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Research article

Set-valued contractions with an application to Fredholm integral inclusions in m_v^b-metric spaces

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Abstract: This study investigates set-valued contractions within the framework of m_v^b-metric spaces, extending classical contraction principles. By introducing and examining the Hausdorff m_v^b-metric, we establish a foundation for set-valued fixed point theorems, thereby contributing significantly to this area of research. Our findings generalize several well-known contraction concepts, including those of Banach, Sehgal, Wardowski, Altun, Bianchini, and Nadler, within the context of m_v^b-metric spaces. These advancements have practical implications, particularly in the study of nonlinear systems and the mathematical model of Fredholm integral inclusions. The results presented here emphasize the growing importance of set-valued fixed points and pave the way for further exploration and application across various scientific and engineering domains.

Keywords: $F-$contraction; Hausdorff m_v^b-metric; Sehgal contraction; nonlinear systems; mathematical model of Fredholm integral inclusion
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1. Introduction

Fixed point theory is a rapidly evolving subfield of mathematics with widespread applicability across various disciplines, including transportation theory, economics, biomathematics, and more. The classic Banach contraction [1] has seen extensive generalizations in numerous directions, as evidenced by research in [2–4]. One fascinating extension is the concept of set-valued contractions according to Nadler [5]. Set-valued mappings offer a broader perspective beyond single-valued mappings and find applications in diverse areas such as game theory, chemical sciences [6], Nash equilibria, and engineering, as highlighted in [7–9]. The notion of Hausdorff distance plays a pivotal role in computer science and mathematics, including applications in fractals, image processing, and optimization theory. The Hausdorff metric [10], which quantifies the greatest distance between a point in one set and its nearest counterpart in another set, holds significant importance.

To extend Banach's contraction theory into more versatile forms, metric spaces have been explored from various angles, leading to concepts such as M_v^b-metric spaces, $b_v(s)-$metric spaces, $v-$generalized metric spaces, and more. The terms $v-$generalized metric, $b-$metric, $b_v(s)-$metric, $M-$metric, rectangular $M-$metric, generalized p_v^b-partial metric, and M_v-metric were introduced by researchers like Branciari [11], Bakhtin [12], Mitrovi and Radenović [13], Asadi et al. [14], Özgür et al. [15], and Asim et al. [16]. In 2021, Joshi et al. [17] introduced the M_v^b-metric as a generalization of the M_v-metric [16], and their work led to solutions for problems like the Cantilever Beam Problem. Additionally, Alam et al. [18–20] applied fixed point results via the M_v^b-metric to address the spread of infectious diseases, nonlinear matrix equations, rockets' ascending motion, and fourth-order differential equations related to beam theory. For a deeper understanding of metric generalizations, Kirk and Shahzed [21] provide valuable references.

Utilizing the ideas of Hausdorff [10], this work aims to define and explore the properties of the Hausdorff m_v^b-metric derived from an m_v^b-metric. We extend the scope of single-valued contractions to set-valued contractions to establish set-valued fixed points. We substantiate these findings with illustrative examples to validate the significance of our hypotheses. To achieve this, we generalize well-known contractions such as Banach [1], Sehgal [22], