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Research article

Spectral solutions for nonlinear static beam and fractional Riccati problems using new Lucas coefficient polynomials

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Abstract: This study proposes novel spectral algorithms employing the Lucas coefficient polynomials to solve two significant nonlinear models: The static beam problem and the fractional Riccati equation. Our suggested approaches are based on developing novel theoretical findings that we derive for the introduced polynomials. These results include formulae for inversion, moment, derivatives, and linearization. Two methodologies are followed to treat the two nonlinear problems. The nonlinear fourth-order integro-differential static beam problem is treated using the collocation method, while the nonlinear fractional Riccati equation is treated using the tau method. Rigorous convergence and error analysis for the Lucas coefficient expansions are given. Compared to previous methods, the suggested algorithms exhibit exponential convergence and high accuracy, as verified by numerical testing. The findings demonstrate that spectral algorithms may effectively handle nonlinear differential equations using Lucas coefficient polynomials.

Keywords: Lucas numbers; spectral methods; nonlinear fractional Riccati equation; nonlinear fourth-order integro-differential equation; convergence analysis
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1. Introduction

Fibonacci and Lucas numbers have several applications in mathematics, science, and engineering due to their intricate recursive structure and inherent properties. These sequences have connections with fields such as number theory and discrete mathematics. In applied sciences, Fibonacci and Lucas numbers describe wave propagation events, patterns in biology (such as phyllotaxis), and population growth. One can refer to [1, 2] for some applications of these numbers. Furthermore, these numbers can be generalized to various polynomials, such as Fibonacci and Lucas polynomials, and their related generalized sequences. These sequences have vital roles in approximation theory and numerical analysis. For example, the authors of [3] suggested an efficient numerical technique to solve fractional differential equations (FDEs) using Fibonacci polynomials. A numerical technique to handle 1D and 2D sinh-Gordon equations utilizing Lucas polynomials was developed in [4]. Using mixed Fibonacci-Lucas polynomials, the authors in [5] presented a numerical solution for mixed-type FDEs. In [6], a collocation approach was presented for solving the time-fractional FitzHugh-Nagumo differential equations (DEs) using the shifted Lucas polynomials. The time-fractional diffusion problem was solved in [7] using the spectral tau method based on Lucas polynomials. For other numerical methods using these polynomials, one can refer to [8–10].

FDEs have been widely used in many scientific and industrial sectors due to their ability to capture memory effects and hereditary traits in intricate systems. The physical sciences use them because they provide more accurate depictions of anomalous diffusion and viscoelastic behavior than traditional models. Fractional-order control systems and circuit modeling are some engineering applications, particularly in systems with frequency-dependent damping or long-term memory. FDEs have several applications in the biological and medical fields, including the simulation of drug distribution, the study of physiological signal dynamics, and the study of the delayed effects of infectious disease transmission. Financial decision-making tools also benefit from FDEs' ability to model memory and jump process settings, which mimic market behavior and pricing decisions. For example, in environmental research, FDEs may be applied to predict groundwater flow and pollutant transport on porous surfaces. They can also be used in image processing to enhance photographs and detect edges more sensitively. These several applications demonstrate the flexibility and effectiveness of fractional models in capturing complex, real-life occurrences. For some applications of the FDEs, one can refer to [11–14].

Various kinds of FDEs have been handled using different numerical and analytical approaches. Authors of [15] used the residual error function and the Laplace transform to find solutions for the fractional Lane-Emden equations. The Adomian decomposition technique was applied to solve a nonlinear system of FDEs in [16]. The differential transform approach was followed in [17] to investigate some fractional dynamical systems. To effectively handle stochastic FDEs, a numerical method using the standard Pell polynomials was suggested in [18]. The authors of [19] introduced a step-by-step technique to solve some FDEs. A fractional-order hybrid Jacobi function-based method was developed in [20] to treat a general class of FDEs. Using the strength of machine learning, a neural network-based approach was suggested in [21] to solve the generalized Caputo-type FDEs. The work in [22] offered a technique for approximating fractal-fractional equations depending on numerical inverse Laplace transforms. Finally, a universal predictor-corrector method was presented in [23], offering a consistent and generic foundation for simulating generalized FDEs.

The fractional Riccati differential equations (FRDEs), extensions of the conventional Riccati differential equations that use fractional derivatives, have recently gained popularity. They have various uses in fields that need modeling of hereditary traits and memory. For example, one may find them employed in control, dynamic games, and financial mathematics; see, for example, [24, 25]. The authors of [26] used the decomposition technique to solve the FRDEs, laying the groundwork in the field. The approach proposed in [27] was followed to solve Riccati equations of fractional order, where the authors employed a numerical technique to enhance computing performance. To solve the FRDEs, a numerical solution based on Haar wavelets was proposed in [28]. Furthermore, the Muntz-Legendre wavelet collocation technique was introduced to solve such equations in [29]. The authors of [30] suggested a new Laplace-reproducing kernel Hilbert space approach. In [31], a finite difference method was proposed to solve the FRDEs. In [32], another technique was followed to treat the FRDEs with the Caputo-Fabrizio fractional derivative. Fractal-fractional Riccati equations with generalized Caputo derivatives were handled in [33] using an orthonormal ultraspherical operational matrix approach. The Laplace residual power series method was applied in [34] to derive analytical solutions for the FRDEs. Finally, a novel Mittag-Leffler-Galerkin approach was presented in [35] to solve the FRDEs efficiently.

A static beam problem is a nonlinear fourth-order integro-differential equation of Kirchhoff type, which simulates the static bending of an extendable beam supported at one end and linked to a fixed nonlinear torsional spring at the other, and is the subject of this work. The deflection of the beam is nonlocal as it depends not only on local derivatives but also on an integral term including the curvature; see [36–38].

Spectral methods are powerful tools for treating DEs. The FDEs and high-order ordinary DEs can be solved effectively using these methods. For smooth problems, their ability to achieve exponential or high-order convergence makes them incredibly precise, which is their main advantage over traditional numerical methods. These methods extend the solution based on global basis functions, usually special functions or special polynomials, giving a precise approximation with minimal degrees of freedom. Biological systems modeling, fluid dynamics, and quantum physics are a few fields that have made good use of spectral methods. For some applications of spectral methods, one can consult [39–41]. The three main spectral methods are the collocation, tau, and Galerkin. Many articles were devoted to these methods to solve several types of DEs. Collocation methods are advantageous because they can treat all types of DEs governed by any governing conditions; see, for instance, [42–44]. The tau and Galerkin methods both need to suggest two sets of functions: trial and test. These two sets are identical in the case of Galerkin, unlike in the tau method; see [45–47].

The following is a list of the primary goals of this article:

- Introducing polynomials whose coefficients are the Lucas numbers.
- Establishing new theoretical results for these polynomials. The derived formulas will be crucial in designing the numerical methods presented.
- Designing a collocation algorithm to solve the static beam problem.
- Designing a tau algorithm to solve the FRDEs.
- Investigating the error and convergence analysis of the proposed expansion.
- Testing our two presented algorithms by providing a few numerical examples.

The impact and the novelty of our paper can be summarized in the following points:

- To the best of our knowledge, this is the first time Lucas coefficient polynomials have been introduced into numerical analysis, opening a new direction beyond the classical Chebyshev, Legendre, or Jacobi approaches.

- The establishment of new theoretical results for these polynomials, including inversion, moment, derivative, and linearization formulas, will be useful in treating other classes of DEs.
- The explicit forms of the operational formulas of the Lucas coefficient polynomials may be used for handling complicated nonlinear problems.
- We expect that the employment of these polynomials in spectral methods will open other insights to introduce generalizations of these polynomials and use them in other important applications.

Here is the outline of the article: The required mathematical preliminaries, including Caputo's fractional derivative, are presented in Section 2. In addition, in this section, we introduce some polynomials whose coefficients are the Lucas polynomials. Section 3 is dedicated to establishing novel theoretical formulations for these polynomials. A spectral collocation technique for solving a nonlinear fourth-order integro-differential beam problem is presented in Section 4. Another spectral technique for solving the nonlinear fractional Riccati problem is shown in Section 5. A thorough convergence and error study of the suggested expansion is presented in Section 6. Several numerical examples are displayed in Section 7 to ensure the efficiency and accuracy of the two presented algorithms. At last, Section 8 wraps up the article with a review of results and possible avenues for further research.

2. Some fundamentals

In this section, an overview of the fractional calculus is given. In addition, some properties of Lucas coefficient polynomials are presented.

2.1. Fractional derivative in the sense of Caputo

Definition 2.1. *In Caputo's sense, the fractional derivative $D^{\gamma}\xi(s)$ is defined as [48]*

$$D^{\gamma}\xi(s) = \frac{1}{\Gamma(r-\gamma)}\int_0^s (s-t)^{r-\gamma-1}\xi^{(r)}(t)dt, \quad \gamma > 0, \quad s > 0,\ r-1 < \gamma \le r, \quad r \in \mathbb{N}. \tag{2.1}$$

For D^{γ} with $r-1 < \gamma \le r, \quad r \in \mathbb{N}$, the following identities hold:

$$D^{\gamma}C = 0, \quad C \text{ is a constant}, \tag{2.2}$$

$$D^{\gamma}s^r = \begin{cases} 0, & \text{if } r \in \mathbb{N}_0 \quad \textit{and} \quad r < \lceil\gamma\rceil, \\ \frac{r!}{\Gamma(r-\gamma+1)}\,s^{r-\gamma}, & \text{if } r \in \mathbb{N}_0 \quad \textit{and} \quad r \ge \lceil\gamma\rceil, \end{cases} \tag{2.3}$$

where $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and $\lceil\gamma\rceil$ is the ceiling function.

