Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(9): 20825–20842.
DOI: 10.3934/math.2025930
Received: 12 June 2025
Revised: 02 September 2025
Accepted: 05 September 2025
Published: 10 September 2025

Research article

Soft structural Oxtoby–Rose operators and their generated topologies

Zanyar A. Ameen[1,*] **and Ohud F. Alghamdi**[2]

[1] Department of Mathematics, College of Science, University of Duhok, Duhok 42001, Iraq
[2] Department of Mathematics, Faculty of Science, Al-Baha University, Al-Baha, Saudi Arabia

* **Correspondence:** Email: zanyar.ameen@uod.ac; +9647504727411.

Abstract: In this study, we introduce the notion of soft Oxtoby–Rose operators within the framework of abstract measurable soft spaces, extend the classical concept of lower-density operators, and explore their essential characteristics. Subsequently, we delve into the so-called soft Oxtoby–Rose topologies (soft OR-topologies), the soft topologies generated by soft Oxtoby–Rose operators. We examine the key features and definitions associated with soft OR-topologies. Specifically, we demonstrate that within soft OR-topologies, Baire category soft sets, soft locally closed sets, and Borel soft sets are all equivalent. We wrap up this research by analyzing various soft topological properties linked to soft OR-topologies.

Keywords: soft Oxtoby–Rose operator; lower-density soft operator; soft topology; soft density topology
Mathematics Subject Classification: 54A10, 54A99, 03E99

1. Introduction

The issue of the design of complicated systems incorporating uncertainty is often encountered by researchers in lots of areas, like system engineering, medical science, AI, economics, and other fields. In addressing such situations, traditional approaches such as probability theory, fuzzy sets [1], and rough sets [2] are widely employed, but due to parameter constraints, they do not always offer acceptable solutions. Molodtsov [3] probed the theory of soft sets, an innovative method for addressing uncertainty that deals with the limitations of previous methods. The utilization of soft sets is employed in this method, leading to a set of parameterized subsets from a single universal set. Unlike prior techniques, soft set theory does not impose exact limits on objects, and parameters are picked in a number of forms, like words, phrases, numbers, and maps. Consequently, in practice the theory is very flexible and can be used easily. Molodtsov has also employed his theory to a variety of subjects, which demonstrates its flexibility and wide-ranging relevance. Several scholars offer diverse real-world implementations of soft sets, including those related to the COVID-19 outbreak, decision-making processes, information systems, and prostate cancer [4–8]. A systematic literature review on soft set theory was given by Alcantud et al. in [9]. Further extensions and hybrid models of soft sets are proposed in [10–12] followed by some applications [13, 14].

Classical topology is a well-known and significant area of mathematics that deals with the application of set theory concepts and topological structures. Soft topology [15] is a recent topic of topology research that brings together the concepts of the theory of soft sets and classical (general) topology. The authors in [15] defined the most fundamental properties of soft topological spaces, like soft open sets, soft closed sets, soft neighborhoods, soft separation axioms, soft regular spaces, and soft normal spaces. Further, separability [16], compactness [17], doorness and submaximality [18], and connectedness of soft topological spaces [19] are examples of how traditional topological ideas have been extended and made more broadly applicable in soft set environments.

Another ongoing field of research is strategies for constructing topologies on a recognizable universal set in soft settings. Terepeta [20] created two efficient methods that allow one to transform crisp topologies into soft topologies. To create soft topologies from crisp topologies, Terepeta's processes were improved by Alcantud [21]. The preceding techniques have been examined with additional details in [22].

Kandil et al. [23] presented a technique for producing soft ideal topologies, using the generalized local function as a basis. Another different technique of generating soft ideal topologies was presented in [24]. This technique was based on determining cluster soft points. The cluster soft set, which is the collection of all cluster soft points of some soft set, behaves like a soft operator of the soft set. In this note, we introduce a novel soft operator on a measurable soft space called lower-density. This operator is a generalization of the lower-density operator, which itself extends both the Lebesgue lower-density operator [25, 26] and the lower-density soft operator [27], noting that such an operator can be regarded as the set of all density points. For more details on the classical lower-density operators and density topologies, we refer the reader to check [28–33]. Furthermore, we study the soft topologies produced by soft Oxtoby–Rose operators on (abstract) measurable soft spaces in the sense of category.

The remainder of the paper is organized as follows: We provide a survey of the available concepts on soft sets and soft topologies in Section 2. The introduction of soft Oxtoby–Rose operators along with fundamental characteristics is covered in Section 3. We define the term "soft OR-topology" in Section 4. Next, we determine sufficient and necessary requirements for the density of a soft topology. Additionally, we derive many characterizations for soft OR-topologies. A few soft topological properties of this group of soft topologies are established in this section. We provide an overview of the study in Section 5.

2. Notations and background

Throughout this note, we refer to our universe as the set X, a nonempty set of parameters as $\widetilde{\varrho}$, and any index set as Λ.

Definition 2.1. *[3] A soft set over X is defined to be the pair*

$$(Z, \varrho) = \{(\gamma, Z(\gamma)) : \gamma \in \varrho, Z(\gamma) \in 2^X\},$$

where $\emptyset \neq \varrho \subseteq \widetilde{\varrho}$ *and* $Z : \varrho \to 2^X$ *is a function.*

Definition 2.2. *[34] Let* $(Z,\varrho) \in \mathbb{S}(\widetilde{X})$*. Then* (Z,ϱ) *is called a null soft set with respect to* ϱ*, denoted by* $\widetilde{\varnothing}$*, if* $Z(\gamma) = \emptyset$ *for each* $\gamma \in \varrho$*, and it is an absolute soft set with respect to* ϱ*, denoted by* $\widetilde{X}$*, if* $Z(\gamma) = X$ *for each* $\gamma \in \varrho$*.*

$\mathbb{S}(\widetilde{X})$ represents the class of all soft subsets over X related to a set of parameters ϱ.

Remark 2.3. *A soft set* $(Z,\varrho) \in \mathbb{S}(\widetilde{X})$ *extends to the soft set* $(Z,\widetilde{\varrho})$ *by setting* $Z(\gamma) = \emptyset$ *for each* $\gamma \in \widetilde{\varrho} - \varrho$*.*

Definition 2.4. *[7] Let* $(Z,\varrho) \in \mathbb{S}(\widetilde{X})$*. The complement* $(Z,\varrho)^c$ *of* (Z,ϱ) *is a soft set* (Z^c,ϱ)*, where* $Z^c : \varrho \to 2^X$ *is such that* $Z^c(\gamma) = X - Z(\gamma)$ *for each* $\gamma \in \varrho$*.*

Definition 2.5. *[35] Let* $(Z,\varrho) \in \mathbb{S}(\widetilde{X})$*. Then* (Z,ϱ) *is said to be finite (resp. countable) if* $Z(\gamma)$ *is finite (resp. countable) for each* $\gamma \in \varrho$*. Otherwise, it is called infinite (resp. uncountable).*

Definition 2.6. *[7, 36] Let* $(Z,\varrho), (L,\varrho) \in \mathbb{S}(\widetilde{X})$*. Then* (Z,ϱ) *is a soft subset of* (L,ϱ)*, denoted by* $(Z,\varrho)\widetilde{\subseteq}(L,\varrho)$*, if* $Z(\gamma) \subseteq L(\gamma)$ *for each* $\gamma \in \varrho$*. And* (Z,ϱ) *is equal to* (L,ϱ) *if* $(Z,\varrho)\widetilde{\subseteq}(L,\varrho)$ *and* $(L,\varrho)\widetilde{\subseteq}(Z,\varrho)$*.*

Definition 2.7. *[20, 34] Given a collection* $\{(O_\lambda,\varrho) : \lambda \in \Lambda\}\widetilde{\subseteq}\mathbb{S}(\widetilde{X})$*. Then, the intersection of* (O_λ,ϱ) *is a soft set* (Z,ϱ) *having the property that* $Z(\gamma) = \bigcap_{\lambda\in\Lambda} O_\lambda(\gamma)$ *for each* $\gamma \in \varrho$*, which is denoted by* $(Z,\varrho) = \widetilde{\bigcap}_{\lambda\in\Lambda}(O_\lambda,\varrho)$*, and the union of* (O_λ,ϱ) *is a soft set* (Z,ϱ) *having the property that* $Z(\gamma) = \bigcup_{\lambda\in\Lambda} O_\lambda(\gamma)$ *for each* $\gamma \in \varrho$*, which is denoted by* $(Z,\varrho) = \widetilde{\bigcup}_{\lambda\in\Lambda}(O_\lambda,\varrho)$*.*

Definition 2.8. *[24, 34] Let* $(Z,\varrho), (L,\varrho) \in \mathbb{S}(\widetilde{X})$*. The set difference between* (Z,ϱ) *and* (L,ϱ) *is defined to be the soft set* $(R,\varrho) = (L,\varrho) - (Z,\varrho)$*, where* $R(\gamma) = L(\gamma) - Z(\gamma)$ *for each* $\gamma \in \varrho$*. The symmetric difference between* (Z,ϱ) *and* (L,ϱ) *is defined to be the soft set* $(R,\varrho) = (L,\varrho)\widetilde{\Delta}(Z,\varrho)$*, where* $R(\gamma) = L(\gamma)\Delta Z(\gamma)$ *for each* $\gamma \in \varrho$*.*

Definition 2.9. *[37] Let* $(Z,\varrho) \in \mathbb{S}(\widetilde{X})$*. Then* (Z,ϱ) *is called a soft point, symbolized with* x_γ*, if there exists* $x \in X$ *and* $\gamma \in \varrho$ *such that* $Z(\gamma) = \{x\}$ *and* $Z(\delta) = \emptyset$ *for each* $\delta \in \varrho - \{\gamma\}$*. The expression* $x_\gamma \in (Z,\varrho)$ *implies* $x \in Z(\gamma)$*.*

$\mathbb{P}(\widetilde{X})$ represents the family of all soft points in X linked to ϱ.

Definition 2.10. *[15] A soft topology over X is a family* $\mathfrak{T} \subseteq \mathbb{S}(\widetilde{X})$ *that includes* $\widetilde{\varnothing}$*,* $\widetilde{X}$ *and satisfies the condition that it is closed under arbitrary unions and finite intersections.*

The 3-tuple $(X,\mathfrak{T},\varrho)$ is defined to be a soft topological space. An element of $\mathfrak{T}$ that is known as a soft $\mathfrak{T}$-open set or simply a soft open set. The complement of a soft $\mathfrak{T}$-open set or simply a soft open set, is referred to as a soft $\mathfrak{T}$-closed set, or simply a soft closed set. The class denoted by $\mathfrak{T}^c$ contains all soft closed sets in $(X,\mathfrak{T},\varrho)$. The collection of soft topologies over X is represented by $\mathbb{T}(\widetilde{X})$. (see [38]).

