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*Research article*

# An error bound estimation for the positive semi-definite tensor complementarity problem and its applications

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**Abstract:** For the positive semi-definite tensor complementarity problem (TCP), based on the natural residual function, we first established an error bound estimation for the positive semi-definite TCP without the fractional term of the residual function. Compared with the existing results, the requirements imposed on the TCP such as being an $m$-uniform $P$-function and being $m$-monotone were removed. As an application of the error bound obtained, we showed the global $R$-linear convergence rate of the proposed self-adaptive projection algorithm for solving the TCP via an equivalent transformation of this problem. Meanwhile, we also obtained an $\epsilon$-optimal solution in a finite number of iterations. Finally, numerical results were reported to demonstrate the efficiency of the proposed method.

**Keywords:** positive semi-definite TCP; error bound; self-adaptive projection algorithm; global $R$-linear convergence rate
**Mathematics Subject Classification:** 90C30, 90C33

## 1. Introduction

Let the sets $T_{m,n}$ and $R^n$ consist of all real $m$-order $n$-dimensional tensors and all real $n$-dimensional vectors, respectively. Given a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in T_{m,n}$ and $q \in R^n$, the tensor complementarity problem, denoted by TCP, is to find a vector $x \in R^n$ such that

$$x \geq 0, \quad \mathcal{A}x^{m-1} + q \geq 0, \quad x^\top(\mathcal{A}x^{m-1} + q) = 0, \tag{1.1}$$

where $\mathcal{A}x^{m-1}$ is a vector in $R^n$ with its $i$th component as

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m},$$

and it is a homogeneous polynomial of degree $m-1$. We denote by $X^*$ the solution set of this TCP, and suppose that it is nonempty.

The TCP was first introduced and used by Song and Qi [1], and was researched initially by Song and Qi [2] and Che, Qi, and Wei [3]. The TCP has growing applications [4, 5], for example, in game theory [6], nonlinear compressed sensing [7], and so on.

The TCP is a natural generalization of the linear complementarity problem (LCP). Inspired by the fundamental role played by the structured matrices in the properties of solutions of LCPs, a lot of theoretical results about the properties of the solution set of the TCP have been obtained [8]. The TCP is also a special case of the nonlinear complementarity problem (NCP), due to the multilinearity and homogeneity of the mapping $\mathcal{A}x^{m-1}$. TCPs have their own special and interesting properties that are not covered by the theory of the general NCP. More interestingly, by making use of properties of structured tensors, some error bounds obtained usually go beyond the frameworks of error bounds of the NCP. For example, Zheng et al. [9] presented some error bounds for the TCP with a $P$-tensor. Ling et al. [10] studied a Hölderian local error bound of generalized TCPs with an $m$-monotone function. Ling et al. [11] analyzed the error bounds of the polynomial complementarity problem with an $m$-uniform $P$-function. Hu et al. [12] established a local fractional-type error bound based on the natural residual function for the quadratic complementarity problem with an $R_0$-tensor.

Recently, some numerical methods for solving the TCP have been presented [13]. For instance, the homotopy continuation algorithm [14], the potential reduction algorithm [15], the nonsmooth Newton algorithm [16], the smoothing Newton method [6], the total least squares method [17], the nonlinear gradient dynamic method [18], and the Levenberg-Marquar algorithm [10]. In particular, the projection methods have been successfully developed to solve the TCP recently (see [19–21]).

Motived by these, we further consider an error bound without the fractional term for the positive semi-definite TCP based on the natural residual function, where the requirement of some abstract conditions, e.g., an $m$-uniform $P$-function [11] and being $m$-monotone [10], are removed. At the same time, based on the error bound established, we further consider a new projection method to solve the TCP in this paper.

The remainder of this paper is organized as follows. In Section 2, we recall some basic definitions of some structured tensors and some related properties, and develop an equivalent transformation of problem (1.1). In Section 3, based on the natural residual function, we establish an error bound for the positive semi-definite TCP by making some characterization of the structured tensors. As a specific application of the error bound obtained, we show the global linear convergence rate of the proposed self-adaptive projection method for the TCP in Section 4, and an approximate optimal solution is obtained in a finite number of iterations. In Section 5, some numerical experiments on the TCP are constructed to verify the effectiveness of the obtained results. Some remarks and conclusions are given in Section 6.

Some notations used in this paper are in order. We denote the index set $\{1, 2, \cdots, n\}$ as $[n]$. For vector $x \in R^n$ and number $r$, we use $x^{[r]}$ to denote column vector $(x_1^r, x_2^r, \cdots, x_n^r)^\top$. For convenience, in some places, we also use $\langle x, y\rangle$ to represent the standard Euclidean inner product of the two column vectors $x$ and $y$ in $R^n$. We denote by $R^n_+$ the nonnegative quadrant in $R^n$, and by $x_+$ the orthogonal projection of vector $x \in R^n$ onto $R^n_+$, i.e., $(x_i)_+ := \max\{x_i, 0\}$, $i \in [n]$. We denote by $\|\cdot\|$ the Euclidean 2-norm. We denote by $det(M)$ the determinant of the square matrix $M$ and, by $I_n$, the identity matrix of order $n$. For any subset $S \subseteq [m]$, we denote by $A_S$ the submatrix of $A$ obtained by removing all rows $i \notin S$ of $A$. Analogously, for any vector $x \in R^m$ and any subset $S \subseteq [m]$, we denote by $x_S$ the vector with components $x_i$, $i \in S$ (with the $x_i$'s arranged in the same order as in $x$).

## 2. Preliminaries

In this section, we give some definitions and some related properties [22], and present an equivalent formulation of problem (1.1).

**Definition 2.1.** *Let $\mathcal{A} \in T_{m,n}$ be positive semidefinite (definite) if for any $x \in R^n$ and $x \neq 0$, one has*

$$x^\top \mathcal{A} x^{m-1} = \sum_{i_1,i_2,\cdots,i_m=1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \geq 0 (> 0).$$

**Definition 2.2.** *Let $\mathcal{A} \in T_{m,n}$ be symmetric if and only if $a_{i_1 i_2 \cdots i_m}$ is invariant by any permutation $\pi$, that is, $a_{i_1 i_2 \cdots i_m} = a_{\pi(i_1 i_2 \cdots i_m)}$ where all $i_k \in [n]$ with $k \in [m]$. Denote by $ST_{m,n}$ the set of all m-order n-dimensional symmetric tensors.*

In the following, the definition of the diagonalizable tensors [23, 24] is given as follows.

**Definition 2.3.** *Let $\mathcal{A} \in ST_{m,n}$ be diagonalizable if and only if $\mathcal{A}$ can be represented as*

$$\{\mathcal{A} \in ST_{m,n} \mid \mathcal{A} = \mathcal{D} \times_1 B \times_2 B \cdots \times_m B\},$$

*where $B \in R^{n \times n}$ with $det(B) \neq 0$ and $\mathcal{D}$ is a diagonal tensor. Denote by $D_{m,n}$ all m-order n-dimensional diagonalizable tensors.*

*Obviously, $D_{m,n} \subseteq ST_{m,n}$. When $\mathcal{A} \in D_{m,n}$ is positive semi-definite, and $F(x) \triangleq \mathcal{A} x^{m-1} + q$, the following theorem gives a property of the Jacobian matrix $\nabla F(x)$ at the $x(x \neq 0)$ point (Theorem 4.4 [3]).*

**Theorem 2.1.** *Assume that $\mathcal{A} \in D_{m,n}$ is positive semi-definite. Then, the Jacobian matrix $\nabla F(x)$ is positive semi-definite with nonzero vectors $x \in R^n$.*

To establish an equivalent formulation of (1.1), we give the following assumption.

**Assumption 2.1.** *Assume that tensor $\mathcal{A} \in D_{m,n}$ is positive semi-definite.*

*To proceed, we establish an optimization problem as follows:*

$$\begin{aligned} \min \quad & g(x) \triangleq \frac{1}{m} x^\top \mathcal{A} x^{m-1} + x^\top q, \\ \text{s.t.} \quad & x \in R^n_+. \end{aligned} \tag{2.1}$$

*By Theorem 2.1, we know that $g(x)$ is convex, and the feasible set is a polyhedron. Then problem (2.1) is a standard convex optimization. Thus the stationary set of (2.1) coincides with the solution set of problem (1.1).*

## 3. Error bound for the positive semi-definite TCP

In this section, using properties of structured tensors, we establish an error bound estimation for the positive semi-definite TCP via the natural residual function. To this end, we recall a useful definition and some properties, which are the basis of our analysis.