2.2. Lucas numbers and Lucas coefficient polynomials

The Lucas numbers satisfy the following recurrence relation:

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2,\ L_1 = 1. \tag{2.4}$$

A few Lucas numbers are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123$.
The Benit form of these numbers can be written as

$$L_n = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n}, \quad n \ge 0. \tag{2.5}$$

Here, we consider the polynomials with Lucas number coefficients defined as

$$\mathsf{L}_i^*(t) = \sum_{r=0}^{i} L_r\, t^{i-r}, \tag{2.6}$$

where L_r are the standard Lucas numbers.
Formula (2.6) can also be written in the form

$$\mathsf{L}_i^*(t) = \sum_{r=0}^{i} L_{i-r}\, t^{r}. \tag{2.7}$$

From the representation in (2.7), it is clear that $\mathsf{L}_i^*(t)$ satisfies the following recurrence relation:

$$\mathsf{L}_i^*(t) - t\,\mathsf{L}_{i-1}^*(t) - L_i = 0. \tag{2.8}$$

Remark 2.1. *To the best of our knowledge, the literature lacks many essential formulas useful for treating numerically different types of DEs. We will develop new formulas for the introduced polynomials in the upcoming section.*

3. New formulas of the Lucas coefficient polynomials

Now, we aim to derive some new formulas regarding the polynomials $\mathsf{L}_i^*(t)$, which will be pivotal in deriving our proposed numerical algorithms in the upcoming section. More precisely, we will derive the following formulas:

- The inversion formula for the representation in (2.6).
- The moment formula for the polynomials $\mathsf{L}_i^*(t)$.
- The expression of the high-order derivatives of these polynomials in terms of their original ones.
- The linearization formula for the polynomials $\mathsf{L}_i^*(t)$.
- Some specific definite integral formulas involving these polynomials and their derivatives.

The first important theorem is concerned with the inversion formula of the Lucas coefficient polynomials. To prove this formula, the following lemma is required.

Lemma 3.1. *The following identity holds for every non-negative integer i:*

$$\sum_{r=0}^{i} 2^r L_r = \frac{1}{5} 2^{i+1} \left(2L_{i+2} - L_{i+1}\right). \tag{3.1}$$

Proof. Through induction, we will go forward. For $i = 0$, both the left and right sides are 2. Let formula (3.1) be valid. Our aim now is to prove the following identity:

$$\sum_{r=0}^{i+1} 2^r L_r = \frac{1}{5} 2^{i+2} \left(2L_{i+3} - L_{i+2}\right). \tag{3.2}$$

Now, we can write

$$\sum_{r=0}^{i+1} 2^r L_r = \sum_{r=0}^{i} 2^r L_r + 2^{i+1} L_{i+1}. \tag{3.3}$$

Based on (3.1), the last formula turns into

$$\sum_{r=0}^{i+1} 2^r L_r = \frac{1}{5} 2^{i+1} \left(2L_{i+2} - L_{i+1}\right) + 2^{i+1} L_{i+1}, \tag{3.4}$$

which simplifies to

$$\sum_{r=0}^{i+1} 2^r L_r = \frac{1}{5} 2^{i+2} \left(L_{i+2} + 2L_{i+1}\right). \tag{3.5}$$

Using the well-known recurrence relation of Lucas numbers in (2.4), it can be shown that

$$L_{i+2} + 2L_{i+1} = 2L_{i+3} - L_{i+2},$$

and accordingly, we get

$$\sum_{r=0}^{i+1} 2^r L_r = \frac{1}{5} 2^{i+2} \left(2L_{i+3} - L_{i+2}\right). \tag{3.6}$$

Lemma 3.1 is now proved. □

Theorem 3.1. *The inversion formula of* $\mathsf{L}_i^*(t)$ *is*

$$t^i = \sum_{r=0}^{i} S_{r,i}\, \mathsf{L}_r^*(t), \quad i \geq 1, \tag{3.7}$$

where

$$S_{r,i} = \begin{cases} \frac{1}{2}, & \text{if } r = i, \\ \frac{-1}{4}, & \text{if } r = i-1, \\ \frac{-5}{2^{i-r+1}}, & \text{if } 0 \leq r \leq i-2. \end{cases} \tag{3.8}$$

Proof. We are going to prove the following identity:

$$t^i = \frac{1}{2}\, \mathsf{L}_i^*(t) - \frac{1}{4}\, \mathsf{L}_{i-1}^*(t) - 5 \sum_{r=0}^{i-2} 2^{-3-r}\, \mathsf{L}_{i-r-2}^*(t), \tag{3.9}$$

which is an alternative form of (3.7).
We proceed by induction. First, for $i = 1$, noting that $\mathsf{L}_0^*(t) = 2$, $\mathsf{L}_1^*(t) = 2t + 1$, it is easy to see that

$$t = \frac{1}{2}\, \mathsf{L}_1^*(t) - \frac{1}{4}\, \mathsf{L}_0^*(t).$$

Thus (3.9) is valid for $i = 1$. Now, assume that formula (3.9) holds; then, to complete the proof, we will prove the following identity:

$$t^{i+1} = \frac{1}{2} \mathsf{L}_{i+1}^*(t) - \frac{1}{4} \mathsf{L}_i^*(t) - 5 \sum_{r=0}^{i-1} 2^{-3-r} \mathsf{L}_{i-r-1}^*(t). \tag{3.10}$$

Multiplying formula (3.9) by t, we get

$$t^{i+1} = \frac{1}{2}\, t\,\mathsf{L}_i^*(t) - \frac{1}{4}\, t\,\mathsf{L}_{i-1}^*(t) - 5 \sum_{r=0}^{i-2} 2^{-3-r} t\,\mathsf{L}_{i-r-2}^*(t). \tag{3.11}$$

Using the recurrence relation (2.8), we have the following three expressions:

$$t\,\mathsf{L}_i^*(t) = \mathsf{L}_{i+1}^*(t) - L_{i+1},$$
$$t\,\mathsf{L}_{i-1}^*(t) = \mathsf{L}_i^*(t) - L_i,$$
$$t\,\mathsf{L}_{i-r-2}^*(t) = \mathsf{L}_{i-r-1}^*(t) - L_{i-r-1}.$$

Inserting the above three expressions into (3.11), we get

$$t^{i+1} = \frac{1}{2}\left(\mathsf{L}_{i+1}^*(t) - L_{i+1}\right) - \frac{1}{4}\left(\mathsf{L}_i^*(t) - L_i\right) - 5 \sum_{r=0}^{i-2} 2^{-3-r}\left(\mathsf{L}_{i-r-1}^*(t) - L_{i-r-1}\right). \tag{3.12}$$

The last formula takes the form

$$t^{i+1} = \sum_1 + \sum_2, \tag{3.13}$$

where

$$\sum_1 = \frac{1}{2}\mathsf{L}_{i+1}^*(t) - \frac{1}{4}\,\mathsf{L}_i^*(t) - 5 \sum_{r=0}^{i-1} 2^{-3-r}\,\mathsf{L}_{i-r-1}^*(t), \tag{3.14}$$

$$\sum_2 = -\frac{1}{2}L_{i+1} + \frac{1}{4}L_i + \frac{5}{8}\sum_{r=0}^{i-1} 2^{-r} L_{i-r-1}. \tag{3.15}$$

Now, we will show that $\sum_2 = 0$. From (3.15), We can write

$$\sum_2 = -\frac{1}{2}L_{i+1} + \frac{1}{4}L_i + 5 \cdot 2^{-i-2} \sum_{r=0}^{i-1} 2^r\, L_r. \tag{3.16}$$

Lemma 3.1 enables one to obtain the following identity:

$$\sum_{r=0}^{i-1} 2^r L_r = \frac{1}{5} 2^i \left(2L_{i+1} - L_i\right), \tag{3.17}$$

and accordingly, after simplifying, we get

$$\sum_2 = 0. \tag{3.18}$$

Now, from formulas (3.13), (3.14), and (3.18), we get

$$t^{i+1} = \frac{1}{2}\mathsf{L}_{i+1}^*(t) - \frac{1}{4}\mathsf{L}_i^*(t) - 5 \sum_{r=0}^{i-1} 2^{-3-r}\mathsf{L}_{i-r-1}^*(t). \tag{3.19}$$

This completes the proof. □

Remark 3.1. *An alternative form for* (3.7) *is*

$$t^i = \sum_{r=0}^{i} G_{r,i}\, \mathsf{L}^*_{i-r}(t), \tag{3.20}$$

with

$$G_{r,i} = \begin{cases} \frac{1}{2}, & \text{if } r = 0, \\ -\frac{1}{4}, & \text{if } r = 1, \\ -\frac{5}{2^{r+1}}, & \text{if } 2 \le r \le i. \end{cases} \tag{3.21}$$

Theorem 3.2. *Let m and j be two non-negative integers. The following moment formula is valid:*

$$t^m\, \mathsf{L}^*_j(t) = \sum_{p=0}^{j+m} M_{p,j,m}\, \mathsf{L}^*_{j+m-p}(t), \tag{3.22}$$

with

$$M_{p,j,m} = \sum_{r=0}^{\min(p,j)} L_r\, G_{p-r,j+m+r}, \tag{3.23}$$

and $G_{r,i}$ are as given in (3.21).

