degenerate if and only if $G - x$ is weakly f-degenerate.

If $f, f' \in \mathbb{Z}^G$, $f \leq f'$, and G is weakly f-degenerate, then it is obvious that G is weakly f'-degenerate. However, a DeleteSave move (v, A) legal for (G, f) may not be legal for (G, f'). This happens if $f(v) \geq \sum_{u\in A} f(u) + |A|$, but $f'(v) \leq \sum_{u\in A} f'(u) + |A|$. In this case, one may replace A by A', where $A' = A - \{u : f'(u) > f(u)\}$.

Now, we could start the proof of Theorem 5. If Theorem 5(a) is false, then we choose G as a counterexample with a minimum $|E(G)|$ and $f \in \mathbb{Z}^G$ is defined as $f(e) = 5$ for each edge e.

Let $H = (L(G))^2$. As $\Delta(G) \leq 3$, we have $|N_H(e)| \leq 12$ for any $e \in V(H) = E(G)$, where $N_H(e)$ denotes the set of all neighbors of e in H. For a removal scheme Ω, let $(H_\Omega, f_\Omega) = \Omega(H, f)$. For $e \in H_\Omega$, denote $N_\Omega(e)$ be the set of the neighbors e' of e removed by the move of Ω, i.e., $N_\Omega(e) = N_H(e) - N_{H_\Omega}(e)$. Next, we will prove some claims for G that will lead to a contradiction.

Claim 1. For any 1-vertex u in G, the neighbor v of u is 3_0-vertex, and v has only one 1-neighbor.

Proof. Suppose that G has a 1-vertex u and the neighbor v of u is not 3_0-vertex. If v is 1-vertex, then $G = \{uv\}$. Obviously, H is weakly f-degenerate by a deletion move, which is a contradiction.

Otherwise, v is either 2-vertex or 3-vertex, which has at least one 2-neighbor or 3-vertex, which has another 1-neighbor. Let $G_1 = G - u$, $G_2 = G[\{uv\}]$, and $H_i = (L(G_i))^2$ for $i \in [2]$. Note that $H_1 = H[E(G) - \{uv\}]$ and $H_2 = H[\{uv\}]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H_1 by minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H_2$. Due to $|N_\Omega(uv)| \leq 5$, we have $f_\Omega(uv) \geq 0$. Obviously, $\theta = (uv)$ is a legal deletion move for (H_Ω, f_Ω). By Proposition 1, H is weakly f-degenerate, which is a contradiction. □

Claim 2. For any 2-vertex in G, it has at most one 2-neighbor.

Proof. Suppose that G has one 2-vertex u, and v and w are two 2-neighbors of u.

Let $G_1 = G - u$, $G_2 = G[E(u)]$, and $H_i = (L(G_i))^2$ for $i \in [2]$. Note that $H_1 = H[E(G) - E(u)]$ and $H_2 = H[E(u)]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H_1 by minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H_2$. For each $e \in E(u)$, as $|N_\Omega(e)| \leq 4$, we could obtain $f_\Omega(e) \geq 1$. By Proposition 1, H_Ω is weakly f_Ω-degenerate and H is weakly f-degenerate, which is a contradiction. □

Claim 3. G doesn't contain a path $x_1x_2x_3x_4x_5x_6$ such that x_i is an old 3-vertex for any $i \in [6]$.

Proof. Suppose that G has such a path that $x_1x_2x_3x_4x_5x_6$, a_i is 1-neighbor of x_i for any $i \in [6]$, and as the neighbors of x_1 and x_6, u and v, respectively, are 3-vertices by Claim 1 (see Figure 2). We want to stress in advance that the process of following proof works, regardless of whether u and v, are one same vertex.

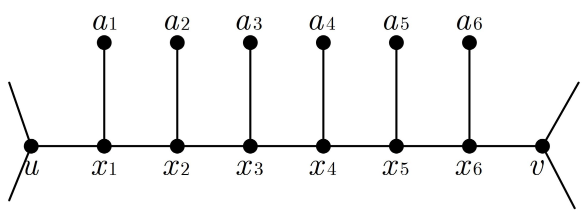

Figure 2. The configuration for Claim 3.

We could choose $U = \{x_2, x_3, x_4, x_5\}$ and set $E' = E(U) \cup \{x_1a_1\}$; then, let $G' = G - E'$ $G'' = G[E']$, $H' = (L(G'))^2$, and $H'' = (L(G''))^2$. Note that $H' = H[E(G) - E']$ and $H'' = H[E']$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H' by minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H''$. It is easy to check that $|N_\Omega(x_5x_6)| \leq 4$, $|N_\Omega(e)| \leq 3$ for $e \in \{x_1a_1, x_1x_2\}$, $|N_\Omega(e)| \leq 2$ for $e \in \{x_4x_5, x_5a_5\}$, $|N_\Omega(e)| \leq 1$ for $e \in \{x_2a_2, x_2x_3\}$, and $|N_\Omega(e)| = 0$ for $e \in \{x_3a_3, x_3x_4, x_4a_4\}$. Thus,

$$f_\Omega(e) \geq \begin{cases} 1, & \text{if } e = x_5x_6; \\ 2, & \text{if } e \in \{x_1a_1, x_1x_2\}; \\ 3, & \text{if } e \in \{x_4x_5, x_5a_5\}; \\ 4, & \text{if } e \in \{x_2a_2, x_2x_3\}; \\ 5, & \text{if } e \in \{x_3a_3, x_3x_4, x_4a_4\}. \end{cases}$$