**Proposition 3.1.** *[25] Assume that $K$ is a nonempty closed convex subset of $R^n$, and for given $z \in R^n$, set $P_K(z) = \arg\min\{\|z - y\| \mid y \in K\}$. Then, one has*

$$\|P_K(z_1) - P_K(z_2)\| \leq \|z_1 - z_2\|, \ \forall z_1, z_2 \in R^n.$$

*For (1.1) and $x \in R^n$, we define the following natural residue function:*

$$r(x, \rho) = \|\min\{x, \rho(\mathcal{A}x^{m-1} + q)\}\| = \|x - P_{R^n_+}\{x - \rho(\mathcal{A}x^{m-1} + q)\}\|, \tag{3.1}$$

*where $\rho > 0$ is some constant. The natural residue function is intimately related to the solution of (1.1) as given by the following result (Lemma 3.1 [26]).*

**Proposition 3.2.** *The vector $x^*$ is a solution of (1.1) if and only if $r(x^*, \rho) = 0$ with some $\rho > 0$.*

**Proposition 3.3.** *(Lemma 2.5.1 [27]) Assume that $\Omega$ is a nonempty closed convex subset of $R^n$, and $f : \Omega \to R^n$ is differentiable. Then, for any $x, y \in \Omega$ and each vector $\beta \in R^n$, there exist vector $\xi = x + t(y - x) \in \Omega$ and constant $0 < t < 1$ such that*

$$\beta^\top [f(x) - f(y)] = \beta^\top \nabla f(\xi)(y - x). \tag{3.2}$$

*Take $\beta = f(x) - f(y)$ in (3.2), and it is not difficult to see that*

$$\|f(x) - f(y)\| \leq \|\nabla f(\xi)(y - x)\|. \tag{3.3}$$

Based on Propositions 3.1–3.2 and (3.3), we can establish the following error bound for the TCP.

**Theorem 3.1.** *Suppose Assumption 2.1 holds. Then, there exists a positive constant $\tau$ such that*

$$dist(x, X^*) \leq \tau r(x, \rho), \quad \forall x \in \{x \in R^n \mid \|x\| \leq c_1\}, \tag{3.4}$$

*where $dist(x, X^*)$ denotes the distance from point $x$ to the solution set $X^*$, and $c_1$ and $\rho$ are both positive constants.*

*Proof.* Suppose that the theorem is false. Then, there exist the sequence $\{x^k\}$ and the positive sequence $\{\tau_k\}$ such that $\|x^k\| \leq c_1$, $\tau_k \to \infty$ as $k \to \infty$ and

$$dist(x^k, X^*) > \tau_k r(x^k, \rho). \tag{3.5}$$

Hence,

$$\frac{r(x^k, \rho)}{dist(x^k, X^*)} < \frac{1}{\tau_k} \to 0 \ \text{ as } k \to \infty. \tag{3.6}$$

Since $\{x^k\}$ is bounded and $r(x, \rho)$ is continuous, by (3.6), we have $r(x^k, \rho) \to 0$ as $k \to \infty$. Since $\{x^k\}$ is bounded again, there exists a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $\lim_{k_i \to \infty} x^{k_i} = \bar{x}$ with $r(\bar{x}, \rho) = 0$. Hence $\bar{x} \in X^*$.

On the other hand, we first defined vector function

$$f(x) = x - \rho\mathcal{A}x^{m-1}.$$

Using Proposition 3.3 and (3.3) for vector function $f(x)$ with $x^{k_i}$ and $\bar{x}$, there exist constant $0 < t_{k_i} < 1$ and vector $\xi_{k_i} = \bar{x} + t_{k_i}(x^{k_i} - \bar{x})$ such that

$$\|f(x^{k_i}) - f(\bar{x})\| \leq \|\nabla f(\xi_{k_i})(x^{k_i} - \bar{x})\| = \|(I - \rho\nabla F(\xi_{k_i}))(x^{k_i} - \bar{x})\|. \tag{3.7}$$

Since $\mathcal{A} \in D_{m,n}$ is positive semi-definite, by Theorem 2.1, then the matrix $\nabla F(\bar{x})$ is positive semi-definite, Therefore, one has $[\nabla F(\bar{x})]_{ii} \geq 0$ for any $i \in [n]$. In the following, we define two index sets

$$S = \{i \in [n] \mid [\nabla F(\bar{x})]_{ii} = 0\}, \quad \bar{S} = \{i \in [n] \mid [\nabla F(\bar{x})]_{ii} > 0\}.$$

For any $i \in S$, i.e., $[\nabla F(\bar{x})]_{ii} = 0$, together with Theorem 2.1, it follows that $[\nabla F(\bar{x})]_S = 0$.

For any $i \in \bar{S}$, i.e., $[\nabla F(\bar{x})]_{ii} > 0$, it follows that there exists $\rho > 0$ such that

$$\|I_{\bar{S}} - \rho[\nabla F(\bar{x})]_{\bar{S}}\| < 1. \tag{3.8}$$

Based on the discussion above, combining $x^{k_i} \to \bar{x}$ with $r(x^{k_i}, \rho) \to 0$, and letting $y^{k_i} = x^{k_i} - \bar{x}$ for simplicity, one has

$$\begin{aligned}
&\lim_{\|y_{\bar{S}}^{k_i}\|\to 0}\lim_{\|y_S^{k_i}\|\to 0} \frac{r(x^{k_i},\rho)}{\|y^{k_i}\|} \\
&= \lim_{\|y_{\bar{S}}^{k_i}\|\to 0}\lim_{\|y_S^{k_i}\|\to 0} \frac{\|x^{k_i} - P_{R_+^n}(x^{k_i} - \rho(\mathcal{A}(x^{k_i})^{m-1} + q)\|}{\|y^{k_i}\|} \\
&= \lim_{\|y_{\bar{S}}^{k_i}\|\to 0}\lim_{\|y_S^{k_i}\|\to 0} \frac{\|[x^{k_i} - P_{R_+^n}(x^{k_i} - \rho(\mathcal{A}(x^{k_i})^{m-1} + q) - [\bar{x} - P_{R_+^n}(\bar{x} - \rho(\mathcal{A}\bar{x}^{m-1} + q))]\|}{\|y^{k_i}\|} \\
&= \lim_{\|y_{\bar{S}}^{k_i}\|\to 0}\lim_{\|y_S^{k_i}\|\to 0} \frac{\|y^{k_i} - [P_{R_+^n}(x^{k_i} - \rho(\mathcal{A}(x^{k_i})^{m-1} + q)) - P_{R_+^n}(\bar{x} - \rho(\mathcal{A}\bar{x}^{m-1} + q))]\|}{\|y^{k_i}\|} \\
&\geq \lim_{\|y_{\bar{S}}^{k_i}\|\to 0}\lim_{\|y_S^{k_i}\|\to 0} \frac{\|y^{k_i}\| - \|P_{R_+^n}(x^{k_i} - \rho(\mathcal{A}(x^{k_i})^{m-1} + q)) - P_{R_+^n}(\bar{x} - \rho(\mathcal{A}\bar{x}^{m-1} + q))\|}{\|y^{k_i}\|} \\
&\geq \lim_{\|y_{\bar{S}}^{k_i}\|\to 0}\lim_{\|y_S^{k_i}\|\to 0} \frac{\|y^{k_i}\| - \|(x^{k_i} - \rho(\mathcal{A}(x^{k_i})^{m-1} + q)) - (\bar{x} - \rho(\mathcal{A}\bar{x}^{m-1} + q))\|}{\|y^{k_i}\|} \\
&= \lim_{\|y_{\bar{S}}^{k_i}\|\to 0}\lim_{\|y_S^{k_i}\|\to 0} \frac{\|y^{k_i}\| - \|(x^{k_i} - \rho\mathcal{A}(x^{k_i})^{m-1}) - (\bar{x} - \mathcal{A}\bar{x}^{m-1})\|}{\|y^{k_i}\|} \\
&\geq \lim_{\|y_{\bar{S}}^{k_i}\|\to 0}\lim_{\|y_S^{k_i}\|\to 0} \frac{\|y^{k_i}\| - \|(I - \rho\nabla F(\xi_{k_i}))y^{k_i}\|}{\|y^{k_i}\|} \\
&= \lim_{\|y_{\bar{S}}^{k_i}\|\to 0}\lim_{\|y_S^{k_i}\|\to 0} \Big[\frac{\|y^{k_i}\| - [\|(I - \rho\nabla F(\xi_{k_i}))_S y^{k_i}\|^2 + \|(I - \rho\nabla F(\xi_{k_i}))_{\bar{S}} y^{k_i}\|^2]^{\frac{1}{2}}}{\|y^{k_i}\|}\Big] \\
&\geq \lim_{\|y_{\bar{S}}^{k_i}\|\to 0}\lim_{\|y_S^{k_i}\|\to 0} \Big[\frac{\|y^{k_i}\| - [\|(I - \rho\nabla F(\xi_{k_i}))_S y^{k_i}\|^2 + \|(I - \rho\nabla F(\xi_{k_i}))_{\bar{S}}\|^2\|y^{k_i}\|^2]^{\frac{1}{2}}}{\|y^{k_i}\|}\Big] \\
&\geq 1 - \|I_{\bar{S}} - \rho[\nabla F(\bar{x})]_{\bar{S}}\| > 0,
\end{aligned} \tag{3.9}$$

where the second inequality follows from Proposition 3.1, the third inequality by (3.7), the fourth inequality by the Cauchy-Schwarz inequality, the fifth inequality by the fact that $\xi_{k_i} \to \bar{x}$ as $k_i \to \infty$ and $[\nabla F(\bar{x})]_S = 0$, and the last inequality by (3.8).

