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Research article

Fixed point theorems in extended b-suprametric spaces with applications

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Abstract: The primary objective of this study is to establish novel fixed point theorems for single-valued mappings satisfying generalized contraction conditions in the framework of extended b-suprametric spaces. These results not only generalize and unify existing fixed point results in the literature but also provide a broader context for analysis. Building upon this foundation, we further derive corresponding fixed point theorems in suprametric and b-suprametric spaces as special cases of our main results. To highlight the relevance and originality of the proposed theorems, we present a concrete example that validates the applicability and sharpness of the key findings. Finally, we demonstrate the practical significance of our theoretical contributions by applying the main result to establish the existence of solutions for a nonlinear Volterra-Fredholm integral equation.

Keywords: fixed points; single-valued mappings; extended b-suprametric spaces; nonlinear Volterra-Fredholm integral equation
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1. Introduction

Fixed point (FP) theory is a vast and diverse field with three principal branches: metric, topological, and discrete FP theory. Among these, metric FP theory serves as a foundational pillar, concentrating on proving the existence and uniqueness of FPs for self-mappings defined on metric spaces (MSs). This theory is intrinsically tied to the notions of distance and convergence, which are the defining characteristics of MSs. The concept of a MS, developed by Maurice Fréchet [1] in 1906, establishes a solid foundation for quantifying distances between elements within a set. A MS is formally defined as a set equipped with a distance function, or metric, that satisfies specific axioms such as non-negativity, symmetry, and the triangle inequality. Over time, this foundational idea has undergone significant development and generalization, leading to the creation of more complex mathematical structures that extend beyond classical MSs. One of the notable generalizations is the partial metric space (PMS), introduced by Matthews [2], which allows the self-distance of a point to be non-zero. This relaxation of the traditional MS axioms is particularly in areas that deal with computation, semantics, and formal methods. Similarly, Bakhtin [3] introduced b-metric spaces (b-MSs), where the triangle inequality is modified by incorporating a constant factor on the right-hand side. Further generalizations include rectangular metric spaces (RMSs), developed by Branciari [4], where the classical triangle inequality is replaced by a rectangular condition. This modification has found applications in the analysis of nonlinear problems and optimization models. Kamran et al. [5] generalized the concept of b-MSs by replacing the constant $b \geq 1$ with a control function $\varphi : \mathcal{N} \times \mathcal{N} \to [1, +\infty)$, thereby introducing the notion of extended b-metric spaces (Eb-MSs). Later on, Samreen et al. [6] developed the concept of α-admissibility in Eb-MSs and proved FP theorems by employing suitable contractive conditions. Among the more recent advancements is the concept of supra metric spaces (SMSs), introduced by Berzig [7]. SMSs are characterized by a relaxed version of the triangle inequality, allowing for a broader range of applications. The framework of SMSs has also been applied to nonlinear integral and matrix equations, demonstrating its versatility in addressing complex mathematical challenges. Later on, Berzig [8, 9] extended the notion of SMS by generalizing the triangle inequality axiom and introduced a new metric structure, termed b-SMSs. Panda et al. [10] proposed the concept of extended suprametric spaces (extended SMS) and established FP results for mappings satisfying generalized contractive conditions. Recently, Panda et al. [11] advanced this framework by introducing extended b-suprametric spaces (extended b-SMSs) to prove certain generalized FP theorems.

On the other hand, FP theorems serve as powerful analytical tools in the study of nonlinear functional analysis and are extensively employed to investigate the existence and uniqueness of solutions to various types of integral equations, including Volterra, Fredholm, and mixed-type equations, by transforming the original problem into an equivalent FP problem in an appropriate metric or topological space. Karapınar et al. [12] explored a novel method for solving the Fredholm integral equation by applying FP theory in the framework of extended b-MSs. Berzig [7] applied the FP results in SMSs to investigate the existence of solutions to some nonlinear integral equations. Panda et al. [10] established the existence of solutions for Ito-Doob-type stochastic integral equations by applying a FP theorem in the setting of extended SMSs. Abdou [13] explored the application of FP theorems to obtain solutions of nonlinear mixed Volterra-Fredholm integral equations. The reader is encouraged to consult [14–17] for a more comprehensive discussion.

In this study, we investigate the concept of extended b-SMSs and establish FP theorems for single-valued mappings under generalized contraction conditions. Our results not only extend and unify several existing FP theorems in the literature but also offer a more comprehensive analytical framework. As a natural consequence, we derive corollaries that specialize to extended SMSs, b-SMSs, and SMSs, demonstrating the versatility of our approach. To underscore the novelty and applicability of our findings, we provide a concrete example that illustrates the validity and optimality of the proposed theorems. Furthermore, we emphasize the practical utility of our theoretical contributions by applying the main results to establish the existence of solutions for a nonlinear Volterra-Fredholm integral equation, thereby bridging abstract theory with real-world applications.

2. Preliminaries

The concept of MS was introduced by Fréchet [1] in 1906 and defined as follows:

Definition 1. *([1]) Let $\mathcal{N} \neq \emptyset$ and $d : \mathcal{N} \times \mathcal{N} \to \mathbb{R}^+$ be a function that fulfills the following axioms:*

(i) $0 \leq d(\imath, \varsigma)$ *and* $d(\imath, \varsigma) = 0 \Leftrightarrow \imath = \varsigma$,
(ii) $d(\imath, \varsigma) = d(\varsigma, \imath)$,
(iii) $d(\imath, \varsigma) \leq d(\imath, \nu) + d(\nu, \varsigma)$,

for all $\imath, \varsigma, \nu \in \mathcal{N}$, *then* $(\mathcal{N}, d)$ *is referred to as a MS.*

Berzig [7] introduced the concept of SMS in this way.

Definition 2. *([7]) Let $\mathcal{N} \neq \emptyset$ and $\curlywedge$ be a non-negative real constant. Consider a function $d : \mathcal{N}\times\mathcal{N} \to \mathbb{R}^+$ that satisfies the following properties:*

(i) $0 \leq d(\imath, \varsigma)$ *and* $d(\imath, \varsigma) = 0 \Longleftrightarrow \imath = \varsigma$,
(ii) $d(\imath, \varsigma) = d(\varsigma, \imath)$,
(iii) $d(\imath, \varsigma) \leq d(\imath, \nu) + d(\nu, \varsigma) + \curlywedge d(\imath, \nu) d(\nu, \varsigma)$,

for all $\imath, \varsigma, \nu \in \mathcal{N}$, *then* $(\mathcal{N}, d)$ *is called a SMS.*

Example 1. Let $\mathcal{N} = \{0, 1, 2\}$ and define the distance function $d : \mathcal{N} \times \mathcal{N} \to \mathbb{R}^+$ as follows:

$$d(0,1) = d(1,0) = 0.5,$$
$$d(0,2) = d(2,0) = 1,$$
$$d(1,2) = d(2,1) = 2,$$

and

$$d(0,0) = d(1,1) = d(2,2) = 0.$$

Let us choose $\curlywedge = 1.5$. Then $(\mathcal{N}, d)$ is a SMS but not a MS because the triangle of MS is not satisfied; that is,

$$2 = d(1,2) > d(1,0) + d(0,2) = 0.5 + 1.$$

Panda et al. [10] introduced the concept of extended suprametric spaces and formalized its structure in the following manner.

Definition 3. *([10]) Let $\mathcal{N} \neq \emptyset$ and $\curlywedge : \mathcal{N} \times \mathcal{N} \to [1, \infty)$ be a function. Consider a function $d : \mathcal{N} \times \mathcal{N} \to \mathbb{R}^+$ that satisfies the following properties:*

(i) $0 \leq d(\imath, \varsigma)$ *and* $d(\imath, \varsigma) = 0 \Longleftrightarrow \imath = \varsigma$,
(ii) $d(\imath, \varsigma) = d(\varsigma, \imath)$,
(iii) $d(\imath, \varsigma) \leq d(\imath, \nu) + d(\nu, \varsigma) + \curlywedge(\imath, \varsigma) d(\imath, \nu) d(\nu, \varsigma)$,

for all $\imath, \varsigma, \nu \in \mathcal{N}$, *then* $(\mathcal{N}, d)$ *is called an extended SMS.*

Example 2. Let $\mathcal{N} = \ell^\infty(\mathbb{R})$, the set of all bounded real sequences $\imath = \{\imath_n\}_{n=1}^\infty$. Define

$$d(\imath, \varsigma) = \sup_{n\geq 1} |\imath_n - \varsigma_n|^p, \quad \text{for some } p \in (0,1),$$

and define $\curlywedge : \mathcal{N} \times \mathcal{N} \to [1, \infty)$ by

$$\curlywedge(\imath, \varsigma) = 1 + \sup_{n\geq 1} |\imath_n| + \sup_{n\geq 1} |\varsigma_n|.$$

Then $(\mathcal{N}, d)$ is extended SMS.

Berzig [9] defined the notion of b-suprametric space (b-SMS) in this way.

Definition 4. *([9]) Let $\mathcal{N} \neq \emptyset$, $b \geq 1$ and $\curlywedge$ be a non-negative real constant. Consider a function $d : \mathcal{N} \times \mathcal{N} \to \mathbb{R}^+$ that satisfies the following properties:*

(i) $0 \leq d(\imath, \varsigma)$ *and* $d(\imath, \varsigma) = 0 \iff \imath = \varsigma$,
(ii) $d(\imath, \varsigma) = d(\varsigma, \imath)$,
(iii) $d(\imath, \varsigma) \leq b\,(d(\imath, \nu) + d(\nu, \varsigma)) + \curlywedge d(\imath, \nu) d(\nu, \varsigma)$,

for all $\imath, \varsigma, \nu \in \mathcal{N}$, *then* $(\mathcal{N}, d)$ *is called a b-SMS.*

In a recent contribution, Panda et al. [11] defined the concept of extended b-suprametric spaces (extended b-SMS) as follows.

Definition 5. *([11]) Let $\mathcal{N} \neq \emptyset$, $b \geq 1$ and $\curlywedge : \mathcal{N} \times \mathcal{N} \to [1, \infty)$ be a function. Consider a function $d : \mathcal{N} \times \mathcal{N} \to \mathbb{R}^+$ that satisfies the following properties:*

(i) $0 \leq d(\imath, \varsigma)$ *and* $d(\imath, \varsigma) = 0 \iff \imath = \varsigma$,
(ii) $d(\imath, \varsigma) = d(\varsigma, \imath)$,
(iii) $d(\imath, \varsigma) \leq b\,(d(\imath, \nu) + d(\nu, \varsigma)) + \curlywedge(\imath, \varsigma) d(\imath, \nu) d(\nu, \varsigma)$,

for all $\imath, \varsigma, \nu \in \mathcal{N}$, *then* $(\mathcal{N}, d)$ *is called an extended b-SMS.*

Example 3. ([11]) Take $\mathcal{N} = \ell_p(\mathbb{R})$ where as $p \in (0, 1)$, and $\ell_p(\mathbb{R}) = \{\{x_n\} \subset \mathbb{R}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty\}$ and $d : \mathcal{N} \times \mathcal{N} \to \mathbb{R}^+$is provided that

$$d\left(\{x_n\}, \{y_n\}\right) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}} \text{ for all } \{x_n\}, \{y_n\} \in \mathcal{N}.$$

Then $(\mathcal{N}, d)$ is an extended b-SMS with $b = 2^{\frac{1}{p}}$.

Example 4. Let $\mathcal{N} = C[0, 1]$, the space of continuous real-valued functions on $[0, 1]$. Define

$$d\left(f, g\right) = \left(\int_0^1 |f(t) - g(t)|^p \, dt\right)^{\frac{1}{p}}, \ p \in (0, 1).$$

Then ($\mathcal{N}$,d) is an extended b-SMS with $b = 2^{\frac{1}{p}}$.

Remark 1. *(i) If we define the function $\curlywedge : \mathcal{N} \times \mathcal{N} \to [1, \infty)$ by $\curlywedge(\imath, \varsigma) = \curlywedge \geq 1$ in Definition 5, then we obtain the notion of a b-SMS.*

(ii) By setting $b = 1$ in Definition 5, we derive the notion of an extended SMS.