We only need to prove that H'' is weakly f_Ω-degenerate. Set

$$f_0(e) = \begin{cases} 1, & \text{if } e = x_5x_6; \\ 2, & \text{if } e \in \{x_1a_1, x_1x_2\}; \\ 3, & \text{if } e \in \{x_4x_5, x_5a_5\}; \\ 4, & \text{if } e \in \{x_2a_2, x_2x_3\}; \\ 5, & \text{if } e \in \{x_3a_3, x_3x_4, x_4a_4\}. \end{cases}$$

As $f_\Omega \geq f_0$, it is sufficient to prove that H'' is weakly f_0-degenerate. We will give a legal removal scheme $\Omega' = \mathsf{DeleteSave}(\theta_1, \theta_2, \ldots, \theta_{10})$ for (H'', f_0) such that all vertices of H'' are removed. For $i \in [10]$, let (H_i, f_i) be the resulting pair after applying θ_i over (H_{i-1}, f_{i-1}), where $H_0 = H''$. For $i \in \{0, 1, \cdots, 10\}$, the operation θ_i and the function value of $f_i(e)$ for $e \in V(H_i)$ are shown in the $(i+1)$-th row of the following Table 1, where θ_0 is undefined. The cell of the table is empty if the corresponding edge is removed.

Table 1. The legal removal scheme for (H_0, f_0) in Claim 3.

θ_i \ $f_i(e)$	x_1a_1	x_1x_2	x_2a_2	x_2x_3	x_3a_3	x_3x_4	x_4a_4	x_4x_5	x_5a_5	x_5x_6
	2	2	4	4	5	5	5	3	3	1
(x_3x_4, x_2x_3)	2	1	3	4	4		4	2	2	0
(x_5x_6)	2	1	3	4	4		3	1	1	
(x_3a_3, x_2a_2)	2	0	3	3			2	0	1	
(x_4x_5)	2	0	3	2			1		0	
(x_5a_5)	2	0	3	2			0			
(x_4a_4)	2	0	3	1						
(x_1x_2)	1		2	0						
(x_2x_3)	0		1							
(x_1a_1)			0							
(x_2a_2)										

From this Table 1, it is easy to verify that H'' is weakly f_0-degenerate. Hence, H is weakly f-degenerate by Proposition 1, which is a contradiction. □

Next, the discharging method is used to lead a contradiction. First, we provide the initial charge for every vertex u in G by $ch(u) = d_G(u) - \frac{33}{16}$. It is easy to see that the sum of all charges $\Sigma_{u \in V(G)} ch(u)$ is negative due to $\text{mad}(G) < \frac{33}{16}$. All discharging rules are shown as follows:

R1. For each young 3-vertex u, $ux_1x_2\ldots x_t$ is a path where each x_i is 2-vertex for $i \in [t]$, and the neighbor v of x_t is not 2-vertex; then, u sends $\frac{1}{32}$ to each x_i for $i \in [t]$.

R2. For each young 3-vertex u, $ux_1x_2\ldots x_t$ is a path where each x_i is an old 3-vertex for $i \in [t]$, and there exists one 2^+-neighbor v of x_t which is not old 3-vertex; then, u sends $\frac{1}{16}$ to each x_i for $i \in [t]$.

R3. For each old 3-vertex u, u sends $\frac{17}{16}$ to the 1-neighbor of u.

After these discharging rules (R1–R3), the final charge of each vertex u denoted by $ch^*(u)$ of G will be considered. By Claim 1, we should notice the following: in R1, as one neighbor of x_t, v is also a young 3-vertex; and in R2, as one 3-neighbor of x_t, v is not contained in such a path and is one young 3-vertex.

For the 1-vertex u, by R3, it receives $\frac{17}{16}$ from one old 3-vertex. Therefore, $ch^*(u) = -\frac{17}{16}+\frac{17}{16} = 0$. For the 2-vertex u, by R1, it receives $\frac{1}{32}\times 2$ from some young 3-vertices. Therefore, $ch^*(u) = -\frac{1}{16}+\frac{1}{32}\times 2 = 0$. For the old 3-vertex u, it sends $\frac{17}{16}$ to its 1-neighbor by R3; at the same time, it will receive $\frac{1}{16}\times 2$ from some young 3-vertices by R2. Therefore, $ch^*(u) = \frac{15}{16} - \frac{17}{16} + \frac{1}{16}\times 2 = 0$.

Now, we only need to consider each young 3-vertex. Assume that u is a young 3-vertex; we use s to denote the number of 2-vertices that receive charges from u, and t to denote the number of bad 3-vertices that receive charges from u. By Claim 2 and Claim 3, we have $s \le 6$ and $t \le 15$.

- If $s = 0$, then $t \le 15$. Therefore, we have $ch^*(u) \ge \frac{15}{16} - \frac{1}{16}\times 15 = 0$.
- If $1 \le s \le 2$, then $t \le 10$. Therefore, we have $ch^*(u) \ge \frac{15}{16} - \frac{1}{16}\times 10 - \frac{1}{32}\times 2 > 0$.
- If $3 \le s \le 4$, then $t \le 5$. Therefore, we have $ch^*(u) \ge \frac{15}{16} - \frac{1}{16}\times 5 - \frac{1}{32}\times 4 > 0$.
- Otherwise, $5 \le s \le 6$ and $t = 0$. Therefore, we have $ch^*(u) \ge \frac{15}{16} - \frac{1}{32}\times 6 > 0$.

