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Research article

A new two-dimensional macroscopic local fractional viscous diffusive model of vehicular traffic flow

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Abstract: In this paper, we proposed a two-dimensional macroscopic local fractional viscous-diffusive model of vehicular traffic flow, derived from the two-dimensional fractal Navier-Stokes equation. The model was formulated by combining the derived momentum equation with the fractal LWR framework. The proposed fractal dynamic velocity equation incorporates convection, anticipation, relaxation, diffusion, and viscosity as governing parameters. A stability analysis was performed, and an analytical solution to the model was obtained. Certain aspects of multilane traffic flow are further explored through illustrative examples. The solutions are presented and discussed using graphical representations, which demonstrate how non-differentiable traffic density and speed functions evolve dynamically. The study highlights the influence of viscous-diffusive effects on traffic flow, showing that road narrowing increases viscosity and diffusion, thereby reducing traffic speed, whereas these factors have a comparatively smaller impact on wider roads.

Keywords: local fractional calculus; macroscopic model; two-dimensional LWR model; fractal Laplace transform; traffic flow model
Mathematics Subject Classification: 26A27, 26A33, 35C10, 35R11, 74G10

1. Introduction

Traffic models serve as essential tools for managing and controlling traffic, with the primary objective of optimizing transportation systemss and generating economic and environmental benefits, such as reducing congestion and pollution. Based on the scale of observation, traffic models are generally classified into three categories: Microscopic, macroscopic, and kinetic models. Microscopic models describe the dynamics of individual vehicles, simulating the behavior of a single vehicle–driver unit and capturing the complex interactions that occur within traffic flow. In these models, microscopic characteristics such as vehicle position and velocity are treated as dynamic variables. By contrast, macroscopic models consider aggregated factors, such as vehicle density and average flow, and represent traffic dynamics through first- or higher-order continuum equations, drawing analogies with the motion of continuous media such as fluids. Kinetic models occupy an intermediate position between the two: They can be derived from microscopic descriptions, while macroscopic models can, in turn, be obtained from kinetic formulations.

The two primary categories of macroscopic traffic models are first-order models, which rely on scalar hyperbolic equations, and second-order models, which are made up of hyperbolic equation systemss [1, 2]. As one of the most popular first-order models for traffic analysis, the traffic flow equation by Lighthill and Whitham [3] and Richards [4] is commonly utilised. It is referred to as the Lighthill–Whitham–Richards (LWR) model and represented by a single equation; a continuity equation. Nonetheless, the following grounds led to criticism of the LWR model: The model is unable to explain how traffic breaks down; models have infinite acceleration; the finite speed adaptation and reaction times govern the traffic capacity and the characteristic waves; since the LWR model does not account for inertial effects, a vehicle's velocity must be adjusted instantly and drivers have trouble adapting their speed to the average velocity because they are unable to anticipate changes in traffic circumstances in advance. These inadequacies led to developing the dynamic velocity equation and LWR to model these drawbacks [5]. The works of Payne [6] and Whitham [7] are the source of this speed equation. Subsequently, the backward travelling wave characteristics of the second-order isotropic equation revealed even another fault [8]. As a result, models with anisotropic properties were created. The Zhang models [9, 10], Aw–Rascle model [11], and Jiang–Wu–Zhu model [12] are the most often used of these. In order to capture drivers' physiological reactions [13] and reactions to traffic cues [14], the Payne–Whitham model has been further modified. Furthermore, the Jiang–Wu–Zhu framework has been improved to account for diffusion [15]. However, viscosity for multi-lane traffic flow has not been specifically taken into account in these second-order models. Diffusion models were used in an initial attempt to simulate resistance [16]. Unlike viscosity, recognised within a two-dimensional spatial domain, these models have been provided in a one-dimensional spatial domain. A two-dimensional macroscopic model that considers viscosity is presented by Fosu and Oduro [17].

The LWR model continues to be a widely used tool for simulating traffic flow, even with the introduction of second-order formulations designed to overcome the limitations of the first-order equation, owing to its effectiveness and ability to capture the qualitative behavior of road traffic. However, when physical variables in traditional traffic models, such as density or speed, are regarded as non-differentiable functions of space and time defined on Cantorian (fractal) sets, the conventional conservation law no longer holds, rendering the classical LWR framework inadequate. Wang et al. [18] consequently modified the dynamical LWR equation fractally to tackle this situation in the framework of local fractional calculus and enabling local fractional derivatives within the local fractional conservation laws. This LWR model's fractal version is written as

$$\frac{\partial^\sigma \wp}{\partial \tau^\sigma} + \frac{\partial^\sigma \Theta}{\partial \omega^\sigma} = 0,\ 0 < \sigma \leq 1, \tag{1.1}$$

where $\wp$ represents non-differentiable traffic density, Θ is vehicle flux which is a function of density, and σ is the order of fractal derivative. Many fields, notably the behaviour of fluids, applied mathematical applications, signal processing, and the theory of quantum mechanics, have employed local fractional calculus or fractal calculus to solve fractional issues [19–21]. It is one of the best and most effective methods for managing continuous, fractal, and non-differentiable functions that are currently accessible. Recently, Sun determined the approximate solution of time fractional nonlinear MKDV equation within local fractional operators [22], Singh et al. analysed the fractal view of local fractional Fokker–Planck equation to study particle's Brownian motion [23]. Fractal Navier–Stokes equations were introduced by Yang et al. [24], whereas Zhao et al. introduced the fractal Maxwell's systems of equations [25]. Numerical methods have been employed in the past literature to analyse and solve the traffic flow models. Lebacque [26] showed that the LWR equation can be solved for any feasible Riemann issue with the help of a Godunov-type approach. Helbing and Treiber [27] have solved non-equilibrium models using McCormack's method.

The second-order dynamic velocity equation incorporates convection, anticipation, relaxation, diffusion, and viscosity as its constitutive components. Here, we examine whether these dynamical factors are explicitly addressed or remain unresolved within specific model formulations. It is observed that, while convection, anticipation, and relaxation are consistently included, diffusion and viscosity are rarely considered. In this work, a new fractal two-dimensional momentum equation is developed by integrating all of these variables within the framework of local fractional calculus, thereby incorporating local fractional derivatives. The terms in this fractal momentum equation are derived from fluid dynamics theory, with the non-slip condition serving as a guiding principle.

A random multi-lane flow is depicted loosely in the Figure 1. It is anticipated that each lane's flow velocity will vary from the next. Let $v_1, v_2, ..., v_n$ be the velocities for the lanes $1, 2, ..., n$, respectively. As one approaches the inner lanes, the speed rises. The fastest-moving vehicles are in lane n, whereas the slowest-moving vehicles are in lane 1. The two dimensions ω and μ are shown in the figure. It is evident from the figure that the flow is only in ω direction. There is no flow in μ direction. The Newtonian law of viscosity can be used to model the inter-lane traffic resistance under this characterisation as

$$\ell = \kappa \frac{\partial^\sigma v}{\partial \mu^\sigma}, \; 0 < \sigma \leq 1, \tag{1.2}$$

where, ℓ represents the whole shear effect and κ is viscosity coefficient. This interprets the traffic's resistance to shear forces. When there is minimal interaction between two nearby vehicles, the rate is lower.

We propose a two-dimensional local fractional viscous diffusive model of vehicular traffic flow by combining the fractal continuity equation with the fractal momentum equation. It should be noted that each of the terms being considered describes some actual traffic occurrences. Further, the proposed fractal framework is analytically solved by employing the local fractional Laplace variational iteration method (LFLVIM) and the non-differentiable solutions are derived.

The remaining part of the section is summarized as follows: Fundamentals of local fractional calculus are discussed in Section 2. The model is derived in Section 3. Stability analysis is carried out in Section 4. In Section 5, we deal with the illustration of the method used. Exemplary instances are discussed in Section 6, and the conclusion is drawn in Section 7.

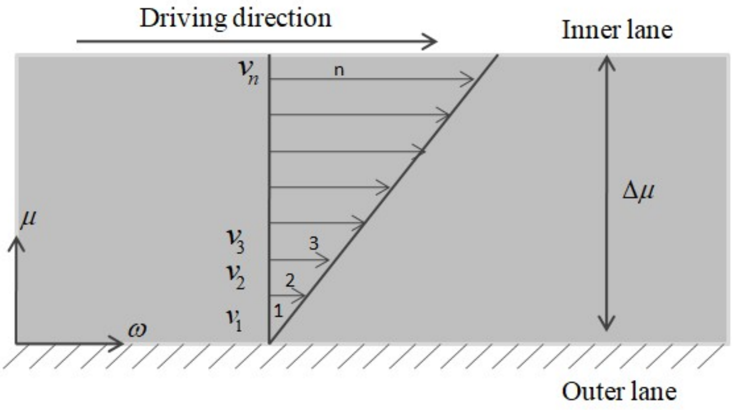

Figure 1. Velocity differentials across traffic flow.

2. Local fractional calculus and properties

The ideas of local fractional calculus and local fractional Laplace transform (LFLT) are covered in this section.

Definition 2.1 ([28]). *A function $\vartheta(j)$ for $j \in (k,l)$ is termed as a local fractional continuous (LFC) at $j = j_o$ if*

$$|\vartheta(j) - \vartheta(j_o)| < \varsigma^{\sigma}, 0 < \sigma \leq 1, \tag{2.1}$$

for $\varsigma, \delta > 0$ and $|j - j_o| < \delta$. Moreover, if this happens $\forall j \in (k,l)$, then $\vartheta(j)$ is LFC function on (k,l) and denoted as $\vartheta(j) \in C_{\sigma}(k,l)$.

Definition 2.2 ([28]). *The local fractional derivative of function $\vartheta(j) \in C_{\sigma}(k,l)$ at $j = j_o$ is expressed as*

$$D_j^{\sigma}\vartheta(j_o) = \vartheta^{(\sigma)}(j_0) = \frac{d^{\sigma}\vartheta(j_o)}{dj^{\sigma}} = \lim_{j \to j_o} \frac{\Delta^{\sigma}(\vartheta(j) - \vartheta(j_o))}{(j - j_o)^{\sigma}}, \tag{2.2}$$

where

$$\Delta^{\sigma}(\vartheta(j) - \vartheta(j_o)) \cong \Gamma(1+\sigma)(\vartheta(j) - \vartheta(j_o)), \tag{2.3}$$

and σ, $(0 < \sigma < 1)$ is order of fractional derivative. The expression for Gamma function, $\Gamma(\mu)$, is given as

$$\Gamma(\mu) = \int_0^{\infty} l^{\mu-1}\exp(l), \ \Re(\mu) > 0. \tag{2.4}$$

Definition 2.3 ([28]). *Let the closed interval $[k,l]$ be partitioned as (c_d, c_{d+1}), $d = 0, 1, \cdots, H-1$ with $c_H = y$, $\Delta c_d = c_{d+1} - c_d$ and $\Delta c = \max\{\Delta c_0, \Delta c_1, ...\}$. Then, the local fractional integral of function $\vartheta(j)$ in $[k,l]$ is given as*

$${}_kJ_l^{\sigma}\vartheta(j) = \frac{1}{\Gamma(1+\sigma)}\int_k^l \vartheta(c)(dc)^{\sigma} = \frac{1}{\Gamma(1+\sigma)}\lim_{\Delta c \to 0}\sum_{d=0}^{H-1}\vartheta(c_d)(\Delta c_d)^{\sigma}. \tag{2.5}$$

Definition 2.4 ([28]). *Over fractal space, the Mittage-Leffler function is expressed as*

$$E_{\sigma}\left(\jmath^{\sigma}\right)=\sum_{H=0}^{\infty}\frac{\jmath^{H\sigma}}{\Gamma\left(1+H\sigma\right)},\ 0<\sigma\leq 1. \tag{2.6}$$

Definition 2.5 ([28]). *Over fractal space, the local fractional derivative of following functions is defined as*

$$D_{j}^{\sigma}j^{L\sigma}=\frac{\Gamma\left(1+L\sigma\right)}{\Gamma\left(1+\left(L-1\right)\sigma\right)}j^{(L-1)\sigma}, \tag{2.7}$$

$$D_{j}^{\sigma}E_{\sigma}\left(Lj^{\sigma}\right)=LE_{\sigma}\left(Lj^{\sigma}\right). \tag{2.8}$$

Definition 2.6 ([29, 30] LFLT). *Let a function $\vartheta:\Re\rightarrow C$ is continuously non-differentiable, then LFLT of ϑ is defined as*

$$\mathrm{T}_{\sigma}\left\{\vartheta\left(j\right)\right\}=\vartheta_{s}^{\mathrm{T},\sigma}\left(s\right)=\frac{1}{\Gamma\left(1+\sigma\right)}\int_{0}^{\infty}\vartheta\left(j\right)E_{\sigma}\left(-j^{\sigma}s^{\sigma}\right)\left(dj\right)^{\sigma},\ 0<\sigma\leq 1, \tag{2.9}$$

where T_{σ} symbolizes local fractional Laplace operator.