Definition 2.11. *[15] Let* $(L,\varrho) \neq \widetilde{\varnothing}$ *be a soft subset of* $(X,\mathfrak{T},\varrho)$*. Then* $\mathfrak{T}_{(L,\varrho)} = \{(G,\varrho)\widetilde{\cap}(L,\varrho) : (G,\varrho) \in \mathfrak{T}\}$ *is said to be a relative soft topology over Y and* $(Y,\mathfrak{T}_{(L,\varrho)},\varrho)$ *is a soft topological subspace of* $(X,\mathfrak{T},\varrho)$*.*

Definition 2.12. *[39] Let* $(G,\varrho) \in \mathbb{S}(\widetilde{X})$ *and* $\mathfrak{T} \in \mathbb{T}(\widetilde{X})$*. A soft neighborhood of* $x_\gamma \in \mathbb{P}(\widetilde{X})$ *is a soft set* (G,ϱ) *whenever there is* $(H,\varrho) \in \mathfrak{T}(x_\gamma)$ *such that* $x_\gamma \in (H,\varrho)\widetilde{\subseteq}(G,\varrho)$*, where* $\mathfrak{T}(x_\gamma)$ *is the collection of all soft* $\mathfrak{T}$*-open sets containing* x_γ*.*

Definition 2.13. *[40] A (countable) soft base for a soft topology* $\mathcal{T}$ *is a (countable) subfamily* $\mathcal{B} \subseteq \mathcal{T}$ *such that every member of* $\mathcal{T}$ *is a union of members of* $\mathcal{B}$*.*

Definition 2.14. *[38] Let $\mathcal{G}\widetilde{\subseteq}\mathbb{S}(\widetilde{X})$. The intersection of all soft topologies over X containing $\mathcal{G}$ is called a soft topology produced by $\mathcal{G}$ and is denoted by $\mathcal{T}(\mathcal{G})$.*

Lemma 2.15. *[15] Let $(X, \mathcal{T}, \varrho)$ be a soft topological space. The collection $\mathcal{T}(\gamma) = \{U(\gamma) : (G, \varrho) \in \mathcal{T}\}$ is a topology on X for each $\gamma \in \varrho$.*

Definition 2.16. *[15] Let $(Z, \varrho) \in \mathbb{S}(\widetilde{X})$ and $\mathcal{T} \in \mathbb{T}(\widetilde{X})$. Then $Int_{\mathcal{T}}(Z, \varrho) = \widetilde{\bigcup}\{(G, \varrho) : (G, \varrho)\widetilde{\subseteq}(Z, \varrho), (G, \varrho) \in \mathcal{T}\}$ and $Cl_{\mathcal{T}}(Z, \varrho) = \widetilde{\bigcap}\{(G, \varrho) : (Z, \varrho)\widetilde{\subseteq}(G, \varrho), (G, \varrho) \in \mathcal{T}^c\}$ are respectively defined to be the soft interior and the soft closure of (Z, ϱ).*

We can use $Int(Z, \varrho)$ and $Cl(Z, \varrho)$ for the soft interior and soft closure of (Z, ϱ), if there is no possibility of misunderstanding.

Remark 2.17. *[41] Let $(Z, \varrho) \in \mathbb{S}(\widetilde{X})$ and $\mathcal{T} \in \mathbb{T}(\widetilde{X})$. Then*

$$Int((Z, \varrho)^c) = (Cl(Z, \varrho))^c \text{ and } Cl((Z, \varrho)^c) = (Int(Z, \varrho))^c.$$

Definition 2.18. *[23] A soft σ-ideal over X is a non-null family $\mathfrak{I}\widetilde{\subseteq}\mathbb{S}(\widetilde{X})$ that satisfies the following:*

(1) If $(O_1, \varrho), (O_2, \varrho), \cdots \in \mathfrak{I}$, then $\widetilde{\bigcup}_{n=1}^{\infty}(O_n, \varrho) \in \mathfrak{I}$.

(2) If $(Z, \varrho) \in \mathfrak{I}$ and $(L, \varrho)\widetilde{\subseteq}(Z, \varrho)$, then $(L, \varrho) \in \mathfrak{I}$.

Whenever (1) holds true for finite members, then $\mathfrak{I}$ is called a soft ideal over X. The family of all soft ideals over X is denoted by $\mathbb{I}(\tilde{X})$.

Definition 2.19. *Let $(Z, \varrho), (L, \varrho) \in \mathbb{S}(\widetilde{X})$ and let $\mathcal{T} \in \mathbb{T}(\widetilde{X})$. Then (Z, ϱ) is called soft regular open [42] if $Int(Cl(Z, \varrho)) = (Z, \varrho)$; soft nowhere $\mathcal{T}$-dense (or shortly soft nowhere dense) [43] if $Int(Cl(Z, \varrho)) = \widetilde{\varnothing}$; soft locally $\mathcal{T}$-closed (or shortly soft locally closed) [44] if $(Z, \varrho) = (G, \varrho)\widetilde{\bigcap}(E, \varrho)$ for some $(G, \varrho) \in \mathcal{T}$ and $(E, \varrho) \in \mathcal{T}^c$; soft $\mathcal{T}$-dense (or shortly soft dense) in (L, ϱ) [45] if $(L, \varrho)\widetilde{\subseteq}Cl(Z, \varrho)$; a soft set of the first-category [45] if $(Z, \varrho) = \widetilde{\bigcup}_{i=1}^{\infty}(A_i, \varrho)$, where each (A_i, ϱ) is a soft nowhere $\mathcal{T}$-dense set; otherwise, it is a soft set of the second category [45].*

The family of all soft regular open sets (resp. soft nowhere $\mathcal{T}$-dense sets, first category soft sets, soft locally $\mathcal{T}$-closed sets) in $(X, \mathcal{T})$ is denoted by $\mathbb{RO}(\mathcal{T})$ (resp. $\mathbb{N}(\mathcal{T})$, $\mathbb{M}(\mathcal{T})$, $\mathbb{LC}(\mathcal{T})$).

Definition 2.20. *[24, 46] Let $(Z, \varrho) \in \mathbb{S}(\widetilde{X})$, $\mathcal{T} \in \mathbb{T}(\widetilde{X})$, and $\mathfrak{I} \in \mathbb{I}(\widetilde{X})$. The point $x_\gamma \in \mathbb{P}(\widetilde{X})$ is called a cluster soft point of (Z, ϱ) if $(Z, \varrho)\widetilde{\bigcap}(G, \varrho) \notin \mathfrak{I}$ for each $(G, \varrho) \in \mathcal{T}(x_\gamma)$. The cluster soft set of (Z, ϱ), denoted by $\mathfrak{c}_{(\mathcal{T},\mathfrak{I})}(Z, \varrho)$ or simply $\mathfrak{c}(Z, \varrho)$, is defined to be the collection of all the cluster soft points of (Z, ϱ).*

If $\mathfrak{I} = \mathbb{M}(\mathcal{T})$, then the cluster soft point x_γ is called a second category soft point of (Z, ϱ). Otherwise, x_γ is a first-category soft point. The family of all first (resp. second) category soft points of (Z, ϱ) is denoted by $C_1(Z, \varrho)$ (resp. $C_2(Z, \varrho)$).

The cluster soft topology is given by $\mathcal{T}_{\mathfrak{c}}(\mathfrak{I}) = \{(Z, \varrho) \in \mathbb{S}(\widetilde{X}) : \mathfrak{c}((Z, \varrho)^c)\widetilde{\subseteq}(Z, \varrho)^c\}$. It is identical with the soft ideal topology introduced in [23].

Definition 2.21. *A collection Σ of soft sets over X is called a soft σ-algebra [47] if Σ comprises $\widetilde{\varnothing}$ and is closed under countable unions and the complement. And Σ is called a soft algebra [48] if it is closed under finite unions of its elements.*

The smallest soft σ-algebra over X containing the family $\mathcal{C}\widetilde{\subseteq}\mathbb{S}(\widetilde{X})$ is called the soft σ-algebra produced by $\mathcal{C}$. The same is true for soft algebras.

Definition 2.22. *[45] Let $(Z,\varrho) \in \mathbb{S}(\widetilde{X})$, $\Upsilon \in \mathbb{T}(\widetilde{X})$, and $\Im \in \mathbb{I}(\widetilde{X})$. Then (Z,ϱ) is said to be a soft Υ-open set modulo $\Im$ if $(Z,\varrho) = (H,\varrho)\widetilde{\Delta}(P,\varrho)$ for some $(H,\varrho) \in \Upsilon$ and $(P,\varrho) \in \Im$. The family of all soft $\mathcal{T}$-open sets modulo $\Im$ is symbolized by $\mathbb{B}_0(\Upsilon,\Im)$.*

If $\Im$ is equal to the soft σ-ideal $\mathbb{M}(\Upsilon)$, then $\mathbb{B}_0(\Upsilon,\mathbb{M}(\Upsilon))$ is the soft σ-algebra of Baire property soft sets.

Definition 2.23. *[45] The soft σ-algebra produced by a soft topology $\Upsilon \in \mathbb{T}(\widetilde{X})$ is called a Borel soft σ-algebra. A Borel soft set is the member of this soft σ-algebra.*

Lemma 2.24. *[46] Let $\Upsilon \in \mathbb{T}(\widetilde{X})$. For $(Z,\varrho) \in \mathbb{M}(\Upsilon)$, there exists a soft F_σ set $(L,\varrho) \in \mathbb{M}(\Upsilon)$ such that $(Z,\varrho)\widetilde{\subseteq}(L,\varrho)$.*

Lemma 2.25. *Let $\Upsilon \in \mathbb{T}(\widetilde{X})$ and $(Z,\varrho),(L,\varrho) \in \mathbb{S}(\widetilde{X})$. Then*

(1) $C_2((Z,\varrho) - C_2(Z,\varrho)) = \widetilde{\emptyset}$.

(2) $(Z,\varrho)\widetilde{\subseteq}(L,\varrho) \implies C_2(Z,\varrho)\widetilde{\subseteq}C_2(L,\varrho)$.

Proof. Theorem 3 (1) and [45, Proposition 12 (2)]. □

Theorem 2.26. *[45] Let $(Z,\varrho) \in \mathbb{S}(\widetilde{X})$ and $\Upsilon \in \mathbb{T}(\widetilde{X})$. The following properties are equivalent:*

(1) $(Z,\varrho) \in \mathbb{B}_0(\Upsilon,\mathbb{M}(\Upsilon))$.

(2) $(Z,\varrho) = [(G,\varrho) - (M,\varrho)]\widetilde{\cup}(N,\varrho)$, where $(G,\varrho) \in \Upsilon$ and $(N,\varrho),(M,\varrho) \in \mathbb{M}(\Upsilon)$.

(3) $C_2(Z,\varrho) - (Z,\varrho) \in \mathbb{M}(\Upsilon)$.

Definition 2.27. *A soft topological space (X,Υ,ϱ) is said to be soft T_1 [16] if every x_γ in (X,Υ,ϱ) is soft closed.*

Definition 2.28. *A soft topological space (X,Υ,ϱ) is said to be soft compact [17] (resp. soft Lindelöf [49]) if every soft open cover of (X,Υ,ϱ) has a finite (resp. countable) subcover.*

Definition 2.29. *A soft topological space (X,Υ,ϱ) is said to be soft second countable [49] if (X,Υ,ϱ) has a countable soft base. It is said to be soft first countable [49] if every x_γ in (X,Υ,ϱ) has a countable soft base.*

Definition 2.30. *[45] A soft topological space (X,Υ,ϱ) is said to a be soft Baire space if every non-null open set is of the second category.*

Definition 2.31. *[50] A soft topological space (X,Υ,ϱ) is said to be soft nodec if every soft nowhere dense set is soft closed.*

3. Soft Oxtoby–Rose operators

In what follows, by the quadruple $(X, \Sigma, \mathfrak{I}, \varrho)$, we mean a measurable soft space, shortly MSS; where Σ is a soft σ-algebra over X and $\mathfrak{I}$ is a proper soft σ-ideal in Σ, i.e., $\mathfrak{I}\widetilde{\subset}\Sigma$ and $\widetilde{X} \notin \mathfrak{I}$. We denote the expression $(Z,\varrho)\widetilde{\Delta}(L,\varrho) \in \mathfrak{I}$ by $(Z,\varrho) \approx (L,\varrho)$

Definition 3.1. *Let $(X, \Sigma, \mathfrak{I}, \varrho)$ be an MSS. The mapping $\varphi : \Sigma \to \mathbb{S}(\widetilde{X})$ is named a soft Oxtoby–Rose operator on $(X, \Sigma, \mathfrak{I}, \varrho)$; shortly, a soft OR-operator, if it satisfies the following conditions, for each $(Z,\varrho), (L,\varrho) \in \Sigma$:*

(C_1) $\varphi(\widetilde{X}) = \widetilde{X}$ *and* $\varphi(\varnothing) = \widetilde{\varnothing}$;

(C_2) $\varphi((Z,\varrho)\widetilde{\cap}(L,\varrho)) = \varphi(Z,\varrho)\widetilde{\cap}\varphi(L,\varrho)$;

(C_3) $(Z,\varrho) \approx (L,\varrho) \Longrightarrow \varphi(Z,\varrho) = \varphi(L,\varrho)$; *and*

(C_4) $(Z,\varrho) \approx \varphi(Z,\varrho)$.

Notice that soft OR-operators under the name of lower-density soft operators on different domain structures were discussed in [27]. The concept of almost lower-density soft operators is presented in [51].