Hence, for each vertex u, $ch^*(u)$ is non-negative. It implies that $0 \le \Sigma_{u\in V(G)}ch^*(u) = \Sigma_{u\in V(G)}ch(u) < 0$, which is a contradiction. The proof of Theorem 5(a) is finished.

3. The proof of Theorem 5(b)

If Theorem 5(b) is false, then we choose G as a counterexample with a minimum $|E(G)|$ and $f \in \mathbb{Z}^G$ is defined as $f(e) = 6$ for each edge e. Let $H = (L(G))^2$. Next, some claims of G are given in order to obtain a contradiction.

Claim 4. G has no 1-vertex.

Proof. Suppose G has one 1-vertex u, and the neighbor of u is v. Let $G_1 = G - u$, $G_2 = G[\{uv\}]$, and $H_i = (L(G_i))^2$ for $i \in [2]$. Note that $H_1 = H[E(G) - E(u)]$ and $H_2 = H[E(u)]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H_1 due to minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H_2$. For $e = uv$, we have $|N_\Omega(e)| \le 6$; thus, $f_\Omega(e) \ge 0$. Since the deletion move (uv) can delete only one vertex in H_2, H_2 is weakly f_Ω-degenerate. By Proposition 1, H is weakly f-degenerate, which is a contradiction. □

Claim 5. For any 2-vertex u in G, it has at least one 3-neighbor.

Proof. Due to Claim 4, we could suppose that G has one 2-vertex u and its neighbors x and y of it are 2-vertices. Let $G_1 = G-u$, $G_2 = G[\{ux, uy\}]$, and $H_i = (L(G_i))^2$ for $i \in [2]$. Note that $H_1 = H[E(G)-E(u)]$ and $H_2 = H[E(u)]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H_1 by minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H_2$. Due to $d(x) = d(y) = 2$ in G, it is easy to obtain $|N_\Omega(e)| \leq 4$, for any $e \in E(u)$. Therefore, $f_\Omega(e) \geq 2$, for any $e \in E(u)$. A legal removal scheme $\mathsf{DeleteSave}((ux), (uy))$ could remove all vertices in H_Ω. By Proposition 1, H_Ω is weakly f_Ω-degenerate and H is weakly f-degenarate, which is a contradiction. □

Claim 6. For any 2_1-vertex u in G, it does not have any 3_k-neighbor; here, $k \in \{2, 3\}$.

Proof. Suppose u is one 2_1-vertex in G, which has one 3_k-neighbor y and another 2-neighbor x of u; here, $k \in \{2, 3\}$.

Let $G_1 = G - u$, $G_2 = G[E(u)]$, and $H_i = (L(G_i))^2$ for $i \in [2]$. Note that $H_1 = H[E(G) - E(u)]$ and $H_2 = H[E(u)]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H_1 by minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H_2$. As $|N_\Omega(ux)| \leq 5$ and $|N_\Omega(uy)| \leq 6$, thus $f_\Omega(ux) \geq 1$ and $f_\Omega(uy) \geq 0$. Then, we choose one removal scheme $\Omega' = \mathsf{DeleteSave}((uy), (ux))$. It is easy to verify that Ω' is a legal removal scheme for (H_Ω, f_Ω) such that all vertices of H_Ω are removed. Therefore, H is weakly f-degenerate by Proposition 1, which is a contradiction. □

Claim 7. For any 2_2-vertex u in G, it has at most one 3_3-neighbor.

Proof. Suppose that G has one 2_2-vertex u, and x and y are 3_3-neighbors of u. a, b, and u are 2-neighbors of x, while c, d, and u are 2-neighbors of y (Figure 3).

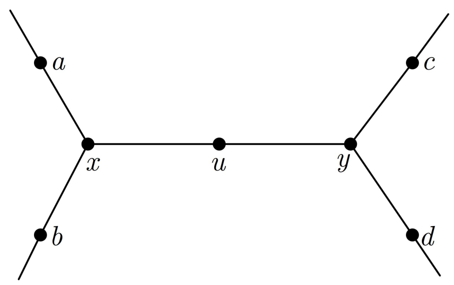

Figure 3. The configuration for Claim 7.

We could choose $U = \{x, u, y\}$ and let $G' = G - E(U)$, $G'' = G[E(U)]$, $H' = (L(G'))^2$, and $H'' = (L(G''))^2$. Note that $H' = H[E(G) - E(U)]$ and $H'' = H[E(U)]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H' by minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H''$. It is easy to check that $|N_\Omega(e)| \leq 2$ for $e \in \{ux, uy\}$ and $|N_\Omega(e)| \leq 4$ for $e \in \{xa, xb, yc, yd\}$. Therefore, $f_\Omega(e) \geq 4$ for $e \in \{ux, uy\}$ and $f_\Omega(e) \geq 2$ for $e \in \{xa, xb, yc, yd\}$. We only need to prove that H'' is weakly f_Ω-degenerate.

Let

$$f_0(e) = \begin{cases} 4, & \text{if } e \in \{ux, uy\}; \\ 2, & \text{if } e \in \{xa, xb, yc, yd\}. \end{cases}$$

As $f_\Omega \geq f_0$, it suffices to obtain a legal removal scheme $\Omega_0 = \mathsf{DeleteSave}(\theta_1, \theta_2, \ldots, \theta_6)$ for (H_0, f_0) that removes all the vertices of H_0; here, $H_0 = H''$. For $i \in [6]$, let (H_i, f_i) be the resulting pair after applying θ_i over (H_{i-1}, f_{i-1}). For $i \in \{0, 1, \cdots, 6\}$, the operation θ_i and the function value of $f_i(e)$ for $e \in V(H_i)$ are shown in the $(i+1)$-th row of Table 2, where θ_0 is undefined. The cell of the table is empty if the corresponding edge is removed.