Further, local fractional inverse Laplace transform is read as

$$\mathrm{T}_{\sigma}{}^{-1}\left\{\vartheta_{s}^{\mathrm{T},\sigma}\left(s\right)\right\}=\vartheta\left(j\right)=\frac{1}{(2\pi)^{\sigma}}\int_{\eta-i\infty}^{\eta+i\infty}\vartheta_{s}^{\mathrm{T},\sigma}\left(s\right)E_{\sigma}\left(j^{\sigma}s^{\sigma}\right)\left(ds\right)^{\sigma},\ 0<\sigma\leq 1, \tag{2.10}$$

where $\mathrm{T}_{\sigma}{}^{-1}$ symbolizes the fractal inverse Laplace operator with the property $E_{\sigma}\left((2\pi)^{\sigma}i^{\sigma}\right)=1$, and $\mathrm{Re}\left(s\right)=\eta>0$.

In addition, the sufficient condition for convergence is expressed as

$$\frac{1}{\Gamma\left(1+\sigma\right)}\int_{0}^{\infty}|\vartheta\left(j\right)|(dj)^{\sigma}<k<\infty. \tag{2.11}$$

proposition 1 ([29, 30] LFLT of fractal order derivative). *A function $\vartheta\left(j\right)$ with fractal order derivative $p\sigma$, $0<\sigma\leq 1$ follows*

$$\mathrm{T}_{\sigma}\left\{\vartheta^{p\sigma}\left(j\right)\right\}=s^{p\sigma}\mathrm{T}_{\sigma}\left\{\vartheta\left(j\right)\right\}-s^{(p-1)\sigma}\vartheta\left(0\right)-s^{(p-2)\sigma}\vartheta^{(\sigma)}\left(0\right)-\cdots-\vartheta^{((p-1)\sigma)}\left(0\right), \tag{2.12}$$

for any positive integer p.

Some properties of LFLT,

$$\mathrm{T}_{\sigma}\left\{ek\left(j\right)+fl\left(j\right)\right\}=e\mathrm{T}_{\sigma}\left\{k\left(j\right)\right\}+f\mathrm{T}_{\sigma}\left\{l\left(j\right)\right\},\ e,f\in C, \tag{2.13}$$

for $r\in C$,

$$\mathrm{T}_{\sigma}\left\{\frac{j^{r\sigma}}{\Gamma\left(1+r\sigma\right)}\right\}=\frac{1}{s^{\sigma(r+1)}}, \tag{2.14}$$

$$\mathrm{T}_{\sigma}\left\{\frac{j^{h\sigma}}{\Gamma\left(1+h\sigma\right)}E_{\sigma}\left(r^{\sigma}j^{\sigma}\right)\right\}=\frac{1}{(s-r)^{\sigma(h+1)}}, \tag{2.15}$$

$$\mathrm{T}_{\sigma}\left\{E_{\sigma}\left(-rj^{\sigma}\right)+\frac{rj^{\sigma}}{\Gamma\left(1+\sigma\right)}-1\right\}=\frac{r^{2}}{\left(s^{\sigma}+r\right)s^{2\sigma}}. \tag{2.16}$$

3. Model derivation

The local fractional two-dimensional systems of the macroscopic traffic flow model is created by two equations: fractal LWR equation and momentum equation. The local fractional LWR model is read as

$$\frac{\partial^\sigma \wp}{\partial \tau^\sigma} + \frac{\partial^\sigma \Theta}{\partial \omega^\sigma} = 0, \; 0 < \sigma \leq 1, \tag{3.1}$$

where $\wp$ represents non-differentiable traffic density, $\Theta = \wp v$ is vehicle flow rate with velocity v, and σ is the order of fractal derivative.

As previously mentioned, the momentum equation was developed to address the limitations of the LWR model. We derive a new local fractional dynamic velocity equation from the reduced fractal Navier-Stokes equation. The two-dimensional local fractional Navier-Stokes equation is given as

$$\wp\left(\frac{\partial^\sigma v}{\partial \tau^\sigma} + v\frac{\partial^\sigma v}{\partial \omega^\sigma} + \zeta\frac{\partial^\sigma v}{\partial \mu^\sigma}\right) = h_\omega - \frac{\partial^\sigma P}{\partial \omega^\sigma} + \xi\left(\frac{\partial^{2\sigma} v}{\partial \omega^{2\sigma}} + \frac{\partial^{2\sigma} v}{\partial \mu^{2\sigma}}\right), \tag{3.2}$$

$$\wp\left(\frac{\partial^\sigma \zeta}{\partial \tau^\sigma} + v\frac{\partial^\sigma \zeta}{\partial \omega^\sigma} + \zeta\frac{\partial^\sigma \zeta}{\partial \mu^\sigma}\right) = h_\mu - \frac{\partial^\sigma P}{\partial \omega^\sigma} + \xi\left(\frac{\partial^{2\sigma} \zeta}{\partial \omega^{2\sigma}} + \frac{\partial^{2\sigma} \zeta}{\partial \mu^{2\sigma}}\right), \tag{3.3}$$

where ξ denotes viscosity rate, P represents pressure term, h_ω and h_μ are the gravitational forces, $\wp$ is traffic density, v and ζ represent speed in ω and μ directions, respectively. According to the mechanisms of traffic flow (see Figure 1), it can be observed that there is no flow in μ direction, thus, speed ζ and its fractal derivatives become zero. As a result, Eq (3.3) is omitted and the above systems reduces to

$$\wp\left(\frac{\partial^\sigma v}{\partial \tau^\sigma} + v\frac{\partial^\sigma v}{\partial \omega^\sigma}\right) = h_\omega - \frac{\partial^\sigma P}{\partial \omega^\sigma} + \xi\left(\frac{\partial^{2\sigma} v}{\partial \omega^{2\sigma}} + \frac{\partial^{2\sigma} v}{\partial \mu^{2\sigma}}\right). \tag{3.4}$$

The term $\frac{\partial^{2\sigma} v}{\partial \mu^{2\sigma}}$ is decomposed as a constant and a fractal derivative term, that is

$$\frac{\partial^{2\sigma} v}{\partial \mu^{2\sigma}} \approx s_\mu \frac{\partial^\sigma v}{\partial \mu^\sigma}, \tag{3.5}$$

where s_μ is traffic sensitivity [17]. The purpose of s_μ is to simulate the sensitivity that ensures safe distance among vehicles travelling in the same direction on adjacent lanes. We introduce a diffusive term as a part of the equation, that is

$$D\frac{\partial^{2\sigma} v}{\partial \omega^{2\sigma}}. \tag{3.6}$$

The velocities of two adjacent vehicles decrease in tandem with a decrease in the inter-lane distance. In another way, when traffic gets viscous, the flow decreases. To consider this, a negative sign is used to represent this relationship. Thus, Eq (3.4) becomes

$$\wp\left(\frac{\partial^\sigma v}{\partial \tau^\sigma} + v\frac{\partial^\sigma v}{\partial \omega^\sigma}\right) = h_\omega - \frac{\partial^\sigma P}{\partial \omega^\sigma} - \xi\left(D\frac{\partial^{2\sigma} v}{\partial \omega^{2\sigma}} + s_\mu\frac{\partial^\sigma v}{\partial \mu^\sigma}\right). \tag{3.7}$$

The pressure term in the equation indicates how drivers assess whatever is ahead and react to the downstream density. The relaxation term takes the place of the net influence of pressure force as in

Jiang's model [12] to ensure that vehicles do not crash with one another. The relaxation term is defined as

$$\frac{V(\wp) - v}{\tau}. \tag{3.8}$$

Another significant term is the anticipation term, given as

$$c\frac{\partial^{\sigma} v}{\partial \omega^{\sigma}}, \tag{3.9}$$

which is utilised to address the issue of drivers responding backward to stimuli. Here, driver anticipation rate is denoted as c. This quantity is integrated to fractal version into this new formulation from the classical model [12]. Further, it is assumed that h_{ω} is zero. Thus, the following is the expression for the new dynamic local fractional velocity equation

$$\frac{\partial^{\sigma} v}{\partial \tau^{\sigma}} + v\frac{\partial^{\sigma} v}{\partial \omega^{\sigma}} = \frac{V(\wp) - v}{\tau} + c\frac{\partial^{\sigma} v}{\partial \omega^{\sigma}} - \frac{\xi}{\wp}\left(D\frac{\partial^{2\sigma} v}{\partial \omega^{2\sigma}} + s_{\mu}\frac{\partial^{\sigma} v}{\partial \mu^{\sigma}}\right). \tag{3.10}$$

Thus, Eq (3.10) together with Eq (3.1) constituted a new two-dimensional local fractional viscous-diffusive traffic flow model.

4. Stability criterion

The stability requirement of the proposed model is assessed by employing the linearization technique. Let us assume a homogeneous solution $\wp(\omega, \mu, \tau) = \wp_e$ and $v(\omega, \mu, \tau) = V_e(\wp)$. From these stationary solutions, any deviation are expressed as

$$\delta\wp = \wp(\omega, \mu, \tau) - \wp_e \ \ and \ \ \delta v = v(\omega, \mu, \tau) - V_e(\wp). \tag{4.1}$$

Therefore, the proposed local fractional two-dimensional viscous-diffusive traffic flow model is linearised as

$$\frac{\partial^{\sigma}(\delta\wp)}{\partial \tau^{\sigma}} + V_e\frac{\partial^{\sigma}(\delta\wp)}{\partial \omega^{\sigma}} + \wp_e\frac{\partial^{\sigma}(\delta\wp)}{\partial \omega^{\sigma}} = 0, \tag{4.2}$$

$$\frac{\partial^{\sigma}(\delta v)}{\partial \tau^{\sigma}} + V_e\frac{\partial^{\sigma}(\delta v)}{\partial \omega^{\sigma}} - c\frac{\partial^{\sigma}(\delta v)}{\partial \omega^{\sigma}} = \frac{1}{\tau}\left(\frac{dV_e}{d\wp}.\delta\wp - \delta v\right) - \frac{\xi D}{\wp_e}\frac{\partial^{2\sigma}(\delta v)}{\partial \omega^{2\sigma}} - \frac{\xi s_{\mu}}{\wp_e}\frac{\partial^{\sigma}(\delta v)}{\partial \mu^{\sigma}}. \tag{4.3}$$

It becomes clear by observing vehicle trajectories that traffic moves in a wave-like manner. This proposal enables the theory of waves to be used to infer the propagation of flow disturbance. Therefore, in order to determine whether a disturbance will intensify or abate over time, the fundamental simple wave functions are used.

$$\delta\wp = \hat{\wp}E_{\sigma}(is_1\omega^{\sigma} + is_2\mu^{\sigma} + (\eta - i\gamma)\tau^{\sigma}) \ and \ \delta v = \hat{v}E_{\sigma}(is_1\omega^{\sigma} + is_2\mu^{\sigma} + (\eta - i\gamma)\tau^{\sigma}), \tag{4.4}$$

where s_1 and s_2 represent spatial wave numbers. These, in turn, define the wavelength along the lateral and longitudinal axes. γ denotes frequency of the wave, η is wave dumping, $\hat{\wp}$ and $\hat{v}$ represent altitude at time τ.