Lemma 3.2. *Let $\varphi : \Sigma \to \mathbb{S}(\widetilde{X})$ be a soft OR-operator on an MSS $(X, \Sigma, \mathfrak{I}, \varrho)$ and let $(Z,\varrho), (L,\varrho) \in \Sigma$. Then the following properties hold:*

(1) $\varphi(Z,\varrho) = \varphi(\varphi(Z,\varrho))$.

(2) $(Z,\varrho)^c \approx [\varphi(Z,\varrho)]^c$.

(3) $\varphi(Z,\varrho)\widetilde{\cap}\varphi[(Z,\varrho)^c] = \widetilde{\varnothing}$.

(4) If $\varphi(Z,\varrho) = \varphi(L,\varrho)$, *then* $(Z,\varrho) \approx (L,\varrho)$.

(5) If $(Z,\varrho)\widetilde{\subseteq}(L,\varrho)$, *then* $\varphi(Z,\varrho)\widetilde{\subseteq}\varphi(L,\varrho)$.

(6) $\varphi(Z,\varrho)\widetilde{\cup}\varphi(L,\varrho)\widetilde{\subseteq}\varphi[(Z,\varrho)\widetilde{\cup}(L,\varrho)]$.

(7) $\varphi(Z,\varrho)\widetilde{\subseteq}\varphi(L,\varrho)$ *iff* $(Z,\varrho) - (L,\varrho) \in \mathfrak{I}$.

(8) $\mathfrak{I} = \{(Z,\varrho) \in \Sigma : \varphi(Z,\varrho) = \widetilde{\varnothing}\}$.

Proof. (1) Suppose $(Z,\varrho) \in \Sigma$. By (C_4), $(Z,\varrho) \approx \varphi(Z,\varrho)$, and so, by (C_3), $\varphi(Z,\varrho) = \varphi(\varphi(Z,\varrho))$.

(2) Suppose $(Z,\varrho) \in \Sigma$. By considering

$$\begin{aligned}(Z,\varrho)^c\widetilde{\Delta}[\varphi(Z,\varrho)]^c &= [(Z,\varrho)^c - [\varphi(Z,\varrho)]^c]\widetilde{\cup}[[\varphi(Z,\varrho)]^c - (Z,\varrho)^c] \\ &= [(Z,\varrho)^c\widetilde{\cap}\varphi(Z,\varrho)]\widetilde{\cup}[[\varphi(Z,\varrho)]^c\widetilde{\cap}(Z,\varrho)] \\ &= [\varphi(Z,\varrho) - (Z,\varrho)]\widetilde{\cup}[(Z,\varrho) - \varphi(Z,\varrho)] \\ &= (Z,\varrho)\widetilde{\Delta}\varphi(Z,\varrho) \in \mathfrak{I}.\end{aligned}$$

Thus, we have $(Z,\varrho)^c \approx [\varphi(Z,\varrho)]^c$.

(3) Suppose $(Z,\varrho) \in \Sigma$. Then, by (C_2), $\varphi(Z,\varrho)\widetilde{\cap}\varphi[(Z,\varrho)^c] = \varphi[(Z,\varrho)\widetilde{\cap}(Z,\varrho)^c] = \varphi(\widetilde{\varnothing}) = \widetilde{\varnothing}$.

(4) Suppose $(Z,\varrho),(L,\varrho) \in \Sigma$. Since $(Z,\varrho) \approx \varphi(Z,\varrho)$ and $\varphi(Z,\varrho) = \varphi(L,\varrho)$, then $(Z,\varrho) \approx \varphi(L,\varrho)$. However, $\varphi(L,\varrho) \approx (L,\varrho)$ implies $(Z,\varrho) \approx (L,\varrho)$.

(5) Suppose $(Z,\varrho),(L,\varrho) \in \Sigma$ such that $(Z,\varrho)\widetilde{\subseteq}(L,\varrho)$. Then $(Z,\varrho) = (Z,\varrho)\widetilde{\cap}(L,\varrho)$ and therefore, $\varphi(Z,\varrho) = \varphi(Z,\varrho)\widetilde{\cap}\varphi(L,\varrho)\ \widetilde{\subseteq}\varphi(L,\varrho)$. Hence, $\varphi(Z,\varrho)\widetilde{\subseteq}\varphi(L,\varrho)$.

(6) Suppose $(Z,\varrho),(L,\varrho) \in \Sigma$. Since $(Z,\varrho)\widetilde{\subseteq}(Z,\varrho)\widetilde{\cup}(L,\varrho)$ and $(L,\varrho)\widetilde{\subseteq}(Z,\varrho)\widetilde{\cup}(L,\varrho)$, then by (5), $\varphi(Z,\varrho)\widetilde{\subseteq}\varphi[(Z,\varrho)\widetilde{\cup}(L,\varrho)]$ and $\varphi(L,\varrho)\widetilde{\subseteq}\varphi[(Z,\varrho)\widetilde{\cup}(L,\varrho)]$. Thus, $\varphi(Z,\varrho)\widetilde{\cup}\varphi(L,\varrho)\ \widetilde{\subseteq}\ \varphi[(Z,\varrho)\widetilde{\cup}(L,\varrho)]$.

(7) Suppose $(Z,\varrho),(L,\varrho) \in \Sigma$. If $\varphi(Z,\varrho)\widetilde{\subseteq}\varphi(L,\varrho)$, then $((Z,\varrho) - \varphi(Z,\varrho))\widetilde{\cup}((L,\varrho) - \varphi(L,\varrho)) \in \mathfrak{I}$. But $(Z,\varrho) - (L,\varrho)\widetilde{\subseteq}((Z,\varrho) - \varphi(Z,\varrho))\widetilde{\cup}((L,\varrho) - \varphi(L,\varrho))$. Thus, $(Z,\varrho) - (L,\varrho) \in \mathfrak{I}$.

Conversely, if $(Z,\varrho) - (L,\varrho) \in \mathfrak{I}$, then $(Z,\varrho) = (Z,\varrho)\widetilde{\cap}(L,\varrho)$. Therefore, $\varphi(Z,\varrho) = \varphi(Z,\varrho)\widetilde{\cap}\ \varphi(L,\varrho)\widetilde{\subseteq}\varphi(L,\varrho)$ and so, $\varphi(Z,\varrho)\widetilde{\subseteq}\varphi(L,\varrho)$.

(8) Suppose $(Z,\varrho) \in \mathfrak{I}$. Since $(Z,\varrho) \approx \widetilde{\varnothing}$, by (C_3), then $\varphi(Z,\varrho) = \varphi(\widetilde{\varnothing}) = \widetilde{\varnothing}$. Thus, $\varphi(Z,\varrho) = \widetilde{\varnothing}$.

□

4. Soft Oxtoby–Rose topologies

In this section, we introduce the so-called "soft Oxtoby–Rose topology", for short soft OR-topology, and investigate its main properties.

Definition 4.1. *Let φ be a soft OR-operator on an MSS $(X,\Sigma,\mathfrak{I},\varrho)$. If the family*

$$\mathcal{T}_\varphi = \{(Z,\varrho) \in \Sigma : (Z,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho)\}$$

forms a soft topology over X, then $\mathcal{T}_\varphi$ is called the soft OR-topology produced by φ.

Proposition 4.2. *Let φ be a soft OR-operator on an MSS $(X,\Sigma,\mathfrak{I},\varrho)$. The soft OR-topology $\mathcal{T}_\varphi$ produced by φ is equivalent to the following family:*

$$\mathcal{T}^0_\varphi = \{\varphi(Z,\varrho) - (L,\varrho) : (Z,\varrho) \in \Sigma, (L,\varrho) \in \mathfrak{I}\}.$$

Proof. We need to prove that $\mathcal{T}_\varphi = \mathcal{T}^0_\varphi$. Suppose $(Z,\varrho) \in \Sigma, (L,\varrho) \in \mathfrak{I}$. Then, $\varphi(Z,\varrho) - (L,\varrho) \in \mathcal{T}^0_\varphi$. Set $(Z,\varrho) = \varphi(Z,\varrho) - (L,\varrho)$. Therefore, $(Z,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho)$ and thus, $\mathcal{T}^0_\varphi\widetilde{\subseteq}\mathcal{T}_\varphi$.

Conversely, let $(Z,\varrho) \in \Sigma$. Assume $(Z,\varrho) \in \mathcal{T}_\varphi$. Then, $(Z,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho)$. Put $(Z,\varrho) = \varphi(Z,\varrho) - (\varphi(Z,\varrho) - (Z,\varrho))$. Since $(L,\varrho) = \varphi(Z,\varrho) - (Z,\varrho) \in \mathfrak{I}$, then $(Z,\varrho) \in \mathcal{T}^0_\varphi$. Accordingly, $\mathcal{T}_\varphi = \mathcal{T}^0_\varphi$. □

Remark 4.3. *It is worth noting that, generally, $\mathcal{T}_\varphi$ may not be a soft topology as shown in the next example. However, it constitutes a soft base for some soft topology (compare this remark with [24, Theorem 4.5]).*

Example 4.4. *Consider the soft topology $\mathcal{T}$ on the class of real numbers $\mathbb{R}$ produced by the soft base:*

$$\mathcal{B} = \{(\gamma,(s,t)) : s,t \in \mathbb{R}; s < t, \gamma \in \varrho\}.$$

The soft identity mapping φ on the MSS $(\mathbb{R}, \mathbb{B}(\mathfrak{T}), \mathfrak{I}_0, \varrho)$ is a soft OR-operator, where $\mathfrak{I}_0 = \{\widetilde{\varnothing}\}$. However, $\mathfrak{T}_\varphi$ is not a soft topology over $\mathbb{R}$ because it is not closed under arbitrary unions. Take the soft set (V, ϱ), where V is a Vitali (non-Borel) set (see [52, page 259]). Clearly, $(V, \varrho) \notin \mathbb{B}(\mathfrak{T})$ and so $(V, \varrho) \notin \mathfrak{T}_\varphi$. However,

$$(V, \varrho) = \widetilde{\bigcup}_{x_\gamma \in (V,\varrho)} \{x_\gamma\},$$

and each $\{x_\gamma\} \in \mathfrak{T}_\varphi$ (as $\{x_\gamma\} \in \mathbb{B}(\mathfrak{T})$).

Notice that the following are natural examples of soft OR-topologies:

Example 4.5. *Let $\mathbb{S}(\widetilde{X})$ be the soft σ-algebra of all soft sets over an infinite set X and let $\mathfrak{I}_{x_\gamma} = \{(Z, \varrho) \in \mathbb{S}(\widetilde{X}) : x_\gamma \notin (Z, \varrho)\}$ be the soft σ-ideal for some fixed soft point x_γ. The soft operator φ on the MSS $(X, \mathbb{S}(\widetilde{X}), \mathfrak{I}_{x_\gamma}, \varrho)$ given by*

$$\varphi(Z, \varrho) = \begin{cases} \widetilde{X}, & \text{if } (Z, \varrho) \notin \mathfrak{I}_{x_\gamma}, \\ \widetilde{\varnothing}, & \text{if } (Z, \varrho) \in \mathfrak{I}_{x_\gamma}, \end{cases}$$

is a soft OR-operator. Then

$$\begin{aligned} \mathfrak{T}_\varphi &= \{(Z, \varrho) \in \mathbb{S}(\widetilde{X}) : (Z, \varrho) \widetilde{\subseteq} \varphi(Z, \varrho)\} \\ &= \{(Z, \varrho) \in \mathbb{S}(\widetilde{X}) : x_\gamma \in (Z, \varrho)\} \widetilde{\cup} \{\widetilde{\varnothing}\} \end{aligned}$$

is a soft topology on $(X, \mathbb{S}(\widetilde{X}), \mathfrak{I}_{x_\gamma}, \varrho)$.