Furthermore, by (3.6), we further obtain

$$\lim_{\|y_{\bar{S}}^{k_i}\|\to 0} \lim_{\|y_S^{k_i}\|\to 0} \frac{r(x^{k_i},\rho)}{\|y^{k_i}\|} \leq \lim_{k_i\to\infty} \frac{r(x^{k_i})}{dist(x^{k_i},X^*)} = 0. \tag{3.10}$$

This contradicts (3.9). Thus, (3.4) holds. □

The TCP is a subclass of generalized tensor complementarity problems (GTCP) [10] and polynomial complementarity problems (PCP) [11]. When the GTCP or PCP reduce to the TCP, from the conclusion above, an error bound for the TCP is established via the natural residual function. The requirement of some abstract conditions, e.g., being $m$-monotone [10] and an $m$-uniform $P$-function [11], are both removed.

In addition, we first consider the GTCP [10] of finding $x \in R^n$ such that

$$x \in \mathcal{K}, \quad \mathcal{A}x^{m-1} + q \in \mathcal{K}^*, \quad x^\top(\mathcal{A}x^{m-1} + q) = 0,$$

where $\mathcal{K} = \{v \in R^n \mid Bv \geq 0,\ Cv = 0\}$ is a polyhedral cone in $R^m$ for some matrices $B \in R^{s\times n}, C \in R^{t\times n}$, and $\mathcal{K}^* = \{u \in R^n \mid u = B^\top\lambda_1 + C^\top\lambda_2,\ \lambda_1 \in R^s_+,\ \lambda_2 \in R^t\}$ is its dual cone. By the discussion in Section 1 of [26] for the GTCP, we conclude that the GTCP can be equivalently transformed into the following variational inequality problem of finding $x^* \in R^n$ such that

$$(x - x^*)^\top(\mathcal{A}(x^*)^{m-1} + q) \geq 0, \quad \forall\ x \in \mathcal{K}.$$

The problem can further be formulated as the following fixed-point problem [26] via a projection operator:

$$x - P_{\mathcal{K}}[x - \rho(\mathcal{A}x^{m-1} + q)] = 0,$$

where $\rho > 0$ is a constant. Substituting $r(x^{k_i},\rho) = x^{k_i} - P_{R^n_+}(x^{k_i} - \rho(\mathcal{A}(x^{k_i})^{m-1} + q)$ in (3.9) with $r(x^{k_i},\rho) = x^{k_i} - P_{\mathcal{K}}(x^{k_i} - \rho(\mathcal{A}(x^{k_i})^{m-1} + q)$, it is straightforward to verify that (3.9) holds. Thus, the proof uses a similar technique to that of Theorem 3.1 above. Under the assumption of Theorem 3.1, we conclude that there exists a positive constant $\tau_1$ such that

$$dist(x, X^*) \leq \tau_1\|x - P_{\mathcal{K}}[x - \rho(\mathcal{A}x^{m-1} + q)]\|.$$

In what follows, we consider the PCP of finding $x \in R^n$ such that

$$x \geq 0, \quad F(x) \triangleq \sum_{k=1}^{m/2} \mathcal{A}_k x^{m-(2k-1)} + q \geq 0, \quad x^\top F(x) = 0,$$

where $\Lambda \triangleq (\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_{m/2}) \in T_{m,n} \times T_{m-2,n} \times \cdots \times T_{2,n}$, $m$ is even, and $\mathcal{A}_{m/2}$ is a square matrix. Suppose that $\mathcal{A}_k (k \in [m/2])$ satisfies the hypotheses of Theorem 3.1, and let $g_k(x) = \mathcal{A}_k x^{m-(2k-1)}$. It follows from Theorem 2.1 that the Jacobian matrix $\nabla g_k(x) = (m - (2k - 1))\mathcal{A}_k x^{m-2k}$ is positive semi-definite with nonzero vectors $x$. Therefore, we can further obtain that the Jacobian matrix

$$\nabla F(x) = (m - 1)\mathcal{A}_1 x^{m-2} + (m - 3)\mathcal{A}_2 x^{m-4} + \cdots + \mathcal{A}_{m/2}$$

is also positive semi-definite. This, together with letting $f(x) = x - \rho(\sum_{k=1}^{m/2} \mathcal{A}_k x^{m-(2k-1)} + q)$ in the proof of Theorem 3.1, and using use a similar argument to that of Theorem 3.1 above, we obtain that there exists a positive constant $\tau_2$ such that

$$dist(x, X^*) \leq \tau_2 \| \min\{x, \rho(\sum_{k=1}^{m/2} \mathcal{A}_k x^{m-(2k-1)} + q)\}\|.$$

At the end of this section, a more explicit comparison with existing bounds and the corresponding required assumptions are summarized in Table 1, and we also summarize key differences.

$$s_1(x) = \| \min\{x, [\mathcal{A}(x-\hat{x})^{m-1}]^{[\frac{1}{m-1}]} + (\mathcal{A}\hat{x}^{m-1} + q)^{[\frac{1}{m-1}]}\}\|_\infty,$$

where $x \in R^n$, and $\hat{x} \in X^*$ satisfies $dist(x, X^*) = \|x - \hat{x}\|_\infty$.

$$s_2(x) = \hat{c}(x)[\|C(x)\| + \|V(\mathcal{A}x^{m-1} + q)\| + \| \min\{Bx, U(\mathcal{A}x^{m-1} + q)\}\|]^{\frac{1}{m-1}},$$

where

$$\hat{c}(x) = \left\{\frac{1}{\rho} \max\left\{1, \|G\| c(\mathcal{A}) \frac{\|x\|^{m-1} - \|x^*\|^{m-1}}{\|x\| - \|x^*\|}, s(\|B\| + \|U\| c(\mathcal{A}) \frac{\|x\|^{m-1} - \|x^*\|^{m-1}}{\|x\| - \|x^*\|})\right\}\right\}^{\frac{1}{m-1}}, x^* \in X^*.$$

$$s_3(x) = \| \min\{x, \sum_{k=1}^{m-1} \mathcal{A}_k x^{m-k} + q\}\|_\infty^{\frac{1}{m-1}} \left(1 + \frac{\tilde{c}}{\|x^*\| - \|x\|} \left(\sum_{k=1}^{m-1} \|x^*\|^{m-k} - \sum_{k=1}^{m-1} \|x\|^{m-k}\right)\right)^{\frac{1}{m-1}},$$

where $x^*$ is the unique solution of the PCP.

$$s_4(x) = \|x - P_{R_+^n}[x - (\mathcal{A}x^{m-1} + q)]\|^{\max\{\frac{1}{3 \cdot 6^{5n-1}}, \frac{1}{2 \cdot 9^{3n-1}}\}}.$$

$$r(x, \rho) = \| \min\{x, \rho \mathcal{A}x^{m-1} + q\}\|.$$

It can be easily seen from Table 1 that our established error bound estimation via the natural residual function has no fractional term, which is a key difference compared with other existing error bounds. Thus, this is a new result for the TCP.

**Table 1.** Error bounds for the TCP (PCP or GTCP).

| Assumptions | Error bounds | Problems | References |
|---|---|---|---|
| $P$-tensor | $s_1(x)$ | TCP | Theorem 3.2 [9] |
| $m$-monotone | $s_2(x)$ | GTCP | Theorem 4.1 [10] |
| $m$-uniform $P$-function | $s_3(x)$ | PCP | Theorem 5.4 [11] |
| $R_0$-tensor | $s_4(x)$ | Quadratic CP | Theorem 3.1 [12] |
| Assumption 2.1 | $r(x, \rho)$ | TCP | Theorem 3.1 in this paper |

## 4. Algorithm and linear convergence

In this section, we develop a new method for solving model (1.1), and establish its global $R$-linear convergence rate based on the obtained error bound in Section 3.

Now, our algorithm is stated formally as follows.