Proof. From the power form representation in (2.6), we can write

$$t^m\, \mathsf{L}^*_j(t) = \sum_{r=0}^{j} L_r\, t^{j+m-r}. \tag{3.24}$$

Inserting the inversion formula (3.20) into the last formula yields the following formula:

$$t^m\, \mathsf{L}^*_j(t) = \sum_{r=0}^{j} L_r \sum_{s=0}^{j+m-r} G_{s,j+m+r}\, \mathsf{L}^*_{j+m-r-s}(t). \tag{3.25}$$

The above formula can be transformed into the following one with some algebraic manipulations:

$$t^m\, \mathsf{L}^*_j(t) = \sum_{p=0}^{j+m} M_{p,j,m}\, \mathsf{L}^*_{j+m-p}(t), \tag{3.26}$$

with

$$M_{p,j,m} = \sum_{r=0}^{\min(p,j)} L_r\, G_{p-r,j+m+r}.$$

This ends the proof. □

Theorem 3.3. *Consider two positive integers r, q with $r \ge q$. The following derivative expression holds:*

$$D^q \mathsf{L}^*_r(t) = \sum_{m=0}^{r-q} \bar{G}^q_{r,m}\, \mathsf{L}^*_m(t), \tag{3.27}$$

where

$$\bar{G}^{q}_{r,m} = \begin{cases} \dfrac{1}{2}\, L_{r-q-m}\,(m+1)_q - \dfrac{1}{4}\, L_{-m-q+r-1}\,(m+2)_q \\ -5 \displaystyle\sum_{k=0}^{r-q-m-2} 2^{-1+k+m+q-r}\, L_k\,(1-k-q+r)_q, & \text{if } 0 \le m \le r-q-2, \\ \dfrac{-q\,(r-1)!}{2\,(r-q)!}, & \text{if } m = r-q-1, \\ (1-q+r)_q, & \text{if } m = r-q. \end{cases}$$

Proof. If we differentiate the representation in (2.6) with respect to t, then we get

$$D^q \mathsf{L}^*_r(t) = \sum_{k=0}^{r-q} L_k\,(r-k-q+1)_q\, t^{r-k-q}. \tag{3.28}$$

Based on the inversion formula (3.7), the following formula can be obtained:

$$D^q \mathsf{L}^*_r(t) = \sum_{k=0}^{r-q} L_k\,(r-k-q+1)_q \left(\frac{1}{2} L^*_{r-k-q}(t) - \frac{1}{4}\, L^*_{r-k-q-1}(t) - 5 \sum_{m=2}^{r-k-q} \frac{L^*_{r-k-q-m}(t)}{2^{m+1}} \right), \tag{3.29}$$

which can be written as

$$\begin{aligned} D^q \mathsf{L}^*_r(t) = {} & \frac{1}{2} \sum_{k=0}^{r-q} L_k\,(r-k-q+1)_q\, L^*_{r-k-q}(t) - \frac{1}{4} \sum_{k=0}^{r-q} L_k\,(r-k-q+1)_q\, L^*_{r-k-q-1}(t) \\ & - 5 \sum_{k=0}^{r-q} L_k\,(r-k-q+1)_q \sum_{m=2}^{r-k-q} \frac{L^*_{r-k-q-m}(t)}{2^{m+1}}. \end{aligned} \tag{3.30}$$

Rearranging the terms in the last sum of (3.30) yields the following formula:

$$\begin{aligned} D^q \mathsf{L}^*_r(t) = {} & \frac{1}{2} \sum_{k=0}^{r-q} L_k\,(r-k-q+1)_q\, L^*_{r-k-q}(t) - \frac{1}{4} \sum_{k=0}^{r-q} L_k\,(r-k-q+1)_q\, L^*_{r-k-q-1}(t) \\ & - 5 \sum_{s=0}^{r-q-2} \sum_{k=0}^{s} 2^{-3+k-s}\, L_k\,(1-k-q+r)_q\, L^*_{r-q-s-2}(t), \end{aligned} \tag{3.31}$$

which can be written again as

$$\begin{aligned} D^q \mathsf{L}^*_r(t) = {} & \frac{1}{2} \sum_{m=0}^{r-q} L_{r-q-m}\,(m+1)_q\, L^*_m(t) - \frac{1}{4} \sum_{m=0}^{r-q-1} L_{-1-m-q+r}\,(m+2)_q\, L^*_m(t) \\ & - 5 \sum_{m=0}^{r-q-2} \sum_{k=0}^{r-q-m-2} 2^{-1+k+m+q-r}\, L_k\,(1-k-q+r)_q\, L^*_m(t). \end{aligned} \tag{3.32}$$

Finally, we can write

$$D^q \mathsf{L}^*_r(t) = \sum_{m=0}^{r-q} \bar{G}^{q}_{r,m}\, \mathsf{L}^*_m(t),$$

with

$$\bar{G}^{q}_{r,m}=\begin{cases}\frac{1}{2}L_{r-q-m}(1+m)_q-\frac{1}{4}L_{-m-q+r-1}(2+m)_q\\ -5\sum_{k=0}^{r-q-m-2}2^{-1+k+m+q-r}L_k(1-k-q+r)_q, & \text{if } 0\le m\le r-q-2,\\ \frac{-q\,(r-1)!}{2\,(r-q)!}, & \text{if } m=r-q-1,\\ (1-q+r)_q, & \text{if } m=r-q.\end{cases}$$

This finalizes the proof of the theorem. □

Corollary 3.1. *The first-, second-, and fourth-order derivatives of $\mathsf{L}^*_r(t)$ may be represented in the following form:*

$$\frac{d\,\mathsf{L}^*_r(t)}{d\,t}=\sum_{m=0}^{r-1}\sigma_{r,m}\,\mathsf{L}^*_m(t), \tag{3.33}$$

$$\frac{d^2\,\mathsf{L}^*_r(t)}{d\,t^2}=\sum_{m=0}^{r-2}\bar{G}^2_{r,m}\,\mathsf{L}^*_m(t), \tag{3.34}$$

$$\frac{d^4\,\mathsf{L}^*_r(t)}{d\,t^4}=\sum_{m=0}^{r-4}\bar{G}^4_{r,m}\,\mathsf{L}^*_m(t), \tag{3.35}$$

where

$$\sigma_{r,m}=\begin{cases}r, & \textit{if } m=r-1,\\ -\frac{1}{2}, & \textit{if } m=r-2,\\ -\frac{1}{4}(m+2)\,L_{r-m-2}+\frac{1}{2}(m+1)\,L_{r-m-1}-5\sum_{k=0}^{r-m-3}2^{k+m-r}(-k+r)L_k, & \textit{if } 0\le m\le r-3,\end{cases} \tag{3.36}$$

$$\bar{G}^2_{r,m}=\begin{cases}r(r-1), & \textit{if } m=r-2,\\ 1-r, & \textit{if } m=r-3,\\ \frac{1}{4}(m+2)\,(-(m+3)L_{r-m-3}+2(m+1)L_{r-m-2})\\ -5\sum_{k=0}^{r-m-4}2^{1+k+m-r}(k-r)(1+k-r)L_k, & \textit{if } 0\le m\le r-4,\end{cases} \tag{3.37}$$

$$\bar{G}^4_{r,m}=\begin{cases}(r-3)(r-2)(r-1)r, & \textit{if } m=r-4,\\ -2(r-3)(r-2)(r-1), & \textit{if } m=r-5,\\ \frac{1}{4}(m+2)(m+3)(m+4)\,(-(m+5)L_{r-m-5}+2(m+1)L_{r-m-4})\\ -5\sum_{k=0}^{r-m-6}2^{3+k+m-r}(k-r)(1+k-r)(2+k-r)(3+k-r)L_k, & \textit{if } 0\le m\le r-6.\end{cases} \tag{3.38}$$

Proof. The results of Corollary 3.1 may be directly obtained by setting, respectively, $q = 1, 2,$ and 4 in Theorem 3.3. □

Now, the following corollary exhibits the general operational matrix of derivatives of the Lucas coefficient polynomials. It can be established based on the explicit general derivative formula in (3.27).

Corollary 3.2. *If we consider the following vector:*

$$\boldsymbol{L}(t) = [\mathsf{L}_0^*(t), \mathsf{L}_1^*(t), \ldots, \mathsf{L}_N^*(t)]^T, \tag{3.39}$$

then, the qth-order derivative of the vector $\boldsymbol{L}(t)$ *can be written in the following matrix form:*

$$\frac{d^q\, \boldsymbol{L}(t)}{d\, t^q} = \bar{\boldsymbol{G}}^{\boldsymbol{q}}\, \boldsymbol{L}(t), \tag{3.40}$$

where $\bar{\boldsymbol{G}}^{\boldsymbol{q}} = (\bar{G}^q_{r,m})$ *is the operational matrix of derivatives of order* $(N+1)^2$.
For example, the matrix $\bar{\boldsymbol{G}}^{\boldsymbol{4}}$ *takes the following form for* $N = 6$*:*

$$\bar{\boldsymbol{G}}^{\boldsymbol{4}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 2 & 0 & 0 & 0 & 0 & 0 \\ -\frac{11}{4} & -\frac{1}{2} & 3 & 0 & 0 & 0 & 0 \\ -\frac{31}{8} & -\frac{11}{4} & -\frac{1}{2} & 4 & 0 & 0 & 0 \\ -\frac{111}{16} & -\frac{31}{8} & -\frac{11}{4} & -\frac{1}{2} & 5 & 0 & 0 \\ -\frac{351}{32} & -\frac{111}{16} & -\frac{31}{8} & -\frac{11}{4} & -\frac{1}{2} & 6 & 0 \end{pmatrix}. \tag{3.41}$$

Theorem 3.4. *The following linearization formula holds for* $\mathsf{L}_i^*(t)$*:*

$$\mathsf{L}_i^*(t)\, \mathsf{L}_j^*(t) = \sum_{p=0}^{i+j} V_{p,i,j}\, \mathsf{L}_{i+j-p}^*(t), \tag{3.42}$$

where

$$V_{p,i,j} = \sum_{r=0}^{\min(i,p)} L_r \sum_{m=0}^{\min(p-r,j)} L_m\, G_{p-m-r,j+i-r+m}, \tag{3.43}$$

and $G_{r,i}$ *are given in* (3.21).