Example 4.6. *Let $(\mathbb{R}, \Sigma, \mathfrak{I}_\omega, \varrho)$ be an MSS, where Σ is any soft σ-algebra over the real number system $\mathbb{R}$ and $\mathfrak{I}_\omega$ is a soft σ-ideal over $\mathbb{R}$ which includes all soft subsets that are countable. Let a soft operator φ be defined on the MSS $(\mathbb{R}, \Sigma, \mathfrak{I}_\omega, \varrho)$ by*

$$\varphi(Z, \varrho) = \begin{cases} \widetilde{\mathbb{R}}, & \text{if } (Z, \varrho) \approx \widetilde{\mathbb{R}}, \\ \widetilde{\varnothing}, & \text{if } (Z, \varrho) \not\approx \widetilde{\mathbb{R}}. \end{cases}$$

Clearly, φ is a soft OR-operator. Then

$$\begin{aligned} \mathfrak{T}_\varphi &= \{(Z, \varrho) \in \Sigma : (Z, \varrho) \widetilde{\subseteq} \varphi(Z, \varrho)\} \\ &= \{(Z, \varrho) \in \Sigma : (Z, \varrho) \approx \widetilde{\mathbb{R}}\} \widetilde{\cup} \{\widetilde{\varnothing}\} \end{aligned}$$

is a soft topology on $(\mathbb{R}, \Sigma, \mathfrak{I}_\omega, \varrho)$.

Proposition 4.7. *Let $\mathfrak{T}_\varphi$ be a soft OR-topology produced by a soft OR-operator φ on an MSS $(X, \Sigma, \mathfrak{I}, \varrho)$. For any $(Z, \varrho) \in \Sigma$, we have*

(1) $Int_{\mathfrak{T}_\varphi}(Z, \varrho) = (Z, \varrho) \widetilde{\cap} \varphi(Z, \varrho)$.

(2) $Cl_{\mathfrak{T}_\varphi}(Z, \varrho) = (Z, \varrho) \widetilde{\cup} [\varphi((Z, \varrho)^c)]^c$.

Proof. Suppose $(Z, \varrho) \in \Sigma$.

(1) Let us start with

$$(Z, \varrho) \widetilde{\Delta} [(Z, \varrho) \widetilde{\cap} \varphi(Z, \varrho)] = [(Z, \varrho) - [(Z, \varrho) \widetilde{\cap} \varphi(Z, \varrho)]] \widetilde{\cup} [[(Z, \varrho) \widetilde{\cap} \varphi(Z, \varrho)] - (Z, \varrho)]$$

$$= [(Z,\varrho)\widetilde{\cap}[(Z,\varrho)^c\widetilde{\cup}(\varphi(Z,\varrho))^c]]\widetilde{\cup}[[(Z,\varrho)\widetilde{\cap}\varphi(Z,\varrho)]\widetilde{\cap}(Z,\varrho)^c]$$
$$= [(Z,\varrho)\widetilde{\cap}(\varphi(Z,\varrho))^c]\widetilde{\cup}\widetilde{\varnothing}$$
$$= (Z,\varrho) - \varphi(Z,\varrho).$$

Since $(Z,\varrho) - \varphi(Z,\varrho)\widetilde{\subseteq}(Z,\varrho) \approx \varphi(Z,\varrho)$ implies $(Z,\varrho) - \varphi(Z,\varrho) \in \mathfrak{I}$. By (C_3),

$$(Z,\varrho)\widetilde{\cap}\varphi(Z,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho) = \varphi[(Z,\varrho)\widetilde{\cap}\varphi(Z,\varrho)]. \tag{4.1}$$

This implies that $(Z,\varrho)\widetilde{\cap}\varphi(Z,\varrho) \in \mathfrak{T}_\varphi$ and therefore, $(Z,\varrho)\widetilde{\cap}\varphi(Z,\varrho)\widetilde{\subseteq}Int_{\mathfrak{T}_\varphi}(Z,\varrho)$. Then, by Definition 2.16, $Int_{\mathfrak{T}_\varphi}(Z,\varrho) = \widetilde{\cup}(G_\lambda,\varrho)$ such that $(G_\lambda,\varrho)\widetilde{\subseteq}\varphi(G_\lambda,\varrho)$ and $(G_\lambda,\varrho)\widetilde{\subseteq} (Z,\varrho)$ for each λ. Since φ is monotone, then we have $(G_\lambda,\varrho)\widetilde{\subseteq}\varphi(G_\lambda,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho)$ and therefore $(G_\lambda,\varrho)\widetilde{\subseteq}(Z,\varrho)\widetilde{\cap}\varphi(Z,\varrho)$ for each λ. Thus, $Int_{\mathfrak{T}_\varphi}(Z,\varrho)\widetilde{\subseteq}(Z,\varrho)\widetilde{\cap}\varphi(Z,\varrho)$. Hence, $Int_{\mathfrak{T}_\varphi}(Z,\varrho) = (Z,\varrho)\widetilde{\cap}\varphi(Z,\varrho)$.

(2) By using Remark 2.17, one can have

$$Cl_{\mathfrak{T}_\varphi}(Z,\varrho) = [Int_{\mathfrak{T}_\varphi}[(Z,\varrho)^c]]^c = [(Z,\varrho)^c\widetilde{\cap}\varphi((Z,\varrho)^c)]^c = (Z,\varrho)\widetilde{\cup}[\varphi((Z,\varrho)^c)]^c.$$

□

Proposition 4.8. *Let $\mathfrak{T}_\varphi$ be a soft OR-topology produced by a soft OR-operator φ on an MSS $(X,\Sigma,\mathfrak{I},\varrho)$. Then $\varphi(Z,\varrho)\widetilde{\subseteq}Cl_{\mathfrak{T}_\varphi}(Z,\varrho)$ for each $(Z,\varrho) \in \Sigma$.*

Proof. Let $(Z,\varrho) \in \Sigma$. By Lemma 3.2 item (3), $\varphi(Z,\varrho)\widetilde{\cap}\varphi((Z,\varrho)^c) = \widetilde{\varnothing}$, which implies $\varphi(Z,\varrho)\widetilde{\subseteq} [\varphi((Z,\varrho)^c)]^c$ and, by Proposition 4.7 item (2), $[\varphi((Z,\varrho)^c)]^c\widetilde{\subseteq}Cl_{\mathfrak{T}_\varphi}(Z,\varrho)$. Therefore, $\varphi(Z,\varrho)\widetilde{\subseteq} Cl_{\mathfrak{T}_\varphi}(Z,\varrho)$. □

Proposition 4.9. *Let $\mathfrak{T}_\varphi$ be a soft OR-topology produced by a soft OR-operator φ on an MSS $(X,\Sigma,\mathfrak{I},\varrho)$. Then $\varphi(Z,\varrho) \in \mathbb{RO}(\mathfrak{T}_\varphi)$ for each $(Z,\varrho) \in \Sigma$.*

Proof. Let $(Z,\varrho) \in \Sigma$. Now,

$$\begin{aligned}
Int_{\mathfrak{T}_\varphi}[Cl_{\mathfrak{T}_\varphi}[\varphi(Z,\varrho)]] &= Int_{\mathfrak{T}_\varphi}[\varphi(Z,\varrho)\widetilde{\cup}[\varphi([\varphi(Z,\varrho)]^c)]^c] \\
&= Int_{\mathfrak{T}_\varphi}[\varphi(Z,\varrho)\widetilde{\cup}[\varphi((Z,\varrho)^c)]^c] \quad \text{(by Lemma 3.2 item (2))} \\
&= Int_{\mathfrak{T}_\varphi}[[\varphi((Z,\varrho)^c)]^c] \quad \text{(by Lemma 3.2 item (3))} \\
&= [\varphi((Z,\varrho)^c)]^c\widetilde{\cap}\varphi([\varphi((Z,\varrho)^c)]^c) \\
&= [\varphi((Z,\varrho)^c)]^c\widetilde{\cap}\varphi([(Z,\varrho)^c]^c) \\
&= [\varphi((Z,\varrho)^c)]^c\widetilde{\cap}\varphi(Z,\varrho) \\
&= \varphi(Z,\varrho).
\end{aligned}$$

Therefore, $\varphi(Z,\varrho) \in \mathbb{RO}(\mathfrak{T}_\varphi)$. □

This conclusion can be employed immediately:

Corollary 4.10. *Let $\mathfrak{T}_\varphi$ be a soft OR-topology produced by a soft OR-operator φ on an MSS $(X,\Sigma,\mathfrak{I},\varrho)$. Then $(Z,\varrho) \in \mathbb{RO}(\mathfrak{T}_\varphi)$ iff $\varphi(Z,\varrho) = (Z,\varrho)$.*

Definition 4.11. *[27] A collection $\mathcal{S}$ of soft sets over X is termed a soft π-system provided that $\mathcal{S}$ satisfies the following axioms:*

(1) $(Z,\varrho) \in \mathcal{S}$ *and* $(L,\varrho) \in \mathcal{S}$ *imply* $(Z,\varrho)\widetilde{\cap}(L,\varrho) \in \mathcal{S}$.

(2) $\mathcal{S} \neq \widetilde{\varnothing}$.

We call $\mathcal{S}$ *a strong soft* π*-system if* $\widetilde{X} \in \mathcal{S}$.

According to the following theorem, the range of φ may be determined with Σ.

Theorem 4.12. *Assume* φ *is a soft OR-operator on an MSS* $(X,\Sigma,\mathfrak{I},\varrho)$*. If there is a strong soft* π*-system* $\mathcal{S}$ *on X such that* $\mathfrak{I}\widetilde{\subseteq}\mathcal{S}\widetilde{\subseteq}\Sigma$ *and a soft OR-operator* $\phi : \mathcal{S} \to \Sigma$ *on* $(X,\mathcal{S},\mathfrak{I},\varrho)$*, then* $\mathcal{T}_\phi = \mathcal{T}_\varphi$.

Proof. Set $\mathcal{S} = \{(Z,\varrho) \in \Sigma : \varphi(Z,\varrho) \in \Sigma\}$. By (C_1) and (C_2), $\mathcal{S}$ is a strong soft π-system on X including $\widetilde{\varnothing}, \widetilde{X}$ such that $\mathfrak{I}\widetilde{\subseteq}\mathcal{S}\widetilde{\subseteq}\Sigma$. If we restrict φ to $\mathcal{S}$ and denote it by $\phi = \varphi|_{\mathcal{S}}$. Apparently, ϕ is a soft OR-operator on $(X,\mathcal{S},\mathfrak{I},\varrho)$. To prove that $\mathcal{T}_\phi = \mathcal{T}_\varphi$, it suffices to show only $\mathcal{T}_\varphi\widetilde{\subseteq}\mathcal{T}_\phi$ since the reverse is always possible. If $(Z,\varrho) \in \mathcal{T}_\varphi$, then $(Z,\varrho) \in \Sigma$ and $(Z,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho)$. Since $\varphi(Z,\varrho) \approx (Z,\varrho)$, then $\varphi(Z,\varrho) = (Z,\varrho)\widetilde{\cup}[\varphi(Z,\varrho) - (Z,\varrho)]$ and therefore, $(Z,\varrho) \in \mathcal{S}$. This means that $(Z,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho) = \phi(Z,\varrho)$ and hence, $(Z,\varrho) \in \mathcal{T}_\phi$. Thus, $\mathcal{T}_\varphi\widetilde{\subseteq}\mathcal{T}_\phi$. □

Proposition 4.13. *Let* $\mathcal{T}_\varphi$ *be a soft OR-topology produced by a soft OR-operator* φ *on an MSS* $(X,\Sigma,\mathfrak{I},\varrho)$*. Then* $(Z,\varrho) \in \mathfrak{I}$ *iff* $(Z,\varrho) \in \mathcal{T}^c_\varphi\widetilde{\cap}\mathbb{N}(\mathcal{T}_\varphi)$.

Proof. Suppose that $(Z,\varrho) \in \mathfrak{I}$. Obviously, $(Z,\varrho)^c \in \Sigma$. Since $(Z,\varrho)^c \approx \widetilde{X}$, then $\varphi((Z,\varrho)^c) = \varphi(\widetilde{X})$. Therefore, $(Z,\varrho)^c\widetilde{\subseteq}\widetilde{X} = \varphi(\widetilde{X}) = \varphi((Z,\varrho)^c)$, and so, $(Z,\varrho)^c \in \mathcal{T}_\varphi$. Hence, $(Z,\varrho) \in \mathcal{T}^c_\varphi$. We now prove that $(Z,\varrho) \in \mathbb{N}(\mathcal{T}_\varphi)$. If $(G,\varrho) \in \mathcal{T}_\varphi$ such that $(G,\varrho)\widetilde{\subseteq}(Z,\varrho)$, then $(G,\varrho)\widetilde{\subseteq}\varphi(G,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho)\widetilde{\subseteq} [\widetilde{X} - \varphi((G,\varrho)^c)] = \widetilde{\varnothing}$. This implies that (G,ϱ) must be null. Therefore, $(Z,\varrho) \in \mathbb{N}(\mathcal{T}_\varphi)$. Thus, $(Z,\varrho) \in \mathcal{T}^c_\varphi\widetilde{\cap}\mathbb{N}(\mathcal{T}_\varphi)$.