Table 2. The legal removal scheme for (H_0, f_0) in Claim 7.

θ_i \ $f_i(e)$	ux	uy	xa	xb	yc	yd
	4	4	2	2	2	2
(ux, xa)		3	2	1	1	1
(uy, yc)			1	0	1	0
(xb)			0		1	0
(xa)					1	0
(yd)					0	
(yc)						

By Table 2, all vertices in H'' are removed by the legal removal scheme Ω_0. Therefore, H is weakly f-degenerate by Proposition 1, which is a contradiction. □

We will use the discharging method to prove Theorem 5(b). First, every vertex u of G gets an initial charge by $ch(u) = d_G(u) - \frac{27}{11}$. Due to $\text{mad}(G) < \frac{27}{11}$, the sum of all charges is negative. Next, some discharging rules are given as follows.

R1. For 2_1-vertex u, the only 3-neighbor sends $\frac{5}{11}$ to u.

R2. For 2_2-vertex u, the two 3-neighbors of u are v and w.

R2.1. If there exists one neighbor of u as a 3_3-vertex (we assume that 3_3-vertex is v), then v sends $\frac{2}{11}$ to u and w sends $\frac{3}{11}$ to u.

R2.2. Otherwise, every neighbor of u sends $\frac{5}{22}$ to u.

After these discharging rules (R1, R2, R2.1, and R2.2), the sum doesn't change. Let us consider the final charge for each vertex u denoted by $ch^*(u)$. By the way, $ch(u) = -\frac{5}{11}$ if $d_G(u) = 2$, and $ch(u) = \frac{6}{11}$ if $d_G(u) = 3$.

In order to quickly calculate $ch^*(u)$, we consider which kind of vertex is available for different rules in advance. For rule R1, by Claim 6, the only 3-neighbor is 3_1-vertex. For rule R2.1, by Claim 7, the 3-neighbor w is not 3_3-vertex. For rule R2.2, every 3-neighbor is either 3_1-vertex or 3_2-vertex.

Due to Claim 4, G doesn't have a 1-vertex. We only need to consider the 2-vertex and 3-vertex. For the 2-vertex u, by Claim 5, u is not 2_0-vertex. If u is 2_1-vertex, then we have $ch^*(u) = -\frac{5}{11} + \frac{5}{11} = 0$ by rule R1. Otherwise, if u is 2_2-vertex, by R2, then either $ch^*(u) = -\frac{5}{11} + \frac{2}{11} + \frac{3}{11} = 0$ or $ch^*(u) = -\frac{5}{11} + \frac{5}{22} \times 2 = 0$.

Next, 3-vertex u will be considered. If u is 3_3-vertex, then u sends $\frac{2}{11}$ to each 2_2-neighbor by R2.1. It is easy to obtain $ch^*(u) \geq \frac{6}{11} - \frac{2}{11} \times 3 = 0$.

When u is 3_2-vertex, we assume that v and w are two 2-neighbors of u and x is the 3-neighbor of u. u sends at most $\frac{3}{11}$ to each 2_2-neighbor by R2. Hence, there is $ch^*(u) \geq \frac{6}{11} - \frac{3}{11} \times 2 = 0$.

When u is 3_1-vertex, u sends at most $\frac{5}{11}$ to the only 2-neighbor of u by R1 and R2. Therefore, $ch^*(u) \geq \frac{6}{11} - \frac{5}{11} > 0$. Otherwise, if u is 3_0-vertex, then it doesn't send any charge to any neighbor. Therefore, $ch^*(u) = ch(u) > 0$.

After all discharging rules, for each vertex u in G, we find that $ch^*(u)$ is non-negative. Therefore, the final sum is also non-negative. Thus, we have $0 \leq \Sigma_{u \in V(G)} ch^*(u) = \Sigma_{u \in V(G)} ch(u) < 0$, which is a contradiction. Hence, the counterexample G of Theorem 5(b) doesn't exist.

4. The proof of Theorem 5(c)

In this section, we will give the proof of Theorem 5(c). If Theorem 5(c) is false, then we choose G as a counterexample with a minimum $|E(G)|$ and $f \in \mathbb{Z}^G$ is defined as $f(e) = 7$ for each edge e. Let $H = (L(G))^2$. Next, some claims of G are given in order to obtain a contradiction.

Claim 8. *G has no 1-vertex.*

Proof. This proof is similar to the proof of Claim 4; therefore, we omit it here. □

Claim 9. *For any 2-vertex u in G, all neighbors of u are 3-vertices.*

Proof. By Claim 8, we could suppose that the 2-vertex u has a 2-neighbor v and another neighbor w. Let $G_1 = G - u$, $G_2 = G[E(u)]$, and $H_i = (L(G_i))^2$ for $i \in [2]$. Note that $H_1 = H[E(G) - E(u)]$ and $H_2 = H[E(u)]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H_1 by minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then $H_\Omega = H_2$. Considering $|N_\Omega(e)|$ for $e \in \{uv, uw\} = E(u)$, we can obtain $|N_\Omega(uv)| \leq 5$ and $|N_\Omega(uw)| \leq 7$. Therefore, $f_\Omega(uv) \geq 2$ and $f_\Omega(uw) \geq 0$.