(iii) If we define $\curlywedge : \mathcal{N} \times \mathcal{N} \to [1, \infty)$ by $\curlywedge(\imath, \varsigma) = \curlywedge \geq 1$ and also take $b = 1$, then we arrive at the notion of a SMS.

Theorem 1. *([11]) Let $(\mathcal{N}, d)$ be a complete extended b-SMS with $b \geq 1$ and $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ be a self mapping. Suppose that there exists a constant $\varpi \in [0, 1)$ such that*

$$d(\mathcal{R}\imath, \mathcal{R}\varsigma) \leq \frac{\varpi}{b} d(\imath, \varsigma),$$

holds for all $\imath, \varsigma \in \mathcal{N}$. Then $\mathcal{R}$ possesses a unique FP, and for every $\imath_0 \in \mathcal{N}$ the iterative sequence defined by $\imath_n = \mathcal{R}\imath_{n-1}$, $\forall\, n \in \mathbb{N}$ converges to this FP.

Lemma 1. *([11]) Let* $(\mathcal{N}, d)$ *be an extended b-SMS and* $\{\imath_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{N}$*. Then*

(i) $\{\imath_n\}_{n\in\mathbb{N}}$ *is said to converge to* $\imath$ *if, for all* $\epsilon > 0$*, the ball* $B(\imath,\epsilon)$ *contained all but a finite number of terms of the sequence. In this case* $\imath$ *is a limit point of* $\{\imath_n\}_{n\in\mathbb{N}}$*, and we write* $\lim_{n\to\infty} d(\imath_n, \imath) = 0$.

(ii) $\{\imath_n\}_{n\in\mathbb{N}}$ *is said to be Cauchy if, for all* $\epsilon > 0$*, there exists some natural number k such that* $d(\imath_n, \imath_m) < \epsilon$*, for all* $n, m \geq k$.

(iii) $(\mathcal{N}, d)$ *is complete if every Cauchy sequence is convergent.*

3. Main results

In this section, we present our main results concerning FP theorems for single-valued mappings satisfying generalized contraction conditions in the setting of extended b-suprametric spaces. These theorems serve as a unifying framework that extends several well-known results in FP theory.

Theorem 2. *Let* $(\mathcal{N}, d)$ *be a complete extended b-SMS with* $b \geq 1$ *and consider a mapping* $\mathcal{R} : \mathcal{N} \to \mathcal{N}$*. Suppose that there exist the functions* $\alpha, \beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ *and a constant* $\varpi \in [0, 1)$ *such that* $\alpha(\imath, \mathcal{R}\imath)\alpha(\varsigma, \mathcal{R}\varsigma) \geq \beta(\imath, \mathcal{R}\imath)\beta(\varsigma, \mathcal{R}\varsigma)$ *implies*

$$d(\mathcal{R}\imath, \mathcal{R}\varsigma) \leq \frac{\varpi}{b}\mathcal{M}(\imath, \varsigma), \tag{3.1}$$

where

$$\mathcal{M}(\imath, \varsigma) = \max\left\{d(\imath, \varsigma), \min\left\{\frac{d(\imath,\mathcal{R}\imath)d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\imath, \varsigma)}, \frac{d(\imath,\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\imath)}{1 + d(\imath, \varsigma)}\right\}\right\}, \tag{3.2}$$

holds for all $\imath, \varsigma \in \mathcal{N}$*. Assume that the following conditions hold:*

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ *is an* α*-admissible mapping with respect to* β*,*

(ii) there exists $\imath_0 \in \mathcal{N}$ *such that* $\alpha(\imath_0, \mathcal{R}\imath_0) \geq \beta(\imath_0, \mathcal{R}\imath_0)$*,*

(iii) either $\mathcal{R}$ *is continuous, or if* $\{\imath_n\}$ *is a sequence in* $\mathcal{N}$ *such that* $\imath_n \to \imath$, $\alpha(\imath_n, \imath_{n+1}) \geq \beta(\imath_n, \imath_{n+1})$*, then* $\alpha(\imath, \mathcal{R}\imath) \geq \beta(\imath, \mathcal{R}\imath)$.

Then $\mathcal{R}$ *possesses a FP.*

Moreover, if $\alpha(\imath, \mathcal{R}\imath)\alpha(\varsigma, \mathcal{R}\varsigma) \geq \beta(\imath, \mathcal{R}\imath)\beta(\varsigma, \mathcal{R}\varsigma)$ *for* $\imath, \varsigma \in Fix(\mathcal{R})$*, then the FP is unique.*

Proof. Let $\imath_0 \in \mathcal{N}$ such that $\alpha(\imath_0, \imath_1) = \alpha(\imath_0, \mathcal{R}\imath_0) \geq \beta(\imath_0, \mathcal{R}\imath_0) = \beta(\imath_0, \imath_1)$. Define a sequence $\{\imath_n\}$ in $\mathcal{N}$ by $\imath_{n+1} = \mathcal{R}^n\imath_0 = \mathcal{R}\imath_n$, $\forall\, n \in \mathbb{N}$. Since the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ is an α-admissible mapping with respect to β, so we have

$$\alpha(\imath_1, \imath_2) = \alpha(\mathcal{R}\imath_0, \mathcal{R}\imath_1) \geq \beta(\mathcal{R}\imath_0, \mathcal{R}\imath_1) = \beta(\imath_1, \imath_2).$$

Continuing in this way, we get

$$\alpha(\imath_{n-1}, \imath_n) = \alpha(\mathcal{R}\imath_{n-2}, \mathcal{R}\imath_{n-1}) \geq \beta(\mathcal{R}\imath_{n-2}, \mathcal{R}\imath_{n-1}) = \beta(\imath_{n-1}, \imath_n),$$

and

$$\alpha(\imath_n, \imath_{n+1}) = \alpha(\mathcal{R}\imath_{n-1}, \mathcal{R}\imath_n) \geq \beta(\mathcal{R}\imath_{n-1}, \mathcal{R}\imath_n) = \beta(\imath_n, \imath_{n+1}),$$

$\forall\, n \in \mathbb{N}$. We can express these inequalities in this way

$$\alpha(\imath_{n-1}, \imath_n) = \alpha(\imath_{n-1}, \mathcal{R}\imath_{n-1}) \geq \beta(\imath_{n-1}, \mathcal{R}\imath_{n-1}) = \beta(\imath_{n-1}, \imath_n), \tag{3.3}$$

and

$$\alpha(\imath_n, \imath_{n+1}) = \alpha(\imath_n, \mathcal{R}\imath_n) \geq \beta(\imath_n, \mathcal{R}\imath_n) = \beta(\imath_n, \imath_{n+1}), \tag{3.4}$$

$\forall\, n \in \mathbb{N}$. By (3.3) and (3.4), we have

$$\alpha(\iota_{n-1}, \mathcal{R}\iota_{n-1})\alpha(\iota_n, \mathcal{R}\iota_n) \geq \beta(\iota_{n-1}, \mathcal{R}\iota_{n-1})\beta(\iota_{n-1}, \mathcal{R}\iota_{n-1}), \tag{3.5}$$

$\forall\, n \in \mathbb{N}$. Clearly, if $\exists\, n_0 \in \mathbb{N}$ for which $\iota_{n_0+1} = \iota_{n_0}$, then $\mathcal{R}\iota_{n_0} = \iota_{n_0}$ and the proof is completed. Hence, we assume that $\iota_{n+1} \neq \iota_n$ or $d(\mathcal{R}\iota_{n-1}, \mathcal{R}\iota_n) > 0$ for every $n \in \mathbb{N}$. By (3.1) and (3.2), we have

$$d(\iota_n, \iota_{n+1}) = d(\mathcal{R}\iota_{n-1}, \mathcal{R}\iota_n) \leq \frac{\varpi}{b}\mathcal{M}(\iota_{n-1}, \iota_n), \tag{3.6}$$

where

$$\begin{aligned}
\mathcal{M}(\iota_{n-1}, \iota_n) &= \max\left\{d(\iota_{n-1}, \iota_n), \min\left\{\begin{array}{c} \frac{d(\iota_{n-1}, \mathcal{R}\iota_{n-1})d(\iota_n, \mathcal{R}\iota_n)}{1+d(\iota_{n-1}, \iota_n)}, \\ \frac{d(\iota_{n-1}, \mathcal{R}\iota_n)d(\iota_n, \mathcal{R}\iota_{n-1})}{1+d(\iota_{n-1}, \iota_n)} \end{array}\right\}\right\} \\
&= \max\left\{d(\iota_{n-1}, \iota_n), \min\left\{\begin{array}{c} \frac{d(\iota_{n-1}, \iota_n)d(\iota_n, \iota_{n+1})}{1+d(\iota_{n-1}, \iota_n)}, \\ \frac{d(\iota_{n-1}, \iota_{n+1})d(\iota_n, \iota_n)}{1+d(\iota_{n-1}, \iota_n)} \end{array}\right\}\right\} \\
&= d(\iota_{n-1}, \iota_n),
\end{aligned}$$

$\forall\, n \in \mathbb{N}$. By (3.6), we have

$$d(\iota_n, \iota_{n+1}) \leq \frac{\varpi}{b}d(\iota_{n-1}, \iota_n) < d(\iota_{n-1}, \iota_n).$$

Thus, regardless of the given integer κ, the sequence $\{d(\iota_n, \iota_{n+1}\}$ is non-increasing and meets the following:

$$d(\iota_n, \iota_{n+1}) \leq \left(\frac{\varpi}{b}\right)^{n-\kappa} d(\iota_\kappa, \iota_{\kappa+1}), \forall n > \kappa. \tag{3.7}$$

Therefore, $\lim_{n\to\infty} d(\iota_n, \iota_{n+1}) = 0$, which yields that for $\epsilon > 0, \kappa \in \mathbb{N}$ such that for all $n \geq \kappa$, we have

$$d(\iota_n, \iota_{n+1}) < \epsilon. \tag{3.8}$$

We shall now demonstrate the Cauchy nature of the series $\{\iota_n\}$. BY utilizing (3.7), (3.8) and triangular inequality, we have

$$\begin{aligned}
&d\left(\iota_p, \iota_q\right) \leq b\left[d\left(\iota_p, \iota_{p+1}\right) + d\left(\iota_{p+1}, \iota_q\right)\right] + \wedge\left(\iota_p, \iota_q\right)d\left(\iota_p, \iota_{p+1}\right)d\left(\iota_{p+1}, \iota_q\right) \\
&\leq b\left[\left(\frac{\varpi}{b}\right)^{p-k} d(\iota_k, \iota_{k+1}) + d\left(\iota_{p+1}, \iota_q\right)\right] + \wedge\left(\iota_p, \iota_q\right)\left(\frac{\varpi}{b}\right)^{p-k} d(\iota_k, \iota_{k+1})\, d\left(\iota_{p+1}, \iota_q\right) \\
&\leq b\left[\left(\frac{\varpi}{b}\right)^{p-k}\epsilon + d\left(\iota_{p+1}, \iota_q\right)\right] + \wedge\left(\iota_p, \iota_q\right)\left(\frac{\varpi}{b}\right)^{p-k}\epsilon d\left(\iota_{p+1}, \iota_q\right) \\
&\leq b\left(\frac{\varpi}{b}\right)^{p-k}\epsilon + \left[b + \wedge\left(\iota_p, \iota_q\right)\left(\frac{\varpi}{b}\right)^{p-k}\epsilon\right]d\left(\iota_{p+1}, \iota_q\right),
\end{aligned} \tag{3.9}$$

where,

$$\begin{aligned}
d\left(\iota_{p+1}, \iota_q\right) &\leq b\left[d\left(\iota_{p+1}, \iota_{p+2}\right) + d\left(\iota_{p+2}, \iota_q\right)\right] + \wedge\left(\iota_{p+1}, \iota_q\right)d\left(\iota_{p+1}, \iota_{p+2}\right)d\left(\iota_{p+2}, \iota_q\right) \\
&\leq b\left[\left(\frac{\varpi}{b}\right)^{p-k+1}\epsilon + d\left(\iota_{p+2}, \iota_q\right)\right] + \wedge\left(\iota_{p+1}, \iota_q\right)\left(\frac{\varpi}{b}\right)^{p-k+1}\epsilon d\left(\iota_{p+2}, \iota_q\right)
\end{aligned}$$

$$\leq b\left(\frac{\varpi}{b}\right)^{p-k+1}\epsilon+\left[b+\wedge\left(\iota_{p+1},\iota_q\right)\left(\frac{\varpi}{b}\right)^{p-k+1}\epsilon\right]d\left(\iota_{p+2},\iota_q\right). \tag{3.10}$$