Conversely, assume $(Z,\varrho) \in \mathcal{T}^c_\varphi\widetilde{\cap}\mathbb{N}(\mathcal{T}_\varphi)$. Since $(Z,\varrho)^c \in \mathcal{T}_\varphi\widetilde{\subseteq}\Sigma$ and Σ is a soft σ-algebra, then $(Z,\varrho) \in \Sigma$. Therefore, $\widetilde{\varnothing} = Int_\varphi(Z,\varrho) = (Z,\varrho)\widetilde{\cap}\varphi(Z,\varrho)$ implies $(Z,\varrho) = (Z,\varrho) - \varphi(Z,\varrho) \in \mathfrak{I}$. Since $(Z,\varrho) \in \mathbb{N}(\mathcal{T}_\varphi)$ and $(Z,\varrho) = Cl_{\mathcal{T}_\varphi}(Z,\varrho) \in \mathfrak{I}$, by the earlier steps, we have $(Z,\varrho) \in \mathfrak{I}$. We are done. □

Proposition 4.14. *Let* $\mathcal{T}_\varphi$ *be a soft OR-topology produced by a soft OR-operator* φ *on an MSS* $(X,\Sigma,\mathfrak{I},\varrho)$*. If* $(Z,\varrho) \in \mathfrak{I}$*, then* $(Z,\varrho) \in \mathcal{T}^c_\varphi\widetilde{\cap}\mathbb{D}(\mathcal{T}_\varphi)$*, where* $\mathbb{D}(\mathcal{T}_\varphi)$ *is the family of all soft discrete sets in* $(X,\mathcal{T}_\varphi)$*. In addition, the converse holds, whenever* $\mathfrak{I}$ *contains all finite soft sets over X.*

Proof. Suppose $(Z,\varrho) \in \mathfrak{I}$. By Proposition 4.13, $(Z,\varrho) \in \mathcal{T}^c_\varphi$. It remains to show that $(Z,\varrho) \in \mathbb{D}(\mathcal{T}_\varphi)$. Let x_γ be a soft point in (Z,ϱ). Since $(Z,\varrho) - \{x_\gamma\}\widetilde{\subseteq}(Z,\varrho)$ implies $(Z,\varrho) - \{x_\gamma\} \in \mathfrak{I}$. Again, by Proposition 4.13, $(Z,\varrho) - \{x_\gamma\} \in \mathcal{T}^c_\varphi$ and hence, $\{x_\gamma\}$ is a soft $\mathcal{T}_\varphi$-open set in (Z,ϱ). Evidently, $\{x_\gamma\}$ is a soft $\mathcal{T}_\varphi$-closed set in (Z,ϱ). Therefore, $(Z,\varrho) \in \mathbb{D}(\mathcal{T}_\varphi)$.

Conversely, suppose $(Z,\varrho) \in \mathcal{T}^c_\varphi\widetilde{\cap}\mathbb{D}(\mathcal{T}_\varphi)$. Clearly, we have that $(Z,\varrho)^c \in \mathcal{T}_\varphi\widetilde{\subseteq}\Sigma$. Since Σ is a soft σ-algebra, then $(Z,\varrho) \in \Sigma$. But $(Z,\varrho) \in \mathbb{D}(\mathcal{T}_\varphi)$; so, for each $x_\gamma \in (Z,\varrho)$, there exists $(G_{x_\gamma},\varrho) \in \mathcal{T}_\varphi$ with $x_\gamma \in (G_{x_\gamma},\varrho)$ such that $(G_{x_\gamma},\varrho)\widetilde{\cap}(Z,\varrho) = \{x_\gamma\}$. Therefore,

$$x_\gamma \in (G_{x_\gamma},\varrho)\widetilde{\subseteq}\varphi(G_{x_\gamma},\varrho) = \varphi[(G_{x_\gamma},\varrho) - \{x_\gamma\}]\widetilde{\subseteq}\varphi[(Z,\varrho)^c].$$

This means that $(Z,\varrho)\widetilde{\subseteq}\varphi[(Z,\varrho)^c]$; and then, $(Z,\varrho) = \varphi[(Z,\varrho)^c] \approx (Z,\varrho)^c$. By Lemma 3.2 (2), $\varphi[(Z,\rho)^c] \approx (Z,\varrho)^c$, so $(Z,\varrho) \approx (Z,\varrho)^c$. Since $\mathfrak{I}$ contains all finite soft sets, this implies $(Z,\varrho) \in \mathfrak{I}$. □

Proposition 4.15. *Let* $(X,\Sigma,\mathfrak{I},\varrho)$ *be an MSS and let* φ, ϕ *be soft OR-operators on* $(X,\Sigma,\mathfrak{I},\varrho)$*. Then* $\varphi = \phi$ *iff* $\mathcal{T}_\varphi = \mathcal{T}_\phi$.

Proof. Clearly, $\varphi = \phi$ implies $\mathfrak{T}_\varphi = \mathfrak{T}_\phi$.

Conversely, suppose that $(Z,\varrho) \in \Sigma$. By Proposition 4.9, $\varphi(Z,\varrho) \in \mathfrak{T}_\varphi = \mathfrak{T}_\phi$. It follows that $\varphi(Z,\varrho)\widetilde{\subseteq}\phi(\varphi(Z,\varrho))$. By (C_4), $(Z,\varrho) \approx \varphi(Z,\varrho)$, which implies that $\phi(Z,\varrho) = \phi(\varphi(Z,\varrho))$. Therefore, $\varphi(Z,\varrho)\widetilde{\subseteq}\phi(Z,\varrho)$. Similarly, we have that $\phi(Z,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho)$. Hence, $\phi(Z,\varrho) = \varphi(Z,\varrho)$. □

The following definition may be stated based on the uniqueness of the soft OR-topology generated by a soft OR-operator:

Definition 4.16. *Let $(X, \Sigma, \mathfrak{I}, \varrho)$ be an MSS and let $\mathfrak{T} \in \mathbb{T}(\widetilde{X})$. We call $\mathfrak{T}$ a soft OR-topology if there exists a soft OR-operator φ on $(X, \Sigma, \mathfrak{I}, \varrho)$ such that $\mathfrak{T} = \mathfrak{T}_\varphi$.*

Theorem 4.17. *Let $(X, \Sigma, \mathfrak{I}, \varrho)$ be an MSS and let $\mathfrak{T} \in \mathbb{T}(\widetilde{X})$. Then $\mathfrak{T}$ is a soft OR-topology over X iff $\Sigma = \mathbb{B}_0(\mathfrak{T}, \mathfrak{I})$ and $\mathfrak{I} = \mathbb{N}(\mathfrak{T})\widetilde{\cap}\mathfrak{T}^c$.*

Proof. Let $\mathfrak{T}$ be a soft OR-topology over X. By Proposition 4.13, $\mathfrak{I} = \mathbb{N}(\mathfrak{T})\widetilde{\cap}\mathfrak{T}^c$. It is left to prove that $\Sigma = \mathbb{B}_0(\mathfrak{T}, \mathfrak{I})$. Let $(Z,\varrho) \in \Sigma$. Then $(Z,\varrho) = [(Z,\varrho)\widetilde{\cap}\varphi(Z,\varrho)]\widetilde{\cup}[(Z,\varrho) - \varphi(Z,\varrho)]$. Clearly, $(Z,\varrho)\widetilde{\cap}\varphi(Z,\varrho) \in \mathfrak{T}$ and $(Z,\varrho) - \varphi(Z,\varrho) \in \mathfrak{I}$. Since $\mathbb{B}_0(\mathfrak{T}, \mathfrak{I})$ is the soft σ-algebra produced by $\mathfrak{T}\widetilde{\Delta}\mathbb{N}(\mathfrak{T}) = \mathfrak{T}\widetilde{\Delta}\mathfrak{I}$, by [45, Theorem 2], we have $(Z,\varrho) \in \mathbb{B}_0(\mathfrak{T}, \mathfrak{I})$. For the reverse of the inclusion, if $(Z,\varrho) \in \mathbb{B}_0(\mathfrak{T}, \mathfrak{I})$, then $(Z,\varrho) = (G,\varrho)\widetilde{\Delta}(N,\varrho)$, where $(G,\varrho) \in \mathfrak{T}$ and $(N,\varrho) \in \mathbb{N}(\mathfrak{T})$. Since $\mathfrak{T}\widetilde{\subseteq}\Sigma$, so $(Z,\varrho) \in \Sigma$. Consequently, $\Sigma = \mathbb{B}_0(\mathfrak{T}, \mathfrak{I})$.

Conversely, let $\Sigma = \mathbb{B}_0(\mathfrak{T}, \mathfrak{I})$ and $\mathfrak{I}$ contains all soft $\mathfrak{T}$-closed and soft nowhere $\mathfrak{T}$-dense sets. We first prove that there exists a soft OR-operator φ on $(X, \Sigma, \mathfrak{I}, \varrho)$ such that $\mathfrak{T} = \mathfrak{T}_\varphi$. Let $(Z,\varrho) \in \Sigma$. Then $(Z,\varrho) = (G,\varrho)\widetilde{\Delta}(N,\varrho)$, where $(G,\varrho) \in \mathfrak{T}$ and $(N,\varrho) \in \mathbb{N}(\mathfrak{T})$. By Propositions 4.28 and [45, 13], (Z,ϱ) is uniquely represented by $(P,\varrho)\widetilde{\Delta}(M,\varrho)$, where $(P,\varrho) \in \mathbb{RO}(\mathfrak{T})$ and $(M,\varrho) \in \mathbb{N}(\mathfrak{T})$. Define φ as the following:

$$\varphi(Z,\varrho) = (P,\varrho) \text{ for each } (Z,\varrho) \in \Sigma.$$

Then C_1, C_2 are easily satisfied. That is, $\varphi(\widetilde{\varnothing}) = \widetilde{\varnothing}$ and $\varphi(\widetilde{X}) = \widetilde{X}$. Let $(Z,\varrho), (L,\varrho) \in \Sigma$. Then there are $(P,\varrho), (Q,\varrho) \in \mathbb{RO}(\mathfrak{T})$ and $(M,\varrho), (N,\varrho) \in \mathbb{N}(\mathfrak{T})$ such that $(Z,\varrho) = (P,\varrho)\widetilde{\Delta}(M,\varrho)$ and $(L,\varrho) = (Q,\varrho)\widetilde{\Delta}(N,\varrho)$. At present, since

$$\begin{aligned}(Z,\varrho)\widetilde{\cap}(L,\varrho) &= [(P,\varrho)\widetilde{\Delta}(M,\varrho)]\widetilde{\cap}[(Q,\varrho)\widetilde{\Delta}(N,\varrho)] \\ &= [(P,\varrho)\widetilde{\cap}(Q,\varrho)]\widetilde{\Delta}(S,\varrho), \end{aligned} \tag{4.2}$$

for some $(S,\varrho) \in \mathbb{N}(\mathfrak{T})$; where $(S,\varrho) = [(P,\varrho)\widetilde{\cap}(M,\varrho)^c\widetilde{\cap}(N,\varrho)]\widetilde{\cup}[(M,\varrho)\widetilde{\cap}(N,\varrho)]$; then, from Identity (2), we obtain that

$$\varphi(Z,\varrho)\widetilde{\cap}\varphi(L,\varrho) = (P,\varrho)\widetilde{\cap}(Q,\varrho) = \varphi[(P,\varrho)\widetilde{\cap}(Q,\varrho)] = \varphi[(Z,\varrho)\widetilde{\cap}(L,\varrho)].$$