Let $\Omega' = \mathsf{DeleteSave}((uw), (uv))$ be a removal scheme for (H_Ω, f_Ω). It's easy to verify that Ω' is legal and remove all vertices of H_Ω. Therefore, H_Ω is weakly f_Ω-degenerate. This implies that H is weakly f-degenerate by Proposition 1, which is a contradiction. □

Claim 10. *G has no 3_3-vertex.*

Proof. Suppose that G has a 3_3-vertex u, and v, z, and w are three 2-neighbors of u. Let $G_1 = G - u$, $G_2 = G[E(u)]$, and $H_i = (L(G_i))^2$ for $i \in [2]$. Note that $H_1 = H[E(G) - E(u)]$ and $H_2 = H[E(u)]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H_1 by minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H_2$. Similarily, we can obtain $|N_\Omega(e)| \leq 5$ for $e \in \{uv, uz, uw\} = E(u)$. Therefore, $f_\Omega(e) \geq 2$ for $e \in E(u)$. Due to Proposition 1, H is weakly f-degenerate, which is a contradiction. □

Claim 11. *For any 2-vertex u in G, u has at most one 3_2-neighbor.*

Proof. Suppose that G has a 2-vertex u, and v and w are two 3_2-neighbors of u. v has one 2-neighbor a and one 3-neighbor z, and w has one 2-neighbor b and one 3-neighbor y (Figure 4).

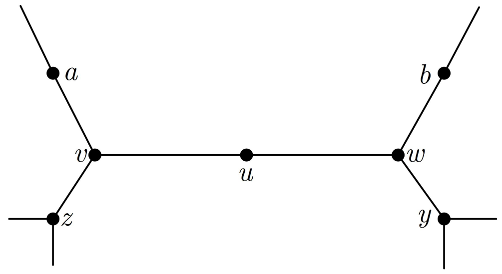

Figure 4. The configuration in Claim 11.

We could choose $U = \{v, u, w\}$ and let $G' = G - E(U)$, $G'' = G[E(U)]$, $H' = (L(G'))^2$, and $H'' = (L(G''))^2$. Note that $H' = H[E(G) - E(U)]$ and $H'' = H[E(U)]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H' by minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H''$.

It is easy to check that $|N_\Omega(e)| \leq 7$ for $e \in \{zv, wy\}$, $|N_\Omega(e)| \leq 5$ for $e \in \{av, wb\}$, and $|N_\Omega(e)| \leq 3$ for $e \in \{vu, uw\}$. Therefore, $f_\Omega(e) \geq 0$ for $e \in \{zv, wy\}$, $f_\Omega(e) \geq 2$ for $e \in \{av, wb\}$, and $f_\Omega(e) \geq 4$ for $e \in \{vu, uw\}$. It is sufficient to prove that H'' is weakly f_Ω-degenerate. Set

$$f_0(e) = \begin{cases} 0, & \text{if } e \in \{zv, wy\}; \\ 2, & \text{if } e \in \{av, wb\}; \\ 4, & \text{if } e \in \{vu, uw\}. \end{cases}$$

Due to $f_\Omega \geq f_0$, we need to show that H'' is weakly f_0-degenerate by a legal removal scheme $\Omega_0 = \mathsf{DeleteSave}(\theta_1, \theta_2, \ldots, \theta_6)$ for (H_0, f_0), which removes all the vertices of H_0; here, $H_0 = H''$. For $i \in [6]$, let (H_i, f_i) be the resulting pair after applying θ_i over (H_{i-1}, f_{i-1}). For $i \in \{0, 1, \cdots, 6\}$, the operation θ_i and function value of $f_i(e)$ for $e \in V(H_i)$ are shown in the $(i+1)$-th row of Table 3, where θ_0 is undefined. The cell of the table is empty if the corresponding edge is removed.

Table 3. The legal removal scheme for (H_0, f_0) in Claim 11.

θ_i \ $f_i(e)$	zv	av	vu	wy	wb	uw
	0	2	4	0	2	4
(zv)		1	3	0	2	3
(wy)		1	2		1	2
(vu, av)		1			0	1
(wb)		1				0
(uw)		0				
(av)						

Table 3 helps to confirm that this removal scheme Ω_0 is legal and removes all vertices in H_0. Therefore, H is weakly f-degenerate by Proposition 1, which is a contradiction. □

Similarily, we use the discharging method to finish this proof to obtain a contradiction. First, let $ch(u) = d_G(u) - \frac{13}{5}$ for each vertex of G. Hence, the sum of all charges is negative due to $\text{mad}(G) < \frac{13}{5}$. Next, some discharging rules are given as follows:

R1. For the 2_2-vertex u, the two neighbors of u are v and w.

R1.1 If there exists one neighbor of u as a 3_2-vertex (we assume that this 3_2-vertex is v), then v sends $\frac{1}{5}$ to u, and w sends $\frac{2}{5}$ to u.

R1.2 Otherwise, each neighbor of u sends $\frac{3}{10}$ to u.

Before considering the final charge of every vertex u in G denoted by $ch^*(u)$, let us figure out which kind of vertex is applied for which rules. By Claims 8–10, G has no 1-vertex, 2_0-vertex, 2_1-vertex, and 3_3-vertex. For rule R1.1, due to Claim 11, the only possibility is that v is 3_2-vertex and w is 3_1-vertex. For rule R1.2, v and w are both 3_1-vertices.