From the above two inequalities (3.9) and (3.10), we get

$$\begin{aligned} d\left(\iota_p,\iota_q\right) \leq\ & b\left(\frac{\varpi}{b}\right)^{p-k}\epsilon+b\left(\frac{\varpi}{b}\right)^{p-k+1}\epsilon\left[b+\wedge\left(\iota_p,\iota_q\right)\left(\frac{\varpi}{b}\right)^{p-k}\epsilon\right] \\ & +\left[b+\wedge\left(\iota_p,\iota_q\right)\left(\frac{\varpi}{b}\right)^{p-k}\epsilon\right]\left[b+\wedge\left(\iota_{p+1},\iota_q\right)\left(\frac{\varpi}{b}\right)^{p-k+1}\epsilon\right]d\left(\iota_{p+2},\iota_q\right). \end{aligned}$$

Employing (3.8) in each of the terms in the sum, we can keep proceeding until we get

$$\begin{aligned} d\left(\iota_p,\iota_q\right) \leq\ & b\left(\frac{\varpi}{b}\right)^{p-k}\epsilon+b\left(\frac{\varpi}{b}\right)^{p-k+1}\epsilon\left[b+\wedge\left(\iota_p,\iota_q\right)\left(\frac{\varpi}{b}\right)^{p-k}\epsilon\right] \\ & +b\left(\frac{\varpi}{b}\right)^{p-k+2}\epsilon\left[b+\wedge\left(\iota_p,\iota_q\right)\left(\frac{\varpi}{b}\right)^{p-k}\epsilon\right]\left[b+\wedge\left(\iota_{p+1},\iota_q\right)\left(\frac{\varpi}{b}\right)^{p-k+1}\epsilon\right] \\ & +\ldots+ \\ \leq\ & b\left(\frac{\varpi}{b}\right)^{p-k}\epsilon\sum_{i=0}^{q-p-1}\left(\frac{\varpi}{b}\right)^{i}\prod_{j=0}^{i-1}\left[b+\epsilon\wedge\left(\iota_{p+j},\iota_q\right)\left(\frac{\varpi}{b}\right)^{p-k+j}\right]. \end{aligned}$$

Since $\frac{\varpi}{b}\in[0,1)$, it follows that

$$d\left(\iota_p,\iota_q\right)\leq b\left(\frac{\varpi}{b}\right)^{p-k}\epsilon\sum_{i=0}^{q-p-1}\left(\frac{\varpi}{b}\right)^{i}\prod_{j=0}^{i-1}\left[b+\epsilon\wedge\left(\iota_{p+j},\iota_q\right)\left(\frac{\varpi}{b}\right)^{j}\right].$$

Now, one can easily verify that the series $\sum_{i=0}^{\infty}U_i$ converges by ratio test, where

$$U_i=\left(\frac{\varpi}{b}\right)^{i}\prod_{j=0}^{i-1}\left[b+\epsilon\wedge\left(\iota_{p+j},\iota_q\right)\left(\frac{\varpi}{b}\right)^{j}\right].$$

Hence, we deduce that $d\left(\iota_p,\iota_q\right)\to 0$, as $p,q\to\infty$, thus $\{\iota_n\}$ is a Cauchy. Since $(\mathcal{N},d)$ is a complete extended b-SMS, $\{\iota_n\}$ converges to some point $\iota\in\mathcal{N}$; that is, $\lim_{n\to\infty}\iota_n=\iota^*$. Assume that $\mathcal{R}$ is continuous. Since $\iota_{n+1}=\mathcal{R}\iota_n$, for all $n\in\mathbb{N}$. So taking the limit as $n\to\infty$, we have

$$\iota^*=\mathcal{R}\iota^*.$$

Secondly, as $\iota_n\to\iota^*$ and $\alpha(\iota_n,\iota_{n+1})\geq\beta(\iota_n,\iota_{n+1})$, then $\alpha(\iota^*,\mathcal{R}\iota^*)\geq\beta(\iota^*,\mathcal{R}\iota^*)$. Thus

$$\alpha(\iota^*,\mathcal{R}\iota^*)\alpha(\iota_n,\mathcal{R}\iota_n)\geq\beta(\iota^*,\mathcal{R}\iota^*)\beta(\iota_n,\mathcal{R}\iota_n).$$

By (3.1) and (3.2), we have

$$d(\iota_{n+1},\iota^*)=d(\mathcal{R}\iota_n,\mathcal{R}\iota^*)\leq\frac{\varpi}{b}\mathcal{M}(\iota_n,\iota^*), \tag{3.11}$$

where

$$\begin{aligned}\mathcal{M}(\imath_n, \imath^*) &= \max\left\{d(\imath_n, \imath^*), \min\left\{\frac{d(\imath_n,\mathcal{R}\imath_n)d(\imath^*, \mathcal{R}\imath^*)}{1+d(\imath_n, \imath^*)}, \frac{d(\imath_n,\mathcal{R}\imath^*)d(\imath^*, \mathcal{R}\imath_n)}{1+d(\imath_n, \imath^*)}\right\}\right\} \\ &= \max\left\{d(\imath_n, \imath^*), \min\left\{\frac{d(\imath_n, \imath_{n+1})d(\imath^*, \mathcal{R}\imath^*)}{1+d(\imath_n, \imath^*)}, \frac{d(\imath_n,\mathcal{R}\imath^*)d(\imath^*, \imath_{n+1})}{1+d(\imath_n, \imath^*)}\right\}\right\}. \end{aligned} \tag{3.12}$$

Taking the limit as $n \to \infty$ in inequality (3.11) and using the inequality (3.12) in it, we conclude that

$$\imath^* = \mathcal{R}\imath^*.$$

Hence $\imath^*$ is a FP of $\mathcal{R}$. To prove the uniqueness of $\imath^*$, suppose, for contradiction, that another FP $\imath'$ of $\mathcal{R}$ exists, i.e., $\imath' = \mathcal{R}\imath'$ with $\imath^* \neq \imath'$, such that

$$\alpha(\imath^*, \mathcal{R}\imath^*)\alpha(\imath', \mathcal{R}\imath') \geq \beta(\imath^*, \mathcal{R}\imath^*)\beta(\imath', \mathcal{R}\imath').$$

Then, from (3.1) and (3.2), we have

$$d(\imath^*, \imath') = d(\mathcal{R}\imath^*, \mathcal{R}\imath') \leq \frac{\varpi}{b}\mathcal{M}\left(\imath^*, \imath'\right), \tag{3.13}$$

where

$$\begin{aligned}\mathcal{M}\left(\imath^*, \imath'\right) &= \max\left\{d(\imath^*, \imath'), \min\left\{\frac{d(\imath^*,\mathcal{R}\imath^*)d(\imath', \mathcal{R}\imath')}{1+d(\imath^*, \imath')}, \frac{d(\imath^*,\mathcal{R}\imath')d(\imath', \mathcal{R}\imath^*)}{1+d(\imath^*, \imath')}\right\}\right\} \\ &= \max\left\{d(\imath^*, \imath'), \min\left\{\frac{d(\imath^*, \imath^*)d(\imath', \imath')}{1+d(\imath^*, \imath')}, \frac{d(\imath^*, \imath')d(\imath', \imath^*)}{1+d(\imath^*, \imath')}\right\}\right\} \\ &= d(\imath^*, \imath').\end{aligned}$$

Then we have only $\mathcal{M}\left(\imath^*, \imath'\right) = d(\imath^*, \imath')$, which implies by (3.13) that

$$d(\imath^*, \imath') \leq \frac{\varpi}{b}d(\imath^*, \imath') < d(\imath^*, \imath'),$$

which is a contradiction because $\frac{\varpi}{b} < 1$. Hence $\imath^* = \imath'$. Hence FP is unique. □

Example 5. Let $\mathcal{N} = \mathbb{R}$ and $d : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ be defined by

$$d(\imath, \varsigma) = |\imath - \varsigma|^2,$$

for all $\imath, \varsigma \in \mathcal{N}$. Define $\curlywedge : \mathcal{N} \times \mathcal{N} \to [1, \infty)$ by

$$\curlywedge(\imath, \varsigma) = 1 + |\imath - \varsigma|.$$

First we show that $(\mathcal{N}, d)$ is a complete extended b-SMS.
(Eb-sup$_1$) Clearly, $|\imath - \varsigma|^2 \geq 0$, so $d(\imath, \varsigma) \geq 0$. Moreover, $d(\imath, \varsigma) = 0$, if and only if $|\imath - \varsigma|^2 = 0$. It is possible only that

$$|\imath - \varsigma| = 0,$$

which implies that $\imath = \varsigma$. Thus we conclude that

$$d(\imath, \varsigma) = 0 \text{ if and only if } \imath = \varsigma.$$

(Eb-sup$_2$) Since $|\imath - \varsigma| = |\varsigma - \imath|$, we have

$$d(\imath, \varsigma) = |\imath - \varsigma|^2 = |\varsigma - \imath|^2 = d(\varsigma, \imath).$$

(Eb-sup$_3$) Now

$$\begin{aligned} d(\imath, \varsigma) &= |\imath - \varsigma|^2 \leq 2\left(|\imath - \nu|^2 + |\nu - \varsigma|^2\right) \\ &\leq 2\left(|\imath - \nu|^2 + |\nu - \varsigma|^2\right) + (1 + |\imath - \varsigma|)\left(|\imath - \nu|^2 |\nu - \varsigma|^2\right) \\ &= b\left(d(\imath, \nu) + d(\nu, \varsigma)\right) + \curlywedge(\imath, \varsigma) d(\imath, \nu) d(\nu, \varsigma), \end{aligned}$$

with $b = 2$. Hence the triangle inequality of extended b-SMS is satisfied. To check completeness, consider any Cauchy sequence $\{\imath_n\}$ in $(\mathcal{N}, d)$. Since $d(\imath_n, \imath_m) = |\imath_n - \imath_m|^2$, the sequence $\{\imath_n\}$ being Cauchy in d implies that $\{\imath_n\}$ is Cauchy in the standard Euclidean metric $d^{'}(\imath, \varsigma) = |\imath - \varsigma|$, because

$$|\imath_n - \imath_m|^2 < \epsilon \Longrightarrow |\imath_n - \imath_m| < \sqrt{\epsilon}.$$

Since $\mathcal{N} = \mathbb{R}$ with the Euclidean metric is complete, so $\{\imath_n\}$ converges to some $\imath \in \mathbb{R}$ in the Euclidean metric. Consequently,

$$d(\imath_n, \imath) = |\imath_n - \imath|^2 \to 0,$$

so $\{\imath_n\}$ converges to $\imath$ in $(\mathcal{N}, d)$. Hence, $(\mathcal{N}, d)$ is a complete extended b-SMS.