Proof. First, the power form representation in (2.6) enables one to write the following formula:

$$\mathsf{L}_i^*(t)\, \mathsf{L}_j^*(t) = \sum_{r=0}^{i} L_r\, t^{i-r} \mathsf{L}_j^*(t).$$

The application of the moment formula (3.22) turns the last formula into the following one:

$$\mathsf{L}_i^*(t)\, \mathsf{L}_j^*(t) = \sum_{r=0}^{i} L_r \sum_{p=0}^{j+i-r} \sum_{m=0}^{\min(p,j)} L_m G_{p-m,j+i-r+m}\, \mathsf{L}_{j+i-p-r}^*(t), \tag{3.44}$$

which can be turned into the following one:

$$\mathsf{L}_i^*(t)\,\mathsf{L}_j^*(t) = \sum_{p=0}^{i+j} V_{p,i,j}\,\mathsf{L}_{i+j-p}^*(t), \tag{3.45}$$

with the linearization coefficients $V_{p,i,j}$ given by

$$V_{p,i,j} = \sum_{r=0}^{\min(i,p)} L_r \sum_{m=0}^{\min(p-r,j)} L_m G_{p-m-r,j+i-r+m}. \tag{3.46}$$

This completes the proof. □

As a direct consequence of the linearization formula of the polynomials $\mathsf{L}_i^*(t)$, the following specific definite integral formula can be derived.

Corollary 3.3. *For any positive real number ℓ, the following integral formula holds:*

$$H_{i,j}^{\ell} = \int_0^{\ell} \mathsf{L}_i^*(t)\,\mathsf{L}_j^*(t)\,dt = \sum_{p=0}^{i+j} V_{p,i,j}\,J_{i+j-p}, \tag{3.47}$$

where $V_{p,i,j}$ are the linearization coefficients defined in (3.43)*, and J_r is given by*

$$J_r = \sum_{k=0}^{r} \frac{L_k\,\ell^{r-k+1}}{r-k+1}. \tag{3.48}$$

Proof. First, it is easy to derive a closed form of $\int_0^{\ell} \mathsf{L}_r^*(t)\,dt$. Integrating both sides of (2.6), we get

$$\int_0^{\ell} \mathsf{L}_r^*(t)\,dt = \sum_{k=0}^{r} L_k \int_0^{\ell} t^{r-k}\,dx, \tag{3.49}$$

and therefore, we can write

$$\int_0^{\ell} \mathsf{L}_r^*(t)\,dt = J_r,$$

with

$$J_r = \sum_{k=0}^{r} \frac{L_k\,\ell^{r-k+1}}{r-k+1}.$$

Now, integrating both sides of the linearization formula (3.42) implies the following integral formula:

$$\int_0^{\ell} \mathsf{L}_i^*(t)\,\mathsf{L}_j^*(t)\,dx = \sum_{p=0}^{j+i} V_{p,i,j} \int_0^{L} \mathsf{L}_{j+i-p}^*(t)\,dt, \tag{3.50}$$

which is equivalent to

$$\int_0^{\ell} \mathsf{L}_i^*(t)\,\mathsf{L}_j^*(t)\,dt = \sum_{p=0}^{i+j} V_{p,i,j}\,J_{i+j-p},$$

and J_r is given in (3.48). □

Corollary 3.4. *For any positive real number ℓ, the following integral formula holds:*

$$\mathcal{A}^{\ell}_{i,j} = \int_0^{\ell} \frac{d\,\mathsf{L}^*_i(t)}{d\,t}\frac{d\,\mathsf{L}^*_j(t)}{d\,t}\,dt = \sum_{m=0}^{i-1}\sum_{n=0}^{j-1} H^{\ell}_{m,n}\,\sigma_{i,m}\,\sigma_{j,n}, \tag{3.51}$$

where $H^{\ell}_{i,j}$ is given in (3.47)*, and $\sigma_{r,m}$ is given in* (3.36).

Proof. From formula (3.33), we can write

$$\frac{d\,\mathsf{L}^*_i(t)}{d\,t} = \sum_{m=0}^{i-1} \sigma_{i,m}\,\mathsf{L}^*_m(t),$$

and accordingly, the following formula can be obtained:

$$\frac{d\,\mathsf{L}^*_i(t)}{d\,t}\,\frac{d\,\mathsf{L}^*_j(t)}{d\,t} = \sum_{m=0}^{i-1}\sum_{n=0}^{j-1} \sigma_{i,m}\,\sigma_{j,n}\,\mathsf{L}^*_m(t)\,\mathsf{L}^*_n(t). \tag{3.52}$$

Integrating both sides of the last formula and making use of formula (3.47) leads to the following formula:

$$\int_0^{\ell} \frac{d\,\mathsf{L}^*_i(t)}{d\,t}\frac{d\,\mathsf{L}^*_j(t)}{d\,t}\,dt = \sum_{m=0}^{i-1}\sum_{n=0}^{j-1} H^{\ell}_{m,n}\,\sigma_{i,m}\,\sigma_{j,n}, \tag{3.53}$$

where $H^{\ell}_{m,n}$ are given in (3.47). This ends the proof. □

4. Collocation approach for the static beam problem

Consider the following static beam problem [36]:

$$\begin{cases} \mathcal{Z}^{(4)}(t) - \lambda\,\mathcal{Z}''(t) - \dfrac{2}{\ell}\displaystyle\int_0^{\ell} [\mathcal{Z}'(t)]^2 dt\,\mathcal{Z}''(t) = f(t), \quad 0 < t < \ell, \\ \mathcal{Z}(0) = \mathcal{Z}(\ell) = \mathcal{Z}''(0) = \mathcal{Z}''(\ell) = 0, \end{cases} \tag{4.1}$$

where $\mathcal{Z}(t)$ represents the static deflection of the beam and λ, ℓ are positive constants. Also, $f(t)$ is a continuous function on $[0, \ell]$.

Let $\mathcal{Z}(t)$ be a square Lebesgue integrable function on $(0, \ell)$, and assume that it can be written as a combination of the linearly independent $\mathsf{L}^*_i(t)$, that is,

$$\mathcal{Z}(t) = \sum_{i=0}^{\infty} \hat{\mathcal{Z}}_i\,\mathsf{L}^*_i(t), \tag{4.2}$$

which can be approximated as

$$\mathcal{Z}(t) \approx \mathcal{Z}_N(t) = \sum_{i=0}^{N} \hat{\mathcal{Z}}_i\,\mathsf{L}^*_i(t). \tag{4.3}$$

The residual $\mathcal{R}(t)$ of Eq (4.1) is given by

$$
\begin{aligned}
\mathcal{R}(t) &= \mathcal{Z}_N^{(4)}(t) - \lambda\,\mathcal{Z}_N''(t) - \frac{2}{\ell}\int_0^{\ell}[\mathcal{Z}_N'(t)]^2 dt\,\mathcal{Z}_N''(t) - f(t)\\
&= \sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,\frac{d^4\,\mathsf{L}_i^*(t)}{d\,t^4} - \lambda\sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,\frac{d^2\,\mathsf{L}_i^*(t)}{d\,t^2} - \frac{2}{\ell}\int_0^{\ell}\sum_{i=0}^{N}\sum_{j=0}^{N}\hat{\mathcal{Z}}_j\,\hat{\mathcal{Z}}_i\,\frac{d\,\mathsf{L}_i^*(t)}{d\,t}\,\frac{d\,\mathsf{L}_j^*(t)}{d\,t}\,dt\times\\
&\quad\sum_{k=0}^{N}\hat{\mathcal{Z}}_k\,\frac{d^2\,\mathsf{L}_k^*(t)}{d\,t^2} - f(t)\\
&= \sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,\frac{d^4\,\mathsf{L}_i^*(t)}{d\,t^4} - \lambda\sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,\frac{d^2\,\mathsf{L}_i^*(t)}{d\,t^2} - \frac{2}{\ell}\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N}\hat{\mathcal{Z}}_j\,\hat{\mathcal{Z}}_i\,\hat{\mathcal{Z}}_k\,\frac{d^2\,\mathsf{L}_k^*(t)}{d\,t^2}\times\\
&\quad\int_0^{\ell}\frac{d\,\mathsf{L}_i^*(t)}{d\,t}\,\frac{d\,\mathsf{L}_j^*(t)}{d\,t}dt\, - f(t).
\end{aligned}
\tag{4.4}
$$

The previous equation can be rewritten after using Corollaries 3.1 and 3.4 as

$$
\begin{aligned}
\mathcal{R}(t) = &\sum_{i=0}^{N}\sum_{m=0}^{i-4}\hat{\mathcal{Z}}_i\,\bar{G}_{i,m}^{4}\,\mathsf{L}_m^*(t) - \lambda\sum_{i=0}^{N}\sum_{m=0}^{i-2}\hat{\mathcal{Z}}_i\,\bar{G}_{i,m}^{2}\,\mathsf{L}_m^*(t)\\
&- \frac{2}{\ell}\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N}\sum_{m=0}^{k-2}\hat{\mathcal{Z}}_j\,\hat{\mathcal{Z}}_i\,\hat{\mathcal{Z}}_k\,\bar{G}_{k,m}^{2}\,\mathsf{L}_m^*(t)\mathcal{A}_{i,j}^{\ell} - f(t).
\end{aligned}
\tag{4.5}
$$

Upon implementing the collocation technique at a set of collocation points $t_s = \frac{\ell\,s}{N+1}$, we get

$$
\mathcal{R}(t_s) = 0, \quad s : 1, ..., N-3. \tag{4.6}
$$

Furthermore, the application of the collocation method to the conditions (4.1) yields

$$
\begin{aligned}
&\sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,\mathsf{L}_i^*(0) = 0,\\
&\sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,\mathsf{L}_i^*(\ell) = 0,\\
&\sum_{i=0}^{N}\sum_{m=0}^{i-2}\bar{G}_{i,m}^{2}\,\hat{\mathcal{Z}}_i\,\mathsf{L}_m^*(0) = 0,\\
&\sum_{i=0}^{N}\sum_{m=0}^{i-2}\bar{G}_{i,m}^{2}\,\hat{\mathcal{Z}}_i\,\mathsf{L}_m^*(\ell) = 0.
\end{aligned}
\tag{4.7}
$$

Therefore, Eqs (4.6) and (4.7) enable us to get a system of $(N+1)$ nonlinear algebraic equations that may be solved using Newton's iterative method to obtain the unknowns $\hat{\mathcal{Z}}_i$.