**Algorithm 4.1.**

**Step 0.** Select $\beta > 0, \eta > 1, \epsilon \geq 0, \gamma \in (0,1), x^0 \in R^n$, and set $k := 0$.

**Step 1.** For the current iterate points $x^{k-1}$, compute

$$x^k := \arg\min_{x \in R^n_+} \left\{ \|x - (x^{k-1} - \frac{1}{L_k} \nabla g(x^{k-1}))\|^2 \right\}, \tag{4.1}$$

where $\nabla g(x^{k-1}) = \mathcal{A}(x^{k-1})^{m-1} + q$, $L_k = \eta^{m_k}\beta$, and $m_k$ is the smallest nonnegative integer $m$ satisfying

$$g(x^k) \leq g(x^{k-1}) - \gamma \langle x^k - x^{k-1}, \nabla g(x^{k-1}) \rangle, \tag{4.2}$$

and

$$0 \leq (1+\gamma)\langle x^k - x^{k-1}, \nabla g(x^{k-1}) \rangle + \frac{\eta^m \beta}{2} \|x^k - x^{k-1}\|^2. \tag{4.3}$$

**Step 2.** If $\|x^k - x^{k-1}\| \leq \epsilon$, stop. Then, $x^k$ is an approximate solution of (1.1). Otherwise, go to Step1 with $k := k+1$.

For $g(x)$ given by (2.1), we let

$$Q_{L_k}(x, x^{k-1}) \triangleq g(x^{k-1}) + \langle x - x^{k-1}, \nabla g(x^{k-1}) \rangle + \frac{L_k}{2} \|x - x^{k-1}\|^2. \tag{4.4}$$

This, together with (4.1), makes it easy to calculate that

$$x^k = \arg\min_{x \in R^n_+} \{ Q_{L_k}(x, x^{k-1}) \}. \tag{4.5}$$

By Taylor's expansion of $g(x)$ at the $x^{k-1}$ point, there exist constant $0 < t < 1$ and vector $\xi = (1-t)x^k + tx^{k-1}$ such that

$$g(x) = g(x^{k-1}) + \langle x - x^{k-1}, \nabla g(x^{k-1}) \rangle + \frac{m-1}{2} (x - x^{k-1})^\top \mathcal{A}\xi^{m-2} (x - x^{k-1}). \tag{4.6}$$

If $L_k \geq (m-1)\|\mathcal{A}\xi^{m-2}\|$, using (4.4), (4.5), and (4.6), we obtain

$$\begin{aligned} g(x^k) &\leq g(x^{k-1}) + \langle x^k - x^{k-1}, \nabla g(x^{k-1}) \rangle + \tfrac{m-1}{2} \|\mathcal{A}\xi^{m-2}\| \|x^k - x^{k-1}\|^2 \\ &\leq Q_{L_k}(x^k, x^{k-1}) \leq Q_{L_k}(x^{k-1}, x^{k-1}) = g(x^{k-1}). \end{aligned} \tag{4.7}$$

Thus, the sequence $\{g(x^k)\}$ is monotonically decreasing.

**Lemma 4.1.** *For any $k \geq 1$, one has*

$$g(x) - g(x^k) \geq \frac{L_k}{2} \|x^k - x^{k-1}\|^2 + L_k \langle x - x^{k-1}, x^{k-1} - x^k \rangle, \quad \forall x \in R^n_+. \tag{4.8}$$

*Proof.* By (4.1), for any $x \in R^n_+$, we get

$$\langle x - x^k, \nabla g(x^{k-1}) + L_k(x^k - x^{k-1})\rangle \geq 0. \tag{4.9}$$

Since $g$ is convex, then one has $g(x) \geq g(x^{k-1}) + \langle x - x^{k-1}, \nabla g(x^{k-1})\rangle$. This, together with (4.9), gives us

$$\begin{aligned} g(x) - g(x^k) &\geq g(x) - Q_{L_k}(x^k, x^{k-1}) \\ &= g(x) - g(x^{k-1}) - \langle x^k - x^{k-1}, \nabla g(x^{k-1})\rangle - \tfrac{L_k}{2}\|x^k - x^{k-1}\|^2 \\ &\geq \langle x - x^{k-1}, \nabla g(x^{k-1})\rangle - \langle x^k - x^{k-1}, \nabla g(x^{k-1})\rangle - \tfrac{L_k}{2}\|x^k - x^{k-1}\|^2 \\ &= \langle x - x^k, \nabla g(x^{k-1})\rangle - \tfrac{L_k}{2}\|x^k - x^{k-1}\|^2 \\ &\geq L_k\langle x - x^k, x^{k-1} - x^k\rangle - \tfrac{L_k}{2}\|x^k - x^{k-1}\|^2 \\ &= L_k\langle (x - x^{k-1}) + (x^{k-1} - x^k), x^{k-1} - x^k\rangle - \tfrac{L_k}{2}\|x^k - x^{k-1}\|^2 \\ &= L_k\langle x - x^{k-1}, x^{k-1} - x^k\rangle + \tfrac{L_k}{2}\|x^k - x^{k-1}\|^2, \end{aligned}$$

where the first inequality follows from the fact that $g(x^k) \leq Q_{L_k}(x^k, x^{k-1})$ by (4.7), and the first equality follows from (4.4) with $x = x^k$. □

In the following convergence analysis, suppose that Algorithm 4.1 generates an infinite sequence, and the global convergence of our proposed algorithm is given.

**Theorem 4.1.** *Under Assumption 2.1, $g(x)$ given by (2.1) is bounded below on $R^n_+$. Then the sequence $\{x^k\}$ generated by Algorithm 4.1 converges globally to a solution of (1.1).*

*Proof.* By Lemma 4.1 with $x = x^{k-1}$, one has

$$2L_k^{-1}(g(x^{k-1}) - g(x^k)) \geq \|x^k - x^{k-1}\|^2.$$

Combining $L_k = \eta^{m_k}\beta$ with $\eta > 1$ yields $L_k \geq \beta$ ($\forall k \geq 1$). This, together with $g(x^{k-1}) - g(x^k) \geq 0$, gives us that

$$2\beta^{-1}(g(x^{k-1}) - g(x^k)) \geq \|x^k - x^{k-1}\|^2. \tag{4.10}$$

By (4.7), we know that sequence $\{g(x^k)\}$ is monotonically decreasing, and $g(x^k)$ is bounded from below, so it converges. It follows from (4.10) that

$$\lim_{k\to\infty} \|x^k - x^{k-1}\| = 0. \tag{4.11}$$

By (4.8) with $x = x^* \in X^*$ and a direct computation yield that

$$\begin{aligned} 0 &\geq \tfrac{2}{L_k}(g(x^*) - g(x^k)) \\ &\geq \|x^k - x^{k-1}\|^2 + 2\langle x^* - x^{k-1}, x^{k-1} - x^k\rangle \\ &= \|x^k - x^{k-1}\|^2 + \langle (x^* - x^k) + (x^k - x^{k-1}), x^{k-1} - x^k\rangle \\ &\quad + \langle x^* - x^{k-1}, (x^{k-1} - x^*) + (x^* - x^k)\rangle \\ &= \langle x^* - x^k, x^{k-1} - x^k\rangle \\ &\quad + \langle x^* - x^{k-1}, x^{k-1} - x^*\rangle + \langle x^* - x^{k-1}, x^* - x^k\rangle \\ &= \langle x^* - x^k, x^{k-1} - x^* + x^* - x^k\rangle \\ &\quad + \langle x^* - x^{k-1}, x^{k-1} - x^*\rangle + \langle x^* - x^{k-1}, x^* - x^k\rangle \\ &= \|x^* - x^k\|^2 - \|x^* - x^{k-1}\|^2. \end{aligned} \tag{4.12}$$

From (4.12), we obtain that the nonnegative sequence $\{\|x^k - x^*\|\}$ is monotonically decreasing, so it converges. Thus, the sequence $\{x^k\}$ is bounded, and there exits a convergent subsequence $\{x^{k_i}\}$ of $\{x^k\}$, where we assume that $\lim_{k\to\infty} x^{k_i} = \bar{x}$. This, together with (4.11), gives us

$$\lim_{k\to\infty} \|x^{k_i-1} - \bar{x}\| \le \lim_{k\to\infty} \|x^{k_i} - x^{k_i-1}\| + \lim_{k\to\infty} \|x^{k_i} - \bar{x}\| = 0. \tag{4.13}$$

By (4.1), one has $\langle x - x^{k_i}, \nabla g(x^{k_i-1}) + L_{k_i}(x^{k_i} - x^{k_i-1})\rangle \ge 0, \forall x \in R^n_+$. Combining this with (4.11) and (4.13) gives $\langle x - \bar{x}, \nabla g(\bar{x})\rangle \ge 0,\ \forall x \in R^n_+$, i.e., $\bar{x}$ is a solution of (1.1). So, the $\bar{x}$ can be used as $x^*$ in the discussion above, and we get that sequence $\{\|x^k - \bar{x}\|\}$ is also convergent. This, together with $\lim_{k\to\infty} \|x^{k_i} - \bar{x}\| = 0$, gives us that $\lim_{k\to\infty} \|x^k - \bar{x}\| = 0$. Thus, the desired result follows. □

Next, we prove that there are upper and lower bounds of the parameter $L_k$, and the iteration stepsize $\frac{1}{L_k}$ can be self-adaptively updated. To this end, we first recall that an operator $T_{\mathcal{A}} : R^n \to R^n$ is defined by, for any $x \in R^n$,

$$T_{\mathcal{A}}(x) \triangleq \begin{cases} \|x\|^{2-m}\mathcal{A}x^{m-1}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and

$$\|T_{\mathcal{A}}\| \triangleq \max_{\|x\|=1} \|T_{\mathcal{A}}(x)\|.$$

The following properities of this operator norm were established by Song and Qi [28, 29].