Now we define $\alpha, \beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ and $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ in this way

$$\alpha(\imath, \varsigma) = 1 + \frac{1}{1 + |\imath - \varsigma|},$$

and

$$\beta(\imath, \varsigma) = \frac{1}{2}\alpha(\imath, \varsigma),$$

and

$$\mathcal{R}(\imath) = \frac{\imath}{2} + 1.$$

Now

$$\alpha(\imath, \mathcal{R}\imath) = 1 + \frac{1}{1 + |\imath - \mathcal{R}\imath|} = 1 + \frac{1}{1 + \left|\imath - \left(\frac{\imath}{2} + 1\right)\right|} = 1 + \frac{1}{1 + \left|\frac{\imath}{2} - 1\right|}.$$

Similarly,

$$\beta(\imath, \mathcal{R}\imath) = \frac{1}{2}\alpha(\imath, \mathcal{R}\imath).$$

Thus the condition $\alpha(\imath, \mathcal{R}\imath)\alpha(\varsigma, \mathcal{R}\varsigma) \geq \beta(\imath, \mathcal{R}\imath)\beta(\varsigma, \mathcal{R}\varsigma)$ simplifies to

$$\alpha(\imath, \mathcal{R}\imath)\alpha(\varsigma, \mathcal{R}\varsigma) \geq \frac{1}{4}\alpha(\imath, \mathcal{R}\imath)\alpha(\varsigma, \mathcal{R}\varsigma).$$

This inequality holds because $\alpha(\iota, \mathcal{R}\iota) \geq 1$ and $\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$. Now if

$$\alpha(\iota, \varsigma) = 1 + \frac{1}{1 + |\iota - \varsigma|} \geq \frac{1}{2}\left(1 + \frac{1}{1 + |\iota - \varsigma|}\right) = \beta(\iota, \varsigma),$$

then

$$\alpha(\mathcal{R}\iota, \mathcal{R}\varsigma) = 1 + \frac{1}{1 + \frac{1}{2}|\iota - \varsigma|} \geq \frac{1}{2}\left(1 + \frac{1}{1 + \frac{1}{2}|\iota - \varsigma|}\right) = \beta(\mathcal{R}\iota, \mathcal{R}\varsigma).$$

Thus, $\mathcal{R}$ is an α-admissible mapping with respect to β. Hence the condition (i) is satisfied. Now choose $\iota_0 = 0$, then

$$\mathcal{R}\iota_0 = \mathcal{R}(0) = 1.$$

Therefore,

$$\begin{aligned} \alpha(\iota_0, \mathcal{R}\iota_0) &= \alpha(0,1) = 1 + \frac{1}{1 + |0 - 1|} \\ &= 1 + \frac{1}{1+1} = \frac{3}{2} \\ &\geq \frac{3}{4} = \frac{1}{2}\left(1 + \frac{1}{1 + |0 - 1|}\right) = \beta(\iota_0, \mathcal{R}\iota_0). \end{aligned}$$

Hence the condition (ii) is satisfied. Moreover, $\mathcal{R}$ is continuous because it is a linear function. Alternatively, if $\{\iota_n\}$ is a sequence in $\mathcal{N}$ such that $\iota_n \to \iota$, $\alpha(\iota_n, \iota_{n+1}) \geq \beta(\iota_n, \iota_{n+1})$, then

$$\alpha(\iota, \mathcal{R}\iota) = 1 + \frac{1}{1 + |\iota - \mathcal{R}\iota|} \geq \frac{1}{2}\left(1 + \frac{1}{1 + |\iota - \mathcal{R}\iota|}\right) = \beta(\iota, \mathcal{R}\iota).$$

Also,

$$d(\mathcal{R}\iota, \mathcal{R}\varsigma) = \left|\frac{\iota}{2} + 1 - \left(\frac{\varsigma}{2} + 1\right)\right|^2 = \frac{|\iota - \varsigma|^2}{4} = \frac{\varpi}{b} d(\iota, \varsigma).$$

Since $d(\iota, \varsigma)$ dominates the other terms of $\mathcal{M}(\iota, \varsigma)$, $\mathcal{M}(\iota, \varsigma) = d(\iota, \varsigma)$. Hence the conditions (3.1) and (3.2) of Theorem 2 are satisfied with $\varpi = \frac{1}{2}$ and the mapping $\mathcal{R}$ possesses 2 as a unique FP.

Corollary 1. *Let $(\mathcal{N}, d)$ be a complete extended b-SMS with $b \geq 1$, and consider a mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. Suppose that there exists a function $\alpha : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ and a constant $\varpi \in [0, 1)$ such that $\alpha(\iota, \mathcal{R}\iota)\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$ implies*

$$d(\mathcal{R}\iota, \mathcal{R}\varsigma) \leq \frac{\varpi}{b}\mathcal{M}(\iota, \varsigma),$$

where

$$\mathcal{M}(\iota, \varsigma) = \max\left\{d(\iota, \varsigma), \min\left\{\frac{d(\iota, \mathcal{R}\iota)d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\iota, \varsigma)}, \frac{d(\iota, \mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\iota)}{1 + d(\iota, \varsigma)}\right\}\right\},$$

holds for all $\iota, \varsigma \in \mathcal{N}$. Assume that the following conditions hold:

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ is an α-admissible mapping,

(ii) there exists $\iota_0 \in \mathcal{N}$ such that $\alpha(\iota_0, \mathcal{R}\iota_0) \geq 1$,

(iii) either $\mathcal{R}$ is continuous, or if ι_n is a sequence in $\mathcal{N}$ such that $\iota_n \to \iota$, $\alpha(\iota_n, \iota_{n+1}) \geq 1$, then $\alpha(\iota, \mathcal{R}\iota) \geq 1$.

Then $\mathcal{R}$ possesses a FP.

Moreover, if $\alpha(\iota, \mathcal{R}\iota)\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$ for $\iota, \varsigma \in Fix(\mathcal{R})$, then the FP is unique.

Proof. Define $\beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ by $\beta(\text{ı}, \varsigma) = 1$ in Theorem 2. □

Corollary 2. *Let $(\mathcal{N}, d)$ be a complete extended b-SMS with $b \geq 1$ and consider a mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. Suppose that there exists a function $\alpha : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ and a constant $\varpi \in [0, 1)$ such that*

$$(d(\mathcal{R}\text{ı}, \mathcal{R}\varsigma) + \eth)^{\alpha(\text{ı},\mathcal{R}\text{ı})\alpha(\varsigma,\mathcal{R}\varsigma)} \leq \frac{\varpi}{b}\mathcal{M}(\text{ı}, \varsigma) + \eth, \tag{3.14}$$

where

$$\mathcal{M}(\text{ı}, \varsigma) = \max\left\{d(\text{ı}, \varsigma), \min\left\{\frac{d(\text{ı},\mathcal{R}\text{ı})d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\text{ı}, \varsigma)}, \frac{d(\text{ı},\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\text{ı})}{1 + d(\text{ı}, \varsigma)}\right\}\right\},$$

holds for all $\text{ı}, \varsigma \in \mathcal{N}$ and $\eth > 0$. Assume that the following conditions hold:

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ is an α-admissible mapping,

(ii) there exists $\text{ı}_0 \in \mathcal{N}$ such that $\alpha(\text{ı}_0, \mathcal{R}\text{ı}_0) \geq 1$,

(iii) either $\mathcal{R}$ is continuous, or if ı_n is a sequence in $\mathcal{N}$ such that $\text{ı}_n \to \text{ı}$, $\alpha(\text{ı}_n, \text{ı}_{n+1}) \geq 1$, then $\alpha(\text{ı}, \mathcal{R}\text{ı}) \geq 1$.

Then $\mathcal{R}$ possesses a FP.

Moreover, if $\alpha(\text{ı}, \mathcal{R}\text{ı})\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$ for $\text{ı}, \varsigma \in Fix(\mathcal{R})$, then the FP is unique.

Proof. Let $\alpha(\text{ı}, \mathcal{R}\text{ı})\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$ for all $\text{ı}, \varsigma \in \mathcal{N}$, then by (3.14), we have

$$(d(\mathcal{R}\text{ı}, \mathcal{R}\varsigma) + \eth)^1 \leq (d(\mathcal{R}\text{ı}, \mathcal{R}\varsigma) + \eth)^{\alpha(\text{ı},\mathcal{R}\text{ı})\alpha(\varsigma,\mathcal{R}\varsigma)} \leq \frac{\varpi}{b}\mathcal{M}(\text{ı}, \varsigma) + \eth,$$

which implies

$$d(\mathcal{R}\text{ı}, \mathcal{R}\varsigma) \leq \frac{\varpi}{b}\mathcal{M}(\text{ı}, \varsigma),$$

where

$$\mathcal{M}(\text{ı}, \varsigma) = \max\left\{d(\text{ı}, \varsigma), \min\left\{\frac{d(\text{ı},\mathcal{R}\text{ı})d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\text{ı}, \varsigma)}, \frac{d(\text{ı},\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\text{ı})}{1 + d(\text{ı}, \varsigma)}\right\}\right\}.$$

Therefore, the conditions of Corollary 1 are satisfied, ensuring that $\mathcal{R}$ has a FP. □

Corollary 3. *Let $(\mathcal{N}, d)$ be a complete extended b-SMS with $b \geq 1$ and consider a mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. Suppose that there exists a function $\alpha : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ and a constant $\varpi \in [0, 1)$ such that $\alpha(\text{ı}, \mathcal{R}\text{ı})\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$ implies*

$$(\alpha(\text{ı}, \mathcal{R}\text{ı})\alpha(\varsigma, \mathcal{R}\varsigma) + 1)^{d(\mathcal{R}\text{ı},\mathcal{R}\varsigma)} \leq 2^{\frac{\varpi}{b}\mathcal{M}(\text{ı},\varsigma)}, \tag{3.15}$$

where

$$\mathcal{M}(\text{ı}, \varsigma) = \max\left\{d(\text{ı}, \varsigma), \min\left\{\frac{d(\text{ı},\mathcal{R}\text{ı})d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\text{ı}, \varsigma)}, \frac{d(\text{ı},\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\text{ı})}{1 + d(\text{ı}, \varsigma)}\right\}\right\},$$

holds for all $\text{ı}, \varsigma \in \mathcal{N}$. Assume that the following conditions hold:

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ is an α-admissible mapping,

(ii) there exists $\text{ı}_0 \in \mathcal{N}$ such that $\alpha(\text{ı}_0, \mathcal{R}\text{ı}_0) \geq 1$,

(iii) either $\mathcal{R}$ is continuous, or if ı_n is a sequence in $\mathcal{N}$ such that $\text{ı}_n \to \text{ı}$, $\alpha(\text{ı}_n, \text{ı}_{n+1}) \geq 1$, then $\alpha(\text{ı}, \mathcal{R}\text{ı}) \geq 1$.

Then $\mathcal{R}$ possesses a FP.

Moreover, if $\alpha(\text{ı}, \mathcal{R}\text{ı})\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$ for $\text{ı}, \varsigma \in Fix(\mathcal{R})$, then the FP is unique.

Proof. Let $\alpha(\imath,\mathcal{R}\imath)\alpha(\varsigma,\mathcal{R}\varsigma)\geq 1$ for all $\imath,\varsigma\in\mathcal{N}$; then by (3.15), we have

$$(1+1)^{d(\mathcal{R}\imath,\mathcal{R}\varsigma)}\leq(\alpha(\imath,\mathcal{R}\imath)\alpha(\varsigma,\mathcal{R}\varsigma)+1)^{d(\mathcal{R}\imath,\mathcal{R}\varsigma)}\leq 2^{\frac{\varpi}{b}\mathcal{M}(\imath,\varsigma)},$$

where

$$\mathcal{M}(\imath,\varsigma)=\max\left\{d(\imath,\varsigma),\min\left\{\frac{d(\imath,\mathcal{R}\imath)d(\varsigma,\mathcal{R}\varsigma)}{1+d(\imath,\varsigma)},\frac{d(\imath,\mathcal{R}\varsigma)d(\varsigma,\mathcal{R}\imath)}{1+d(\imath,\varsigma)}\right\}\right\}.$$

Therefore, the conditions of Corollary 1 are satisfied, ensuring that $\mathcal{R}$ has a FP. □

Corollary 4. *Let $(\mathcal{N},d)$ be a complete extended b-SMS with $b\geq 1$ and consider a mapping $\mathcal{R}:\mathcal{N}\to\mathcal{N}$. Suppose that there exist the function $\beta:\mathcal{N}\times\mathcal{N}\to[0,+\infty)$ and a constant $\varpi\in[0,1)$ such that $\beta(\imath,\mathcal{R}\imath)\beta(\varsigma,\mathcal{R}\varsigma)\leq 1$ implies*

$$d(\mathcal{R}\imath,\mathcal{R}\varsigma)\leq\frac{\varpi}{b}\mathcal{M}(\imath,\varsigma),$$

where

$$\mathcal{M}(\imath,\varsigma)=\max\left\{d(\imath,\varsigma),\min\left\{\frac{d(\imath,\mathcal{R}\imath)d(\varsigma,\mathcal{R}\varsigma)}{1+d(\imath,\varsigma)},\frac{d(\imath,\mathcal{R}\varsigma)d(\varsigma,\mathcal{R}\imath)}{1+d(\imath,\varsigma)}\right\}\right\},$$

holds for all $\imath,\varsigma\in\mathcal{N}$. Assume that the following conditions hold:

(i) the mapping $\mathcal{R}:\mathcal{N}\to\mathcal{N}$ is an β-subadmissible,

(ii) there exists $\imath_0\in\mathcal{N}$ such that $\beta(\imath_0,\mathcal{R}\imath_0)\leq 1$,

(iii) either $\mathcal{R}$ is continuous, or if $\imath_n$ is a sequence in $\mathcal{N}$ such that $\imath_n\to\imath$, $\beta(\imath_n,\imath_{n+1})\leq 1$, then $\beta(\imath,\mathcal{R}\imath)\leq 1$.