Substituting Eq (4.4) and its fractal derivatives into the system (4.3), we have

$$\hat{\wp}\left(\eta - i\gamma\right)\Gamma\left(1+\sigma\right)\tilde{M} + \hat{\wp} i V_e s_1 \Gamma\left(1+\sigma\right)\tilde{M} + \hat{v} i \wp_e s_1 \Gamma\left(1+\sigma\right)\tilde{M} = 0, \tag{4.5}$$

$$\begin{aligned} &\hat{v}\left(\eta - i\gamma\right)\Gamma\left(1+\sigma\right)\tilde{M} + \left(V_e - c\right)\hat{v} i s_1 \Gamma\left(1+\sigma\right)\tilde{M} - \frac{\tilde{M}}{\tau}\left(\frac{dV_e}{d\wp}.\hat{\wp} - \hat{v}\right) \\ &-\frac{\xi D}{\wp_e}\hat{v} s_1^2 \Gamma^2\left(1+\sigma\right)\tilde{M} - \frac{\xi s_\mu}{\wp_e}\hat{v} i s_2 \Gamma\left(1+\sigma\right)\tilde{M} = 0, \end{aligned} \tag{4.6}$$

where

$$\tilde{M} = E_\sigma\left(i s_1 \omega^\sigma + i s_2 \mu^\sigma + \left(\eta - i\gamma\right)\tau^\sigma\right) \neq 0.$$

The above equations can be expressed in a vector form as

$$\begin{bmatrix} \Gamma\left(1+\sigma\right)\tilde{\eta} & \Gamma\left(1+\sigma\right) i \wp_e s_1 \\ -\frac{1}{\tau}\frac{dV_e}{d\wp} & \begin{array}{c}\Gamma\left(1+\sigma\right)\tilde{\eta} - i c s_1 \Gamma\left(1+\sigma\right) + \frac{1}{\tau} - \\ \frac{\xi}{\wp_e}\left(D s_1^2 \Gamma^2\left(1+\sigma\right) + s_\mu i s_2 \Gamma\left(1+\sigma\right)\right)\end{array} \end{bmatrix} \begin{bmatrix} \hat{\wp} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.7}$$

This is in the form $H\hat{x} = 0$. The unknown vector $\hat{x}$ are the amplitudes to be determined. $\tilde{\eta} = \eta - i\left(\gamma - s_1 V_e\right)$ is abbreviation meant to speed up the computing process.

The solution of Eq (4.7) is non trivial if $\det\left(H\right) = 0$. The determinant of matrix H leads to the following quadratic form

$$\tilde{\eta}^2 \Gamma\left(1+\sigma\right) + \tilde{\eta}\left(\frac{1}{\tilde{\tau}} - i\tilde{\beta}\right) + \frac{i\wp_e s_1}{\tau}\frac{dV_e}{d\wp} = 0, \tag{4.8}$$

where

$$\frac{1}{\tilde{\tau}} = \frac{1}{\tau} - \frac{\xi D}{\wp_e} s_1^2 \Gamma^2\left(1+\sigma\right) \text{ and } \tilde{\beta} = c s_1 \Gamma\left(1+\sigma\right) + \frac{\xi s_\mu}{\wp_e} s_2 \Gamma\left(1+\sigma\right).$$

The solution to characteristic Eq (4.8) is

$$\tilde{\eta}_\pm\left(s\right) = \frac{1}{2}\left(-\frac{1}{\tilde{\tau}} + i\tilde{\beta}\right) \pm \sqrt{\frac{1}{4}\left(\frac{1}{\tilde{\tau}^2} - \tilde{\beta}^2\right) + i\left(-\frac{\wp_e s_1 \Gamma\left(1+\sigma\right)}{\tau}\frac{dV_e}{d\wp} - \frac{\tilde{\beta}}{2\tilde{\tau}}\right)}. \tag{4.9}$$

Since the square root term would produce a complex result, as can be observed, this root term is simplified using the following expression

$$\sqrt{\Re \pm i\Im} = \sqrt{\frac{1}{2}\left(\sqrt{\Re^2 + \Im^2} + \Re\right)} \pm i\sqrt{\frac{1}{2}\left(\sqrt{\Re^2 + \Im^2} - \Re\right)}, \tag{4.10}$$

where $\Re$ and $\Im$ represent real and imaginary part, respectively, that is,

$$\Re = \frac{1}{4}\left(\frac{1}{\tilde{\tau}^2} - \tilde{\beta}^2\right) \text{ and } \Im = -\frac{\wp_e s_1 \Gamma\left(1+\sigma\right)}{\tau}\frac{dV_e}{d\wp} - \frac{\tilde{\beta}}{2\tilde{\tau}}. \tag{4.11}$$

Regarding stability, the discriminant resides on the eigenvalues' real component (*Re*). Thus, the real part of the eigenvalues are given as

$$\operatorname{Re}\left(\tilde{\eta}_\pm\left(s\right)\right) = -\frac{1}{2\tilde{\tau}} \pm \sqrt{\frac{1}{2}\left(\sqrt{\Re^2 + \Im^2} + \Re\right)}. \tag{4.12}$$

Based on which is more non-negative, a decision is made between $\mathrm{Re}(\tilde{\eta}_-(s))$ and $\mathrm{Re}(\tilde{\eta}_+(s))$. It is observed that $\mathrm{Re}(\tilde{\eta}_-(s)) < \mathrm{Re}(\tilde{\eta}_+(s))$, suggesting that $\mathrm{Re}(\tilde{\eta}_-(s))$ will automatically meet any condition that satisfies $\mathrm{Re}(\tilde{\eta}_+(s))$. Therefore, $\mathrm{Re}(\tilde{\eta}_+(s))$ is the pertinent eigenvalue to identify transitions from stationary traffic to unstable flow.

That is

$$-\frac{1}{2\tilde{\tau}} + \sqrt{\frac{1}{2}\left(\sqrt{\mathfrak{R}^2 + \mathfrak{I}^2} + \mathfrak{R}\right)} \geq 0, \tag{4.13}$$

this implies

$$\mathfrak{I}^2 \geq \frac{1}{4\tilde{\tau}^4} - \frac{\mathfrak{R}}{\tilde{\tau}^2}. \tag{4.14}$$

Substituting the value of $\mathfrak{R} = \frac{1}{4}\left(\frac{1}{\tilde{\tau}^2} - \tilde{\beta}^2\right)$ and $\pm\left|\mathfrak{I}\right| = \frac{\wp_e s_1 \Gamma(1+\sigma)}{\tau}\left|\frac{dV_e}{d\wp}\right| - \frac{\tilde{\beta}}{2\tilde{\tau}}$ into Eq (4.14), we have

$$\left(\frac{\wp_e s_1 \Gamma(1+\sigma)}{\tau}\left|\frac{dV_e}{d\wp}\right| - \frac{\tilde{\beta}}{2\tilde{\tau}}\right)^2 \geq \frac{1}{4\tilde{\tau}^4} - \frac{1}{4\tilde{\tau}^2}\left(\frac{1}{\tilde{\tau}^2} - \tilde{\beta}^2\right), \tag{4.15}$$

thus, the instability criterion is given as

$$\frac{\wp_e s_1 \Gamma(1+\sigma)}{\tau}\left|\frac{dV_e}{d\wp}\right| \geq \frac{\tilde{\beta}}{\tilde{\tau}} = \Gamma(1+\sigma)\left(cs_1 + \frac{\xi s_\mu}{\wp_e}s_2\right)\left(\frac{1}{\tau} - \frac{\xi D}{\wp_e}s_1^2\Gamma^2(1+\sigma)\right). \tag{4.16}$$

If there is a much greater variation in velocity due to an alteration in density, the instability condition is met. In the synchronised zone (moderate densities) of traffic flow, this situation is clearly visible. In either the free-flow or the crowded zone, the convergence of each vehicle's velocity to the steady-state velocity is achieved. However, the threshold for equilibrium traffic equals the value of the sonic speed c when the diffusion and viscosity rates are absent. The expression is given as

$$\wp_e\left|\frac{dV_e}{d\wp}\right| \geq c. \tag{4.17}$$

5. Local fractional Laplace variational iteration method

Consider a two-dimensional local fractional viscous-diffusive model of traffic flow given by (3.1) and (3.10) along with the initial condition $\wp(\omega,\mu,0) = g(\omega,\mu)$ and $v(\omega,\mu,0) = k(\omega,\mu)$.

Applying the Local fractional variational iteration method, the local fractional correctional functional corresponds to model Eqs (3.1) and (3.10) is expressed as [28]

$$\wp_{n+1}(\omega,\mu,\tau) = \wp_n(\omega,\mu,\tau) + {}_0J_\tau^\sigma\left\{\Lambda_1^\sigma(\omega,\mu,\theta)\left[\frac{\partial^\sigma \wp_n}{\partial\theta^\sigma} + v_n\frac{\partial^\sigma\tilde{\wp}_n}{\partial\omega^\sigma} + \wp_n\frac{\partial^\sigma\tilde{v}_n}{\partial\omega^\sigma}\right]\right\}, \tag{5.1}$$

$$\begin{aligned} v_{n+1}(\omega,\mu,\tau) &= v_n(\omega,\mu,\tau) + {}_0J_\tau^\sigma \\ &\left\{\Lambda_2^\sigma(\omega,\mu,\theta)\left[\frac{\partial^\sigma v_n}{\partial\theta^\sigma} + (v_n - c)\frac{\partial^\sigma\tilde{v}_n}{\partial\omega^\sigma} + \frac{\xi D}{\wp_n}\frac{\partial^{2\sigma}\tilde{v}_n}{\partial\omega^{2\sigma}} + \frac{\xi s_\mu}{\wp_n}\frac{\partial^\sigma\tilde{v}_n}{\partial\mu^\sigma} - \frac{\tilde{V}(\wp_n) - v_n}{\tau}\right]\right\}, \end{aligned} \tag{5.2}$$

where ${}_0J_\tau^\sigma$ denotes fractal integral. $\Lambda_1^\sigma(\omega,\mu,\theta)$ and $\Lambda_2^\sigma(\omega,\mu,\theta)$ represent fractal Lagrange multipliers. $\tilde{\wp}_n$ and $\tilde{v}_n$ are restricted local fractional variations, that is, $\delta^\sigma\tilde{\wp}_n = 0$ and $\delta^\sigma\tilde{v}_n = 0$.

Apply LFLT to (5.1) and (5.2), we have

$$T_\sigma\{\wp_{n+1}\} = T_\sigma\{\wp_n\} + T_\sigma\{\Lambda_1^\sigma\} T_\sigma \left\{ \frac{\partial^\sigma \wp_n}{\partial \theta^\sigma} + v_n \frac{\partial^\sigma \tilde{\wp}_n}{\partial \omega^\sigma} + \wp_n \frac{\partial^\sigma \tilde{v}_n}{\partial \omega^\sigma} \right\}, \tag{5.3}$$

$$\begin{aligned} &T_\sigma\{v_{n+1}\} = T_\sigma\{v_n\} + T_\sigma\{\Lambda_2^\sigma\} \\ &T_\sigma \left\{ \frac{\partial^\sigma v_n}{\partial \theta^\sigma} + (v_n - c) \frac{\partial^\sigma \tilde{v}_n}{\partial \omega^\sigma} + \frac{\xi}{\wp_n} \left(D \frac{\partial^{2\sigma} \tilde{v}_n}{\partial \omega^{2\sigma}} + s_\mu \frac{\partial^\sigma \tilde{v}_n}{\partial \mu^\sigma} \right) - \frac{\tilde{V}(\wp_n) - v_n}{\tau} \right\}. \end{aligned} \tag{5.4}$$

Taking fractal variations to (5.3) and (5.4), we obtain

$$\delta^\sigma T_\sigma\{\wp_{n+1}\} = \delta^\sigma T_\sigma\{\wp_n\} + T_\sigma\{\Lambda_1^\sigma\} \delta^\sigma T_\sigma \left\{ \frac{\partial^\sigma \wp_n}{\partial \theta^\sigma} + v_n \frac{\partial^\sigma \tilde{\wp}_n}{\partial \omega^\sigma} + \wp_n \frac{\partial^\sigma \tilde{v}_n}{\partial \omega^\sigma} \right\}, \tag{5.5}$$

$$\begin{aligned} &\delta^\sigma T_\sigma\{v_{n+1}\} = \delta^\sigma T_\sigma\{v_n\} + T_\sigma\{\Lambda_2^\sigma\} \\ &\delta^\sigma T_\sigma \left\{ \frac{\partial^\sigma v_n}{\partial \theta^\sigma} + (v_n - c) \frac{\partial^\sigma \tilde{v}_n}{\partial \omega^\sigma} + \frac{\xi}{\wp_n} \left(D \frac{\partial^{2\sigma} \tilde{v}_n}{\partial \omega^{2\sigma}} + s_\mu \frac{\partial^\sigma \tilde{v}_n}{\partial \mu^\sigma} \right) - \frac{\tilde{V}(\wp_n) - v_n}{\tau} \right\}, \end{aligned} \tag{5.6}$$

which leads

$$\delta^\sigma T_\sigma\{\wp_n\} \left(1 + T_\sigma\{\Lambda_1^\sigma\} s^\sigma\right) = 0, \tag{5.7}$$

$$\delta^\sigma T_\sigma\{v_n\} \left(1 + T_\sigma\{\Lambda_2^\sigma\} \left(s^\sigma + \frac{1}{\tau}\right)\right) = 0. \tag{5.8}$$