For the 2_2-vertex u, by rule R1, $ch^*(u) = -\frac{3}{5} + \frac{1}{5} + \frac{2}{5} = -\frac{3}{5} + \frac{3}{10} \times 2 = 0$. For the 3_2-vertex u, u sends $\frac{1}{5}$ to each 2-neighbor of u by R1.1. Therefore, $ch^*(u) = \frac{2}{5} - \frac{1}{5} \times 2 = 0$. For the 3_1-vertex u, u sends at most $\frac{2}{5}$ to the only 2-neighbor, and we obtain $ch^*(u) \geq \frac{2}{5} - \frac{2}{5} = 0$. For the 3_0-vertex u, there is no change for the charge of it. Therefore, $ch^*(u) = ch(u) = \frac{2}{5} > 0$.

Therefore, the final charge of each vertex is non-negative. This implies that $0 \leq \Sigma_{u \in V(G)} ch^*(u) = \Sigma_{u \in V(G)} ch(u) < 0$, which is a contradiction. The proof of Theorem 5(c) is finished.

5. The Proof of Theorem 5(d)

In this section, we provide the proof of Theorem 5(d). If Theorem 5(d) is false, then we choose G as a counterexample with a minimum $|E(G)|$ and $f \in \mathbb{Z}^G$ is defined as $f(e) = 8$ for each edge e. Let $H = (L(G))^2$. Similarily, some claims of G are given as follows.

Claim 12. G has no 1-vertex.

Proof. This proof is similar to the proof of Claim 4; therefore, we omit it here. □

Claim 13. For any 2-vertex u in G, u doesn't have a 2-neighbor.

Proof. Suppose that G has one 2-vertex u that has one 2-neighbor v, and the other neighbor of u is w.

Let $G_1 = G - u$, $G_2 = G[E(u)]$, and $H_i = (L(G_i))^2$ for $i \in [2]$. Note that $H_1 = H[E(G) - E(u)]$ and $H_2 = H[E(u)]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H_1 by minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H_2$. Considering $|N_\Omega(e)|$ for $e \in E(u)$, we can obtain $|N_\Omega(vu)| \leq 5$ and $|N_\Omega(uw)| \leq 7$. Therefore, $f_\Omega(vu) \geq 3$ and $f_\Omega(uw) \geq 1$. Hence, by Proposition 1, H is weakly f-degenerate, which is a contradiction. □

Claim 14. For any 3-vertex u in G, it has at most one 2-neighbor.

Proof. Suppose that G has one 3-vertex u, where v and w are two 2-neighbors of u, and z is the other neighbor of u.

Let $G_1 = G - u$, $G_2 = G[E(u)]$, and $H_i = (L(G_i))^2$ for $i \in [2]$. Note that $H_1 = H[E(G) - E(u)]$ and $H_2 = H[E(u)]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H_1 by minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H_2$. Due to $|N_\Omega(e)| \leq 6$ for $e \in \{uv, uw\}$ and $|N_\Omega(uz)| \leq 8$, we can obtain $f_\Omega(e) \geq 2$ for $e \in \{uv, uw\}$ and $f_\Omega(uz) \geq 0$.

Then, let $\Omega' = \mathsf{DeleteSave}((uz), (uv), (uw))$. It is easy to check that Ω' is a legal removal scheme for (H_2, f_Ω) such that all vertices in H_2 are removed. Hence, H is weakly f-degenerate, which is a contradiction. □

Claim 15. For any 3_1-vertex u in G, it doesn't have a 3_1-neighbor.

Proof. Suppose that G has one 3_1-vertex u which has a 3_1-neighbor v; z and b are the other two neighbors of u, w and a are the other two neighbors of v, x is the other 3-neighbor of w, and y is the other 3-neighbor of z by Claim 13 (see Figure 5(a)).

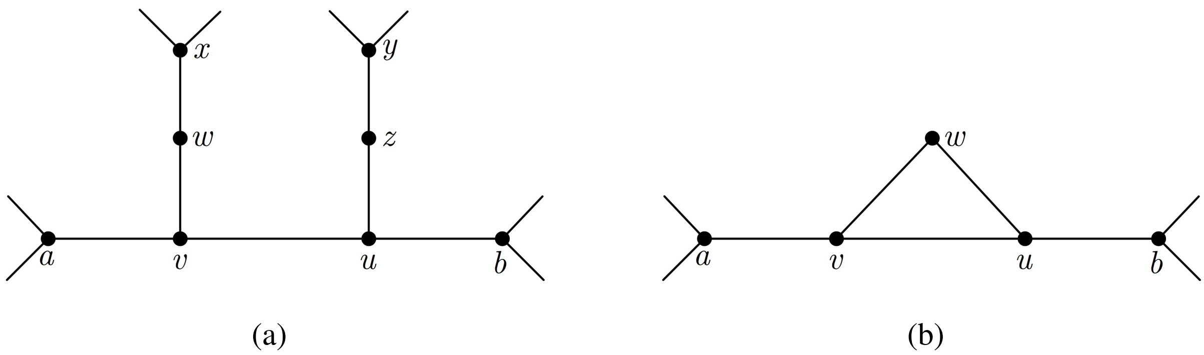

Figure 5. Two configurations of Claim 15.