Then $\mathcal{R}$ possesses a FP. Moreover, if $\beta(\imath,\mathcal{R}\imath)\beta(\varsigma,\mathcal{R}\varsigma)\leq 1$ for $\imath,\varsigma\in Fix(\mathcal{R})$, then the FP is unique.

Proof. Define $\alpha:\mathcal{N}\times\mathcal{N}\to[0,+\infty)$ by $\alpha(\imath,\varsigma)=1$ in Theorem 2. □

Corollary 5. *Let $(\mathcal{N},d)$ be a complete extended b-SMS with $b\geq 1$ and consider a mapping $\mathcal{R}:\mathcal{N}\to\mathcal{N}$. Suppose that there exist the function $\beta:\mathcal{N}\times\mathcal{N}\to[0,+\infty)$ and a constant $\varpi\in[0,1)$ such that*

$$d(\mathcal{R}\imath,\mathcal{R}\varsigma)+\Game\leq\left(\frac{\varpi}{b}\mathcal{M}(\imath,\varsigma)+\Game\right)^{\beta(\imath,\mathcal{R}\imath)\beta(\varsigma,\mathcal{R}\varsigma)},\tag{3.16}$$

where

$$\mathcal{M}(\imath,\varsigma)=\max\left\{d(\imath,\varsigma),\min\left\{\frac{d(\imath,\mathcal{R}\imath)d(\varsigma,\mathcal{R}\varsigma)}{1+d(\imath,\varsigma)},\frac{d(\imath,\mathcal{R}\varsigma)d(\varsigma,\mathcal{R}\imath)}{1+d(\imath,\varsigma)}\right\}\right\},$$

holds for all $\imath,\varsigma\in\mathcal{N}$ and $\Game>0$. Assume that the following conditions hold:

(i) the mapping $\mathcal{R}:\mathcal{N}\to\mathcal{N}$ is an β-subadmissible,

(ii) there exists $\imath_0\in\mathcal{N}$ such that $\beta(\imath_0,\mathcal{R}\imath_0)\leq 1$,

(iii) either $\mathcal{R}$ is continuous or if $\{\imath_n\}$ is a sequence in $\mathcal{N}$ such that $\imath_n\to\imath$, $\beta(\imath_n,\imath_{n+1})\leq 1$, then $\beta(\imath,\mathcal{R}\imath)\leq 1$.

Then $\mathcal{R}$ possesses a FP.

Moreover, if $\beta(\imath,\mathcal{R}\imath)\beta(\varsigma,\mathcal{R}\varsigma)\leq 1$ for $\imath,\varsigma\in Fix(\mathcal{R})$, then the FP is unique.

Proof. Let $\beta(\imath,\mathcal{R}\imath)\beta(\varsigma,\mathcal{R}\varsigma)\leq 1$ for all $\imath,\varsigma\in\mathcal{N}$; then by (3.16), we have

$$d(\mathcal{R}\imath,\mathcal{R}\varsigma)+\Game\leq\left(\frac{\varpi}{b}\mathcal{M}(\imath,\varsigma)+\Game\right)^{\beta(\imath,\mathcal{R}\imath)\beta(\varsigma,\mathcal{R}\varsigma)}\leq\frac{\varpi}{b}\mathcal{M}(\imath,\varsigma)+\Game,$$

implies that

$$d(\mathcal{R}\text{ı}, \mathcal{R}\varsigma) \leq \frac{\varpi}{b}\mathcal{M}(\text{ı}, \varsigma),$$

where

$$\mathcal{M}(\text{ı}, \varsigma) = \max\left\{d(\text{ı}, \varsigma), \min\left\{\frac{d(\text{ı},\mathcal{R}\text{ı})d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\text{ı}, \varsigma)}, \frac{d(\text{ı},\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\text{ı})}{1 + d(\text{ı}, \varsigma)}\right\}\right\}.$$

Therefore, the conditions of Corollary 4 are satisfied, ensuring that $\mathcal{R}$ has a unique FP. □

Corollary 6. *Let $(\mathcal{N}, d)$ be a complete extended b-SMS with $b \geq 1$ and consider a mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. Suppose that there exist the function $\beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ and a constant $\varpi \in [0, 1)$ such that*

$$2^{d(\mathcal{R}\text{ı},\mathcal{R}\varsigma)} \leq (\beta(\text{ı}, \mathcal{R}\text{ı})\beta(\varsigma, \mathcal{R}\varsigma) + 1)^{\frac{\varpi}{b}\mathcal{M}(\text{ı},\varsigma)},$$

where

$$\mathcal{M}(\text{ı}, \varsigma) = \max\left\{d(\text{ı}, \varsigma), \min\left\{\frac{d(\text{ı},\mathcal{R}\text{ı})d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\text{ı}, \varsigma)}, \frac{d(\text{ı},\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\text{ı})}{1 + d(\text{ı}, \varsigma)}\right\}\right\},$$

holds for all $\text{ı}, \varsigma \in \mathcal{N}$. Assume that the following conditions hold:

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ is an β-subadmissible,

(ii) there exists $\text{ı}_0 \in \mathcal{N}$ such that $\beta(\text{ı}_0, \mathcal{R}\text{ı}_0) \leq 1$,

(iii) either $\mathcal{R}$ is continuous, or if $\{\text{ı}_n\}$ is a sequence in $\mathcal{N}$ such that $\text{ı}_n \to \text{ı}$, $\beta(\text{ı}_n, \text{ı}_{n+1}) \leq 1$, then $\beta(\text{ı}, \mathcal{R}\text{ı}) \leq 1$.

Then $\mathcal{R}$ possesses a FP.

Moreover, if $\beta(\text{ı}, \mathcal{R}\text{ı})\beta(\varsigma, \mathcal{R}\varsigma) \leq 1$ for $\text{ı}, \varsigma \in Fix(\mathcal{R})$, then the FP is unique.

Corollary 7. *Let $(\mathcal{N}, d)$ be a complete extended b-SMS with $b \geq 1$ and consider a mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. Suppose that there exist the function $\beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ and a constant $\varpi \in [0, 1)$ such that*

$$d(\mathcal{R}\text{ı}, \mathcal{R}\varsigma) \leq \beta(\text{ı}, \mathcal{R}\text{ı})\beta(\varsigma, \mathcal{R}\varsigma)\frac{\varpi}{b}\mathcal{M}(\text{ı}, \varsigma), \tag{3.17}$$

where

$$\mathcal{M}(\text{ı}, \varsigma) = \max\left\{d(\text{ı}, \varsigma), \min\left\{\frac{d(\text{ı},\mathcal{R}\text{ı})d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\text{ı}, \varsigma)}, \frac{d(\text{ı},\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\text{ı})}{1 + d(\text{ı}, \varsigma)}\right\}\right\},$$

holds for all $\text{ı}, \varsigma \in \mathcal{N}$. Assume that the following conditions hold:

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ is an β-subadmissible,

(ii) there exists $\text{ı}_0 \in \mathcal{N}$ such that $\beta(\text{ı}_0, \mathcal{R}\text{ı}_0) \leq 1$,

(iii) either $\mathcal{R}$ is continuous, or if $\{\text{ı}_n\}$ is a sequence in $\mathcal{N}$ such that $\text{ı}_n \to \text{ı}$, $\beta(\text{ı}_n, \text{ı}_{n+1}) \leq 1$, then $\beta(\text{ı}, \mathcal{R}\text{ı}) \leq 1$.

Then $\mathcal{R}$ possesses a FP.

Moreover, if $\beta(\text{ı}, \mathcal{R}\text{ı})\beta(\varsigma, \mathcal{R}\varsigma) \leq 1$ for $\text{ı}, \varsigma \in Fix(\mathcal{R})$, then the FP is unique.

Proof. Let $\beta(\text{ı}, \mathcal{R}\text{ı})\beta(\varsigma, \mathcal{R}\varsigma) \leq 1$ for all $\text{ı}, \varsigma \in \mathcal{N}$; then by (3.17), we have

$$d(\mathcal{R}\text{ı}, \mathcal{R}\varsigma) \leq \beta(\text{ı}, \mathcal{R}\text{ı})\beta(\varsigma, \mathcal{R}\varsigma)\frac{\varpi}{b}\mathcal{M}(\text{ı}, \varsigma) \leq \frac{\varpi}{b}\mathcal{M}(\text{ı}, \varsigma),$$

where

$$\mathcal{M}(\text{ı}, \varsigma) = \max\left\{d(\text{ı}, \varsigma), \min\left\{\frac{d(\text{ı},\mathcal{R}\text{ı})d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\text{ı}, \varsigma)}, \frac{d(\text{ı},\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\text{ı})}{1 + d(\text{ı}, \varsigma)}\right\}\right\}.$$

Therefore, the conditions of Corollary 4 are satisfied, ensuring that $\mathcal{R}$ has a unique FP. □

4. Fixed point results in extended suprametric spaces

By taking $b = 1$ in Definition 5, the notion of an extended b-SMS reduces to that of an extended SMS. Under this specialization, we derive the following results.

Corollary 8. *Let $(\mathcal{N}, d)$ be a complete extended SMS, and consider a mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. Suppose that there exist the functions $\alpha, \beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ and a constant $\varpi \in [0, 1)$ such that $\alpha(\iota, \mathcal{R}\iota)\alpha(\varsigma, \mathcal{R}\varsigma) \geq \beta(\iota, \mathcal{R}\iota)\beta(\varsigma, \mathcal{R}\varsigma)$ implies*

$$d(\mathcal{R}\iota, \mathcal{R}\varsigma) \leq \varpi \mathcal{M}(\iota, \varsigma),$$

where

$$\mathcal{M}(\iota, \varsigma) = \max\left\{d(\iota, \varsigma), \min\left\{\frac{d(\iota,\mathcal{R}\iota)d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\iota, \varsigma)}, \frac{d(\iota,\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\iota)}{1 + d(\iota, \varsigma)}\right\}\right\},$$

holds for all $\iota, \varsigma \in \mathcal{N}$. Assume that the following conditions hold:

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ is an α-admissible mapping with respect to β,

(ii) there exists $\iota_0 \in \mathcal{N}$ such that $\alpha(\iota_0, \mathcal{R}\iota_0) \geq \beta(\iota_0, \mathcal{R}\iota_0)$,

(iii) either $\mathcal{R}$ is continuous, or if $\{\iota_n\}$ is a sequence in $\mathcal{N}$ such that $\iota_n \to \iota$, $\alpha(\iota_n, \iota_{n+1}) \geq \beta(\iota_n, \iota_{n+1})$, then $\alpha(\iota, \mathcal{R}\iota) \geq \beta(\iota, \mathcal{R}\iota)$.