5. Tau approach for the nonlinear fractional Riccati problem

Our aim in this section is to solve the following nonlinear fractional Riccati equation [49]:

$$\begin{cases} D^{\beta}\mathcal{Z}(t) = g(t) + \nu_1\,\mathcal{Z}(t) + \nu_2\,\mathcal{Z}^2(t), \quad t\in[0,1],\\ \mathcal{Z}(0)=\varepsilon, \end{cases} \tag{5.1}$$

where $0<\beta\le 1$, ν_1, ν_2, ε are known constants, and $g(t)$ is a continuous function on [0,1]. The residual $\mathbf{Re}(t)$ of Eq (5.1) can be expressed as

$$\begin{aligned} \mathbf{Re}(t) &= D^{\beta}\mathcal{Z}_N(t) - g(t) - \nu_1\,\mathcal{Z}_N(t) - \nu_2\,\mathcal{Z}_N^2(t)\\ &= \sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,D^{\beta}\mathsf{L}_i^*(t) - g(t) - \nu_1\sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,\mathsf{L}_i^*(t) - \nu_2\sum_{i=0}^{N}\sum_{j=0}^{N}\hat{\mathcal{Z}}_j\,\hat{\mathcal{Z}}_i\,\mathsf{L}_i^*(t)\mathsf{L}_j^*(t)\\ &= \sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,D^{\beta}\mathsf{L}_i^*(t) - g(t) - \nu_1\sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,\mathsf{L}_i^*(t) - \nu_2\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{p=0}^{i+j}\hat{\mathcal{Z}}_j\,\hat{\mathcal{Z}}_i\,V_{p,i,j}\,\mathsf{L}_{i+j-p}^*(t). \end{aligned} \tag{5.2}$$

We apply the Tau method to get

$$(\,\mathbf{Re}(t),\mathsf{L}_r^*(t)\,) = 0, \quad r=0,1,\ldots,N-1, \tag{5.3}$$

which can be written alternatively in the following form:

$$\begin{aligned} \sum_{i=0}^{N}\hat{\mathcal{Z}}_i\left(D^{\beta}\,\mathsf{L}_i^*(t),\mathsf{L}_r^*(t)\right) - \nu_1\sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,(\mathsf{L}_i^*(t)\,,\mathsf{L}_r^*(t))&\\ - \nu_2\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{p=0}^{i+j}\hat{\mathcal{Z}}_j\,\hat{\mathcal{Z}}_i\,V_{p,i,j}\,(\mathsf{L}_{i+j-p}^*(t)\,,\mathsf{L}_r^*(t)\,) &= (g(t)\,,\mathsf{L}_r^*(t)\,), \end{aligned} \tag{5.4}$$

or in the following alternative form:

$$\sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,w_{i,r} - \nu_1\sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,y_{i,r} - \nu_2\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{p=0}^{i+j}\hat{\mathcal{Z}}_j\,\hat{\mathcal{Z}}_i\,V_{p,i,j}\,y_{i+j-p,r} = g_r, \tag{5.5}$$

where

$$g_r = (g(t)\,,\mathsf{L}_r^*(t)\,), \tag{5.6}$$

$$w_{i,r} = \left(D^{\beta}\,\mathsf{L}_i^*(t),\mathsf{L}_r^*(t)\right), \tag{5.7}$$

$$y_{i,r} = (\mathsf{L}_i^*(t)\,,\mathsf{L}_r^*(t))\,. \tag{5.8}$$

Moreover, the initial condition in (5.1) leads to

$$\sum_{i=0}^{N}\hat{\mathcal{Z}}_i\,\mathsf{L}_i^*(0) = \epsilon. \tag{5.9}$$

Finally, the system in (5.5) and (5.9) can be solved with the aid of the Gauss elimination method to get the unknown expansion coefficients $\hat{\mathcal{K}}_i$.

Theorem 5.1. *The elements $y_{i,r}$, $w_{i,r}$ are given explicitly by the following formulas:*

$$y_{i,r} = \int_0^1 \mathsf{L}_i^*(t)\,\mathsf{L}_r^*(t)\,dt = H^1_{i,r}, \tag{5.10}$$

$$w_{i,r} = \int_0^1 [D^\beta\,\mathsf{L}_i^*(t)]\,\mathsf{L}_r^*(t)\,dt = \sum_{k=1}^{i}\sum_{n=0}^{r} \frac{k!\,L_{i-k}\,L_{r-n}}{(1-\beta+k+n)\,\Gamma(k-\beta+1)}, \tag{5.11}$$

$$\tag{5.12}$$

where $H^\ell_{i,r}$ is given in (3.47).

Proof. The first part is a direct result from Corollary 3.3 after putting $\ell = 1$ in Eq (3.47).
To prove the second part, $w_{i,r}$ can be written as

$$\begin{aligned} w_{i,r} &= \int_0^1 [D^\beta\,\mathsf{L}_i^*(t)]\,\mathsf{L}_r^*(t)\,dt \\ &= \sum_{k=1}^{i}\sum_{n=0}^{r} \frac{k!\,L_{i-k}\,L_{r-n}}{\Gamma(k-\beta+1)} \int_0^1 t^{k-\beta+n}\,dt, \end{aligned} \tag{5.13}$$

which immediately gives the following relation:

$$w_{i,r} = \sum_{k=1}^{i}\sum_{n=0}^{r} \frac{k!\,L_{i-k}\,L_{r-n}}{(1-\beta+k+n)\,\Gamma(k-\beta+1)}, \tag{5.14}$$

which gives the desired result. □

6. Convergence and error analysis

This section is interested in analyzing in detail the expansion in terms of $\mathsf{L}_i^*(t)$. Thus, some important lemmas and theorems are presented and proved.

Lemma 6.1. *Let $t \in [0, \ell]$, $\quad \ell > 0$. This inequality holds:*

$$|\mathsf{L}_i^*(t)| \leq 2(\ell\,\mathfrak{R})^i, \qquad \forall\ i \geq 0, \tag{6.1}$$

where $\mathfrak{R} = \frac{1+\sqrt{5}}{2}$ which represents the golden ratio.

Proof. The application of Eq (2.7) enables us to write

$$|\mathsf{L}_i^*(t)| = \sum_{r=0}^{i} |L_{i-r}|\,|t^r|. \tag{6.2}$$

By virtue of the following inequalities $|L_{i-r}| \leq 2(\mathfrak{R})^i$, $\quad |t^i| \leq \ell^i$, the previous equation can be written as

$$\begin{aligned} |\mathsf{L}_i^*(t)| &= 2\sum_{r=0}^{i} \ell^r\,\mathfrak{R}^{i-r} \\ &= \frac{2\left(2^{-i}\left(\sqrt{5}+1\right)^{i+1} - 2\,\ell^{i+1}\right)}{-2\,\ell + \sqrt{5}+1}. \end{aligned} \tag{6.3}$$

Finally, the last equation can be estimated by

$$\frac{2\left(2^{-i}\left(\sqrt{5}+1\right)^{i+1}-2\,\ell^{i+1}\right)}{-2\,\ell+\sqrt{5}+1}\le 2(\ell\,\mathfrak{R})^{i},\qquad \forall\, i\ge 0. \tag{6.4}$$

This ends the proof. □

Lemma 6.2. *Let $\mathcal{Z}(t)$ be an infinitely differentiable function at the origin. It can be expanded as*

$$\mathcal{Z}(t)=\sum_{i=0}^{\infty}\sum_{m=i}^{\infty}\frac{\mathcal{Z}^{(m)}(0)\,S_{i,m}}{m!}\,\mathsf{L}_i^{*}(t), \tag{6.5}$$

where $S_{i,m}$ is defined in (3.8).

Proof. Let $\mathcal{Z}(t)$ can be expanded as

$$\mathcal{Z}(t)=\sum_{k=0}^{\infty}\frac{\mathcal{Z}^{(k)}(0)}{k!}\,t^{k}. \tag{6.6}$$

Based on the expression (3.7), $\mathcal{Z}(t)$ can be rewritten as

$$\mathcal{Z}(t)=\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{\mathcal{Z}^{(k)}(0)\,S_{r,k}}{k!}\,\mathsf{L}_r^{*}(t), \tag{6.7}$$

which can also be represented in the form

$$\mathcal{Z}(t)=\sum_{i=0}^{\infty}\sum_{m=i}^{\infty}\frac{\mathcal{Z}^{(m)}(0)\,S_{i,m}}{m!}\,\mathsf{L}_i^{*}(t). \tag{6.8}$$

This completes the proof. □

Theorem 6.1. *If $\mathcal{Z}(t)$ is defined on $[0,\ell]$ and $|\mathcal{Z}^{(i)}(0)|\le\lambda^{i},\ i>0$, where $\lambda>0$ and $\mathcal{Z}(t)=\sum_{i=0}^{\infty}\hat{\mathcal{Z}}_i\,\mathsf{L}_i^{*}(t)$, then we get*

$$|\hat{\mathcal{Z}}_i|<\frac{5\,e^{\lambda/2}\,\lambda^{i}}{2\,i!}. \tag{6.9}$$

Moreover, the series converges absolutely.