**Lemma 4.2.** *(Theorem 4.3 [28], Lemmas 2.3, 2.4 [29]) Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$*. Then*

(*i*) $\|T_{\mathcal{A}}(x)\| \le \|T_{\mathcal{A}}\|\|x\|$;

(*ii*) $\|T_{\mathcal{A}}\| \le \max_{i\in[n]} n^{\frac{m-2}{2}} \left(\sum_{i=1}^n \left(\sum_{i_2,\cdots,i_m=1}^n |a_{ii_2\cdots i_m}|\right)^2\right)^{\frac{1}{2}}$.

**Lemma 4.3.** *For $L_k$ defined in Algorithm 4.1, we have $\beta \le L_k < \eta M_0, \forall k \ge 1$, where $M_0 > 0$, $\beta > 0$, and $\eta > 1$ are constants.*

*Proof.* By $L_k = \eta^{m_k}\beta$ and $\eta > 1$, we obtain $L_k \ge \beta(\forall k \ge 1)$. In addition, by Theorem 4.1, we have that the sequence $\{x^k\}$ is bounded, i.e., there exists a constant $c_0 > 0$ such that $\|x^k\| \le c_0$. Combining this with (4.6) gives $\|\xi\| \le (1-t)\|x^k\| + t\|x^{k-1}\| = c_0$. It follows from the definition of $T_{\mathcal{A}}(x)$ that

$$(m-1)\|\mathcal{A}\xi^{m-2}\| = (m-1)\|\xi\|^{m-2}\|T_{\mathcal{A}}(\xi)\| \le (m-1)\|\xi\|^{m-1}\|T_{\mathcal{A}}\| \le (m-1)c_0^{m-1}\|T_{\mathcal{A}}\| \triangleq M_0, \tag{4.14}$$

where the first inequality is by Lemma 4.2 (*i*). Combining (4.14) with (4.6) gives

$$g(x^k) \le g(x^{k-1}) + \langle x^k - x^{k-1}, \nabla g(x^{k-1})\rangle + \frac{M_0}{2}\|x^k - x^{k-1}\|^2. \tag{4.15}$$

Since inequalities (4.2) and (4.3) are satisfied for some $\eta^m\beta \ge M_0$, this, together with the definition of $m_k$, we ensures that $L_k/\eta = \eta^{m_k-1}\beta$ must violate it, i.e., $L_k/\eta < M_0$ for every $k \ge 1$, and the desired result follows. □

**Lemma 4.4.** *Suppose that Assumption 2.1 holds, and the sequence $\{x^k\}$ is generated by Algorithm 4.1. Then, for any $k \ge 1$, we have*

$$g(x^k) - g(x^*) \le \frac{\eta M_0}{2k}\|x^* - x^0\|^2, \tag{4.16}$$

*where $M_0$ is defined in (4.14), $x^* \in X^*$.*

*Proof.* In order to make the description more concise, in (4.12), we replace $x^k$ with $x^t$, and we have

$$\frac{2}{L_t}(g(x^*) - g(x^t)) \geq \|x^* - x^t\|^2 - \|x^* - x^{t-1}\|^2. \tag{4.17}$$

Since $x^* \in X^*$, i.e., $x^*$ is a solution of (2.1), it follows that $g(x^t) - g(x^*) \geq 0$. Combining this with Lemma 4.3, we obtain

$$\frac{2}{\eta M_0}(g(x^*) - g(x^t)) \geq \frac{2}{L_t}(g(x^*) - g(x^t)) \geq \|x^* - x^t\|^2 - \|x^* - x^{t-1}\|. \tag{4.18}$$

Summing (4.18) over $t = 1, 2, \cdots, k$ gives

$$\frac{2}{\eta M_0}(kg(x^*) - \sum_{t=1}^{k} g(x^t)) \geq \|x^* - x^k\|^2 - \|x^* - x^0\|^2. \tag{4.19}$$

By (4.10), and replacing $x^k$ with $x^t$, one has

$$2\beta^{-1}(g(x^{t-1}) - g(x^t)) \geq \left\|x^t - x^{t-1}\right\|^2. \tag{4.20}$$

Multiplying both sides of (4.20) by $\frac{\beta(t-1)}{\eta M_0}$, we get

$$\frac{2}{\eta M_0}((t-1)g(x^{t-1}) - tg(x^t) + g(x^t)) \geq \frac{\beta(t-1)}{\eta M_0}\|x^t - x^{t-1}\|^2. \tag{4.21}$$

Summing (4.21) over $t = 1, 2, \cdots, k$ gives

$$\frac{2}{\eta M_0}(-kg(x^k) + \sum_{t=1}^{k} g(x^t)) \geq \frac{\beta}{\eta M_0}\sum_{t=1}^{k}(t-1)\|x^t - x^{t-1}\|^2. \tag{4.22}$$

Adding (4.19) and (4.22), one has

$$\frac{2k}{\eta M_0}(g(x^*) - g(x^k)) \geq \|x^* - x^k\|^2 + \frac{\beta}{\eta M_0}\sum_{t=1}^{k}(t-1)\|x^t - x^{t-1}\|^2 - \|x^* - x^0\|^2 \geq -\|x^* - x^0\|^2.$$

This, together with $g(x^*) - g(x^k) \leq 0$, gives us that (4.16) holds. □

**Remark 4.1.** *By (4.16), it is easy to see that getting an $\epsilon$-optimal solution requires the number of iterations to be at most $[c/\epsilon] + 1$, where $c = \frac{1}{2}\eta M_0\|x^0 - x^*\|^2$.*

To proceed, we establish the linear convergence rate of our proposed method. To this end, we give some lemmas as follows.

**Lemma 4.5.** *Suppose that Assumption 2.1 holds, and the sequence $\{x^k\}$ is generated by Algorithm 4.1. Then, one has*

$$dist(x^k, X^*) \leq \tau(1 + \frac{M_0}{\beta})\|x^k - x^{k-1}\|, \tag{4.23}$$

*where positive constant $\tau$ is given by Theorem 3.1.*

*Proof.* By (4.1), we obtain $\langle x - x^k, (x^k - x^{k-1}) + L_k^{-1}\nabla g(x^{k-1})\rangle \geq 0, \forall x \in R_+^n$. It follows that

$$x^k - P_{R_+^n}\{x^{k-1} - \frac{1}{L_k}\nabla g(x^{k-1})\} = 0. \tag{4.24}$$

For $\mathcal{A}x^{m-1}$, using (3.3), there exists an $n$-vector $\xi^k$ lying on the line segment joining $x^{k-1}$ with $x^k$ such that

$$\|\mathcal{A}(x^k)^{m-1} - \mathcal{A}(x^{k-1})^{m-1}\| \leq (m-1)\|\mathcal{A}(\xi^k)^{m-2}(x^k - x^{k-1})\|.$$

This, together with a similar argument to that for (4.14), gives us

$$(m-1)\|\mathcal{A}(\xi^k)^{m-2}(x^k - x^{k-1})\| \leq (m-1)\|\mathcal{A}(\xi^k)^{m-2}\|\|x^k - x^{k-1}\| \leq M_0\|x^k - x^{k-1}\|,$$

i.e.,

$$\|\mathcal{A}(x^k)^{m-1} - \mathcal{A}(x^{k-1})^{m-1}\| \leq M_0\|x^k - x^{k-1}\|. \tag{4.25}$$