Then $\mathcal{R}$ possesses a FP.

Moreover, if $\alpha(\iota, \mathcal{R}\iota)\alpha(\varsigma, \mathcal{R}\varsigma) \geq \beta(\iota, \mathcal{R}\iota)\beta(\varsigma, \mathcal{R}\varsigma)$ for $\iota, \varsigma \in Fix(\mathcal{R})$, then the FP is unique.

Corollary 9. *Let $(\mathcal{N}, d)$ be a complete extended SMS and consider a mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. Suppose that there exists a function $\alpha : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ and a constant $\varpi \in [0, 1)$ such that $\alpha(\iota, \mathcal{R}\iota)\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$ implies*

$$d(\mathcal{R}\iota, \mathcal{R}\varsigma) \leq \varpi \mathcal{M}(\iota, \varsigma),$$

where

$$\mathcal{M}(\iota, \varsigma) = \max\left\{d(\iota, \varsigma), \min\left\{\frac{d(\iota,\mathcal{R}\iota)d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\iota, \varsigma)}, \frac{d(\iota,\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\iota)}{1 + d(\iota, \varsigma)}\right\}\right\},$$

holds for all $\iota, \varsigma \in \mathcal{N}$. Assume that the following conditions hold:

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ is an α-admissible mapping,

(ii) there exists $\iota_0 \in \mathcal{N}$ such that $\alpha(\iota_0, \mathcal{R}\iota_0) \geq 1$,

(iii) either $\mathcal{R}$ is continuous, or if $\{\iota_n\}$ is a sequence in $\mathcal{N}$ such that $\iota_n \to \iota$, $\alpha(\iota_n, \iota_{n+1}) \geq 1$, then $\alpha(\iota, \mathcal{R}\iota) \geq 1$.

Then $\mathcal{R}$ possesses a FP.

Moreover, if $\alpha(\iota, \mathcal{R}\iota)\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$ for $\iota, \varsigma \in Fix(\mathcal{R})$, then the FP is unique.

Proof. Define $\beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ by $\beta(\iota, \varsigma) = 1$ in Corollary 8. □

Corollary 10. *Let $(\mathcal{N}, d)$ be a complete extended SMS, and consider a mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. Suppose that there exist the function $\beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ and a constant $\varpi \in [0, 1)$ such that $\beta(\iota, \mathcal{R}\iota)\beta(\varsigma, \mathcal{R}\varsigma) \leq 1$ implies*

$$d(\mathcal{R}\iota, \mathcal{R}\varsigma) \leq \varpi \mathcal{M}(\iota, \varsigma),$$

where

$$\mathcal{M}(\iota, \varsigma) = \max\left\{d(\iota, \varsigma), \min\left\{\frac{d(\iota,\mathcal{R}\iota)d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\iota, \varsigma)}, \frac{d(\iota,\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\iota)}{1 + d(\iota, \varsigma)}\right\}\right\},$$

holds for all $\imath, \varsigma \in \mathcal{N}$. *Assume that the following conditions hold:*

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ *is an* β*-subadmissible*,

(ii) there exists $\imath_0 \in \mathcal{N}$ *such that* $\beta(\imath_0, \mathcal{R}\imath_0) \leq 1$,

(iii) either $\mathcal{R}$ *is continuous, or if* $\{\imath_n\}$ *is a sequence in* $\mathcal{N}$ *such that* $\imath_n \to \imath$, $\beta(\imath_n, \imath_{n+1}) \leq 1$, *then* $\beta(\imath, \mathcal{R}\imath) \leq 1$.

Then $\mathcal{R}$ *possesses a FP.*

Moreover, if $\beta(\imath, \mathcal{R}\imath)\beta(\varsigma, \mathcal{R}\varsigma) \leq 1$ *for* $\imath, \varsigma \in Fix(\mathcal{R})$, *then the FP is unique.*

Proof. Define $\alpha : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ by $\alpha(\imath, \varsigma) = 1$ in Corollary 8. □

5. Existence of fixed points in b-suprametric spaces

If we define the function $\curlywedge : \mathcal{N} \times \mathcal{N} \to [1, \infty)$ by $\curlywedge(\imath, \varsigma) = \curlywedge \geq 1$ in Definition 5, the general framework of an extended b-SMS simplifies to the classical b-SMS. This specialization allows us to derive the following results.

Corollary 11. *Let* $(\mathcal{N}, d)$ *be a complete b-SMS with* $b \geq 1$ *and consider a mapping* $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. *Suppose that there exist the functions* $\alpha, \beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ *and a constant* $\varpi \in [0, 1)$ *such that* $\alpha(\imath, \mathcal{R}\imath)\alpha(\varsigma, \mathcal{R}\varsigma) \geq \beta(\imath, \mathcal{R}\imath)\beta(\varsigma, \mathcal{R}\varsigma)$ *implies*

$$d(\mathcal{R}\imath, \mathcal{R}\varsigma) \leq \frac{\varpi}{b}\mathcal{M}(\imath, \varsigma),$$

where

$$\mathcal{M}(\imath, \varsigma) = \max\left\{d(\imath, \varsigma), \min\left\{\frac{d(\imath, \mathcal{R}\imath)d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\imath, \varsigma)}, \frac{d(\imath, \mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\imath)}{1 + d(\imath, \varsigma)}\right\}\right\},$$

holds for all $\imath, \varsigma \in \mathcal{N}$. *Assume that the following conditions hold:*

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ *is an* α*-admissible mapping with respect to* β,

(ii) there exists $\imath_0 \in \mathcal{N}$ *such that* $\alpha(\imath_0, \mathcal{R}\imath_0) \geq \beta(\imath_0, \mathcal{R}\imath_0)$,

(iii) either $\mathcal{R}$ *is continuous, or if* $\{\imath_n\}$ *is a sequence in* $\mathcal{N}$ *such that* $\imath_n \to \imath$, $\alpha(\imath_n, \imath_{n+1}) \geq \beta(\imath_n, \imath_{n+1})$, *then* $\alpha(\imath, \mathcal{R}\imath) \geq \beta(\imath, \mathcal{R}\imath)$.

Then $\mathcal{R}$ *possesses a FP.*

Moreover, if $\alpha(\imath, \mathcal{R}\imath)\alpha(\varsigma, \mathcal{R}\varsigma) \geq \beta(\imath, \mathcal{R}\imath)\beta(\varsigma, \mathcal{R}\varsigma)$ *for* $\imath, \varsigma \in Fix(\mathcal{R})$, *then the FP is unique.*

Corollary 12. *Let* $(\mathcal{N}, d)$ *be a complete b-SMS with* $b \geq 1$ *and consider a mapping* $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. *Suppose that there exists a function* $\alpha : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ *and a constant* $\varpi \in [0, 1)$ *such that* $\alpha(\imath, \mathcal{R}\imath)\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$ *implies*

$$d(\mathcal{R}\imath, \mathcal{R}\varsigma) \leq \frac{\varpi}{b}\mathcal{M}(\imath, \varsigma),$$

where

$$\mathcal{M}(\imath, \varsigma) = \max\left\{d(\imath, \varsigma), \min\left\{\frac{d(\imath, \mathcal{R}\imath)d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\imath, \varsigma)}, \frac{d(\imath, \mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\imath)}{1 + d(\imath, \varsigma)}\right\}\right\},$$

holds for all $\imath, \varsigma \in \mathcal{N}$. *Assume that the following conditions hold:*

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ *is an* α*-admissible mapping*,

(ii) there exists $\imath_0 \in \mathcal{N}$ *such that* $\alpha(\imath_0, \mathcal{R}\imath_0) \geq 1$,

(iii) either $\mathcal{R}$ *is continuous, or if* $\{\imath_n\}$ *is a sequence in* $\mathcal{N}$ *such that* $\imath_n \to \imath$, $\alpha(\imath_n, \imath_{n+1}) \geq 1$, *then* $\alpha(\imath, \mathcal{R}\imath) \geq 1$.

Then $\mathcal{R}$ possesses a FP.

Moreover, if $\alpha(\text{ı}, \mathcal{R}\text{ı})\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$ for $\text{ı}, \varsigma \in Fix(\mathcal{R})$, then the FP is unique.

Proof. Define $\beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ by $\beta(\text{ı}, \varsigma) = 1$ in Corollary 11. □

Corollary 13. *Let $(\mathcal{N}, d)$ be a complete b-SMS with $b \geq 1$ and consider a mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. Suppose that there exist the function $\beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ and a constant $\varpi \in [0, 1)$ such that $\beta(\text{ı}, \mathcal{R}\text{ı})\beta(\varsigma, \mathcal{R}\varsigma) \leq 1$ implies*

$$d(\mathcal{R}\text{ı}, \mathcal{R}\varsigma) \leq \frac{\varpi}{b}\mathcal{M}(\text{ı}, \varsigma),$$

where

$$\mathcal{M}(\text{ı}, \varsigma) = \max\left\{d(\text{ı}, \varsigma), \min\left\{\frac{d(\text{ı},\mathcal{R}\text{ı})d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\text{ı}, \varsigma)}, \frac{d(\text{ı},\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\text{ı})}{1 + d(\text{ı}, \varsigma)}\right\}\right\},$$

holds for all $\text{ı}, \varsigma \in \mathcal{N}$. Assume that the following conditions hold:

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ is an β-subadmissible,

(ii) there exists $\text{ı}_0 \in \mathcal{N}$ such that $\beta(\text{ı}_0, \mathcal{R}\text{ı}_0) \leq 1$,

(iii) either $\mathcal{R}$ is continuous, or if $\{\text{ı}_n\}$ is a sequence in $\mathcal{N}$ such that $\text{ı}_n \to \text{ı}$, $\beta(\text{ı}_n, \text{ı}_{n+1}) \leq 1$, then $\beta(\text{ı}, \mathcal{R}\text{ı}) \leq 1$.

Then $\mathcal{R}$ possesses a FP.

Moreover, if $\beta(\text{ı}, \mathcal{R}\text{ı})\beta(\varsigma, \mathcal{R}\varsigma) \leq 1$ for $\text{ı}, \varsigma \in Fix(\mathcal{R})$, then the FP is unique.

Proof. Define $\alpha : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ by $\alpha(\text{ı}, \varsigma) = 1$ in Corollary 11. □

6. Fixed point findings in suprametric spaces

By defining the function $\curlywedge : \mathcal{N} \times \mathcal{N} \to [1, \infty)$ by $\curlywedge(\text{ı}, \varsigma) = \curlywedge \geq 1$ and setting the constant $b = 1$ in Definition 5, the structure of an extended b-SMS reduces to that of a standard SMS. This particular case enables us to derive the following results.

Corollary 14. *Let $(\mathcal{N}, d)$ be a complete SMS and consider a mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. Suppose that there exist the functions $\alpha, \beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ and a constant $\varpi \in [0, 1)$ such that $\alpha(\text{ı}, \mathcal{R}\text{ı})\alpha(\varsigma, \mathcal{R}\varsigma) \geq \beta(\text{ı}, \mathcal{R}\text{ı})\beta(\varsigma, \mathcal{R}\varsigma)$ implies*

$$d(\mathcal{R}\text{ı}, \mathcal{R}\varsigma) \leq \varpi\mathcal{M}(\text{ı}, \varsigma),$$

where

$$\mathcal{M}(\text{ı}, \varsigma) = \max\left\{d(\text{ı}, \varsigma), \min\left\{\frac{d(\text{ı},\mathcal{R}\text{ı})d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\text{ı}, \varsigma)}, \frac{d(\text{ı},\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\text{ı})}{1 + d(\text{ı}, \varsigma)}\right\}\right\},$$

holds for all $\text{ı}, \varsigma \in \mathcal{N}$. Assume that the following conditions hold:

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ is an α-admissible mapping with respect to β,

(ii) there exists $\text{ı}_0 \in \mathcal{N}$ such that $\alpha(\text{ı}_0, \mathcal{R}\text{ı}_0) \geq \beta(\text{ı}_0, \mathcal{R}\text{ı}_0)$,

(iii) either $\mathcal{R}$ is continuous, or if $\{\text{ı}_n\}$ is a sequence in $\mathcal{N}$ such that $\text{ı}_n \to \text{ı}$, $\alpha(\text{ı}_n, \text{ı}_{n+1}) \geq \beta(\text{ı}_n, \text{ı}_{n+1})$, then $\alpha(\text{ı}, \mathcal{R}\text{ı}) \geq \beta(\text{ı}, \mathcal{R}\text{ı})$.