Proof. The application of Lemma 6.1 together with using the assumptions of the theorem leads to

$$\hat{\mathcal{Z}}_i=\sum_{m=i}^{\infty}\frac{\mathcal{Z}^{(m)}(0)\,S_{i,m}}{m!}. \tag{6.10}$$

Taking |.| for the previous equation, we get

$$\begin{aligned}|\hat{\mathcal{Z}}_i|&=\sum_{m=i}^{\infty}\frac{\left|\mathcal{Z}^{(m)}(0)\right|\,|S_{i,m}|}{m!}\\&\le\sum_{m=i}^{\infty}\frac{5\,\lambda^{m}}{m!\,2^{m-i+1}}\\&=\frac{5\,e^{\lambda/2}\,2^{i-1}\left(\Gamma(i)-\Gamma\left(i,\frac{\lambda}{2}\right)\right)}{\Gamma(i)},\end{aligned} \tag{6.11}$$

where $\Gamma(.)$ and $\Gamma(.,.)$ represent the well-known gamma and upper incomplete gamma functions [50]. Now, with the aid of the following inequality:

$$\frac{\Gamma(i)-\Gamma\left(i,\frac{\lambda}{2}\right)}{\Gamma(i)} < \frac{\left(\frac{\lambda}{2}\right)^i}{i!}, \quad \forall i>0, \tag{6.12}$$

we can write Eq (6.11) as

$$|\hat{\mathcal{Z}}_i| < \frac{5\,e^{\lambda/2}\,\lambda^i}{2\,i!}. \tag{6.13}$$

To prove the second part of this theorem, using inequalities (6.1) and (6.9), we can write

$$\begin{aligned}\left|\sum_{i=0}^{\infty}\hat{\mathcal{Z}}_i\,\mathsf{L}_i^*(t)\right| &= \sum_{i=0}^{\infty}\left|\hat{\mathcal{Z}}_i\right|\,\left|\mathsf{L}_i^*(t)\right| \\ &< \sum_{i=0}^{\infty}\frac{5\,e^{\lambda/2}\,(\lambda\,\ell\,\mathfrak{R})^i}{i!} \\ &= 5\,\lambda\,\mathfrak{R}\,\ell\,e^{\lambda\left(\mathfrak{R}\,\ell+\frac{1}{2}\right)},\end{aligned} \tag{6.14}$$

so the series converges absolutely. □

Theorem 6.2. *The following upper estimate holds on the truncation error if $\mathcal{Z}(t)$ meets the hypothesis of Theorem 6.1.*

$$|\mathcal{Z}(t)-\mathcal{Z}_N(t)| < \frac{e^{(1+\sqrt{5})\,\lambda}(\lambda\,\mathfrak{R}\,\ell)^N}{N!}. \tag{6.15}$$

Proof. We can write

$$\begin{aligned}|\mathcal{Z}(t)-\mathcal{Z}_N(t)| &= \sum_{i=N+1}^{\infty}\left|\hat{\mathcal{Z}}_i\right|\,\left|\mathsf{L}_i^*(t)\right| \\ &< \sum_{i=N+1}^{\infty}\frac{5\,e^{\lambda/2}\,(\lambda\,\ell\,\mathfrak{R})^i}{i!} \\ &= \frac{5\,\lambda^N\,e^{\lambda\left(\mathfrak{R}\,\ell+\frac{1}{2}\right)}(\mathfrak{R}\,\ell)^N(\lambda\,\mathfrak{R}\,\ell)^{1-N}(\Gamma(N)-\Gamma(N,\lambda\,\mathfrak{R}\,\ell))}{\Gamma(N)}.\end{aligned} \tag{6.16}$$

Using inequality (6.12), the previous equation can be written as

$$|\mathcal{Z}(t)-\mathcal{Z}_N(t)| < \frac{e^{(1+\sqrt{5})\,\lambda}(\lambda\,\mathfrak{R}\,\ell)^N}{N!}. \tag{6.17}$$

□

Theorem 6.3. *(Stability) Under the assumptions of Theorem 6.1, we have*

$$|\mathcal{Z}_{N+1}(t)-\mathcal{Z}_N(t)| < \frac{5\,e^{\lambda/2}\,(\lambda\,\ell\,\mathfrak{R})^{N+1}}{(N+1)!}. \tag{6.18}$$

Proof. Based on definition of $\mathcal{Z}_N(t)$, we can write

$$\mathcal{Z}_{N+1}(t) - \mathcal{Z}_N(t) = \hat{\mathcal{Z}}_{N+1} \mathsf{L}^*_{N+1}(t). \tag{6.19}$$

Taking $|.|$ for both sides of the previous equation, we get

$$|\mathcal{Z}_{N+1}(t) - \mathcal{Z}_N(t)| = \left|\hat{\mathcal{Z}}_{N+1}\right| \left|\mathsf{L}^*_{N+1}(t)\right|, \tag{6.20}$$

which can be written after using Theorem 6.1 and Lemma 6.1 as

$$|\mathcal{Z}_{N+1}(t) - \mathcal{Z}_N(t)| = \left|\hat{\mathcal{Z}}_{N+1}\right| \left|\mathsf{L}^*_{N+1}(t)\right| < \frac{5\, e^{\lambda/2}\, (\lambda\, \ell\, \mathfrak{R})^{N+1}}{(N+1)!}. \tag{6.21}$$

This concludes the proof of this theorem. □

7. Illustrative examples

This section is confined to presenting some examples to test the two collocation and tau algorithms designed to solve the two nonlinear models in this paper. Comparisons with some other methods in the literature are also presented.

Example 7.1. *[36] Consider the following equation:*

$$\begin{cases} \mathcal{Z}^{(4)}(t) - 2\,\mathcal{Z}''(t) - \dfrac{2}{\pi}\displaystyle\int_0^{\pi} [\mathcal{Z}'(t)]^2 dt\, \mathcal{Z}''(t) = -4\, \sin(t), \quad 0 < t < \pi, \\ \mathcal{Z}(0) = \mathcal{Z}(\pi) = \mathcal{Z}''(0) = \mathcal{Z}''(\pi) = 0, \end{cases} \tag{7.1}$$

where the exact solution of this problem is $\mathcal{Z}(t) = -\sin(t)$.

Table 1 presents a comparison of L_∞-errors between our method and the method in [36]. Figures 1 and 2 show the maximum absolute errors (MAEs) and L_∞-errors at different values of N. Table 2 shows the absolute errors (AEs) and relative absolute errors (RAE) at $N = 18$. These results confirm that the proposed method efficiently decreases errors over the whole domain and demonstrates a strong correlation between the approximate and precise solutions.

Table 1. Comparison of L_∞-errors for Example 7.1.

	Presented method at $N = 18$	Method in [36] at $N = 14$
L_∞-errors	4.62164×10^{-12}	1.8089×10^{-8}

Table 2. Errors of Example 7.1 at $N = 18$.

t	AE	RAE	Approximate soulution
$\frac{\pi}{10}$	5.65603×10^{-13}	1.83033×10^{-12}	-0.309017
$\frac{2\pi}{10}$	1.17029×10^{-12}	1.99101×10^{-12}	-0.587785
$\frac{3\pi}{10}$	1.84275×10^{-12}	2.27776×10^{-12}	-0.809017
$\frac{4\pi}{10}$	2.58771×10^{-12}	2.72088×10^{-12}	-0.951057
$\frac{5\pi}{10}$	3.37019×10^{-12}	3.37019×10^{-12}	-1.000000
$\frac{6\pi}{10}$	4.09428×10^{-12}	4.30498×10^{-12}	-0.951057
$\frac{7\pi}{10}$	4.56946×10^{-12}	5.64816×10^{-12}	-0.809017
$\frac{8\pi}{10}$	4.45866×10^{-12}	7.58552×10^{-12}	-0.587785
$\frac{9\pi}{10}$	3.1849×10^{-12}	1.03065×10^{-11}	-0.309017

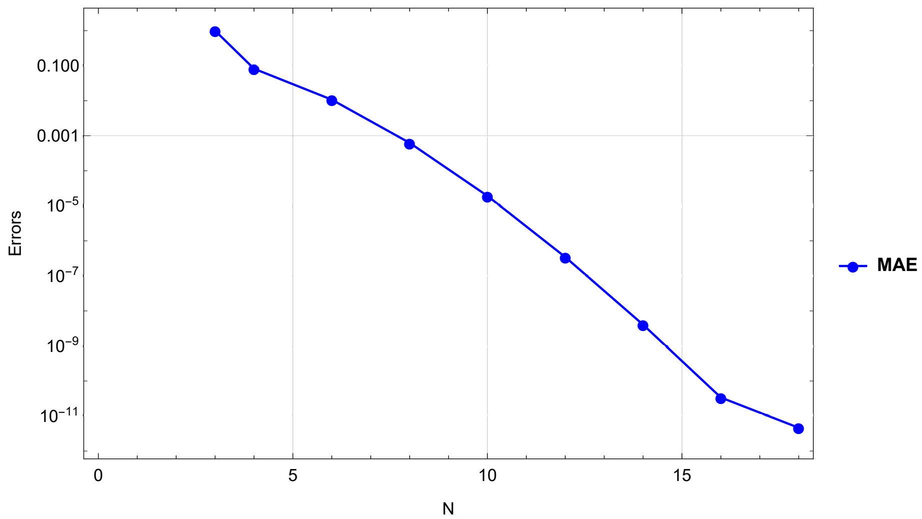

Figure 1. MAE of Example 7.1.

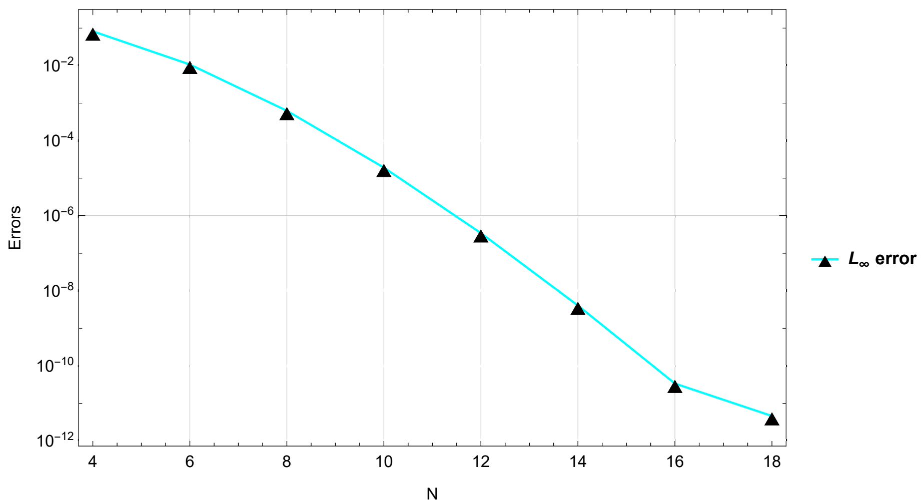

Figure 2. L_∞-errors of Example 7.1.