Thus, the LFLT of fractal Lagrange multipliers are given as

$$T_\sigma\{\Lambda_1^\sigma\} = -\frac{1}{s^\sigma}, \tag{5.9}$$

$$T_\sigma\{\Lambda_2^\sigma\} = -\frac{1}{s^\sigma + \frac{1}{\tau}}. \tag{5.10}$$

Therefore, the successive iterative formula is defined as

$$T_\sigma\{\wp_{n+1}\} = T_\sigma\{\wp_n\} - \frac{1}{s^\sigma} T_\sigma \left\{ \frac{\partial^\sigma \wp_n}{\partial \theta^\sigma} + v_n \frac{\partial^\sigma \tilde{\wp}_n}{\partial \omega^\sigma} + \wp_n \frac{\partial^\sigma \tilde{v}_n}{\partial \omega^\sigma} \right\}, \tag{5.11}$$

$$\begin{aligned} &T_\sigma\{v_{n+1}\} = T_\sigma\{v_n\} \\ &- \frac{1}{s^\sigma + \frac{1}{\tau}} T_\sigma \left\{ \frac{\partial^\sigma v_n}{\partial \theta^\sigma} + (v_n - c) \frac{\partial^\sigma \tilde{v}_n}{\partial \omega^\sigma} + \frac{\xi}{\wp_n} \left(D \frac{\partial^{2\sigma} \tilde{v}_n}{\partial \omega^{2\sigma}} + s_\mu \frac{\partial^\sigma \tilde{v}_n}{\partial \mu^\sigma} \right) - \frac{\tilde{V}(\wp_n) - v_n}{\tau} \right\}, \end{aligned} \tag{5.12}$$

along with the initial approximations

$$T_\sigma\{\wp_0\} = T_\sigma\{\wp(\omega, \mu, 0)\} = \wp_0(\omega, \mu, s), \tag{5.13}$$

$$T_\sigma\{v_0\} = T_\sigma\{v(\omega, \mu, 0)\} = v_0(\omega, \mu, s). \tag{5.14}$$

Thus, the fractal series solution to (3.1) and (3.10) are given as

$$T_\sigma\{\wp\} = \lim_{n\to\infty} T_\sigma\{\wp_n\} \ \textit{and}\ T_\sigma\{v\} = \lim_{n\to\infty} T_\sigma\{v_n\}. \tag{5.15}$$

Hence

$$\wp = \lim_{n\to\infty} T_\sigma^{-1}\{T_\sigma \wp_n\} \ \textit{and}\ v = \lim_{n\to\infty} T_\sigma^{-1}\{T_\sigma v_n\}. \tag{5.16}$$

6. Non-differentiable solutions to two dimensional local fractional viscous-diffusive traffic flow model

In this section, we offer several examples that illustrate LFLVIM solving the systems of a 2D local fractional viscous-diffusive traffic flow model and obtain non-differentiable solutions.

Example 6.1. Consider a 2D local fractional viscous-diffusive model of traffic flow

$$\frac{\partial^{\sigma}\wp}{\partial\tau^{\sigma}} + v\frac{\partial^{\sigma}\wp}{\partial\omega^{\sigma}} = 0, \tag{6.1}$$

$$\frac{\partial^{\sigma}v}{\partial\tau^{\sigma}} + (v - c)\frac{\partial^{\sigma}v}{\partial\omega^{\sigma}} + \frac{\xi}{\wp}\left(D\frac{\partial^{2\sigma}v}{\partial\omega^{2\sigma}} + s_{\mu}\frac{\partial^{\sigma}v}{\partial\mu^{\sigma}}\right) - \frac{\wp - v}{\tau} = 0, \tag{6.2}$$

along with the initial condition

$$\wp(\omega,\mu,0) = E_{\sigma}(\omega^{\sigma}) \;\; and \;\; v(\omega,\mu,0) = \omega^{\sigma}\mu^{\sigma}, \tag{6.3}$$

for all $\tau > 0$ $-\infty < \omega < \infty$, $-b \leq \mu \leq b$. Here, we consider $\tau = 10s\, and\, c = 11m/s$.

In association with (5.11) and (5.12), we have

$$T_{\sigma}\{\wp_{n+1}(\omega,\mu,\tau)\} = T_{\sigma}\{\wp_{n}(\omega,\mu,\tau)\} - \frac{1}{s^{\sigma}}T_{\sigma}\left\{\frac{\partial^{\sigma}\wp_{n}(\omega,\mu,\tau)}{\partial\tau^{\sigma}} + v_{n}\frac{\partial^{\sigma}\wp_{n}(\omega,\mu,\tau)}{\partial\omega^{\sigma}}\right\}, \tag{6.4}$$

$$\begin{aligned} T_{\sigma}\{v_{n+1}(\omega,\mu,\tau)\} = T_{\sigma}\{v_{n}(\omega,\mu,\tau)\} - \frac{1}{s^{\sigma}+\frac{1}{\tau}}T_{\sigma}\Big\{&\frac{\partial^{\sigma}v_{n}(\omega,\mu,\tau)}{\partial\tau^{\sigma}} + (v_{n}(\omega,\mu,\tau) - c) \\ &\times\frac{\partial^{\sigma}v_{n}(\omega,\mu,\tau)}{\partial\omega^{\sigma}} + \frac{\xi}{\wp_{n}}\left(D\frac{\partial^{2\sigma}v_{n}(\omega,\mu,\tau)}{\partial\omega^{2\sigma}} + s_{\mu}\frac{\partial^{\sigma}v_{n}(\omega,\mu,\tau)}{\partial\mu^{\sigma}}\right) \\ &-\frac{\wp_{n}(\omega,\mu,\tau) - v_{n}(\omega,\mu,\tau)}{\tau}\Big\}, \end{aligned} \tag{6.5}$$

with initial approximations

$$T_{\sigma}\{\wp_{0}(\omega,\mu,\tau)\} = \wp_{0}(\omega,\mu,s) = T_{\sigma}\{E_{\sigma}(\omega^{\sigma})\} = \frac{E_{\sigma}(\omega^{\sigma})}{s^{\sigma}}, \tag{6.6}$$

$$T_{\sigma}\{v_{0}(\omega,\mu,\tau)\} = v_{0}(\omega,\mu,s) = T_{\sigma}\{\omega^{\sigma}\mu^{\sigma}\} = \frac{\omega^{\sigma}\mu^{\sigma}}{s^{\sigma}}. \tag{6.7}$$

The first approximation to $\wp(\omega,\mu,\tau)$ is

$$\begin{aligned} T_{\sigma}\{\wp_{1}(\omega,\mu,\tau)\} &= T_{\sigma}\{\wp_{0}(\omega,\mu,\tau)\} - \frac{1}{s^{\sigma}}T_{\sigma}\left\{\frac{\partial^{\sigma}\wp_{0}(\omega,\mu,\tau)}{\partial\tau^{\sigma}} + v_{0}(\omega,\mu,\tau)\frac{\partial^{\sigma}\wp_{0}(\omega,\mu,\tau)}{\partial\omega^{\sigma}}\right\} \\ &= \wp_{0}(\omega,\mu,s) - \frac{1}{s^{\sigma}}\left\{s^{\sigma}\wp_{0}(\omega,\mu,s) - \wp_{0}(\omega,\mu,0) + v_{0}(\omega,\mu,s)\frac{\partial^{\sigma}\wp_{0}(\omega,\mu,s)}{\partial\omega^{\sigma}}\right\} \\ &= \frac{E_{\sigma}(\omega^{\sigma})}{s^{\sigma}} - \frac{\omega^{\sigma}\mu^{\sigma}E_{\sigma}(\omega^{\sigma})}{s^{3\sigma}}, \end{aligned} \tag{6.8}$$

which gives

$$v_{0}(\omega,\mu,\tau) = T_{\sigma}^{-1}\left\{\frac{E_{\sigma}(\omega^{\sigma})}{s^{\sigma}} - \frac{\omega^{\sigma}\mu^{\sigma}E_{\sigma}(\omega^{\sigma})}{s^{3\sigma}}\right\} = E_{\sigma}(\omega^{\sigma})\left\{1 - \frac{\omega^{\sigma}\mu^{\sigma}\tau^{\sigma}}{\Gamma(1+2\sigma)}\right\}. \tag{6.9}$$

The first approximation to $v(\omega,\mu,\tau)$ is

$$
\begin{aligned}
T_\sigma\{v_1(\omega,\mu,\tau)\} &= T_\sigma\{v_0(\omega,\mu,\tau)\} - \frac{1}{s^\sigma+\frac{1}{\tau}}T_\sigma\Big\{\frac{\partial^\sigma v_0(\omega,\mu,\tau)}{\partial\tau^\sigma} + (v_0(\omega,\mu,\tau)-c)\\
&\times\frac{\partial^\sigma v_0(\omega,\mu,\tau)}{\partial\omega^\sigma} + \frac{\xi}{\wp_0}\left(D\frac{\partial^{2\sigma}v_0(\omega,\mu,\tau)}{\partial\omega^{2\sigma}} + s_\mu\frac{\partial^\sigma v_0(\omega,\mu,\tau)}{\partial\mu^\sigma}\right)\\
&- \frac{\wp_0(\omega,\mu,\tau)-v_0(\omega,\mu,\tau)}{\tau}\Big\}\\
&= v_0(\omega,\mu,s) - \frac{1}{s^\sigma+0.1}\Big\{s^\sigma v_0(\omega,\mu,s) - v_0(\omega,\mu,0) + (v_0(\omega,\mu,s)-c)\\
&\times\frac{\partial^\sigma v_0(\omega,\mu,s)}{\partial\omega^\sigma} + \frac{\xi}{\wp_0(\omega,\mu,s)}\left(D\frac{\partial^{2\sigma}v_0(\omega,\mu,s)}{\partial\omega^{2\sigma}} + s_\mu\frac{\partial^\sigma v_0(\omega,\mu,s)}{\partial\mu^\sigma}\right)\\
&- \frac{\wp_0(\omega,\mu,s)-v_0(\omega,\mu,s)}{\tau}\Big\}\\
&= \frac{\omega^\sigma\mu^\sigma}{s^\sigma} - \frac{1}{s^\sigma+0.1}\Big\{\left(\frac{\omega^\sigma\mu^\sigma}{s^\sigma}-11\right)\frac{\mu^\sigma\Gamma(1+\sigma)}{s^\sigma} + \frac{\xi s_\mu\omega^\sigma\Gamma(1+\sigma)}{s^\sigma E_\sigma(\omega^\sigma)}\\
&- \frac{0.1}{s^\sigma}(E_\sigma(\omega^\sigma)-\omega^\sigma\mu^\sigma)\Big\},
\end{aligned}
\tag{6.10}
$$

which implies

$$
\begin{aligned}
v_1(\omega,\mu,\tau) &= \omega^\sigma\mu^\sigma - \omega^\sigma\mu^{2\sigma}\Gamma(1+\sigma)\left[100E_\sigma(-0.1\tau^\sigma) + \frac{10\tau^\sigma}{\Gamma(1+\sigma)} - 100\right]\\
&+ 110\mu^\sigma\Gamma(1+\sigma)[1-E_\sigma(-0.1\tau^\sigma)] + 10s_\mu\xi\Gamma(1+\sigma)\frac{\omega^\sigma}{E_\sigma(\omega^\sigma)}[1-E_\sigma(-0.1\tau^\sigma)]\\
&- (E_\sigma(\omega^\sigma)-\omega^\sigma\mu^\sigma)(1-E_\sigma(-0.1\tau^\sigma)).
\end{aligned}
\tag{6.11}
$$