Then $\mathcal{R}$ possesses a FP.

Moreover, if $\alpha(\text{ı}, \mathcal{R}\text{ı})\alpha(\varsigma, \mathcal{R}\varsigma) \geq \beta(\text{ı}, \mathcal{R}\text{ı})\beta(\varsigma, \mathcal{R}\varsigma)$ for $\text{ı}, \varsigma \in Fix(\mathcal{R})$, then the FP is unique.

Corollary 15. *Let* $(\mathcal{N}, d)$ *be a complete SMS and consider a mapping* $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. *Suppose that there exists a function* $\alpha : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ *and a constant* $\varpi \in [0, 1)$ *such that* $\alpha(\imath, \mathcal{R}\imath)\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$ *implies*

$$d(\mathcal{R}\imath, \mathcal{R}\varsigma) \leq \varpi \mathcal{M}(\imath, \varsigma),$$

where

$$\mathcal{M}(\imath, \varsigma) = \max\left\{d(\imath, \varsigma), \min\left\{\frac{d(\imath,\mathcal{R}\imath)d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\imath, \varsigma)}, \frac{d(\imath,\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\imath)}{1 + d(\imath, \varsigma)}\right\}\right\},$$

holds for all $\imath, \varsigma \in \mathcal{N}$. *Assume that the following conditions hold:*

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ *is an* α*-admissible mapping*,

(ii) there exists $\imath_0 \in \mathcal{N}$ *such that* $\alpha(\imath_0, \mathcal{R}\imath_0) \geq 1$,

(iii) either $\mathcal{R}$ *is continuous, or if* $\{\imath_n\}$ *is a sequence in* $\mathcal{N}$ *such that* $\imath_n \to \imath$, $\alpha(\imath_n, \imath_{n+1}) \geq 1$, *then* $\alpha(\imath, \mathcal{R}\imath) \geq 1$.

Then $\mathcal{R}$ *possesses a FP.*

Moreover, if $\alpha(\imath, \mathcal{R}\imath)\alpha(\varsigma, \mathcal{R}\varsigma) \geq 1$ *for* $\imath, \varsigma \in Fix(\mathcal{R})$, *then the FP is unique.*

Proof. Define $\beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ by $\beta(\imath, \varsigma) = 1$ in Corollary 14. □

Corollary 16. *Let* $(\mathcal{N}, d)$ *be a complete SMS and consider a mapping* $\mathcal{R} : \mathcal{N} \to \mathcal{N}$. *Suppose that there exist the function* $\beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ *and a constant* $\varpi \in [0, 1)$ *such that* $\beta(\imath, \mathcal{R}\imath)\beta(\varsigma, \mathcal{R}\varsigma) \leq 1$ *implies*

$$d(\mathcal{R}\imath, \mathcal{R}\varsigma) \leq \varpi \mathcal{M}(\imath, \varsigma),$$

where

$$\mathcal{M}(\imath, \varsigma) = \max\left\{d(\imath, \varsigma), \min\left\{\frac{d(\imath,\mathcal{R}\imath)d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\imath, \varsigma)}, \frac{d(\imath,\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\imath)}{1 + d(\imath, \varsigma)}\right\}\right\},$$

holds for all $\imath, \varsigma \in \mathcal{N}$. *Assume that the following conditions hold:*

(i) the mapping $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ *is an* β*-subadmissible*,

(ii) there exists $\imath_0 \in \mathcal{N}$ *such that* $\beta(\imath_0, \mathcal{R}\imath_0) \leq 1$,

(iii) either $\mathcal{R}$ *is continuous, or if* $\{\imath_n\}$ *is a sequence in* $\mathcal{N}$ *such that* $\imath_n \to \imath$, $\beta(\imath_n, \imath_{n+1}) \leq 1$, *then* $\beta(\imath, \mathcal{R}\imath) \leq 1$.

Then $\mathcal{R}$ *possesses a FP.*

Moreover, if $\beta(\imath, \mathcal{R}\imath)\beta(\varsigma, \mathcal{R}\varsigma) \leq 1$ *for* $\imath, \varsigma \in Fix(\mathcal{R})$, *then the FP is unique.*

Proof. Define $\alpha : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ by $\alpha(\imath, \varsigma) = 1$ in Corollary 14. □

7. Applications

The study of nonlinear Volterra-Fredholm integral equations plays a fundamental role in various fields such as mathematical physics, engineering, biology, and economics. These equations arise naturally in systems involving memory effects or spatial interactions, such as population dynamics, heat conduction with hereditary effects, and viscoelastic materials.

One of the most effective and widely used tools for investigating the existence and uniqueness of solutions to such integral equations is FP theory. By transforming the integral equation into an equivalent FP problem, we can utilize powerful results from functional analysis to guarantee the existence of solutions under suitable conditions. This approach not only simplifies the analysis but also provides constructive methods for approximating the solution.

In this section, we solve the nonlinear Volterra-Fredholm integral equation of the form

$$\imath(t) = f(t) + \mu_1 \int_a^t K_1(t, s, \imath(s))ds + \mu_2 \int_a^b K_2(t, s, \imath(s))ds, \tag{7.1}$$

where

- $\imath(t)$ is the unknown function defined on the interval $[a, b]$,
- $f(t)$ is a known function defined on $[a, b]$,
- μ_1 and μ_2 are constants (parameters),
- a and b are constants representing the limits of integration,
- $K_1(t, s, \imath(s))$ and $K_2(t, s, \imath(s))$ are kernel functions that depend on t, s and the unknown function $\imath(t)$.

Let $\mathcal{N} = C([a, b], \mathbb{R})$, the set of all continuous real valued functions defined on $[a, b]$. Define $d : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ by

$$d(\imath, \varsigma) = \max_{t\in[a,b]} |\imath(t) - \varsigma(t)|^2,$$

and $\curlywedge : \mathcal{N} \times \mathcal{N} \to [1, \infty)$ by

$$\curlywedge(\imath, \varsigma) = 1 + |\imath(t)| + |\varsigma(t)|,$$

then clearly $(\mathcal{N}, d)$ is a complete extended b-SMS with $b = 2$.

Theorem 3. *Define the mapping*

$$\mathcal{R}(\imath)(t) = f(t) + \mu_1 \int_a^t K_1(t, s, \imath(s))ds + \mu_2 \int_a^b K_2(t, s, \imath(s))ds.$$

Assume the following conditions hold:

(i) The kernel functions $K_1(t, s, \imath(s))$ and $K_2(t, s, \imath(s))$ are continuous in t and s, and satisfy

$$|K_1(t, s, \imath(s))| \leq M_1, \quad |K_2(t, s, \imath(s))| \leq M_2,$$

for some constants $M_1, M_2 > 0$ and for all $t, s \in [a, b]$ and $\imath \in \mathcal{N}$.

(ii) The kernel functions K_1 and K_2 satisfy the Lipschitz condition with respect to $\imath(s)$, that is,

$$|K_1(t, s, \imath(s)) - K_1(t, s, \varsigma(s))| \leq L_1 |\imath(s) - \varsigma(s)|,$$

$$|K_2(t, s, \imath(s)) - K_2(t, s, \varsigma(s))| \leq L_2 |\imath(s) - \varsigma(s)|,$$

for some constants $L_1, L_2 > 0$ and for all $t, s \in [a, b]$ and $\imath, \varsigma \in \mathcal{N}$.

(iii) Suppose that $\varpi = 2(|\mu_1| L_1(b - a) + |\mu_2| L_2(b - a))^2$ such that $\varpi < 1$.

(iv) There exist functions $\alpha, \beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ such that

- $\alpha(\imath, \mathcal{R}\imath)\alpha(\varsigma, \mathcal{R}\varsigma) \geq \beta(\imath, \mathcal{R}\imath)\beta(\varsigma, \mathcal{R}\varsigma)$,
- *$\mathcal{R}$ is an α-admissible with respect to β,*
- *there exists $\imath_0 \in \mathcal{N}$ such that $\alpha(\imath_0, \mathcal{R}\imath_0) \geq \beta(\imath_0, \mathcal{R}\imath_0)$,*

- *either* $\mathcal{R}$ *is continuous, or if* $\{\imath_n\}$ *is a sequence in* $\mathcal{N}$ *such that* $\imath_n \to \imath$, $\alpha(\imath_n, \imath_{n+1}) \geq \beta(\imath_n, \imath_{n+1})$, *then* $\alpha(\imath, \mathcal{R}\imath) \geq \beta(\imath, \mathcal{R}\imath)$.

Under the above conditions, the mapping $\mathcal{R}$ has a unique FP $\imath^* \in \mathcal{N}$, which is the solution to the nonlinear Volterra-Fredholm integral Eq (7.1).

Proof. For $\imath, \varsigma \in \mathcal{N}$, we have

$$\begin{aligned} |\mathcal{R}\imath(t) - \mathcal{R}\varsigma(t)| &= \left|\mu_1 \int_a^t K_1(t,s,\imath(s))ds + \mu_2 \int_a^b K_2(t,s,\imath(s))ds - \left(\mu_1 \int_a^t K_1(t,s,\imath(s))ds + \mu_2 \int_a^b K_2(t,s,\imath(s))ds\right)\right| \\ &\leq |\mu_1| \int_a^t |K_1(t, s, \imath(s)) - K_1(t, s, \varsigma(s))|\, ds + |\mu_2| \int_a^b |K_2(t, s, \imath(s)) - K_2(t, s, \varsigma(s))|\, ds. \end{aligned}$$

By the Lipschitz condition on K_1 and K_2, we have

$$|\mathcal{R}\imath(t) - \mathcal{R}\varsigma(t)| \leq |\mu_1|\, L_1 \int_a^t |\imath(s) - \varsigma(s)|\, ds + |\mu_2|\, L_2 \int_a^b |\imath(s) - \varsigma(s)|\, ds.$$

Since $|\imath(s) - \varsigma(s)| \leq \sqrt{d(\imath, \varsigma)}$, we have

$$|\mathcal{R}\imath(t) - \mathcal{R}\varsigma(t)| \leq |\mu_1|\, L_1 \sqrt{d(\imath, \varsigma)}(t - a) + |\mu_2|\, L_2 \sqrt{d(\imath, \varsigma)}(b - a).$$

Taking the maximum over $t \in [a, b]$, we have

$$d(\mathcal{R}\imath, \mathcal{R}\varsigma) = \max_{t\in[a,b]} |\mathcal{R}\imath(t) - \mathcal{R}\varsigma(t)|^2 \leq (|\mu_1|\, L_1(b - a) + |\mu_2|\, L_2(b - a))^2\, d(\imath, \varsigma) \leq \frac{\varpi}{2} d(\imath, \varsigma).$$

Since $d(\imath, \varsigma) \leq \mathcal{M}(\imath, \varsigma)$, where

$$d(\imath, \varsigma) = \max\left\{d(\imath, \varsigma), \min\left\{\frac{d(\imath,\mathcal{R}\imath)d(\varsigma, \mathcal{R}\varsigma)}{1 + d(\imath, \varsigma)}, \frac{d(\imath,\mathcal{R}\varsigma)d(\varsigma, \mathcal{R}\imath)}{1 + d(\imath, \varsigma)}\right\}\right\},$$

so, the contractive condition of Theorem 2 holds. The remaining conditions of Theorem 2 are satisfied trivially by assumption (iv). Therefore, the mapping $\mathcal{R}$ has a unique FP $\imath^* \in \mathcal{N}$, which is the solution to the nonlinear Volterra-Fredholm integral Eq (7.1). □

Example 6. Let $\mathcal{N} = C([0, 1], \mathbb{R})$, the set of all continuous real valued functions defined on $[0, 1]$. Define $d : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ by

$$d(\imath, \varsigma) = \max_{t\in[0,1]} |\imath(t) - \varsigma(t)|^2,$$

and $\curlywedge : \mathcal{N} \times \mathcal{N} \to [1, \infty)$ by

$$\curlywedge(\imath, \varsigma) = 1 + |\imath(t)| + |\varsigma(t)|,$$

then clearly $(\mathcal{N}, d)$ is a complete extended b-SMS with $b = 2$. Define the mapping

$$\mathcal{R}(\imath)(t) = f(t) + \mu_1 \int_a^t K_1(t, s, \imath(s))ds + \mu_2 \int_a^b K_2(t, s, \imath(s))ds,$$

where

- $f(t) = t$ is a known function defined on $[0, 1]$,

- $\mu_1 = \frac{1}{4}$ and $\mu_2 = \frac{1}{4}$ are constants (parameters),
- $a = 0$ and $b = 1$ are constants representing the limits of integration,
- $K_1(t, s, \text{ı}(s)) = \frac{\text{ı}(s)}{1+\text{ı}^2(s)}$ and $K_2(t, s, \text{ı}(s)) = \frac{\text{ı}(s)}{2+\text{ı}^2(s)}$ are kernel functions that depend on t, s and the unknown function $\text{ı}(t)$.