Example 7.2. *[36] Consider the following equation:*

$$\begin{cases} \mathcal{Z}^{(4)}(t) - \mathcal{Z}''(t) - 2\displaystyle\int_0^1 [\mathcal{Z}'(t)]^2 dt\, \mathcal{Z}''(t) = -t, & 0 < t < 1, \\ \mathcal{Z}(0) = \mathcal{Z}(1) = \mathcal{Z}''(0) = \mathcal{Z}''(1) = 0, \end{cases} \tag{7.2}$$

where the exact solution of this problem is unknown.

Figure 3 shows the stability term $|\mathcal{Z}_{N+1}(t) - \mathcal{Z}_N(t)|$ of the presented method at different values of N. Table 3 shows the residual error (RE) at different values of N and t, which is computed by the following formula:

$$RE = \max_{t\in(0,1)} \left| \mathcal{Z}_N^{(4)}(t) - \mathcal{Z}_N''(t) - 2\int_0^1 [\mathcal{Z}_N'(t)]^2 dt\, \mathcal{Z}_N''(t) + t \right|. \tag{7.3}$$

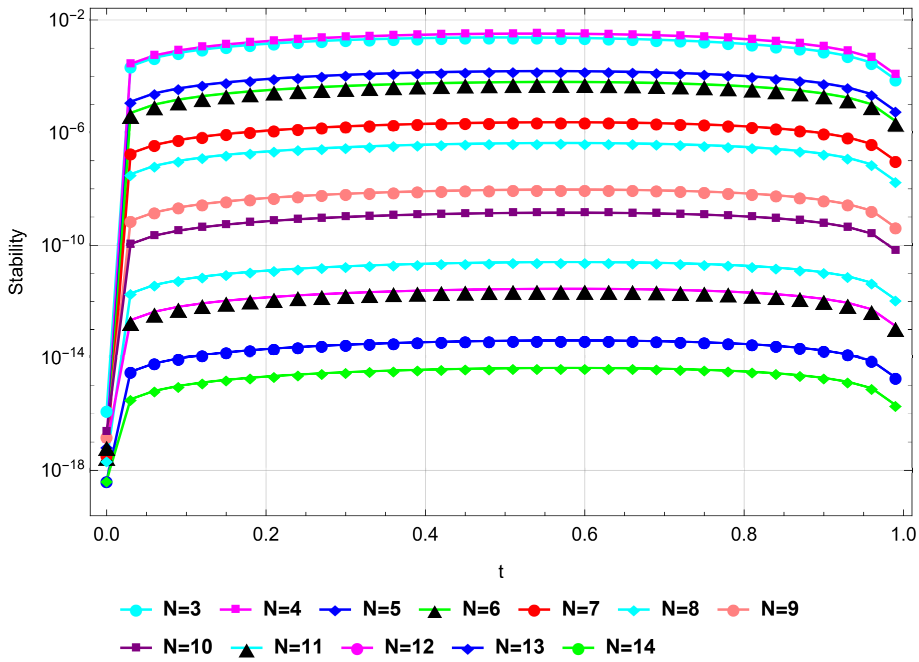

Figure 3. Stability $|\mathcal{Z}_{N+1}(t) - \mathcal{Z}_N(t)|$ of Example 7.2.

Table 3. RE of Example 7.2 at different values of N.

t	$N = 11$	$N = 13$	$N = 15$
0.1	1.00375×10^{-11}	1.20459×10^{-14}	5.55112×10^{-17}
0.2	2.31876×10^{-12}	7.49401×10^{-16}	0
0.3	9.27147×10^{-13}	2.77556×10^{-16}	5.55112×10^{-17}
0.4	4.9355×10^{-13}	2.22045×10^{-16}	5.55112×10^{-17}
0.5	1.66533×10^{-16}	5.55112×10^{-17}	5.55112×10^{-17}
0.6	3.70348×10^{-12}	1.66533×10^{-15}	0
0.7	1.07055×10^{-10}	1.42109×10^{-14}	1.11022×10^{-16}
0.8	3.56228×10^{-9}	2.29738×10^{-12}	1.05471×10^{-14}
0.9	2.97978×10^{-8}	4.5446×10^{-11}	4.07785×10^{-13}

Example 7.3. *Consider the following equation:*

$$\begin{cases} D^{\beta}\mathcal{Z}(t) = g(t) + \mathcal{Z}(t) - \mathcal{Z}^2(t); & t \in [0,1], \\ \mathcal{Z}(0) = 1, \end{cases} \tag{7.4}$$

where $g(t)$ is chosen to meet the exact solution of this problem given by $\mathcal{Z}(t) = e^{\beta t}$.

Table 4 shows the MAE and L_∞-errors at different values of β and N. The table results show that the present method is efficient and accurate. Figure 4 illustrates the AE at different values of N when $\beta = 1$. Figure 5 compares the exact and the approximate solution at $\beta = 0.5$ and $N = 9$. This figure verifies that the suggested approach consistently reduces errors throughout the domain and agrees well with the approximate solution.

Table 4. Error for Example 7.3.

	$\beta = 0.3$		$\beta = 0.6$		$\beta = 0.9$	
N	MAE	L_∞-error	MAE	L_∞-error	MAE	L_∞-error
2	1.03035×10^{-3}	1.03035×10^{-3}	8.06119×10^{-3}	8.06119×10^{-3}	2.45927×10^{-2}	2.45927×10^{-2}
4	7.62294×10^{-7}	7.62294×10^{-7}	2.20413×10^{-5}	2.20413×10^{-5}	1.42101×10^{-4}	1.42101×10^{-4}
6	2.52442×10^{-10}	2.52441×10^{-10}	2.78385×10^{-8}	2.78385×10^{-8}	3.88262×10^{-7}	1.47609×10^{-7}
8	4.61853×10^{-14}	4.59425×10^{-14}	2.01437×10^{-11}	2.01436×10^{-11}	6.15797×10^{-10}	2.43924×10^{-10}
9	3.90799×10^{-14}	3.88834×10^{-14}	4.55191×10^{-13}	4.55178×10^{-13}	1.9527×10^{-11}	8.34109×10^{-12}

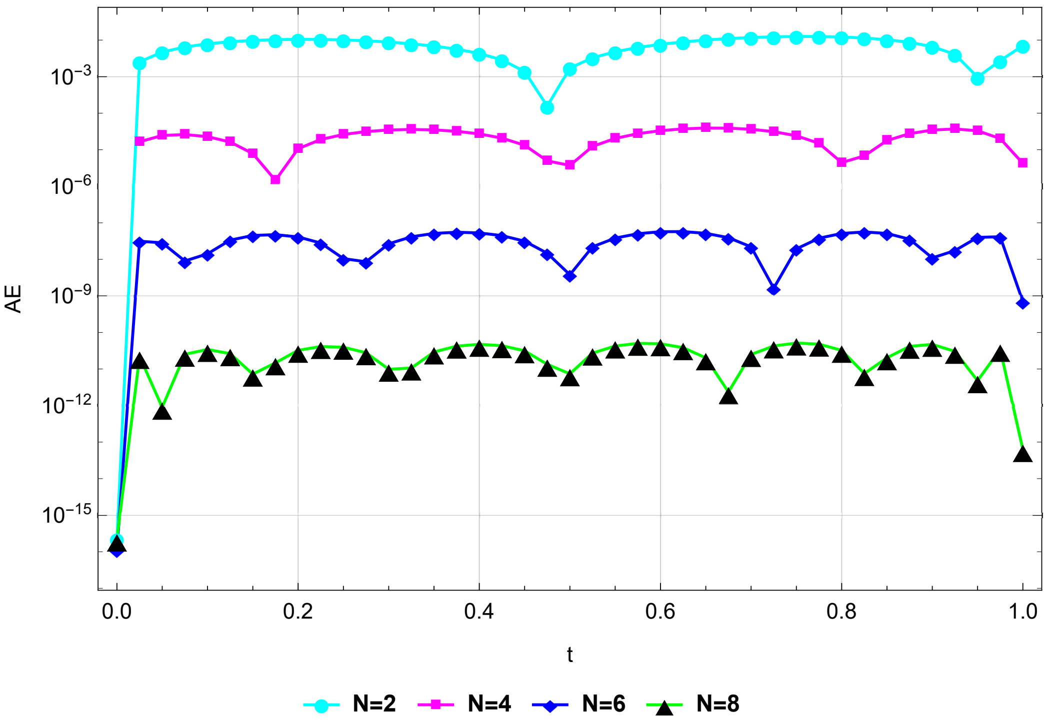

Figure 4. AE of Example 7.3 at $\beta = 1$.