The second approximation to $\wp(\omega,\mu,\tau)$ is

$$
\begin{aligned}
T_\sigma\{\wp_2(\omega,\mu,\tau)\} &= T_\sigma\{\wp_1(\omega,\mu,\tau)\} - \frac{1}{s^\sigma}T_\sigma\left\{\frac{\partial^\sigma\wp_1(\omega,\mu,\tau)}{\partial\tau^\sigma} + v_1(\omega,\mu,\tau)\frac{\partial^\sigma\wp_1(\omega,\mu,\tau)}{\partial\omega^\sigma}\right\}\\
&= \wp_1(\omega,\mu,s) - \frac{1}{s^\sigma}\left\{s^\sigma\wp_1(\omega,\mu,s) - \wp_1(\omega,\mu,0) + v_1(\omega,\mu,s)\frac{\partial^\sigma\wp_1(\omega,\mu,s)}{\partial\omega^\sigma}\right\}\\
&= \omega^\sigma\mu^\sigma E_\sigma(\omega^\sigma)\Big\{-\frac{1}{s^{3\sigma}} + \Gamma(1+\sigma)\frac{\mu^\sigma}{s^{5\sigma}} + \frac{\omega^\sigma\mu^\sigma}{s^{5\sigma}} + \frac{\Gamma(1+\sigma)\mu^\sigma}{s^{4\sigma}(s^\sigma+0.1)}\\
&- \frac{\Gamma^2(1+\sigma)\mu^{3\sigma}}{s^{6\sigma}(s^\sigma+0.1)} - \frac{\Gamma(1+\sigma)\omega^\sigma\mu^{2\sigma}}{s^{6\sigma}(s^\sigma+0.1)} + \frac{11\Gamma(1+\sigma)}{s^{5\sigma}(s^\sigma+0.1)}\Big\} + E_\sigma(\omega^\sigma)\Big\{\frac{1}{s^\sigma}\\
&- \frac{11\Gamma(1+\sigma)\mu^\sigma}{s^{3\sigma}(s^\sigma+0.1)} + \frac{11\Gamma^2(1+\sigma)\mu^{2\sigma}}{s^{5\sigma}(s^\sigma+0.1)}\Big\} + s_\mu\xi\omega^\sigma\Gamma(1+\sigma)\Big\{-\frac{1}{s^{3\sigma}(s^\sigma+0.1)}\\
&+ \frac{\Gamma(1+\sigma)\mu^\sigma}{s^{5\sigma}(s^\sigma+0.1)} + \frac{\omega^\sigma\mu^\sigma}{s^{5\sigma}(s^\sigma+0.1)}\Big\} + 0.1E_\sigma(\omega^\sigma)[E_\sigma(\omega^\sigma)-\omega^\sigma\mu^\sigma]\\
&\times\left\{\frac{1}{s^{3\sigma}(s^\sigma+0.1)} - \frac{\Gamma(1+\sigma)\mu^\sigma}{s^{5\sigma}(s^\sigma+0.1)} + \frac{\omega^\sigma\mu^\sigma}{s^{5\sigma}(s^\sigma+0.1)}\right\},
\end{aligned}
\tag{6.12}
$$

which implies

$$\begin{aligned}
\wp_2(\omega,\mu,\tau) &= \omega^\sigma\mu^\sigma E_\sigma(\omega^\sigma)\Big\{-\frac{\tau^{2\sigma}}{\Gamma(1+2\sigma)}+\frac{\tau^{4\sigma}\mu^\sigma}{\Gamma(1+4\sigma)}\left[\Gamma(1+\sigma)+\omega^\sigma\right]\\
&+\Gamma(1+\sigma)\mu^\sigma\tau^{5\sigma-1}E_{\sigma,5\sigma}(-0.1\tau^\sigma)-\Gamma(1+\sigma)\mu^{2\sigma}\tau^{7\sigma-1}E_{\sigma,7\sigma}(-0.1\tau^\sigma)\\
&\times\left[\Gamma(1+\sigma)\mu^\sigma+\omega^\sigma\right]+11\Gamma(1+\sigma)\tau^{6\sigma-1}E_{\sigma,6\sigma}(-0.1\tau^\sigma)\Big\}+E_\sigma(\omega^\sigma)\\
&\times\Big\{1-11\Gamma(1+\sigma)\mu^\sigma\tau^{4\sigma-1}E_{\sigma,4\sigma}(-0.1\tau^\sigma)+11\Gamma^2(1+\sigma)\mu^{2\sigma}\tau^{6\sigma-1}E_{\sigma,6\sigma}(-0.1\tau^\sigma)\Big\}\\
&+s_\mu\xi(1+\sigma)\omega^\sigma\Big\{-\tau^{4\sigma-1}E_{\sigma,4\sigma}(-0.1\tau^\sigma)+\mu^\sigma\tau^{6\sigma-1}E_{\sigma,6\sigma}(-0.1\tau^\sigma)\left[\Gamma(1+\sigma)+\omega^\sigma\right]\Big\}\\
&+0.1E_\sigma(\omega^\sigma)\left[E_\sigma(\omega^\sigma)-\omega^\sigma\mu^\sigma\right]\Big\{\tau^{4\sigma-1}E_{\sigma,4\sigma}(-0.1\tau^\sigma)-\mu^\sigma\tau^{6\sigma-1}E_{\sigma,6\sigma}(-0.1\tau^\sigma)\\
&\times\left[\Gamma(1+\sigma)-\omega^\sigma\right]\Big\}.
\end{aligned} \tag{6.13}$$

The second approximation to $v(\omega,\mu,\tau)$ is

$$\begin{aligned}
T_\sigma\{v_2(\omega,\mu,\tau)\} &= T_\sigma\{v_1(\omega,\mu,\tau)\}-\frac{1}{s^\sigma+\frac{1}{\tau}}T_\sigma\Big\{\frac{\partial^\sigma v_1(\omega,\mu,\tau)}{\partial\tau^\sigma}+(v_1(\omega,\mu,\tau)-c)\\
&\times\frac{\partial^\sigma v_1(\omega,\mu,\tau)}{\partial\omega^\sigma}+\frac{\xi}{\wp_1}\left(D\frac{\partial^{2\sigma}v_1(\omega,\mu,\tau)}{\partial\omega^{2\sigma}}+s_\mu\frac{\partial^\sigma v_1(\omega,\mu,\tau)}{\partial\mu^\sigma}\right)\\
&-\frac{\wp_1(\omega,\mu,\tau)-v_1(\omega,\mu,\tau)}{\tau}\Big\}\\
&= v_1(\omega,\mu,s)-\frac{1}{s^\sigma+0.1}\Big\{s^\sigma v_1(\omega,\mu,s)-v_1(\omega,\mu,0)+(v_1(\omega,\mu,s)-c)\\
&\times\frac{\partial^\sigma v_1(\omega,\mu,s)}{\partial\omega^\sigma}+\frac{\xi}{\wp_1(\omega,\mu,s)}\left(D\frac{\partial^{2\sigma}v_1(\omega,\mu,s)}{\partial\omega^{2\sigma}}+s_\mu\frac{\partial^\sigma v_1(\omega,\mu,s)}{\partial\mu^\sigma}\right)\\
&-\frac{\wp_1(\omega,\mu,s)-v_1(\omega,\mu,s)}{\tau}\Big\}\\
&= v_1(\omega,\mu,s)-\frac{1}{s^\sigma+0.1}\Big\{\omega^\sigma\mu^\sigma-\frac{\Gamma(1+\sigma)\omega^\sigma\mu^{2\sigma}}{s^\sigma(s^\sigma+0.1)}+\frac{11\Gamma(1+\sigma)\mu^\sigma}{(s^\sigma+0.1)}\\
&+\frac{\Gamma(1+\sigma)\omega^\sigma s_\mu\xi}{E_\sigma(\omega^\sigma)(s^\sigma+0.1)}-\frac{0.1\left[E_\sigma(\omega^\sigma)-\omega^\sigma\mu^\sigma\right]}{(s^\sigma+0.1)}-\omega^\sigma\mu^\sigma+\Big(\frac{\omega^\sigma\mu^\sigma}{s^\sigma}\\
&-\frac{\Gamma(1+\sigma)\omega^\sigma\mu^{2\sigma}}{s^{2\sigma}(s^\sigma+0.1)}+\frac{11\Gamma(1+\sigma)\mu^\sigma}{s^\sigma(s^\sigma+0.1)}+\frac{\Gamma(1+\sigma)\omega^\sigma s_\mu\xi}{E_\sigma(\omega^\sigma)s^\sigma(s^\sigma+0.1)}\\
&-\frac{0.1\left[E_\sigma(\omega^\sigma)-\omega^\sigma\mu^\sigma\right]}{s^\sigma(s^\sigma+0.1)}-11\Big)\Big(\frac{\Gamma(1+\sigma)\mu^\sigma}{s^\sigma}-\frac{\Gamma^2(1+\sigma)\mu^{2\sigma}}{s^{2\sigma}(s^\sigma+0.1)}\\
&+\frac{\Gamma^2(1+\sigma)s_\mu\xi}{E_\sigma(\omega^\sigma)s^\sigma(s^\sigma+0.1)}-\frac{\Gamma(1+\sigma)s_\mu\xi\omega^\sigma}{E_\sigma(\omega^\sigma)s^\sigma(s^\sigma+0.1)}-\frac{0.1\left[E_\sigma(\omega^\sigma)-\Gamma(1+\sigma)\mu^\sigma\right]}{s^\sigma(s^\sigma+0.1)}\Big)\\
&+\frac{\Gamma(1+\sigma)s_\mu\xi\omega^\sigma}{E_\sigma(\omega^\sigma)s^\sigma(s^\sigma+0.1)}\\
&\times\Big[D\left(\frac{\Gamma(1+\sigma)s_\mu\xi}{E_\sigma(\omega^\sigma)s^\sigma(s^\sigma+0.1)}(-2\Gamma(1+\sigma)+\omega^\sigma)-\frac{0.1E_\sigma(\omega^\sigma)}{s^\sigma(s^\sigma+0.1)}\right)\\
&+s_\mu\left(\frac{\Gamma(1+\sigma)\omega^\sigma}{s^\sigma}-\frac{\Gamma(1+2\sigma)\omega^\sigma\mu^\sigma}{s^{2\sigma}(s^\sigma+0.1)}+\frac{11\Gamma^2(1+\sigma)}{s^\sigma(s^\sigma+0.1)}+\frac{0.1\Gamma(1+\sigma)\omega^\sigma}{s^\sigma(s^\sigma+0.1)}\right)\Big]
\end{aligned}$$

$$- 0.1\Big(\frac{E_\sigma(\omega^\sigma)}{s^\sigma} - \frac{E_\sigma(\omega^\sigma)\,\omega^\sigma\mu^\sigma}{s^{3\sigma}} - \frac{\omega^\sigma\mu^\sigma}{s^\sigma} + \frac{\Gamma(1+\sigma)\,\omega^\sigma\mu^{2\sigma}}{s^{2\sigma}(s^\sigma+0.1)} - \frac{11\Gamma(1+\sigma)\,\mu^\sigma}{s^\sigma(s^\sigma+0.1)}$$
$$- \frac{\Gamma(1+\sigma)\,s_\mu\xi\omega^\sigma}{E_\sigma(\omega^\sigma)\,s^\sigma(s^\sigma+0.1)} + \frac{0.1\left[E_\sigma(\omega^\sigma) - \omega^\sigma\mu^\sigma\right]}{s^\sigma(s^\sigma+0.1)}\Big)\Big\}, \tag{6.14}$$