Now we verify all the conditions of above theorem. Since

$$|K_1(t, s, \text{ı}(s))| = \left|\frac{\text{ı}(s)}{1 + (\text{ı}(s))^2}\right| \leq \frac{1}{2}\ ,$$

and

$$|K_2(t, s, \text{ı}(s))| = \left|\frac{\text{ı}(s)}{2 + (\text{ı}(s))^2}\right| \leq \frac{1}{2},$$

so condition (i) is satisfied for $M_1 = \frac{1}{2}$ and $M_2 = \frac{1}{2}$. Since the derivatives of K_1 and K_2 with respect to $\text{ı}(s)$ are

$$\frac{d}{d\text{ı}(s)} \frac{\text{ı}(s)}{1 + \text{ı}^2(s)} = \frac{1 - \text{ı}^2(s)}{(1 + \text{ı}^2(s))^2},$$

and

$$\frac{d}{d\text{ı}(s)} \frac{\text{ı}(s)}{2 + \text{ı}^2(s)} = \frac{2 - \text{ı}^2(s)}{(1 + \text{ı}^2(s))^2}.$$

Now

$$\max_{s\in[0,1]}\left|\frac{1 - \text{ı}^2(s)}{(1 + \text{ı}^2(s))^2}\right| = 1 \ \text{ and } \max_{s\in[0,1]}\left|\frac{2 - \text{ı}^2(s)}{(1 + \text{ı}^2(s))^2}\right| = \frac{1}{2}.$$

Thus condition (ii) is satisfied with $L_1 = 1$ and $L_2 = \frac{1}{2}$. Now

$$\begin{aligned}
\varpi &= 2\left(|\mu_1|\, L_1(b-a) + |\mu_2|\, L_2(b-a)\right)^2 \\
&= 2\left(\frac{1}{4}\cdot 1(1-0) + \frac{1}{4}\cdot\frac{1}{2}(1-0)\right)^2 \\
&= 2\left(\frac{1}{4} + \frac{1}{8}\right)^2 = \frac{9}{32} < 1.
\end{aligned}$$

Define $\alpha, \beta : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ by

$$\alpha(\text{ı}, \varsigma) = 1 + \|\text{ı}(t)\| + \|\varsigma(t)\|,$$

where $\|\text{ı}(t)\| = \max_{t\in[0,1]} |\text{ı}(t)|$ and

$$\beta(\text{ı}, \varsigma) = \frac{1}{2}\alpha(\text{ı}, \varsigma). \tag{7.2}$$

$\mathcal{R}$ is α-admissible with respect to β because

$$\alpha(\text{ı}, \mathcal{R}\text{ı}) = 1 + \|\text{ı}\| + \|\mathcal{R}\text{ı}\| \geq \frac{1}{2}(1 + \|\text{ı}\| + \|\mathcal{R}\text{ı}\|) = \beta(\text{ı}, \mathcal{R}\text{ı}).$$

Choose $\text{ı}_0(t) = 0$. Then

$$\alpha(\text{ı}_0, \mathcal{R}\text{ı}_0) = 1 + \|\text{ı}_0\| + \|\mathcal{R}\text{ı}_0\| = 1 + 0 + \|\mathcal{R}\text{ı}_0\| \geq \frac{1}{2}(1 + \|\text{ı}_0\| + \|\mathcal{R}\text{ı}_0\|) = \beta(\text{ı}_0, \mathcal{R}\text{ı}_0).$$

Moreover, $\mathcal{R}$ is continuous because it is defined by integrals of continuous functions. Alternatively, if $\{\mathfrak{l}_n\}$ is a sequence converging to $\mathfrak{l}$ with $\alpha(\mathfrak{l}_n, \mathfrak{l}_{n+1}) \geq \beta(\mathfrak{l}_n, \mathfrak{l}_{n+1})$, then $\alpha(\mathfrak{l}, \mathcal{R}\mathfrak{l}) \geq \beta(\mathfrak{l}, \mathcal{R})$ holds because $\alpha(\mathfrak{l}, \varsigma) \geq \beta(\mathfrak{l}, \varsigma)$ is always true. Thus, all conditions of the above theorem are satisfied. Therefore, the mapping $\mathcal{R}$ has a unique FP $\mathfrak{l}^* \in \mathcal{N}$, which is the solution to the given nonlinear Volterra-Fredholm integral equation.

8. Conclusions

In this study, we explored the concept of extended b-suprametric spaces and established fixed point theorems for single-valued mappings under generalized contraction conditions. These results extended and unified several known fixed point theorems in the existing literature, thereby offering a broader and more flexible analytical framework. As a natural outcome, we derived corollaries that specialized our general theorems to the classical settings of suprametric spaces and b-metric spaces, demonstrating the adaptability and generality of our approach. To highlight the novelty and validity of the proposed results, we presented an illustrative example that confirmed the effectiveness and sharpness of our theorems. Finally, we demonstrated the practical relevance of our findings by applying the main results to establish the existence of solutions for a nonlinear Volterra-Fredholm integral equation, thereby linking abstract fixed point theory with applied mathematical modeling.

9. Future work and open problems

This work presents novel FP theorems in the framework of extended b-SMSs, laying a foundation for deeper investigation in both theoretical analysis and practical applications. Nonetheless, several unresolved questions persist, highlighting potential directions for future research.

9.1. Generalization to other mathematical structures

- Extending the present findings to broader frameworks, including complex-valued extended b-SMSs, elliptic-valued SMSs, and quaternion-valued SMSs.
- Analyzing the existence of FPs under more relaxed contraction conditions or within hybrid metric structures.
- Studying various classes of mappings, such as cyclic, asymptotic, and quasi-contractive mappings, particularly in the context of multivalued operators.

Developing FP results within these spaces broadens their applicability to areas such as physics, differential equations, and optimization. Moreover, relaxing the contraction conditions increases their relevance to real-world problems where strict contractive assumptions may not be valid.

9.2. Contribution to the study of differential equations

Although this study focuses on the application of FP results to integral equations, a valuable extension lies in exploring their use in differential equations (DEs) and functional differential equations. In particular:

- Analyzing the solutions of fractional differential equations through FP approaches.

- Applying the theory to delay differential equations, especially in systems characterized by memory effects or time-lagged interactions.

The adaptation of FP theory to these classes of differential equations could lead to significant theoretical and computational advancements.

9.3. *Computational methods and algorithmic implementations*

Future research can be directed toward the following promising areas:

- Designing iterative numerical algorithms for approximating FPs, with particular emphasis on applications to Fredholm integral equations and nonlinear partial differential equations. Such schemes can offer practical means to implement and test theoretical results.
- Integrating machine learning techniques to optimize FP computations and applying these methods to real-world problems such as image denoising. This interdisciplinary approach could significantly improve both computational efficiency and practical impact.

These computational strategies would not only validate the theoretical developments presented in this work but also broaden their applicability to various scientific and engineering domains.

Author contributions

Jamshaid Ahmad: Conceptualization, Writing-review and editing, Funding acquisition; Amer H. Albargi: Writing-original draft, Methodology, Formal analysis. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. M. Frechet, Sur quelques points du calcul fonctionnel, *Rend. Circ. Matem. Palermo*, **22** (1906), 1–72. https://doi.org/10.1007/BF03018603
2. S. Matthews, Partial metric topology, *Ann. NY. Acad. Sci.*, **728** (1994), 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x

3. I. A. Bakhtin, The contraction mapping principle in almost metric space, *Funct. Anal., Gos. Ped. Inst., Ulianowsk*, **30** (1989), 26–37.
4. A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen*, **57** (2000), 31–37. https://doi.org/10.5486/PMD.2000.2133
5. T. Kamran, M. Samreen, Q. Ain, A generalization of b-metric space and some fixed point theorems, *Mathematics*, **5** (2017), 19. https://doi.org/10.3390/math5020019
6. M. Samreen, T. Kamran, M. Postolache, Extended b-metric space, extended b-comparison function and nonlinear contractions, *U.P.B. Sci. Bull., Series A*, **80** (2018), 21–28.
7. M. Berzig, First results in suprametric spaces with applications, *Mediterr. J. Math.*, **19** (2022), 226. https://doi.org/10.1007/s00009-022-02148-6
8. M. Berzig, Fixed point results in generalized suprametric spaces, *Topological Algebra and its Applications*, **11** (2023), 20230105. https://doi.org/10.1515/taa-2023-0105
9. M. Berzig, Nonlinear contraction in b-suprametric spaces, *J. Anal.*, **32** (2024), 2401–2414. https://doi.org/10.1007/s41478-024-00732-5
10. S. Panda, R. Agarwal, E. Karapínar, Extended suprametric spaces and Stone-type theorem, *AIMS Mathematics*, **8** (2023), 23183–23199. https://doi.org/10.3934/math.20231179
11. S. Panda, V. Velusamy, I. Khan, S. Niazai, Computation and convergence of fixed-point with an RLC-electric circuit model in an extended b-suprametric space, *Sci. Rep.*, **14** (2024), 9479. https://doi.org/10.1038/s41598-024-59859-x
12. E. Karapınar, P. Kumari, D. Lateef, A new approach to the solution of the Fredholm integral equation via a fixed point on extended b-metric spaces, *Symmetry*, **10** (2018), 512. https://doi.org/10.3390/sym10100512
13. A. Abdou, A fixed point approach to predator-prey dynamics via nonlinear mixed Volterra-Fredholm integral equations in complex-valued suprametric spaces, *AIMS Mathematics*, **10** (2025), 6002–6024. https://doi.org/10.3934/math.2025274
14. H. Ahmad, O. Chauhan, T. Lazar, V. Lazar, Some convergence results on proximal contractions with application to nonlinear fractional differential equation, *AIMS Mathematics*, **10** (2025), 5353–5372. https://doi.org/10.3934/math.2025247
15. M. Younis, H. Ahmad, M. Ozturk, F. Din, M. Qasim, Unveiling fractional-order dynamics: a new method for analyzing Rössler chaos, *J. Comput. Appl. Math.*, **468** (2025), 116639. https://doi.org/10.1016/j.cam.2025.116639
16. S. Panda, V. Vijayakumar, R. Agarwal, Complex-valued suprametric spaces, related fixed point results, and their applications to Barnsley Fern fractal generation and mixed Volterra-Fredholm integral equations, *Fractal Fract.*, **8** (2024), 410. https://doi.org/10.3390/fractalfract8070410
17. S. Zubair, M. Aphane, A. Mukheimer, T. Abdeljawad, A fixed point technique for solving boundary value problems in branciari suprametric spaces, *Results in Nonlinear Analysis*, **7** (2024), 80–93. https://doi.org/10.31838/rna/2024.07.03.008

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