Then, by (3.1), one has

$$\begin{aligned}
r(x^k, \tfrac{1}{L_k}) &= \left\|x^k - P_{R_+^n}\left\{x^k - \tfrac{1}{L_k}\nabla g(x^k)\right\}\right\| \\
&= \left\|\left[x^k - P_{R_+^n}\left\{x^k - \tfrac{1}{L_k}\nabla g(x^k)\right\}\right] - \left[x^k - P_{R_+^n}\left\{x^{k-1} - \tfrac{1}{L_k}\nabla g(x^{k-1})\right\}\right]\right\| \\
&= \left\|P_{R_+^n}\left\{x^k - \tfrac{1}{L_k}\nabla g(x^k)\right\} - P_{R_+^n}\left\{x^{k-1} - \tfrac{1}{L_k}\nabla g(x^{k-1})\right\}\right\| \\
&\leq \left\|\left\{x^k - \tfrac{1}{L_k}(\mathcal{A}(x^k)^{m-1} + q)\right\} - \left\{x^{k-1} - \tfrac{1}{L_k}(\mathcal{A}(x^{k-1})^{m-1} + q)\right\}\right\| \\
&\leq \|x^k - x^{k-1}\| + \tfrac{1}{L_k}\|\mathcal{A}(x^k)^{m-1} - \mathcal{A}(x^{k-1})^{m-1}\| \\
&\leq (1 + \tfrac{M_0}{L_k})\|x^k - x^{k-1}\| \\
&\leq (1 + \tfrac{M_0}{\beta})\|x^k - x^{k-1}\|,
\end{aligned} \tag{4.26}$$

where the second equality is from (4.24), the first inequality is from Proposition 2.1, the third inequality is from (4.25), and the last inequality is from Lemma 4.3. By (4.26) and Theorem 3.1, we have that (4.23) holds. □

**Lemma 4.6.** *Suppose that Assumption 2.1 holds, the sequence $\{x^k\}$ is generated by Algorithm 4.1. Then,*

$$g(x^k) - g(\bar{x}^k) \leq \sigma\|x^k - x^{k-1}\|^2, \tag{4.27}$$

*where $\sigma$ is a positive constant and $\bar{x}^k \in X^*$ is the point closest to $x^k$.*

*Proof.* Applying (2.1) and the mean value theorem, there exists an $n$-vector $z^k$ lying on the line segment joining $\bar{x}^k$ with $x^k$, and one has

$$\begin{aligned}
g(x^k) - g(\bar{x}^k) &= (x^k - \bar{x}^k)^\top \nabla g(z^k) \\
&\le (x^k - \bar{x}^k)^\top \nabla g(z^k) + (\bar{x}^k - x^k)^\top \nabla g(x^{k-1}) + L_k(\bar{x}^k - x^k)^\top (x^k - x^{k-1}) \\
&= (x^k - \bar{x}^k)^\top g(z^k) + (\bar{x}^k - x^k)^\top \nabla g(\bar{x}^k) \\
&\quad + (\bar{x}^k - x^k)^\top (\nabla g(x^{k-1}) - \nabla g(\bar{x}^k)) + L_k(\bar{x}^k - x^k)^\top (x^k - x^{k-1}) \\
&= (x^k - \bar{x}^k)^\top (\mathcal{A}(z^k)^{m-1} - \mathcal{A}(\bar{x}^k)^{m-1}) + L_k(\bar{x}^k - x^k)^\top (x^k - x^{k-1}) \\
&\quad + (\bar{x}^k - x^k)^\top (\mathcal{A}(x^{k-1})^{m-1} - \mathcal{A}(\bar{x}^k)^{m-1}) \\
&= (x^k - \bar{x}^k)^\top (\mathcal{A}(z^k)^{m-1} - \mathcal{A}(\bar{x}^k)^{m-1}) + L_k(\bar{x}^k - x^k)^\top (x^k - x^{k-1}) \\
&\quad + (\bar{x}^k - x^k)^\top (\mathcal{A}(x^{k-1})^{m-1} - \mathcal{A}(x^k)^{m-1}) \\
&\quad + (\bar{x}^k - x^k)^\top (\mathcal{A}(x^k)^{m-1} - \mathcal{A}(\bar{x}^k)^{m-1}) \\
&\le M_0\|x^k - \bar{x}^k\|\|z^k - \bar{x}^k\| + L_k\|\bar{x}^k - x^k\|\|x^k - x^{k-1}\| \\
&\quad + M_0\|\bar{x}^k - x^k\|\|x^{k-1} - x^k\| + M_0\|x^k - \bar{x}^k\|^2 \\
&\le 2M_0\|x^k - \bar{x}^k\|^2 + (\eta + 1)M_0\|\bar{x}^k - x^k\|\|x^k - x^{k-1}\| \\
&= 2M_0 dist(x^k, X^*)^2 + (\eta + 1)M_0\|x^k - x^{k-1}\| dist(x^k, X^*) \\
&\le \sigma\|x^k - x^{k-1}\|^2,
\end{aligned} \tag{4.28}$$

where the first inequality is from (4.9) with $x = \bar{x}^k$, the second inequality is from a similar argument to that for (4.25) and the Cauchy-Schwarz inequality, the third inequality follows from $\|z^k - \bar{x}^k\| \le \|x^k - \bar{x}^k\|$ and Lemma 4.3, and the last inequality is from (4.23) and letting $\sigma = 2M_0[\tau(1 + \frac{M_0}{\beta})]^2 + (\eta + 1)M_0[\tau(1 + \frac{M_0}{\beta})]$. Thus, the desired result follows. □

**Theorem 4.2.** *Suppose that Assumption 2.1 holds, and the generated sequence $\{x^k\}$ by Algorithm 4.1 has global R-linear convergence to a solution of (1.1).*

*Proof.* Applying (4.27) and (4.10), for $\bar{x}^k \in X^*$, we obtain

$$\begin{aligned}
g(x^k) - g(\bar{x}^k) &\le \sigma\|x^k - x^{k-1}\|^2 \le \frac{2\sigma}{\beta}(g(x^{k-1}) - g(x^k)) \\
&= \frac{2\sigma}{\beta}(g(x^{k-1}) - g(\bar{x}^k)) - \frac{2\sigma}{\beta}(g(x^k) - g(\bar{x}^k)),
\end{aligned}$$

i.e.,

$$g(x^{k+1}) - g(\bar{x}^k) \le \theta(g(x^k) - g(\bar{x}^k)), \tag{4.29}$$

where $0 < \theta := \frac{2\sigma}{\beta}/(1 + \frac{2\sigma}{\beta}) < 1$. This, together with (4.10), gives us

$$\begin{aligned}
\|x^k - x^{k-1}\|^2 &\le \frac{2}{\beta}(g(x^k) - g(x^{k-1})) \le \frac{2}{\beta}(g(x^k) - g(\bar{x}^k)) \\
&\le \frac{2}{\beta}\theta(g(x^{k-1}) - g(\bar{x}^k)) \le \cdots\cdots\cdots \\
&\le [\frac{2}{\beta}(g(x^0) - g(\bar{x}^k))]\theta^k.
\end{aligned}$$

This, together with (4.23), gives us

$$dist(x^k, X^*) \le \tau(1 + M_0\beta^{-1})\|x^k - x^{k-1}\| \le \left\{\tau(1 + M_0\beta^{-1})\sqrt{2\beta^{-1}(g(x^0) - g(\bar{x}^k))}\right\}\sqrt{\theta}^k,$$

where $0 < \sqrt{\theta} < 1$. Thus, the sequence $\{x^k\}$ has global $R$-linear convergence to an element of $X^*$. □

## 5. Numerical experiments

Some numerical examples are constructed in this section to show the performance of our proposed method. In our numerical experiments, all codes are run in MATLAB R2024a on the 1.70GHz MateBook D 14 with a 12th Gen Intel(R) Core(TM) i5-1240 processor. Set Tested Prob, Iter Num, CPU T, $r(\bar{x}, \rho)$, Init Point, Par $(\beta, \eta)$ denote the tested problem, iteration number, computational time in seconds, value of the residue $r(\bar{x}, \rho)$ at the approximate solution of the iterative process, initial iteration point, as well as the value of parameters $\beta$ and $\eta$, respectively. For tested Examples 5.1–5.9, we take $\gamma = 0.01, \rho = 0.0001, \epsilon = e - 8$.

**Example 5.1.** *Let* $\mathcal{A} \in T_{4,2}$ *be defined by* $a_{1111} = 6$, $a_{1112} = -1$, $a_{1121} = 0$, $a_{1122} = 0$, $a_{1211} = -2$, $a_{1212} = 0$, $a_{1221} = -1$, $a_{1222} = -2$, $a_{2111} = -2$, $a_{2112} = 0$, $a_{2121} = -2$, $a_{2122} = -1$, $a_{2211} = -1$, $a_{2212} = 0$, $a_{2221} = 0$, $a_{2222} = 7$. *All other entries are zero, and* $q = (1, -1)^\top$.