this implies

$$\begin{aligned} v_2(\omega,\mu,\tau) = T_\sigma^{-1}\Big\{& v_1(\omega,\mu,s) - \frac{1}{s^\sigma+0.1}\Big\{\omega^\sigma\mu^\sigma - \frac{\Gamma(1+\sigma)\,\omega^\sigma\mu^{2\sigma}}{s^\sigma(s^\sigma+0.1)} + \frac{11\Gamma(1+\sigma)\,\mu^\sigma}{(s^\sigma+0.1)} \\ &+ \frac{\Gamma(1+\sigma)\,\omega^\sigma s_\mu\xi}{E_\sigma(\omega^\sigma)(s^\sigma+0.1)} - \frac{0.1\left[E_\sigma(\omega^\sigma) - \omega^\sigma\mu^\sigma\right]}{(s^\sigma+0.1)} - \omega^\sigma\mu^\sigma + \Big(\frac{\omega^\sigma\mu^\sigma}{s^\sigma} - \frac{\Gamma(1+\sigma)\,\omega^\sigma\mu^{2\sigma}}{s^{2\sigma}(s^\sigma+0.1)} \\ &+ \frac{11\Gamma(1+\sigma)\,\mu^\sigma}{s^\sigma(s^\sigma+0.1)} + \frac{\Gamma(1+\sigma)\,\omega^\sigma s_\mu\xi}{E_\sigma(\omega^\sigma)\,s^\sigma(s^\sigma+0.1)} - \frac{0.1\left[E_\sigma(\omega^\sigma) - \omega^\sigma\mu^\sigma\right]}{s^\sigma(s^\sigma+0.1)} - 11\Big) \\ &\times\Big(\frac{\Gamma(1+\sigma)\,\mu^\sigma}{s^\sigma} - \frac{\Gamma^2(1+\sigma)\,\mu^{2\sigma}}{s^{2\sigma}(s^\sigma+0.1)} + \frac{\Gamma^2(1+\sigma)\,s_\mu\xi}{E_\sigma(\omega^\sigma)\,s^\sigma(s^\sigma+0.1)} - \frac{\Gamma(1+\sigma)\,s_\mu\xi\omega^\sigma}{E_\sigma(\omega^\sigma)\,s^\sigma(s^\sigma+0.1)} \\ &- \frac{0.1\left[E_\sigma(\omega^\sigma) - \Gamma(1+\sigma)\,\mu^\sigma\right]}{s^\sigma(s^\sigma+0.1)}\Big) + \frac{\Gamma(1+\sigma)\,s_\mu\xi\omega^\sigma}{E_\sigma(\omega^\sigma)\,s^\sigma(s^\sigma+0.1)} \\ &\times\Big[D\left(\frac{\Gamma(1+\sigma)\,s_\mu\xi}{E_\sigma(\omega^\sigma)\,s^\sigma(s^\sigma+0.1)}\left(-2\Gamma(1+\sigma)+\omega^\sigma\right) - \frac{0.1E_\sigma(\omega^\sigma)}{s^\sigma(s^\sigma+0.1)}\right) \\ &+ s_\mu\left(\frac{\Gamma(1+\sigma)\,\omega^\sigma}{s^\sigma} - \frac{\Gamma(1+2\sigma)\,\omega^\sigma\mu^\sigma}{s^{2\sigma}(s^\sigma+0.1)} + \frac{11\Gamma^2(1+\sigma)}{s^\sigma(s^\sigma+0.1)} + \frac{0.1\Gamma(1+\sigma)\,\omega^\sigma}{s^\sigma(s^\sigma+0.1)}\right)\Big] \\ &- 0.1\Big(\frac{E_\sigma(\omega^\sigma)}{s^\sigma} - \frac{E_\sigma(\omega^\sigma)\,\omega^\sigma\mu^\sigma}{s^{3\sigma}} - \frac{\omega^\sigma\mu^\sigma}{s^\sigma} + \frac{\Gamma(1+\sigma)\,\omega^\sigma\mu^{2\sigma}}{s^{2\sigma}(s^\sigma+0.1)} - \frac{11\Gamma(1+\sigma)\,\mu^\sigma}{s^\sigma(s^\sigma+0.1)} \\ &- \frac{\Gamma(1+\sigma)\,s_\mu\xi\omega^\sigma}{E_\sigma(\omega^\sigma)\,s^\sigma(s^\sigma+0.1)} + \frac{0.1\left[E_\sigma(\omega^\sigma) - \omega^\sigma\mu^\sigma\right]}{s^\sigma(s^\sigma+0.1)}\Big)\Big\}\Big\}. \end{aligned} \tag{6.15}$$

Proceeding in the same way,the fractal series solution of the systems (6.1) and (6.2) is given as

$$T_\sigma\{\wp\} = \lim_{n\to\infty} T_\sigma\{\wp_n\} \;\; and \;\; T_\sigma\{v\} = \lim_{n\to\infty} T_\sigma\{v_n\}. \tag{6.16}$$

Thus, we have

$$\wp = \lim_{n\to\infty} T_\sigma^{-1}\{T_\sigma\wp_n\} \;\; and \;\; v = \lim_{n\to\infty} T_\sigma^{-1}\{T_\sigma v_n\}. \tag{6.17}$$

Example 6.2. Consider a 2D local fractional viscous-diffusive model of traffic flow

$$\frac{\partial^\sigma\wp}{\partial\tau^\sigma} + v\frac{\partial^\sigma\wp}{\partial\omega^\sigma} = 0, \tag{6.18}$$

$$\frac{\partial^\sigma v}{\partial\tau^\sigma} + (v-c)\frac{\partial^\sigma v}{\partial\omega^\sigma} + \frac{\xi}{\wp}\left(D\frac{\partial^{2\sigma} v}{\partial\omega^{2\sigma}} + s_\mu\frac{\partial^\sigma v}{\partial\mu^\sigma}\right) - \frac{\wp^\sigma - v}{\tau} = 0, \tag{6.19}$$

along with the initial condition

$$\wp(\omega,\mu,0) = E_\sigma(\omega^\sigma\mu^\sigma) \;\; and \;\; v(\omega,\mu,0) = \omega^{2\sigma}, \tag{6.20}$$

for all $\tau > 0$ $-\infty < \omega < \infty$, $-b \le \mu \le b$. Here, we consider $\tau = 10s\, and\, c = 11m/s$.

In association with (5.11) and (5.12), we have

$$T_\sigma\{\wp_{n+1}(\omega,\mu,\tau)\} = T_\sigma\{\wp_n(\omega,\mu,\tau)\} - \frac{1}{s^\sigma}T_\sigma\left\{\frac{\partial^\sigma \wp_n(\omega,\mu,\tau)}{\partial\tau^\sigma} + v_n\frac{\partial^\sigma \wp_n(\omega,\mu,\tau)}{\partial\omega^\sigma}\right\}, \tag{6.21}$$

$$\begin{aligned} T_\sigma\{v_{n+1}(\omega,\mu,\tau)\} = T_\sigma\{v_n(\omega,\mu,\tau)\} - \frac{1}{s^\sigma+\frac{1}{\tau}}T_\sigma\Big\{&\frac{\partial^\sigma v_n(\omega,\mu,\tau)}{\partial\tau^\sigma} + (v_n(\omega,\mu,\tau)-c)\\ &\times\frac{\partial^\sigma v_n(\omega,\mu,\tau)}{\partial\omega^\sigma} + \frac{\xi}{\wp_n}\left(D\frac{\partial^{2\sigma} v_n(\omega,\mu,\tau)}{\partial\omega^{2\sigma}} + s_\mu\frac{\partial^\sigma v_n(\omega,\mu,\tau)}{\partial\mu^\sigma}\right)\\ &-\frac{{\wp_n}^\sigma(\omega,\mu,\tau)-v_n(\omega,\mu,\tau)}{\tau}\Big\}, \end{aligned} \tag{6.22}$$

with initial approximations

$$T_\sigma\{\wp_0(\omega,\mu,\tau)\} = \wp_0(\omega,\mu,s) = T_\sigma\{E_\sigma(\omega^\sigma)\} = \frac{E_\sigma(\omega^\sigma\mu^\sigma)}{s^\sigma}, \tag{6.23}$$

$$T_\sigma\{v_0(\omega,\mu,\tau)\} = v_0(\omega,\mu,s) = T_\sigma\left\{\omega^{2\sigma}\right\} = \frac{\omega^\sigma}{s^\sigma}. \tag{6.24}$$

The first approximation to $\wp(\omega,\mu,\tau)$ is

$$\begin{aligned} T_\sigma\{\wp_1(\omega,\mu,\tau)\} &= T_\sigma\{\wp_0(\omega,\mu,\tau)\} - \frac{1}{s^\sigma}T_\sigma\left\{\frac{\partial^\sigma \wp_0(\omega,\mu,\tau)}{\partial\tau^\sigma} + v_0(\omega,\mu,\tau)\frac{\partial^\sigma \wp_0(\omega,\mu,\tau)}{\partial\omega^\sigma}\right\}\\ &= \wp_0(\omega,\mu,s) - \frac{1}{s^\sigma}\left\{s^\sigma\wp_0(\omega,\mu,s) - \wp_0(\omega,\mu,0) + v_0(\omega,\mu,s)\frac{\partial^\sigma \wp_0(\omega,\mu,s)}{\partial\omega^\sigma}\right\}\\ &= \frac{E_\sigma(\omega^\sigma\mu^\sigma)}{s^\sigma} - \frac{\omega^{2\sigma}\mu^\sigma\Gamma(1+\sigma)E_\sigma(\omega^\sigma\mu^\sigma)}{s^{3\sigma}}, \end{aligned} \tag{6.25}$$

this implies

$$\begin{aligned} \wp_1(\omega,\mu,\tau) &= T_\sigma^{-1}\left\{\frac{E_\sigma(\omega^\sigma\mu^\sigma)}{s^\sigma} - \frac{\omega^{2\sigma}\mu^\sigma\Gamma(1+\sigma)E_\sigma(\omega^\sigma\mu^\sigma)}{s^{3\sigma}}\right\}\\ &= E_\sigma(\omega^\sigma\mu^\sigma)\left\{1 - \frac{\omega^{2\sigma}\mu^\sigma\tau^{2\sigma}\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\right\}. \end{aligned} \tag{6.26}$$

The first approximation to $v(\omega,\mu,\tau)$ is

$$\begin{aligned} T_\sigma\{v_1(\omega,\mu,\tau)\} &= T_\sigma\{v_0(\omega,\mu,\tau)\} - \frac{1}{s^\sigma+\frac{1}{\tau}}T_\sigma\Big\{\frac{\partial^\sigma v_0(\omega,\mu,\tau)}{\partial\tau^\sigma} + (v_0(\omega,\mu,\tau)-c)\\ &\times\frac{\partial^\sigma v_0(\omega,\mu,\tau)}{\partial\omega^\sigma} + \frac{\xi}{\wp_0}\left(D\frac{\partial^{2\sigma} v_0(\omega,\mu,\tau)}{\partial\omega^{2\sigma}} + s_\mu\frac{\partial^\sigma v_0(\omega,\mu,\tau)}{\partial\mu^\sigma}\right)\\ &-\frac{{\wp_0}^\sigma(\omega,\mu,\tau)-v_0(\omega,\mu,\tau)}{\tau}\Big\}\\ &= v_0(\omega,\mu,s) - \frac{1}{s^\sigma+0.1}\Big\{s^\sigma v_0(\omega,\mu,s) - v_0(\omega,\mu,0) + (v_0(\omega,\mu,s)-c)\\ &\times\frac{\partial^\sigma v_0(\omega,\mu,s)}{\partial\omega^\sigma} + \frac{\xi}{\wp_0(\omega,\mu,s)}\left(D\frac{\partial^{2\sigma} v_0(\omega,\mu,s)}{\partial\omega^{2\sigma}} + s_\mu\frac{\partial^\sigma v_0(\omega,\mu,s)}{\partial\mu^\sigma}\right) \end{aligned}$$

$$
\begin{aligned}
&- \frac{\wp_0{}^{\sigma}(\omega,\mu,s) - v_0(\omega,\mu,s)}{\tau}\Big\} \\
&= \frac{\omega^{2\sigma}}{s^{\sigma}} - \frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)} \frac{\omega^{3\sigma}}{s^{2\sigma}(s^{\sigma}+0.1)} + \frac{1}{s^{\sigma}(s^{\sigma}+0.1)} \left[\frac{11\Gamma(1+2\sigma)}{\Gamma(1+\sigma)}\omega^{\sigma} - 0.1\omega^{2\sigma} \right] \\
&- \frac{\xi D\Gamma(1+2\sigma)}{(s^{\sigma}+0.1)E_{\sigma}(\omega^{\sigma}\mu^{\sigma})} + \frac{0.1(E_{\sigma}(\omega^{\sigma}\mu^{\sigma}))^{\sigma}}{s^{\sigma^2}(s^{\sigma}+0.1)},
\end{aligned} \tag{6.27}
$$

which gives

$$
\begin{aligned}
v_1(\omega,\mu,\tau) &= \omega^{2\sigma} - \frac{\Gamma(1+2\sigma)\omega^{3\sigma}}{\Gamma(1+\sigma)} \left[100E_{\sigma}(-0.1\tau^{\sigma}) + \frac{10\tau^{\sigma}}{\Gamma(1+\sigma)} - 100 \right] \\
&+ 10(1 - E_{\sigma}(-0.1\tau^{\sigma})) \left[\frac{11\Gamma(1+2\sigma)}{\Gamma(1+\sigma)}\omega^{\sigma} - 0.1\omega^{2\sigma} \right] - \frac{\xi D\Gamma(1+2\sigma)E_{\sigma}(-0.1\tau^{\sigma})}{E_{\sigma}(\omega^{\sigma}\mu^{\sigma})} \\
&+ 0.1(E_{\sigma}(\omega^{\sigma}\mu^{\sigma}))^{\sigma}\tau^{\sigma+\sigma^2-1}E_{\sigma,\sigma+\sigma^2}(-0.1\tau^{\sigma}).
\end{aligned} \tag{6.28}
$$