If $(Z,\varrho) = (P,\varrho)\widetilde{\Delta}(M,\varrho)$, where $(P,\varrho) \in \mathbb{RO}(\mathfrak{T})$ and $(M,\varrho) \in \mathfrak{I}$, then $(Z,\varrho)\widetilde{\Delta}\ (P,\varrho) = (M,\varrho)$, but $\varphi(Z,\varrho) = (P,\varrho)$; hence, $(Z,\varrho)\widetilde{\Delta}\varphi(Z,\varrho) \in \mathbb{N}(\mathfrak{T}) = \mathfrak{I}$. Plainly, if $(Z,\varrho)\widetilde{\Delta}(L,\varrho) \in \mathfrak{I}$, then $\varphi(Z,\varrho) = \varphi(L,\varrho)$. Therefore, φ is a soft OR-operator on $(X, \Sigma, \mathfrak{I}, \varrho)$. We now show that $\mathfrak{T} = \mathfrak{T}_\varphi$. Let $(Z,\varrho) \in \mathfrak{T}_\varphi$. Then $(Z,\varrho) \in \Sigma$ and $(Z,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho)$. By our construction, $(Z,\varrho) = (P,\varrho)\widetilde{\Delta}(M,\varrho)$, where $(P,\varrho) \in \mathbb{RO}(\mathfrak{T})$ and $(M,\varrho) \in \mathfrak{I}$. This implies that $(P,\varrho)\widetilde{\Delta}(M,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho) = (P,\varrho)$. Therefore, $(M,\varrho) - (P,\varrho) = \widetilde{\varnothing}$ and $(P,\varrho) - (M,\varrho) = (Z,\varrho)$. Since each $(M,\varrho) \in \mathfrak{I}\widetilde{\cap}\mathfrak{T}^c$, then $(Z,\varrho) \in \mathfrak{T}$; hence, $\mathfrak{T}_\varphi\widetilde{\subseteq}\mathfrak{T}$. If $(Z,\varrho) \in \mathfrak{T}$; then, by [45, Lemma 18], $(Z,\varrho) = (P,\varrho) - (S,\varrho)$ for some $(P,\varrho) \in \mathbb{RO}(\mathfrak{T})$ and $(S,\varrho) \in \mathbb{N}(\mathfrak{T})\widetilde{\cap}\mathfrak{T}^c$. It follows that $(Z,\varrho) \in \Sigma$ and $(Z,\varrho)\widetilde{\subseteq}(P,\varrho) = \varphi(Z,\varrho)$, and therefore, $(Z,\varrho) \in \mathfrak{T}_\varphi$; hence, $\mathfrak{T}\widetilde{\subseteq}\mathfrak{T}_\varphi$. Thus, $\mathfrak{T} = \mathfrak{T}_\varphi$. □

Corollary 4.18. *Let $(X,\Sigma,\mathfrak{I},\varrho)$ be an MSS and let $\mathcal{T}\in\mathbb{T}(\widetilde{X})$. If $\mathcal{T}$ is a soft OR-topology over X, then $\mathfrak{I}=\mathbb{M}(\mathcal{T})$.*

Proof. Suppose $(Z,\varrho)\in\mathfrak{I}$. By Theorem 4.17, $(Z,\varrho)\in\mathbb{N}(\mathcal{T})$ and $\mathbb{N}(\mathcal{T})\widetilde{\subseteq}\mathbb{M}(\mathcal{T})$ implies $(Z,\varrho)\in\mathbb{M}(\mathcal{T})$. Thus, $\mathfrak{I}\widetilde{\subseteq}\mathbb{M}(\mathcal{T})$. If $(Z,\varrho)\in\mathbb{M}(\mathcal{T})$, then $(Z,\varrho)=\widetilde{\bigcup}_{n=1}^{\infty}(B_n,\varrho)$ for some $(B_n,\varrho)\in\mathbb{N}(\mathcal{T})$. By Theorem 4.17, $Cl_{\mathcal{T}}(B_n,\varrho)\in\mathfrak{I}$. However, $(B_n,\varrho)\widetilde{\subseteq}Cl_{\mathcal{T}}(B_n,\varrho)$ and then $\widetilde{\bigcup}_{n=1}^{\infty}(B_n,\varrho)\widetilde{\subseteq}\widetilde{\bigcup}_{n=1}^{\infty}Cl_{\mathcal{T}}(B_n,\varrho)$. Since $\mathfrak{I}$ is a soft σ-ideal, then $\widetilde{\bigcup}_{n=1}^{\infty}Cl_{\mathcal{T}}(B_n,\varrho)\in\mathfrak{I}$. Consequently, $\widetilde{\bigcup}_{n=1}^{\infty}(B_n,\varrho)\in\mathfrak{I}$, and so $\mathbb{M}(\mathcal{T})\widetilde{\subseteq}\mathfrak{I}$. Thus, $\mathfrak{I}=\mathbb{M}(\mathcal{T})$. □

Next, we find a condition under which $\mathcal{T}_{\varphi}$ forms a soft topology, after giving the following two definitions and a lemma:

Definition 4.19. *Let $(X,\Sigma,\mathfrak{I},\varrho)$ be an MSS and let $(Z,\varrho)\in\mathbb{S}(\widetilde{X})$. A soft set $(L,\varrho)\in\Sigma$ is said to be measurable kernel of (Z,ϱ) if $(L,\varrho)\widetilde{\subseteq}(Z,\varrho)$ and for each $(R,\varrho)\widetilde{\subseteq}(Z,\varrho)-(L,\varrho)$, we have $(R,\varrho)\in\mathfrak{I}$.*

Definition 4.20. *Let $(X,\Sigma,\mathfrak{I},\varrho)$ be an MSS. It is said that $(X,\Sigma,\mathfrak{I},\varrho)$ satisfies the hull property if for each $(Z,\varrho)\in\mathbb{S}(\widetilde{X})$, there exists $(L,\varrho)\in\Sigma$ with $(Z,\varrho)\widetilde{\subseteq}(L,\varrho)$ such that for each $(R,\varrho)\widetilde{\subseteq}(L,\varrho)-(Z,\varrho)$, if $(R,\varrho)\in\Sigma$, then $(R,\varrho)\in\mathfrak{I}$.*

Lemma 4.21. *Let $\mathcal{T}\in\mathbb{T}(\widetilde{X})$ and let $(Z,\varrho)\in\mathbb{S}(\widetilde{X})$ such that $(Z,\varrho)\widetilde{\subseteq}(L,\varrho)$ for some $(L,\varrho)\in\mathbb{B}_0(\mathcal{T},\mathbb{M}(\mathcal{T}))$. If there exists $(W,\varrho)\in\mathbb{B}_0(\mathcal{T},\mathbb{M}(\mathcal{T}))$ such that $(Z,\varrho)\widetilde{\subseteq}(W,\varrho)$, then $(L,\varrho)-(W,\varrho)\in\mathbb{M}(\mathcal{T})$.*

Proof. Assume that $(Z,\varrho)\in\mathbb{S}(\widetilde{X})$. Since, by Lemma 2.25, $(Z,\varrho)-C_2(Z,\varrho)\in\mathbb{M}(\mathcal{T})$, then, by Lemma 2.24, there exists a soft F_{σ}-set $(F,\varrho)\in\mathbb{M}(\mathcal{T})$ such that $(Z,\varrho)\widetilde{\subseteq}(F,\varrho)$. Evidently, $(W,\varrho)=(F,\varrho)\widetilde{\cup}C_2(Z,\varrho)\in\mathbb{B}_0(\mathcal{T},\mathbb{M}(\mathcal{T}))$ such that $(Z,\varrho)\widetilde{\subseteq}(W,\varrho)$. If $(L,\varrho)\in\mathbb{B}_0(\mathcal{T},\mathbb{M}(\mathcal{T}))$ such that $(Z,\varrho)\widetilde{\subseteq}(L,\varrho)$, then, by Lemma 2.25, $C_2(Z,\varrho)\widetilde{\subseteq}C_2(L,\varrho)$. Therefore, we have

$$\begin{aligned}(W,\varrho)-(L,\varrho)&=[(F,\varrho)\widetilde{\cup}C_2(Z,\varrho)]-(L,\varrho)\\&=[(F,\varrho)-(L,\varrho)]\widetilde{\cup}[C_2(Z,\varrho)-(L,\varrho)]\\&\widetilde{\subseteq}(F,\varrho)\widetilde{\cup}[C_2(Z,\varrho)-(L,\varrho)].\end{aligned}$$

Since $C_2(Z,\varrho)-(L,\varrho)\in\mathbb{M}(\mathcal{T})$, by Theorem 2.26 (3), and $(F,\varrho)\in\mathbb{M}(\mathcal{T})$, so $(F,\varrho)\widetilde{\cup}[C_2(Z,\varrho)-(L,\varrho)]\in\mathbb{M}(\mathcal{T})$. It follows that $(W,\varrho)-(L,\varrho)\in\mathbb{M}(\mathcal{T})$. □

Theorem 4.22. *Let φ be a soft OR-operator on an MSS $(X,\Sigma,\mathfrak{I},\varrho)$. Then $(X,\Sigma,\mathfrak{I},\varrho)$ satisfies the hull property iff $\mathcal{T}_{\varphi}$ is a soft OR-topology over X.*

Proof. Suppose $(X,\Sigma,\mathfrak{I},\varrho)$ satisfies the hull property. We have to prove that $\mathcal{T}_{\varphi}$ is a soft topology. Evidently, we have $\widetilde{\emptyset},\widetilde{X}\in\mathcal{T}_{\varphi}$. If $(Z,\varrho),(L,\varrho)\in\mathcal{T}_{\varphi}$, then $(Z,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho)$, $(L,\varrho)\widetilde{\subseteq}\varphi(L,\varrho)$. Therefore, $(Z,\varrho)\widetilde{\cap}(L,\varrho)\widetilde{\subseteq}\varphi(Z,\varrho)\widetilde{\cap}\varphi(L,\varrho)=\varphi[(Z,\varrho)\widetilde{\cap}(L,\varrho)]$ from (C_2); hence, $(Z,\varrho)\widetilde{\cap}(L,\varrho)\in\mathcal{T}_{\varphi}$. We now check that $\widetilde{\bigcup}_{\lambda\in\Lambda}(W_{\lambda},\varrho)\in\mathcal{T}_{\varphi}$ for each $\{(W_{\lambda},\varrho):\lambda\in\Lambda\}\widetilde{\subseteq}\mathcal{T}_{\varphi}$. Let (R,ϱ) be a measurable kernel of $\widetilde{\bigcup}_{\lambda\in\Lambda}(W_{\lambda},\varrho)$. Since $[(W_{\lambda},\varrho)\widetilde{\cap}(R,\varrho)]\approx(W_{\lambda},\varrho)$ for each $\lambda\in\Lambda$, then $\varphi[(W_{\lambda},\varrho)\widetilde{\cap}(R,\varrho)]=\varphi(W_{\lambda},\varrho)$. Now, we obtain that

$$(R,\varrho)\widetilde{\subseteq}\widetilde{\bigcup}_{\lambda\in\Lambda}(W_{\lambda},\varrho)\widetilde{\subseteq}\widetilde{\bigcup}_{\lambda\in\Lambda}\varphi(W_{\lambda},\varrho)=\widetilde{\bigcup}_{\lambda\in\Lambda}\varphi[(W_{\lambda},\varrho)\widetilde{\cap}(R,\varrho)]\widetilde{\subseteq}\varphi(R,\varrho). \tag{4.3}$$

Since $\varphi(R,\varrho)\approx(R,\varrho)=\varphi(R,\varrho)-(R,\varrho)$, then $\widetilde{\bigcup}_{\lambda\in\Lambda}(W_{\lambda},\varrho)-(R,\varrho)\in\mathfrak{I}$, and thus $\widetilde{\bigcup}_{\lambda\in\Lambda}(W_{\lambda},\varrho)=[\widetilde{\bigcup}_{\lambda\in\Lambda}(W_{\lambda},\varrho)-(R,\varrho)]\widetilde{\cup}(R,\varrho)\in\Sigma$, as Σ is closed under finite unions. Furthermore, by Eq (4.3) and Lemma 3.2 (5), we have $\widetilde{\bigcup}_{\lambda\in\Lambda}(W_{\lambda},\varrho)\widetilde{\subseteq}\varphi(\widetilde{\bigcup}_{\lambda\in\Lambda}(W_{\lambda},\varrho))$. Thus, $\widetilde{\bigcup}_{\lambda\in\Lambda}(W_{\lambda},\varrho)\in\mathcal{T}_{\varphi}$.