**Example 5.2.** *Assume that* $\mathcal{A} = (a_{i_1 i_2 i_3}) \in T_{3,2}$ *is given by* $a_{111} = 100, a_{222} = 200, a_{122} = 300, a_{211} = 400$. *All other* $a_{i_1 i_2 i_3} = 0$, *and* $q = (2, -2)^\top$.

**Example 5.3.** *Let* $\mathcal{A} \in T_{4,4}$ *be defined by* $a_{1111} = 15, a_{2222} = 23, a_{3333} = 26, a_{4444} = 18, a_{\pi(1123)} = 1, a_{\pi(2343)} = -1, a_{\pi(1234)} = -1$. *All other entries are zero, where* $\pi(1123)$, $\pi(2343)$, $\pi(1234)$ *are any permutation of permutations* $1123, 2343, 1234$, *respectively, and set* $q = (1, -1, 1, -1)^\top$.

**Example 5.4.** *Suppose that* $\mathcal{A} \in T_{4,4}$ *is defined by* $a_{1111} = 4, a_{2222} = 18, a_{3333} = 35, a_{4444} = 16, a_{\pi(1223)} = 1, a_{\pi(2334)} = -1, a_{\pi(1234)} = -1$. *All other entries are zero, where* $\pi(1223)$, $\pi(2334)$, $\pi(1234)$ *are any permutation of permutations* $1223, 2334, 1234$, *respectively, and* $q = (1, -1, 1, -1)^\top$.

**Example 5.5.** *Let* $\mathcal{A} \in T_{4,2}$ *be defined by* $a_{1111} = 7, a_{1112} = a_{1121} = -2, a_{1122} = 0, a_{1211} = -2, a_{1212} = a_{1221} = 0, a_{1222} = -1, a_{2111} = -2, a_{2112} = a_{2121} = 0, a_{2122} = -1, a_{2211} = 0, a_{2212} = a_{2221} = -1, a_{2222} = 6$. *All other entries are zero and* $q = (1, -1)^\top$.

**Example 5.6.** *Assume that* $\mathcal{A} \in T_{3,2}$ *is given by* $a_{111} = 1, a_{222} = 1, a_{122} = -1$, $a_{211} = -2$. $a_{i_1 i_2 i_3} = 0$ *otherwise, and* $q = (1, -1)^\top$.

**Example 5.7.** *Assume that* $\mathcal{A} \in T_{4,2}$ *is defined by* $a_{1111} = 1, a_{1112} = 2, a_{1122} = 1, a_{2222} = -1$. $a_{i_1 i_2 i_3 i_4} = 0$ *otherwise, and* $q = (0, -1)^\top$.

**Example 5.8.** *Let* $\mathcal{A} \in T_{4,4}$ *be defined by* $a_{1111} = 3$, $a_{2222} = 5$, $a_{3333} = 72$, $a_{4444} = 90, a_{1334} = a_{1343} = a_{1433} = a_{4133} = a_{4313} = a_{4331} = 1, a_{3134} = a_{3143} = a_{3314} = a_{3341} = a_{3413} = a_{3431} = 1, a_{2344} = a_{2434} = a_{2443} = a_{3244} = a_{3424} = a_{3442} = -2, a_{4234} = a_{4243} = a_{4324} = a_{4342} = a_{4423} = a_{4432} = -2$. *All other entries are zero, and* $q = (1, -1, 1, -1)^\top$.

**Example 5.9.** *Let* $\mathcal{A} \in T_{4,2}$ *be defined by* $a_{1111} = 1, a_{1222} = -1, a_{1122} = 1, a_{2222} = 1, a_{2111} = -1$, $a_{2211} = 1$. $a_{i_1 i_2 i_3 i_4} = 0$ *otherwise, and* $q = (-5, -3)^\top$.

We have run the nine examples above, and the numerical results are summarized in Table 2, where the iteration number, the iterative time, as well as the residue of the TCP are displayed. In Figure 1, we also present the trend of $r(x^k, \rho)$ in terms of iteration numbers.

**Table 2.** Numerical results by Algorithm 4.1 for Examples 5.1–5.9.

| Tested prob | Iter num | CPU T | $r(\bar{x}, \rho)$ | Init point | Par $(\beta, \eta)$ |
|---|---|---|---|---|---|
| Example 5.1 | 9 | 0.0283 | 6.6386e-14 | $(0, 0.7)^\top$ | (0.01, 1.35) |
| Example 5.2 | 13 | 0.0211 | 3.5964e-12 | $(0, 0.9)^\top$ | (0.01, 1.5) |
| Example 5.3 | 11 | 0.3571 | 2.1080e-13 | $(0, 0.6, 0, 0.4)^\top$ | (0.001, 1.531) |
| Example 5.4 | 9 | 0.3899 | 1.5664e-13 | $(0, 0.5, 0, 0.5)^\top$ | (0.01, 1.37) |
| Example 5.5 | 6 | 0.0366 | 7.5939e-17 | $(0.9, 1)^\top$ | (0.01, 1.35) |
| Example 5.6 | 18 | 0.0256 | 5.5849e-14 | $(1, 1)^\top$ | (0.01, 1.29) |
| Example 5.7 | 10 | 0.0319 | 5.2004e-14 | $(0, 0)^\top$ | (0.01, 1.31) |
| Example 5.8 | 26 | 0.4539 | 3.7351e-12 | $(0, 1.5, 0, 0.3)^\top$ | (0.001, 1.545) |
| Example 5.9 | 17 | 0.0435 | 2.4357e-12 | $(0, 0)^\top$ | (1.8, 1.09) |

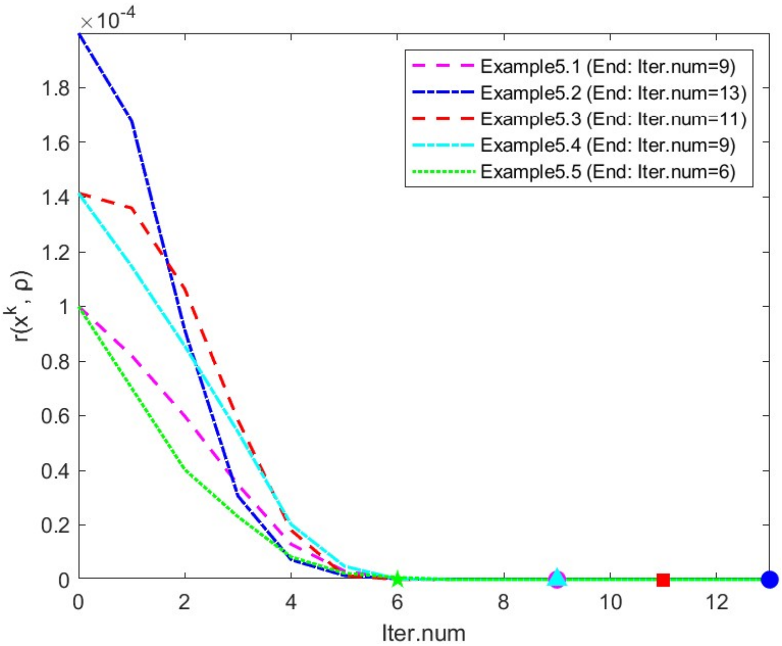


**Figure 1.** The trend of $r(x^k, \rho)$ in terms of the iteration number for Examples 5.1–5.5.

It can be observed from Table 2 that our proposed method can solve the problems above very efficiently. Meanwhile, our proposed method also gives a sufficient high-precision solution for different problems and costs less computational time. This further demonstrates the superiority of our algorithm. In particular, from Column 4 in Table 2 and Figure 1, it is clear that $r(\bar{x}, \rho)$ tends to 0 rapidly. This illustrates linearly convergent behavior in our proposed method. Therefore, this property is very interesting for the TCP.

To proceed, we compare the performance of our proposed method for Example 5.1 with different starting points.

The numerical results obtained are illustrated in Table 3, by which, it can be observed that our proposed method performs well for different starting points. Therefore, Algorithm 4.1 has nice stability and high computation efficiency.