The second approximation to $\wp(\omega,\mu,\tau)$ is given as

$$
\begin{aligned}
T_{\sigma}\{\wp_2(\omega,\mu,\tau)\} &= T_{\sigma}\{\wp_1(\omega,\mu,\tau)\} - \frac{1}{s^{\sigma}}T_{\sigma}\left\{ \frac{\partial^{\sigma}\wp_1(\omega,\mu,\tau)}{\partial\tau^{\sigma}} + v_1(\omega,\mu,\tau)\frac{\partial^{\sigma}\wp_1(\omega,\mu,\tau)}{\partial\omega^{\sigma}} \right\} \\
&= \wp_1(\omega,\mu,s) - \frac{1}{s^{\sigma}}\left\{ s^{\sigma}\wp_1(\omega,\mu,s) - \wp_1(\omega,\mu,0) + v_1(\omega,\mu,s)\frac{\partial^{\sigma}\wp_1(\omega,\mu,s)}{\partial\omega^{\sigma}} \right\} \\
&= \frac{E_{\sigma}(\omega^{\sigma}\mu^{\sigma})}{s^{\sigma}} - \frac{\omega^{2\sigma}\mu^{\sigma}\Gamma(1+\sigma)E_{\sigma}(\omega^{\sigma}\mu^{\sigma})}{s^{3\sigma}} - \frac{\omega^{\sigma}\mu^{\sigma}}{s^{4\sigma}(s^{\sigma}+0.1)}\Big(-\Gamma(1+2\sigma)\omega^{2\sigma} \\
&\times E_{\sigma}(\omega^{\sigma}\mu^{\sigma}) + \xi D\omega^{\sigma}\mu^{\sigma}\Gamma(1+2\sigma)\Gamma^2(1+\sigma) + \xi D\Gamma^2(1+2\sigma)\Big) \\
&+ \frac{\omega^{\sigma}\mu^{\sigma}E_{\sigma}(\omega^{\sigma}\mu^{\sigma})}{s^{5\sigma}(s^{\sigma}+0.1)} \left[\frac{11\Gamma(1+2\sigma)}{\Gamma(1+\sigma)}\omega^{\sigma} - 0.1\omega^{2\sigma} \right] \Big(\omega^{3\sigma}\mu^{\sigma}\Gamma^2(1+\sigma) + \Gamma(1+2\sigma) \Big) \\
&- \frac{\mu^{\sigma}\Gamma(1+\sigma)E_{\sigma}(\omega^{\sigma}\mu^{\sigma})}{s^{3\sigma}(s^{\sigma}+0.1)} \left[\frac{11\Gamma(1+2\sigma)}{\Gamma(1+\sigma)}\omega^{\sigma} - 0.1\omega^{2\sigma} \right] \\
&+ \frac{\xi D\mu^{\sigma}\Gamma(1+\sigma)\Gamma(1+2\sigma)}{s^{2\sigma}(s^{\sigma}+0.1)} - \frac{0.1\mu^{\sigma}\Gamma(1+\sigma)(E_{\sigma}(\omega^{\sigma}\mu^{\sigma}))^{\sigma+1}}{s^{\sigma^2+2\sigma}(s^{\sigma}+0.1)} \\
&+ \frac{\omega^{3\sigma}\mu^{\sigma}E_{\sigma}(\omega^{\sigma}\mu^{\sigma})}{s^{5\sigma}}\Big(\omega^{\sigma}\mu^{\sigma}\Gamma^2(1+\sigma) + \Gamma(1+2\sigma) \Big) \\
&- \frac{\omega^{4\sigma}\mu^{\sigma}\Gamma(1+2\sigma)E_{\sigma}(\omega^{\sigma}\mu^{\sigma})}{s^{6\sigma}(s^{\sigma}+0.1)} \left(\omega^{\sigma}\mu^{\sigma}\Gamma(1+\sigma) + \frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)} \right) \\
&+ \frac{0.1\omega^{\sigma}\mu^{\sigma}(E_{\sigma}(\omega^{\sigma}\mu^{\sigma}))^{\sigma+1}}{s^{\sigma^2+4\sigma}(s^{\sigma}+0.1)} \Big(\Gamma^2(1+\sigma)\omega^{\sigma}\mu^{\sigma} + \Gamma(1+2\sigma) \Big),
\end{aligned} \tag{6.29}
$$

which implies

$$
\begin{aligned}
\wp_2(\omega,\mu,\tau) &= E_{\sigma}(\omega^{\sigma}\mu^{\sigma}) - \frac{\tau^{2\sigma}\omega^{2\sigma}\mu^{\sigma}\Gamma(1+\sigma)E_{\sigma}(\omega^{\sigma}\mu^{\sigma})}{\Gamma(1+2\sigma)} - \tau^{5\sigma-1}E_{\sigma,5\sigma}(-0.1\tau^{\sigma})\omega^{\sigma}\mu^{\sigma} \\
&\times \Big(-\Gamma(1+2\sigma)\omega^{2\sigma}E_{\sigma}(\omega^{\sigma}\mu^{\sigma}) + \xi D\omega^{\sigma}\mu^{\sigma}\Gamma(1+2\sigma)\Gamma^2(1+\sigma) + \xi D\Gamma^2(1+2\sigma) \Big)
\end{aligned}
$$

$$+\tau^{6\sigma-1}E_{\sigma,6\sigma}\left(-0.1\tau^{\sigma}\right)\omega^{\sigma}\mu^{\sigma}E_{\sigma}\left(\omega^{\sigma}\mu^{\sigma}\right)\left[\frac{11\Gamma\left(1+2\sigma\right)}{\Gamma\left(1+\sigma\right)}\omega^{\sigma}-0.1\omega^{2\sigma}\right]\Big(\omega^{3\sigma}\mu^{\sigma}\Gamma^{2}\left(1+\sigma\right)$$
$$+\Gamma\left(1+2\sigma\right)\Big)-\tau^{4\sigma-1}E_{\sigma,4\sigma}\left(-0.1\tau^{\sigma}\right)\mu^{\sigma}\Gamma\left(1+\sigma\right)E_{\sigma}\left(\omega^{\sigma}\mu^{\sigma}\right)\Big[\frac{11\Gamma\left(1+2\sigma\right)}{\Gamma\left(1+\sigma\right)}\omega^{\sigma}$$
$$-0.1\omega^{2\sigma}\Big]+\xi D\mu^{\sigma}\Gamma\left(1+\sigma\right)\Gamma\left(1+2\sigma\right)\left[100E_{\sigma}\left(-0.1\tau^{\sigma}\right)+\frac{10\tau^{\sigma}}{\Gamma\left(1+\sigma\right)}-100\right]$$
$$-0.1\tau^{\sigma^{2}+3\sigma-1}E_{\sigma,\sigma^{2}+3\sigma}\left(-0.1\tau^{\sigma}\right)\mu^{\sigma}\Gamma\left(1+\sigma\right)\left(E_{\sigma}\left(\omega^{\sigma}\mu^{\sigma}\right)\right)^{\sigma+1}+\frac{\tau^{4\sigma}\omega^{3\sigma}\mu^{\sigma}E_{\sigma}\left(\omega^{\sigma}\mu^{\sigma}\right)}{\Gamma\left(1+4\sigma\right)}$$
$$\times\Big(\omega^{\sigma}\mu^{\sigma}\Gamma^{2}\left(1+\sigma\right)+\Gamma\left(1+2\sigma\right)\Big)-\tau^{7\sigma-1}E_{\sigma,7\sigma}\left(-0.1\tau^{\sigma}\right)\omega^{4\sigma}\mu^{\sigma}\Gamma\left(1+2\sigma\right)E_{\sigma}\left(\omega^{\sigma}\mu^{\sigma}\right)$$
$$\times\left(\omega^{\sigma}\mu^{\sigma}\Gamma\left(1+\sigma\right)+\frac{\Gamma\left(1+2\sigma\right)}{\Gamma\left(1+\sigma\right)}\right)+0.1\tau^{\sigma^{2}+5\sigma-1}E_{\sigma,\sigma^{2}+5\sigma}\left(-0.1\tau^{\sigma}\right)\omega^{\sigma}\mu^{\sigma}$$
$$\times\left(E_{\sigma}\left(\omega^{\sigma}\mu^{\sigma}\right)\right)^{\sigma+1}\Big(\Gamma^{2}\left(1+\sigma\right)\omega^{\sigma}\mu^{\sigma}+\Gamma\left(1+2\sigma\right)\Big).\tag{6.30}$$

The second approximation to $v\left(\omega,\mu,\tau\right)$ is expressed as

$$\begin{aligned}T_{\sigma}\left\{v_{2}\left(\omega,\mu,\tau\right)\right\}&=T_{\sigma}\left\{v_{1}\left(\omega,\mu,\tau\right)\right\}-\frac{1}{s^{\sigma}+\frac{1}{\tau}}T_{\sigma}\Big\{\frac{\partial^{\sigma}v_{1}\left(\omega,\mu,\tau\right)}{\partial\tau^{\sigma}}+\left(v_{1}\left(\omega,\mu,\tau\right)-c\right)\\&\times frac\partial^{\sigma}v_{1}\left(\omega,\mu,\tau\right)\partial\omega^{\sigma}+\frac{\xi}{\wp_{1}}\left(D\frac{\partial^{2\sigma}v_{1}\left(\omega,\mu,\tau\right)}{\partial\omega^{2\sigma}}+s_{\mu}\frac{\partial^{\sigma}v_{1}\left(\omega,\mu,\tau\right)}{\partial\mu^{\sigma}}\right)\\&-\frac{{\wp_{1}}^{\sigma}\left(\omega,\mu,\tau\right)-v_{1}\left(\omega,\mu,\tau\right)}{\tau}\Big\}\\&=v_{1}\left(\omega,\mu,s\right)-\frac{1}{s^{\sigma}+0.1}\Big\{s^{\sigma}v_{1}\left(\omega,\mu,s\right)-v_{1}\left(\omega,\mu,0\right)+\left(v_{1}\left(\omega,\mu,s\right)-c\right)\\&\times\frac{\partial^{\sigma}v_{1}\left(\omega,\mu,s\right)}{\partial\omega^{\sigma}}+\frac{\xi}{\wp_{1}\left(\omega,\mu,s\right)}\left(D\frac{\partial^{2\sigma}v_{1}\left(\omega,\mu,s\right)}{\partial\omega^{2\sigma}}+s_{\mu}\frac{\partial^{\sigma}v_{1}\left(\omega,\mu,s\right)}{\partial\mu^{\sigma}}\right)\\&-\frac{{\wp_{1}}^{\sigma}\left(\omega,\mu,s\right)-v_{1}\left(\omega,\mu,s\right)}{\tau}\Big\}.\end{aligned}\tag{6.31}$$

By continuing like this, the fractal series solution of the systems (6.18) and (6.19) is expressed as

$$T_{\sigma}\left\{\wp\right\}=\lim_{n\to\infty}T_{\sigma}\left\{\wp_{n}\right\}\ and\ T_{\sigma}\left\{v\right\}=\lim_{n\to\infty}T_{\sigma}\left\{v_{n}\right\},\tag{6.32}$$

Thus, we have

$$\wp=\lim_{n\to\infty}T_{\sigma}^{-1}\left\{T_{\sigma}\wp_{n}\right\}\ and\ v=\lim_{n\to\infty}T_{\sigma}^{-1}\left\{T_{\sigma}v_{n}\right\}.\tag{6.33}$$

7. Discussion and conclusions

We introduce a new two dimensional macroscopic local fractional viscous diffusive model of vehicular traffic flow. A fractal dynamic momentum equation is derived from 2D fractal Navier-Stokes equation. The convection, anticipation, relaxation, diffusion, and viscosity are the factors that the fractal dynamic velocity equation takes into account. The new viscous diffusive model has been developed by combining the derived momentum equation with the fractal LWR model. The proposed model is solved by employing LFLVIM. The linear stability criterion is established for the new continuum model. The significance of incorporating LFLVIM in the proposed fractal viscous-diffusive model is illustrated with some examples. Numerical simulations for the approximated solutions are carried out for each example under the fractal initial conditions and fractal domain. The parameter values needed to derive the non-differentiable solution and for their numerical simulation are given in Table 1 [17]. The 3D graphical representations are shown for the model with fractal dimension $\sigma = ln2/ln3$ using MATLAB software. These graphics demonstrate the dynamic emergence of the non-differentiable traffic density $\wp(\omega,\mu,\tau)$ and speed function $v(\omega,\mu,\tau)$ for the described traffic flow model. Figures 2 and 3 depict the surface fluctuation of traffic density and speed function, respectively, with respect to time and space, for Example 6.1. It is evident how the length of the road and time affect the traffic density and speed when road width is kept a constant ($\mu = 10m$).