Conversely, suppose $\mathcal{T}_{\varphi}$ is a soft OR-topology over X. By Theorem 4.17 and Corollary 4.18, $\mathfrak{I}=\mathbb{M}(\mathcal{T}_{\varphi})$ and $\Sigma=\mathbb{B}_0(\mathcal{T},\mathbb{M}(\mathcal{T}_{\varphi}))$. Lemma 4.21 guarantees that $(X,\Sigma,\mathfrak{I},\varrho)$ satisfies the hull property. □

Remark 4.23. *Compare the above theorem with [27, Theorem 4.27].*

The following example presents two MSSs to demonstrate whether or not the hull property holds:

Example 4.24. *Consider the MSSs* $(\mathbb{R}, \mathbb{B}_0(\mathfrak{T}), \mathbb{M}(\mathfrak{T}), \varrho)$ *and* $(\mathbb{R}, \mathbb{B}(\mathfrak{T}), \mathfrak{I}_0, \varrho)$*, where* $\mathfrak{T}$ *is the soft topology given in Example 4.4 and* $\mathfrak{I}_0 = \widetilde{\emptyset}$*. By Definition 2.22 and Lemma 4.21, it follows that* $(\mathbb{R}, \mathbb{B}_0(\mathfrak{T}), \mathbb{M}(\mathfrak{T}), \varrho)$ *satisfies the hull property, whereas* $(\mathbb{R}, \mathbb{B}(\mathfrak{T}), \mathfrak{I}_0, \varrho)$ *does not, as shown by Theorem 4.22 and Example 4.4.*

Theorem 4.25. *Let* $\mathfrak{T}_\varphi$ *be a soft OR-topology produced by a soft OR-operator* φ *on an MSS* $(X, \Sigma, \mathfrak{I}, \varrho)$*. Then* $(Z, \varrho) \in \mathbb{LC}(\mathfrak{T}_\varphi)$ *iff* $(Z, \varrho) \in \Sigma$*.*

Proof. Assume $(Z, \varrho) \in \mathbb{LC}(\mathfrak{T}_\varphi)$. Since $\mathfrak{T}_\varphi$ is a soft topology over X, by Proposition 4.2, soft $\mathfrak{T}_\varphi$-open sets are like $(H, \varrho) - (P, \varrho)$, for some $(H, \varrho) \in \mathfrak{T}^0_\varphi$, $(P, \varrho) \in \mathbb{N}(\mathfrak{T}^0_\varphi)$. While, soft $\mathfrak{T}_\varphi$-closed sets are like $(F, \varrho)\widetilde{\cup}(Q, \varrho)$ for some $(F, \varrho)^c \in \mathfrak{T}^0_\varphi$, $(Q, \varrho) \in \mathbb{N}(\mathfrak{T}^0_\varphi)$, where $\mathfrak{T}^0_\varphi$ is a soft topology produced by $\{\varphi(Z, \varrho) : (Z, \varrho) \in \Sigma\}$. Since $\mathbb{B}_0(\mathfrak{T}_\varphi, \mathbb{N}(\mathfrak{T}_\varphi))$ is a soft σ-algebra, then $\mathbb{LC}(\mathfrak{T}^0_\varphi)\widetilde{\subseteq}\mathbb{B}_0(\mathfrak{T}_\varphi, \mathbb{N}(\mathfrak{T}_\varphi))$. However, $\mathfrak{T}^0_\varphi\widetilde{\subseteq}\mathfrak{T}_\varphi$; therefore, $\mathbb{LC}(\mathfrak{T}_\varphi)\widetilde{\subseteq}\mathbb{B}_0(\mathfrak{T}_\varphi, \mathbb{N}(\mathfrak{T}_\varphi)) = \Sigma$. Consequently, $\Sigma = \mathbb{LC}(\mathfrak{T}_\varphi)$.

Conversely, suppose that $(Z, \varrho) \in \Sigma$. Then $(Z, \varrho) = [(Z, \varrho)\widetilde{\cap}\varphi(Z, \varrho)]\widetilde{\cup}[(Z, \varrho) - \varphi(Z, \varrho)]$. By Proposition 4.7 item (1), $(Z, \varrho)\widetilde{\cap}\varphi(Z, \varrho) \in \mathfrak{T}_\varphi$. By (C_4), $(Z, \varrho) - \varphi(Z, \varrho) \in \mathfrak{I}$ and therefore, $(Z, \varrho) - \varphi(Z, \varrho) \in \mathfrak{T}^c_\varphi$. Thus, $(Z, \varrho) \in \mathbb{LC}(\mathfrak{T}_\varphi)$ and so $\Sigma\widetilde{\subseteq}\mathbb{LC}(\mathfrak{T}_\varphi)$. □

Theorem 4.26. *Let* $\mathfrak{T}_\varphi$ *be a soft OR-topology produced by a soft OR-operator* φ *on an MSS* $(X, \Sigma, \mathfrak{I}, \varrho)$*. Then* $\Sigma = \mathbb{B}(\mathfrak{T}_\varphi)$*.*

Proof. Since $\mathfrak{T}_\varphi$ is a soft topology over X, by second part of the proof of Theorems 4.17 and 4.25, $\mathbb{LC}(\mathfrak{T}_\varphi)\widetilde{\subseteq}\mathbb{B}_0(\mathfrak{T}_\varphi, \mathbb{N}(\mathfrak{T}_\varphi)) = \Sigma$. However, $\mathfrak{T}_\varphi\widetilde{\subseteq}\mathbb{LC}(\mathfrak{T}_\varphi)$, which implies that $\mathfrak{T}_\varphi\widetilde{\subseteq}\Sigma$. Since $\mathbb{B}(\mathfrak{T}_\varphi)$ is the lowest soft σ-algebra including $\mathfrak{T}_\varphi$, then $\mathbb{B}(\mathfrak{T}_\varphi)\widetilde{\subseteq}\Sigma$.

Conversely, if $(R, \varrho) \in \Sigma$, by Theorem 4.17, $(R, \varrho) \in \mathbb{B}_0(\mathfrak{T}_\varphi, \mathbb{M}(\mathfrak{T}_\varphi))$. By Theorem 2.26 (2), $(R, \varrho) = [(H, \varrho) - (Z, \varrho)]\widetilde{\cup}\,(L, \varrho)$ for some $(H, \varrho) \in \mathfrak{T}_\varphi$ and $(Z, \varrho), (L, \varrho) \in \mathbb{M}(\mathfrak{T}_\varphi)$. Now, we have

$$(R, \varrho) = [(H, \varrho)\widetilde{\cup}(L, \varrho)]\widetilde{\cap}[(Z, \varrho)^c\widetilde{\cup}(L, \varrho)].$$

Clearly, $(H, \varrho)\widetilde{\cup}(L, \varrho)$ is soft $\mathfrak{T}_\varphi$-closed. However, since $[(Z, \varrho)^c\widetilde{\cup}(L, \varrho)]^c = (Z, \varrho)\widetilde{\cap}(L, \varrho)^c \in \mathbb{M}(\mathfrak{T}_\varphi)$, then $(Z, \varrho)^c\widetilde{\cup}(L, \varrho)$ is soft $\mathfrak{T}_\varphi$-open as $[(Z, \varrho)^c\widetilde{\cup}(L, \varrho)]^c$ is soft $\mathfrak{T}_\varphi$-closed. But $\mathbb{B}(\mathfrak{T}_\varphi)$ contains all soft $\mathfrak{T}_\varphi$-open and soft $\mathfrak{T}_\varphi$-closed sets, and is closed under finite intersections, which means that $(R, \varrho) \in \mathbb{B}(\mathfrak{T}_\varphi)$. Thus, $\Sigma \widetilde{\subseteq} \mathbb{B}(\mathfrak{T}_\varphi)$. In conclusion, $\Sigma = \mathbb{B}(\mathfrak{T}_\varphi)$. □

Remark 4.27. *Compare the above result with [27, Theorem 4.28].*

Proposition 4.28. *Let* $\mathfrak{T}_\varphi$ *be a soft OR-topology produced by a soft OR-operator* φ *on an MSS* $(X, \Sigma, \mathfrak{I}, \varrho)$*. Then*

(1) $(X, \mathfrak{T}_\varphi)$ *is a soft nodec space.*

(2) $(X, \mathfrak{T}_\varphi)$ *is a soft Baire space.*

Proof. (1) Let $(Z, \varrho) \in \mathbb{N}(\mathfrak{T}_\varphi)$. By Proposition 4.13, $\mathbb{N}(\mathfrak{T}_\varphi) = \mathfrak{I}$. But each element in $\mathfrak{I}$ is $\mathfrak{T}^c_\varphi$ implies $(Z, \varrho) \in \mathfrak{T}^c_\varphi$. Thus, $(X, \mathfrak{T}_\varphi)$ is soft nodec.

(2) Let $\widetilde{\varnothing} \neq (Z, \varrho) \in \mathfrak{T}_\varphi$. Evidently, $(Z, \varrho) \notin \mathfrak{I}$. By Corollary 4.18, no $(Z, \varrho) \in \mathbb{M}(\mathfrak{T}_\varphi)$ implies $(Z, \varrho) \in \mathfrak{T}_\varphi$. Therefore, $(X, \mathfrak{T}_\varphi)$ is soft Baire.

□

Proposition 4.29. *Let $\mathfrak{T}_\varphi$ be a soft OR-topology produced by a soft OR-operator φ on an MSS $(X, \Sigma, \mathfrak{I}, \varrho)$ such that $\mathfrak{I}$ contains the whole singleton soft sets. Then*

(1) $(X, \mathfrak{T}_\varphi)$ is not soft compact, if $\mathfrak{I}$ contains an infinite soft set.

(2) $(X, \mathfrak{T}_\varphi)$ is not soft Lindelöf, if $\mathfrak{I}$ contains an uncountable soft set.

(3) $(X, \mathfrak{T}_\varphi)$ is a soft T_1-space.

(4) $(X, \mathfrak{T}_\varphi)$ is not soft separable.

(5) $(X, \mathfrak{T}_\varphi)$ is not soft first countable.

(6) $(X, \mathfrak{T}_\varphi)$ is not soft second countable.

Proof. (1) Assume $(Z, \varrho) \in \mathfrak{I}$ is infinite. For each $y_\gamma \in \mathbb{P}(\widetilde{X})$, $(Z, \varrho) - \{y_\gamma\} \in \mathfrak{I}$ and so $(Z, \varrho) - \{y_\gamma\} \in \mathfrak{T}^c_\varphi$. Therefore, $(Z, \varrho)^c \widetilde{\cup} \{y_\gamma\} \in \mathfrak{T}_\varphi$. This means $\{(Z, \varrho)^c \widetilde{\cup} \{y_\gamma\}\}_{y_\gamma \in (Z,\varrho)}$ is a soft $\mathfrak{T}_\varphi$-open cover of $\widetilde{X}$ with no finite subcover. Hence, $(X, \mathfrak{T}_\varphi)$ is not a soft compact space.

(2) Similar to (1).

(3) The proof is clear since $\{y_\gamma\} \in \mathfrak{I}$ for each $y_\gamma \in \mathbb{P}(\widetilde{X})$ and each element of $\mathfrak{I}$ is soft $\mathfrak{T}_\varphi$-closed. Therefore, $(X, \mathfrak{T}_\varphi)$ is soft T_1.

(4) Since, by Proposition 4.13, $\mathfrak{I} = \mathbb{N}(\mathfrak{T}_\varphi)$, which contains all soft sets that are countable. Consequently, each countable soft set is soft closed. Then, there does not exist a countable soft set which is also a soft $\mathfrak{T}_\varphi$-dense set in $(X, \mathfrak{T}_\varphi)$. Thus, $(X, \mathfrak{T}_\varphi)$ cannot be soft separable.