If $w = z$, then we could consider this situation in Figure 5(b). Let $G_1 = G - w$, $G_2 = G[E(w)]$ and $H_i = (L(G_i))^2$ for $i \in [2]$. Note that $H_1 = H[E(G) - E(w)]$ and $H_2 = H[E(w)]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H_1 by minimality. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H_2$. Due to $|N_\Omega(e)| \leq 5$ for $e \in E(w)$, we obtain $f_\Omega \geq 3$ for $e \in E(w)$. Obviously, by Proposition 1, H_Ω is weakly f_Ω-degenerate and H is weakly f-degenerate, which is a contradiction.

Therefore, $w \neq z$ (the situation in Figure 5(a)). We can choose $U = \{u, v, w, z\}$ and let $G' = G - E(U)$, $G'' = G[E(U)]$, $H' = (L(G'))^2$, and $H'' = (L(G''))^2$. Note that $H' = H[E(G) - E(U)]$ and $H'' = H[E(U)]$. Let Ω be a legal removal scheme for (H, f) that removes all the vertices in H'. Let $(H_\Omega, f_\Omega) = \Omega(H, f)$. Then, $H_\Omega = H''$.

Considering $|N_\Omega(e)|$ for $e \in E(U)$, we can obtain that $|N_\Omega(e)| \leq 6$ for $e \in \{av, wx, zy, ub\}$ and $|N_\Omega(e)| \leq 4$ for $e \in \{vu, vw, uz\}$. Therefore, $f_\Omega(e) \geq 2$ for $e \in \{av, wx, zy, ub\}$ and $f_\Omega(e) \geq 4$ for $e \in \{vu, vw, uz\}$. It suffices to prove that H'' is weakly f_Ω-degenerate.

Let

$$f_0(e) = \begin{cases} 2, & \text{if } e \in \{av, wx, zy, ub\}; \\ 4, & \text{if } e \in \{vu, vw, uz\}. \end{cases}$$

As $f_\Omega \geq f_0$, we only need to prove that H'' is weakly f_0-degenerate by a legal removal scheme $\Omega_0 = \mathsf{DeleteSave}(\theta_1, \theta_2, \ldots, \theta_7)$ such that all vertices in H'' are removed. For $i \in [7]$, let (H_i, f_i) be the resulting pair after applying θ_i over (H_{i-1}, f_{i-1}), where $H_0 = H''$. For $i \in \{0, 1, \cdots, 7\}$, the operation θ_i and the function value of $f_i(e)$ for $e \in V(H_i)$ are shown in the $(i+1)$-th row of Table 4, where θ_0 is undefined. The cell of the table is empty if the corresponding edge is removed.

Table 4. The legal removal scheme for (H_0, f_0) in Claim 15.

θ_i \ $f_i(e)$	av	vw	wx	vu	uz	zy	ub
	2	4	2	4	4	2	2
(vu, zy)	1	3	1		3	2	1
(av)		2	0		2	2	0
(wx)		1			2	2	0
(ub)		0			1	1	
(vw)					0	1	
(uz)						0	
(zy)							

By Table 4, we can verify that Ω_0 is a legal removal scheme for (H_0, f_0) such that all vertices in H_0 are removed. Hence, H is weakly f-degenerate by Proposition 1, which is a contradiction. □

Next, the discharging method is used in order to find a contradiction. For each vertex u in G, let $ch(u) = d_G(u) - \frac{36}{13}$ be an initial charge of it. Therefore, $ch(u) = -\frac{10}{13}$ for the 2-vertex u and $ch(u) = \frac{3}{13}$ for the 3-vertex u. The sum of all charges is negative due to $\text{mad}(G) < \frac{36}{13}$. The discharging rules are shown as follows:

R1. For 2_2-vertex u, each 3-neighbor of u sends $\frac{5}{13}$ to u.

R2. For 3_1-vertex u, each 3_0-neighbor of u sends $\frac{1}{13}$ to u.

Now, we calculate the final charge $ch^*(u)$ for every vertex u in G after these discharging rules (R1 and R2). By Claims 12–14, G has no 1-vertex, 2_1-vertex, 2_0-vertex, 3_3-vertex, and 3_2-vertex.

For each 2_2-vertex u, every 3-neighbor sends $\frac{5}{13}$ to u by R1. Therefore, $ch^*(u) = -\frac{10}{13} + \frac{5}{13} \times 2 = 0$. For each 3_1-vertex u, all 3-neighbors of u are 3_0-vertices by Claim 15. Therefore, $ch^*(u) = \frac{3}{13} - \frac{5}{13} + \frac{1}{13} \times 2 = 0$. For each 3_0-vertex u, it sends $\frac{1}{13}$ to every 3_1-neighbor of u by R2. Thus, $ch^*(u) \geq \frac{3}{13} - \frac{1}{13} \times 3 = 0$. Hence, the final charge of each vertex is non-negative.

Thus, we can easily find that $0 \leq \Sigma_{u \in V(G)} ch^*(u) = \Sigma_{u \in V(G)} ch(u) < 0$, which is a contradiction. The proof of Theorem 5 is completed.

6. Conclusions

According to Theorem 5 and the proposition from [2], the following three corollaries can be implied, concerning coloring parameters for the square of line graph of G denoted by H.

Corollary 1. *Let G be a subcubic graph and $H = (L(G))^2$:*

(a) If $mad(G) < \frac{33}{16}$, then $\chi_P(H) \leq 6$;

(b) If $mad(G) < \frac{27}{11}$, then $\chi_P(H) \leq 7$;

(c) If $mad(G) < \frac{13}{5}$, then $\chi_P(H) \leq 8$;

(d) If $mad(G) < \frac{36}{13}$, then $\chi_P(H) \leq 9$.