**Table 3.** Numerical results by Algorithm 4.1 for Example 5.1 with different starting points.

| Init point | CPU T | $r(\bar{x}, \rho)$ | Approximate solution |
|---|---|---|---|
| $(1, 1)^\top$ | 0.0346 | 3.1784e-14 | $(0, 0.5228)^\top$ |
| $(0, 0)^\top$ | 0.0388 | 8.9982e-14 | $(0, 0.5228)^\top$ |
| $\frac{1}{5}(1, 1)^\top$ | 0.0399 | 8.9982e-14 | $(0, 0.5228)^\top$ |
| $(4, 4)^\top$ | 0.0361 | 8.9982e-14 | $(0, 0.5228)^\top$ |
| rand(2, 1) | 0.0932 | 8.9982e-14 | $(0, 0.5228)^\top$ |
| $2 \times$ rand(2, 1) | 0.0415 | 8.9982e-14 | $(0, 0.5228)^\top$ |
| $5 \times$ rand(2, 1) | 0.0396 | 8.9982e-14 | $(0, 0.5228)^\top$ |

At the end of this section, to further illustrate the effectiveness and stability of the proposed Algorithm 4.1, we test Algorithm 4.1 and the gradient projection (GP) algorithm with the Armijo rule on various TCPs generated in Examples 5.10–5.12 using different choices of $(m, n)$, and compare the two methods through the total number of iterations and the consumed CPU time. The parameters in the two tested algorithms are listed as follows: $\gamma = 0.01$, $\epsilon = e - 5$, $\rho = 0.5$, and we used the command "rand(n,1)" in MATLAB to randomly select the initial iteration point. All algorithms ran 5 times each, and the average of the CPU time and the Iter Num were obtained. The numerical results are reported in Tables 4–6.

**Example 5.10.** *Consider the TCP with tensor* $\mathcal{A} \in T_{4,n}$ *and* $q = (1, 1, \ldots, 1)^\top \in R^n$*, where* $\mathcal{A}$ *has entries* $a_{1111} = a_{nnnn} = 1$*,* $a_{iiii} = 2$ $(i = 2, 3, \cdots, n-1)$*,* $a_{(i+1)iii} = a_{(i-1)iii} = a_{i(i+1)ii} = a_{i(i-1)ii} = a_{ii(i+1)i} = a_{ii(i-1)i} = a_{iii(i+1)} = a_{iii(i-1)} = -\frac{1}{3}$*,* $i \in [4]$*, and* $a_{i_1 i_2 i_3 i_4} = 0$*, otherwise.*

**Example 5.11.** *Assume that* $\mathcal{A} \in T_{m,n}$ *is given by* $a_{ii\cdots i} = i$*,* $i = 1, 2, \cdots, n$*, the other entries are zero, and* $q = (1, 2, \cdots, n)^\top$*.*

**Example 5.12.** *Choose the tensor* $\mathcal{B} \in T_{m,n}$ *as* $b_{i_1 i_2 \ldots i_m} = |\sin(i_1 + i_2 + \cdots + i_m)|$*, and set* $\mathcal{A} = \mathcal{B} + 0.1\mathcal{I}$*. The vector* $q \in R^n$ *is chosen as* $q_k = k$ *for* $k = 1, 2, \cdots, n$*.*

**Table 4.** Comparison of Algorithm 4.1 with the GP algorithm by Example 5.10 with different $(m, n)$.

| $(m, n)$ | Algorithm 4.1 | | GP Algorithm | | Par $(\beta, \eta)$ |
|---|---|---|---|---|---|
| | Iter num | CPU T | Iter mum | CPU T | |
| (4, 5) | 4 | 0.0490 | 7 | 0.0654 | (7, 9) |
| (4, 10) | 4 | 0.5770 | 5 | 0.7134 | (7, 9) |
| (4, 20) | 3 | 8.3675 | 4 | 11.7442 | (7, 9) |
| (4, 30) | 3 | 40.4610 | 4 | 47.7768 | (7, 9) |

**Table 5.** Comparison of Algorithm 4.1 with the GP algorithm by Example 5.11 with different $(m, n)$.

| $(m, n)$ | Algorithm 4.1 | | GP algorithm | | Par $(\beta, \eta)$ |
|---|---|---|---|---|---|
| | Iter num | CPU T | Iter num | CPU T | |
| (4, 3) | 3 | 0.0189 | 7 | 0.0484 | (7, 10) |
| (4, 6) | 3 | 0.3263 | 5 | 0.6116 | (7, 10) |
| (4, 9) | 2 | 1.2026 | 4 | 2.1284 | (7, 10) |
| (4, 12) | 3 | 3.9140 | 6 | 9.4644 | (7, 10) |
| (4, 15) | 3 | 12.5623 | 5 | 20.4580 | (7, 10) |
| (5, 3) | 3 | 0.0552 | 5 | 0.0746 | (7, 10) |
| (5, 6) | 2 | 1.6255 | 3 | 2.3559 | (7, 10) |
| (5, 9) | 3 | 10.0030 | 4 | 24.1041 | (7, 10) |
| (5, 12) | 2 | 48.0638 | 4 | 90.0162 | (7, 10) |
| (6, 3) | 2 | 0.1463 | 4 | 0.2976 | (7, 10) |
| (6, 6) | 2 | 9.3205 | 6 | 23.2068 | (7, 10) |
| (6, 8) | 2 | 47.6230 | 3 | 59.8041 | (7, 10) |
| (8, 2) | 2 | 0.0681 | 3 | 0.0984 | (7, 10) |
| (8, 4) | 2 | 13.8090 | 3 | 21.7760 | (7, 10) |

**Table 6.** Comparison of Algorithm 4.1 with the GP algorithm by Example 5.12 with different $(m, n)$.

| $(m, n)$ | Algorithm 4.1 | | GP algorithm | | Par $(\beta, \eta)$ |
|---|---|---|---|---|---|
| | Iter num | CPU T | Iter num | CPU T | |
| (3, 2) | 6 | 0.0025 | 6 | 0.0031 | (9, 10) |
| (3, 4) | 4 | 0.0056 | 6 | 0.0084 | (9, 10) |
| (3, 10) | 2 | 0.0375 | 2 | 0.0758 | (9, 10) |
| (3, 20) | 2 | 0.2677 | 4 | 0.6373 | (9, 10) |
| (3, 40) | 3 | 3.8369 | 6 | 4.9464 | (9, 10) |
| (4, 2) | 3 | 0.0027 | 5 | 0.0045 | (9, 10) |
| (4, 4) | 3 | 0.0177 | 4 | 0.0254 | (9, 10) |
| (4, 10) | 2 | 0.2683 | 4 | 1.6180 | (9, 10) |
| (4, 20) | 3 | 7.5625 | 5 | 15.3511 | (9, 10) |
| (5, 2) | 3 | 0.0045 | 5 | 0.0076 | (9, 10) |
| (5, 4) | 2 | 0.0485 | 3 | 0.0635 | (9, 10) |
| (5, 10) | 2 | 4.0666 | 3 | 8.3216 | (9, 10) |
| (5, 15) | 2 | 29.6149 | 3 | 46.5032 | (9, 10) |
| (6, 2) | 12 | 0.0095 | 14 | 0.0186 | (9, 10) |
| (6, 4) | 3 | 0.9335 | 5 | 1.5760 | (9, 10) |
| (6, 6) | 2 | 10.1219 | 10 | 13.4285 | (9, 10) |
| (8, 2) | 5 | 0.0275 | 11 | 0.1549 | (9, 10) |
| (8, 4) | 4 | 17.9545 | 5 | 19.4672 | (9, 10) |

From Tables 4–6, we can see that both Algorithm 4.1 and the GP algorithm can effectively find the solution of the TCP, but the CPU times and the number of iterations of Algorithm 4.1 are always less than that of the GP algorithm with different orders $m$ and sizes $n$, which shows that Algorithm 4.1 converges faster than the GP algorithm. This also shows that it has nice stability and high computation efficiency.

## 6. Discussion and conclusions

For the positive semi-definite tensor complementarity problem (TCP), by an equivalent transformation of this problem, we establish an error bound estimation for the positive semi-definite TCP without the fractional term via the natural residual function. The obtained error bound has some clear theoretical advantages over most existing ones for the TCP in [9–11]. For instance: some abstract conditions, e.g., an $m$-uniform $P$-function and being $m$-monotone, are removed. Based on the obtained error bound, we show the global linear convergence rate of the proposed self-adaptive projection algorithm for the TCP. Furthermore, we also obtain an approximate solution of this problem in a finite iteration number. Numerical experiment results are also encouraging.

It is well known that the PCP is a natural generalization of the TCP. Similar to the investigation of TCPs by structured tensors, many theoretical results about the properties of the solution set of the PCP have been discussed by structured tensor tuples, including the uniqueness of the solution, the upper bound (lower bound) of the solution set, error bound theory, and so on [11, 30–32]. Since error bound theory has rich applications in solution methods [33], how to develop an error bound for TCPs (GTCPs or PCPs) under weaker conditions, and apply it to establish the convergence rate of some methods, is a topic to be further investigated.

## Author contributions

Yuanshou Zhang, Hongchun Sun, and Sabir Amina: Study conception and design, write the main manuscript text; Zhiwen Jie: Study conception and design, prepared data collection and analysis. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflict of interest.

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