(5) Pick $y_\gamma \in \mathbb{P}(\widetilde{X})$ and let $\{(O_k, \varrho) : k = 1, 2, \cdots\}$ be a family of soft $\mathfrak{T}_\varphi$-open sets containing y_γ. For any k, let $y^k_\gamma \in (O_k, \varrho)$ with $y^k_\gamma \neq y_\gamma$. Set $(O, \varrho) = (O_1, \varrho) - \{y^k_\gamma : k = 1, 2, \cdots\}$. Then (O, ϱ) is a soft $\mathfrak{T}_\varphi$-open set including y_γ. But, (O, ϱ) does not contain an (O_k, ϱ) for each k. Therefore, $(X, \mathfrak{T}_\varphi)$ cannot be soft first countable.

(6) The proof is followed from (5), since soft second countable space implies soft first countable.

□

5. Conclusions

In this work, we introduced the concept of soft Oxtoby–Rose operators (soft OR-operators) on measurable soft spaces and studied the corresponding soft Oxtoby–Rose topologies (soft OR-topologies) they generate. We established that the proposed soft base yields a soft topology precisely when the underlying measurable soft space satisfies the hull property, and we analyzed the behavior of interior and closure under the soft OR-operator. Moreover, we showed that members of the underlying soft σ-ideal naturally act as soft closed, soft discrete, and soft nowhere dense sets within this framework. A central finding is the equivalence of Baire property soft sets, soft locally closed sets, and Borel soft sets in soft OR-topologies, offering a unified treatment of these classes.

Beyond these characterizations, we identified several distinctive topological properties of soft OR-topologies, which extend classical density-based constructions into the soft set setting. These results provide a foundation for further exploration in the development of soft topology theory. As future work, we plan to introduce a weaker variant of the lower-density soft operator, tentatively called the quasi-lower-density soft operator, and investigate the properties of the soft topologies it generates. We also aim to establish and analyze the connections between these new soft topologies and the soft OR-topologies studied in this paper.

Author contributions

Zanyar A. Ameen: Conceptualization, formal analysis, investigation, methodology, resources, validation, writing-original draft, review & editing, funding acquisition; Ohud F. Alghamdi: Formal analysis, investigation, resources, validation, review & editing, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declare no conflict of interest.

References

1. L. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
2. Z. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956
3. D. Molodtsov, Soft set theory—first results, *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
4. O. Dalkılıç, N. Demirtaş, Algorithms for covid-19 outbreak using soft set theory: Estimation and application, *Soft Comput.*, **27** (2023), 3203–3211. https://doi.org/10.1007/s00500-022-07519-5
5. X. Liu, F. Feng, Q. Wang, R. R. Yager, H. Fujita, J. C. R. Alcantud, Mining temporal association rules with temporal soft sets, *J. Math.*, 2021. https://doi.org/10.1155/2021/7303720
6. P. Maji, A. R. Roy, R. Biswas, An application of soft sets in a decision making problem, *Comput. Math. Appl.*, **44** (2002), 1077–1083. https://doi.org/10.1016/S0898-1221(02)00216-X
7. D. Pei, D. Miao, *From soft sets to information systems*, In: 2005 IEEE international conference on granular computing, **2** (2005), 617–621. https://doi.org/10.1109/GRC.2005.1547365
8. S. Yuksel, T. Dizman, G. Yildizdan, U. Sert, Application of soft sets to diagnose the prostate cancer risk, *J. Inequal. Appl.*, **2013** (2013), 1–11. https://doi.org/10.1186/1029-242X-2013-229

9. J. C. R. Alcantud, A. Z. Khameneh, G. S. García, M. Akram, A systematic literature review of soft set theory, *Neural Comput. Appl.*, **36** (2024), 8951–8975. https://doi.org/10.1007/s00521-024-09552-x
10. J. C. R. Alcantud, The semantics of N-soft sets, their applications, and a coda about three-way decision, *Inform. Sciences*, **606** (2022), 837–852. https://doi.org/10.1007/s00521-024-09552-x
11. A. A. E. Atik, R. A. Gdairi, A. A. Nasef, S. Jafari, M. Badr, Fuzzy soft sets and decision making in ideal nutrition, *Symmetry*, **15** (2023), 1523. https://doi.org/10.3390/sym15081523
12. R. Mareay, Soft rough sets based on covering and their applications, *J. Math. Ind.*, **14** (2024), 4. https://doi.org/10.1186/s13362-024-00142-z
13. J. Sun, J. Zhang, L. Liu, Y. Wu, Q. Shan, Output consensus control of multi-agent systems with switching networks and incomplete leader measurement, *IEEE T. Autom. Sci. Eng.*, **21** (2023), 6643–6652. https://doi.org/10.1109/TASE.2023.3328897
14. J. Sun, G. Wang, J. Yun, L. Liu, Q. Shan, J. Zhang, Cooperative output regulation of multi-agent systems via energy-dependent intermittent event-triggered compensator approach, *J. Franklin I.*, **362** (2025), 107868. https://doi.org/10.1016/j.jfranklin.2025.107868
15. M. Shabir, M. Naz, On soft topological spaces, *Comput. Math. Appl.*, **61** (2011), 1786–1799. https://doi.org/10.1016/j.camwa.2011.02.006
16. S. Bayramov, C. Gunduz, A new approach to separability and compactness in soft topological spaces, *TWMS J. Pure Appl. Math.*, **9** (2018), 82–93.
17. A. Aygünoğlu, H. Aygün, Some notes on soft topological spaces, *Neural Comput. Appl.*, **21** (2012), 113–119. https://doi.org/10.1007/s00521-011-0722-3
18. O. F. Alghamdi, M. H. Alqahtani, Z. A. Ameen, On soft submaximal and soft door spaces, *Contemp. Math.*, **6** (2025), 663–675. https://doi.org/10.37256/cm.6120255321
19. F. Lin, Soft connected spaces and soft paracompact spaces, *Int. J. Math. Comput. Sci.*, **7** (2013), 277–283.
20. M. Terepeta, On separating axioms and similarity of soft topological spaces, *Soft Comput.*, **23** (2019), 1049–1057. https://doi.org/10.1007/s00500-017-2824-z
21. J. C. R. Alcantud, Soft open bases and a novel construction of soft topologies from bases for topologies, *Mathematics*, **8** (2020), 672. https://doi.org/10.3390/math8050672
22. Z. A. Ameen, O. F. Alghamdi, B. A. Asaad, R. A. Mohammed, Methods of generating soft topologies and soft separation axioms, *Eur. J. Pure Appl. Math.*, **17** (2024), 1168–1182. https://doi.org/10.29020/nybg.ejpam.v17i2.5161
23. A. Kandil, O. A. E. Tantawy, S. A. E. Sheikh, A. M. A. E. latif, Soft ideal theory soft local function and generated soft topological spaces, *Appl. Math. Inform. Sci.*, **8** (2014), 1595–1603. http://dx.doi.org/10.12785/amis/080413
24. Z. A. Ameen, S. A. Ghour, Cluster soft sets and cluster soft topologies, *Comput. Appl. Math.*, **42** (2023), 337. https://doi.org/10.1007/s40314-023-02476-7
25. O. Haupt, C. Pauc, La topologie approximative de denjoy envisagée comme vraie topologie, *C. R. Math. Acad. Sci. Paris*, **234** (1952), 390–392.

26. D. Maharam, On a theorem of von neumann, *P. Am. Math. Soc.*, **9** (1958), 987–994. https://doi.org/10.1090/S0002-9939-1958-0105479-6
27. Z. A. Ameen, M. H. Alqahtani, O. F. Alghamdi, Lower density soft operators and density soft topologies, *Heliyon*, **10** (2024), e35280. https://doi.org/10.1016/j.heliyon.2024.e35280
28. F. Tall, The density topology, *Pac. J. Math.*, **62** (1976), 275–284. https://doi.org/10.2140/pjm.1976.62.275
29. W. Poreda, E. W. Bojakowska, W. Wilczynski, A category analogue of the density topology, *Fund. Math.*, **125** (1985), 167–173. https://doi.org/10.4064/fm-125-2-167-173
30. D. Rose, D. Janković, T. Hamlett, Lower density topologies a, *Ann. NY. Acad. Sci.*, **704** (1993), 309–321. https://doi.org/10.1111/j.1749-6632.1993.tb52533.x
31. K. Flak, J. Hejduk, On topologies generated by some operators, *Open Math.*, **11** (2013), 349–356. https://doi.org/10.2478/s11533-012-0077-8
32. J. Hejduk, A. Loranty, On abstract and almost-abstract density topologies, *Acta Math. Hung.*, **155** (2018), 228–240. https://doi.org/10.1007/s10474-018-0838-3
33. J. Hejduk, S. Lindner, A. Loranty, On lower density type operators and topologies generated by them, *Filomat*, **32** (2018), 4949–4957. https://doi.org/10.2298/FIL1814949H
34. M. I. Ali, F. Feng, X. Liu, W. K. Min, M. Shabir, On some new operations in soft set theory, *Comput. Math. Appl.*, **57** (2009), 1547–1553. https://doi.org/10.1016/j.camwa.2008.11.009
35. S. Das, S. Samanta, Soft metric, *Ann. Fuzzy Math. Inform.*, **6** (2013), 77–94.
36. P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, *Comput. Math. Appl.*, **45** (2003), 555–562. https://doi.org/10.1016/S0898-1221(03)00016-6
37. N. Xie, Soft points and the structure of soft topological spaces, *Ann. Fuzzy Math. Inform.*, **10** (2015), 309–322.
38. S. A. Ghour, Z. A. Ameen, Maximal soft compact and maximal soft connected topologies, *Appl. Comput. Intell. S.*, 2022. https://doi.org/10.1155/2022/9860015
39. S. Nazmul, S. Samanta, Neighbourhood properties of soft topological spaces, *Ann. Fuzzy Math. Inform.*, **6** (2013), 1–15. https://doi.org/10.1016/j.fiae.2014.06.006
40. N. Çağman, S. Karataş, S. Enginoglu, Soft topology, *Comput. Math. Appl.*, **62** (2011), 351–358. https://doi.org/10.1016/j.camwa.2011.05.016
41. S. Hussain, B. Ahmad, Some properties of soft topological spaces, *Comput. Math. Appl.*, **62** (2011), 4058–4067. https://doi.org/10.1016/j.camwa.2011.09.051
42. S. Yüksel, N. Tozlu, Z. G. Ergül, Soft regular generalized closed sets in soft topological spaces, *Int. J. Math. Anal.*, **8** (2014), 355–367. https://doi.org/10.12988/ijma.2014.4125
43. M. Riaz, Z. Fatima, Certain properties of soft metric spaces, *J. Fuzzy Math.*, **25** (2017), 543–560.
44. S. A. Ghour, Z. A. Ameen, On soft submaximal spaces, *Heliyon*, **8** (2022), e10574. https://doi.org/10.1016/j.heliyon.2022.e10574
45. Z. A. Ameen, M. H. Alqahtani, Congruence representations via soft ideals in soft topological spaces, *Axioms*, **12** (2023), 1015. https://doi.org/10.3390/axioms12111015

46. Z. A. Ameen, M. H. Alqahtani, Baire category soft sets and their symmetric local properties, *Symmetry*, **15** (2023), 1810. https://doi.org/10.3390/sym15101810

47. A. Z. Khameneh, A. Kilicman, On soft σ-algebras, *Malays. J. Math. Sci.*, **7** (2013), 17–29.

48. M. Riaz, K. Naeem, M. O. Ahmad, Novel concepts of soft sets with applications, *Ann. Fuzzy Math. Inform.*, **13** (2017), 239–251.

49. W. Rong, The countabilities of soft topological spaces, *Int. J. Math. Comput. Sci.*, **6** (2012), 952–955.

50. M. H. Alqahtani, Z. A. Ameen, Soft nodec spaces, *AIMS Math.*, **9** (2024), 3289–3302. https://doi.org/10.3934/math.2024160

51. Z. A. Ameen, O. F. Alghamdi, Soft topologies induced by almost lower density soft operators, *Eng. Let.*, **33** (2025), 712–720.

52. L. Bukovskỳ, *The structure of the real line*, Basel: Springer, **71** (2011). https://doi.org/10.1007/978-3-0348-0006-8_2

AIMS Press