Corollary 2. *Let G be a subcubic graph and $H = (L(G))^2$:*

(a) $mad(G) < \frac{33}{16}$, *then* $\chi_{DP}(H) \leq 6$;

(b) *If* $mad(G) < \frac{27}{11}$, *then* $\chi_{DP}(H) \leq 7$;

(c) *If* $mad(G) < \frac{13}{5}$, *then* $\chi_{DP}(H) \leq 8$;

(d) *If* $mad(G) < \frac{36}{13}$, *then* $\chi_{DP}(H) \leq 9$.

Corollary 3. *Let G be a subcubic graph and $H = (L(G))^2$:*

(a) *If* $mad(G) < \frac{33}{16}$, *then* $\chi_{DPP}(H) \leq 6$;

(b) *If* $mad(G) < \frac{27}{11}$, *then* $\chi_{DPP}(H) \leq 7$;

(c) *If* $mad(G) < \frac{13}{5}$, *then* $\chi_{DPP}(H) \leq 8$;

(d) *If* $mad(G) < \frac{36}{13}$, *then* $\chi_{DPP}(H) \leq 9$.

Author contributions

Conceptualization: Y. Z.; Methodology: J. H.; Validation: J. H.; Investigation: J. H.; Writing-original draft: J. H.; Writing-review and editing: J. H. and Y. Z.; Funding acquisition: Y. Z.. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

References

1. L. D. Andersen, The strong chromatic index of a cubic graph is at most 10, *Discrete Math.*, **108** (1992), 231–252. https://doi.org/10.1016/0012-365X(92)90678-9
2. A. Bernshteyn, E. Lee, Weak degeneracy of graphs, *J. Graph Theory*, **103** (2023), 607–634. https://doi.org/10.1002/jgt.22938
3. M. Bonamy, T. Perrett, L. Postle, Colouring graphs with sparse neighbourhoods: bounds and applications, *J. Comb. Theory Ser. B*, **155** (2022), 278–317. https://doi.org/10.1016/j.jctb.2022.01.009

4. Z. Dvořák, L. Postle, Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8, *J. Comb. Theory Ser. B*, **129** (2018), 38–54. https://doi.org/10.1016/j.jctb.2017.09.001
5. P. Erdős, J. Nešetřil, *Irregularities of partitions*, Ed.G.Halász and VT Sós, 1989, 162–163. https://doi.org/10.1007/978-3-642-61324-1
6. J. L. Fouquet, J. L. Jolivet, Strong edge-coloring of graphs and applications to multi-k-gons, *Ars Comb.*, **16-A** (1983), 141–150.
7. D. Hudák, B. Lužar, R. Soták, R. Škrekovski, Strong edge-coloring of cubic planar graphs, *Discrete Math.*, **324** (2014), 41–49. https://doi.org/10.1016/j.disc.2014.02.002
8. M. Han, T. Wang, J. Wu, H. Zhou, X. Zhu, Weak degeneracy of planar graphs and locally planar graphs, *Electron. J. Comb.*, **30** (2023), 1–17. https://doi.org/10.37236/11749
9. H. Hocquard, M. Montassier, A. Raspaud, P. Valicow, On strong edge-colouring of subcubic graphs, *Discrete Appl. Math.*, **161** (2013), 2467–2469. https://doi.org/10.1016/j.dam.2013.05.021
10. H. Hocquard, P. Valicow, Strong edge colouring of subcubic graphs, *Discrete Appl. Math.*, **159** (2011), 1650–1657. https://doi.org/10.1016/j.dam.2011.06.015
11. P. Horák, H. Qing, W. T. Trotter, Induced matching in cubic graphs, *J. Graph Theory*, **17** (1993), 151–160. https://doi.org/10.1002/jgt.3190170204
12. M. Huang, M. Santana, G. Yu, Strong chromatic index of graphs with maximum degree four, *Electron. J. Comb.*, **25** (2018), 1–24. https://doi.org/10.37236/7016
13. E. Hurley, R. J. Verclos, R. J. Kang, An improved procedure for colouring graphs of bounded local density, In: *Proceedings of the Thirty-Second Annual ACM-SIAM Symposium on Discrete Algorithms*, 2021, 135–148.
14. H. Ma, Z. Miao, H. Zhu, J. H. Zhang, R. Luo, Strong list edge coloring of subcubic graphs, *Math. Prol. Eng.*, 2013, 316501. https://doi.org/10.1155/2013/316501
15. M. Molloy, B. Reed, A bound on the strong chromatic index of a graph, *J. Comb. Theory, Ser. B*, **69** (1997), 103–109. https://doi.org/10.1006/jctb.1997.1724
16. U. Schauz, Mr. Paint and Mrs. Correct, *Electron. J. Comb.*, **16** (2009), 1–18. https://doi.org/10.37236/166
17. B. Zhang, Y. Chang, J. Hu, M. Ma , D. Yang, List strong edge-coloring of graphs with maximum degree 4, *Discrete Math.*, **343** (2020), 111854. https://doi.org/10.1016/j.disc.2020.111854
18. H. Zhu, Z. Miao, On strong list edge coloring of subcubic graphs, *Discrete Math.*, **333** (2014), 6–13. https://doi.org/10.1016/j.disc.2014.06.004
19. X. Zhu, On-line list colouring of graphs, *Electron. J. Combin.*, **16** (2009), 1–16. https://doi.org/10.37236/216