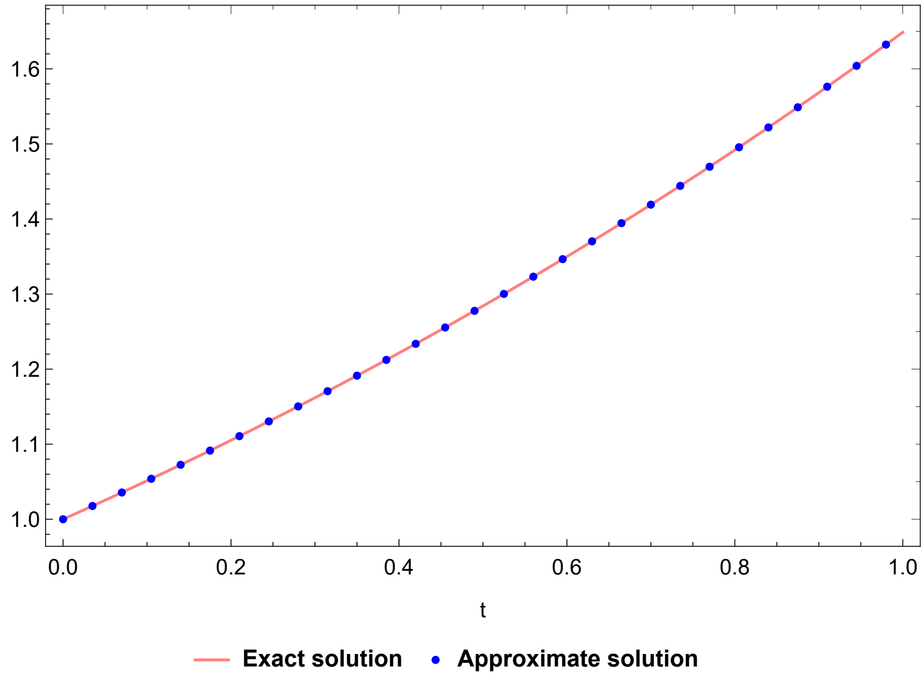

Figure 5. Comparison of exact and approximate solutions of Example 7.3 at $\beta = 0.5$ and $N = 9$.

Example 7.4. *Consider the following equation:*

$$\begin{cases} D^{\beta}\mathcal{Z}(t) = 1 - \mathcal{Z}^2(t); \quad t \in [0,1], \\ \mathcal{Z}(0) = 0, \end{cases} \tag{7.5}$$

where the exact solution of this problem is $\mathcal{Z}(t) = \frac{e^{2t}-1}{e^{2t}+1}$.

Table 5 presents the AE and RAE at $\beta = 1$ *and different values of* N *and* t. *Figure 6 shows the approximate solutions at different values of* β *when* $N = 8$. *This figure verifies that the suggested approach reduces errors consistently throughout the domain and shows a good agreement with the approximate solution. Table 6 presents a comparison between our method at* $N = 9$ *and methods in [51–53] of approximate solutions at* $\beta = 1$. *This table shows that the present method is efficient and accurate.*

Table 5. Errors for Example 7.4.

	$N = 8$		$N = 9$	
t	AE	RAE	AE	RAE
$\frac{1}{10}$	9.24456×10^{-8}	9.27535×10^{-7}	1.30292×10^{-10}	1.30726×10^{-9}
$\frac{2}{10}$	8.01102×10^{-8}	4.05877×10^{-7}	2.56261×10^{-10}	1.29835×10^{-9}
$\frac{3}{10}$	3.65674×10^{-8}	1.25526×10^{-7}	8.68763×10^{-10}	2.98224×10^{-9}
$\frac{4}{10}$	1.19015×10^{-7}	3.1324×10^{-7}	2.41446×10^{-9}	6.35471×10^{-9}
$\frac{5}{10}$	5.78248×10^{-9}	1.2513×10^{-8}	1.97017×10^{-9}	4.26335×10^{-9}
$\frac{6}{10}$	1.18196×10^{-7}	2.20084×10^{-7}	9.90334×10^{-10}	1.84403×10^{-9}
$\frac{7}{10}$	4.61897×10^{-8}	7.64265×10^{-8}	3.27746×10^{-9}	5.42295×10^{-9}
$\frac{8}{10}$	7.66322×10^{-8}	1.15404×10^{-7}	2.86407×10^{-9}	4.31311×10^{-9}
$\frac{9}{10}$	9.51707×10^{-8}	1.32865×10^{-7}	1.94866×10^{-9}	2.72047×10^{-9}

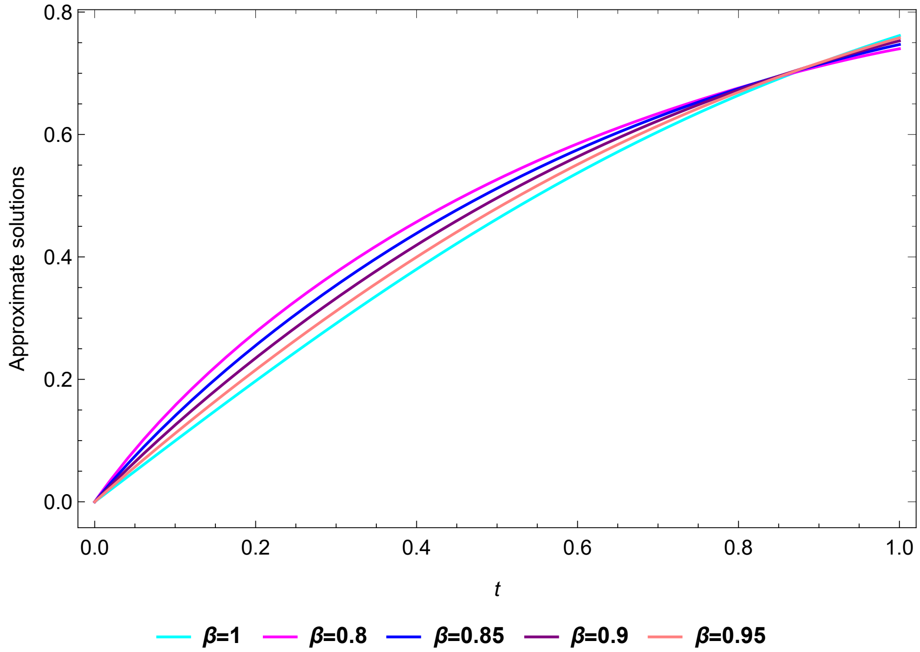

Figure 6. Approximate solutions of Example 7.4.

Table 6. Comparison between approximate solutions of Example 7.4.

t	Presented method	Method in [51]	Method in [52]	Method in [53]	Exact
0.0	0.0	0.0	0.0	0.0	0.0
0.2	0.19737532048	0.197375	0.1973753092	0.1973753160	0.1973753202
0.4	0.37994895984	0.379944	0.3799435784	0.3799469862	0.3799489622
0.6	0.53704956798	0.536857	0.5368572343	0.5369833784	0.5370495669
0.8	0.66403677313	0.661706	0.6617060368	0.6633009217	0.6640367702
1.0	0.76159415595	0.746032	0.746031746	0.7571662667	0.7615941559

Remark 7.1. *We can demonstrate that the theoretical results of the error bound given in Section 6 agree with the numerical results presented in Section 7. As an example, if we set* $\lambda = 0.35$ *and* $\ell = \pi$ *in Eq* (6.15) *of Theorem 6.2, then we can see from the results of Figure 1 that the values of the MAEs do not exceed those of the theoretical bound given in Table 7. Also, if we set* $\lambda = 0.1$ *and* $\ell = 1$ *in Eq* (6.15) *of Theorem 6.2, then we can see from the results of Figure 4 that the values of the MAEs do not exceed those of the theoretical bound given in Table 8.*

Table 7. Theoretical error of Example 7.1.

N	4	6	8	10	12	14	16	18
Error in (6.15)	10^{-1}	10^{-1}	10^{-2}	10^{-3}	10^{-6}	10^{-7}	10^{-9}	10^{-11}

Table 8. Theoretical error of Example 7.3.

N	2	4	6	8
Error in (6.15)	10^{-2}	10^{-5}	10^{-8}	10^{-11}

Remark 7.2. *We can demonstrate that the theoretical results of the stability estimation given in Section 6 agree with the numerical results presented in Section 7. As an example, if we set* $\lambda = 0.5$ *and* $\ell = 1$ *in Eq* (6.15) *of Theorem 6.3, then we can see that the results of Figure 3 do not exceed those of the theoretical bound of stability given in Table 9.*

Table 9. Theoretical bound of stability for Example 7.2.

N	3	4	5	6	7	8	9	10	11	12	13	14
Bound of stability in (6.18)	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-11}	10^{-12}	10^{-13}

8. Concluding remarks

This work presented specific Lucas coefficient polynomial-based spectral methods for numerically solving two nonlinear problems—a fourth-order integro-differential beam model and a fractional Riccati differential equation. We built two efficient spectral frameworks to handle such issues by proving operational identities for the Lucas-based polynomial family. Numerical tests revealed excellent agreement with known solutions and outperformed some current techniques in accuracy. The proposed approach provides a flexible basis for more applications to larger classes of nonlinear, high-order, and fractional differential equations, possibly extending to multidimensional problems. As far as we know, this is the first time these polynomials have been utilized in numerical analysis to produce numerical solutions to differential equations. The proposed method in this paper has been demonstrated for one-dimensional models; extension to multidimensional problems requires further computations and is expected to be explored in future work. In addition, we anticipate that these polynomials may be used to solve various differential equations. All codes were written and debugged by *Mathematica* 11 on an HP Z420 Workstation, Processor: Intel(R) Xeon(R) CPU E5-1620 v2 - 3.70GHz, 16 GB Ram DDR3, and 512 GB storage.

Author contributions

Waleed Mohamed Abd-Elhameed: Conceptualization, methodology, software, validation, formal analysis, investigation, writing—original draft preparation, writing—review and editing, supervision; Shuja'a Ali Alsulami: Methodology, validation, investigation; Omar Mazen Alqubori: Methodology, validation, investigation; Naher Mohammed A. Alsafri: Methodology, validation, investigation; Mohamed Adel: Validation, investigation, funding acquisition; Ahmed Gamal Atta: Conceptualization, methodology, software, validation, formal analysis, investigation, writing—original draft preparation, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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