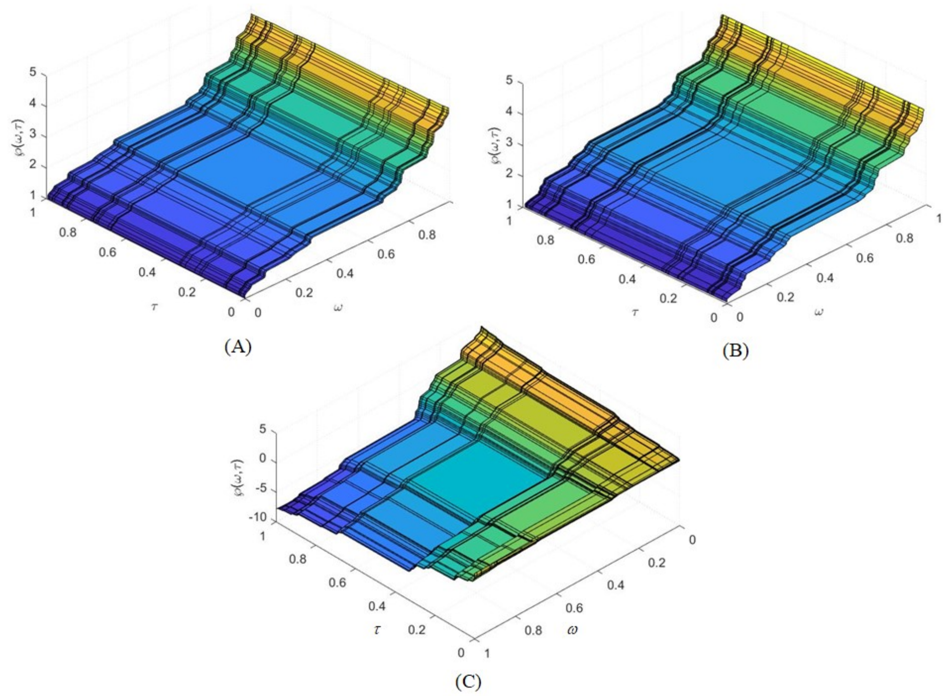

Figure 2. (A)–(C) demonstrate the initial, first and second iteration to traffic density $\wp$ for given systemss (6.1) and (6.2) with fractal order $\sigma = ln2/ln3$.

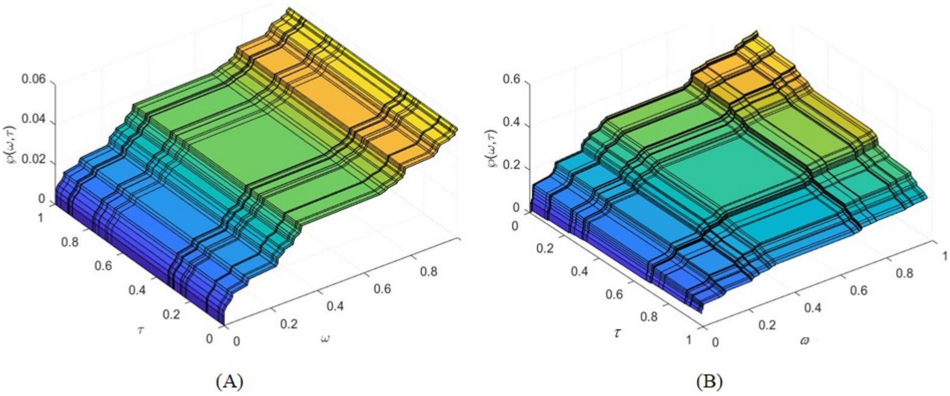

Figure 3. (A) and (B) demonstrate the initial and first iteration to traffic speed v for given systems (6.1) and (6.2) with fractal order $\sigma = ln2/ln3$.

Similarly, the dynamic variations in density and speed of vehicles are shown in Figures 4 and 5, respectively, for Example 6.2. The impact of wider roads overtime on traffic density and speed can be observed in Figure 6. It is prominent that as roads are wider, viscosity and diffusion have less of an impact, which causes vehicle speeds to increase and vehicle densities to drop. A similar pattern can be seen in Figure 7. Thus, it has been found from the simulation results that narrowing the road will raise the viscosity and diffusion rate, eventually leading to a slowdown in traffic. Wider roadways, however, are less affected by diffusion and viscosity.

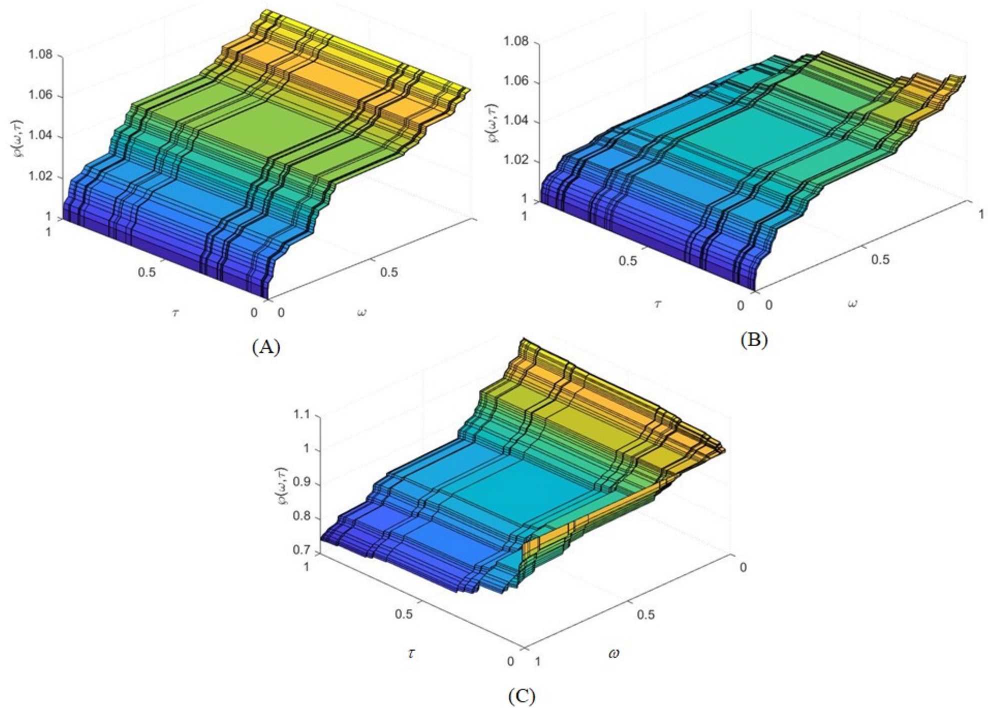

Figure 4. (A)–(C) demonstrate initial, first and second iteration to traffic density $\wp$ for given systems (6.18) and (6.19) with fractal order $\sigma = ln2/ln3$.

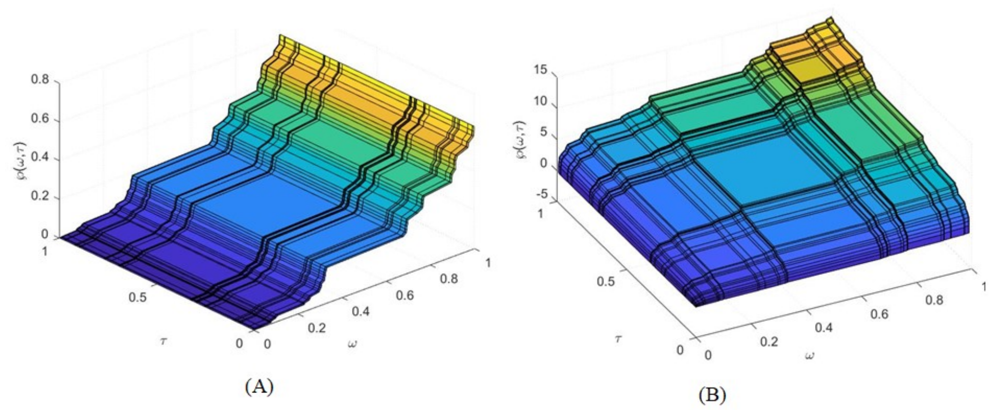

Figure 5. (A) and (B) demonstrate the initial and first iteration to traffic speed v for given systems (6.18) and (6.19) with fractal order $\sigma = ln2/ln3$.

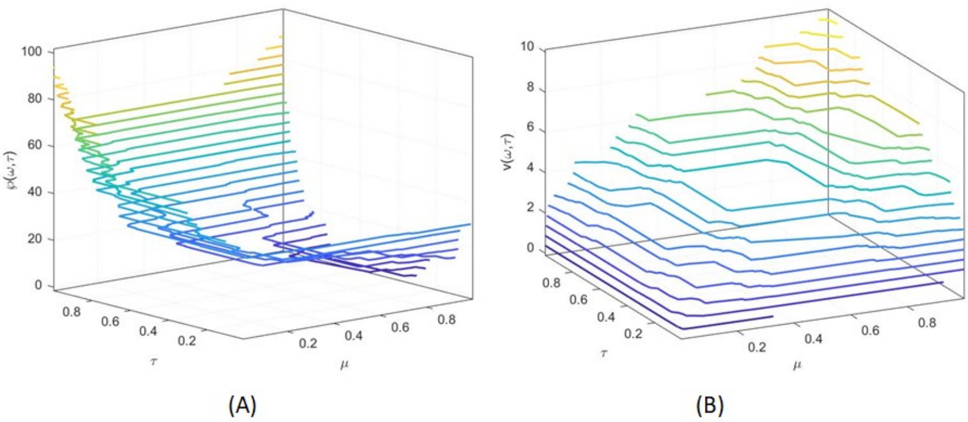

Figure 6. (A) and (B) demonstrate traffic density $\wp$ and traffic speed v, respectively, w.r.t time and road width for given systems (6.1) and (6.2) with fractal order $\sigma = ln2/ln3$.

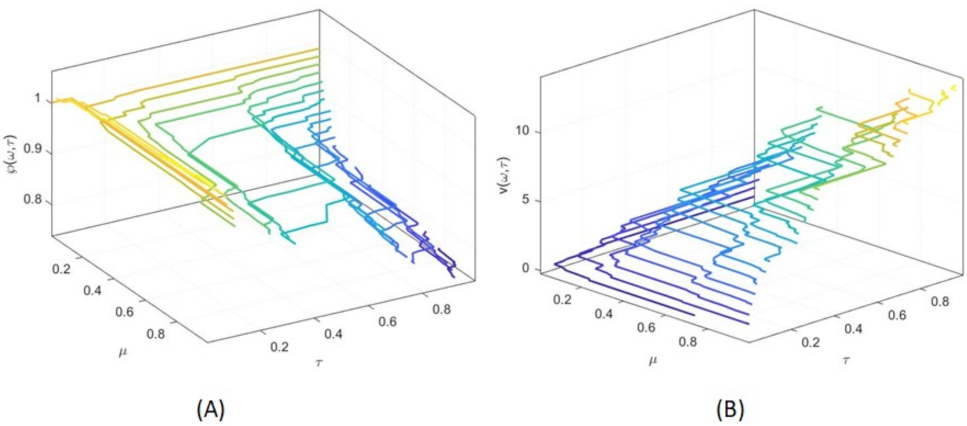

Figure 7. (A) and (B) represent traffic density $\wp$ and traffic speed v, respectively, w.r.t time and road width for given systems (6.18) and (6.19) with fractal order $\sigma = ln2/ln3$.

Table 1. Parameter values used in the study [17].

Parameter	Value
Anticipation rate, c	11 ms^{-1}
Time, τ	10 s
Road width, b	10 m
Diffusion coefficient, D	10 m^2s^{-1}
Viscosity, s_μ	0.011

Author contributions

Conceptualization, Bhawna Pokhriyal and Pranay Goswami; Methodology, Bhawna Pokhriyal, Pranay Goswami, Saad Althobaiti and Kranti Kumar; Formal analysis, Bhawna Pokhriyal and Mohammed Abdalbagi; Investigation, Bhawna Pokhriyal; Resources, Kranti Kumar, Mohammed

Abdalbagi, Saad Althobaiti and Abdalla S. Mahmoud; Writing – original draft, Bhawna Pokhriyal; Writing – review & editing, Pranay Goswami; Visualization, Bhawna Pokhriyal and Saad Althobaiti; Supervision, Pranay Goswami, Kranti Kumar. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflicts of interest to declare here.

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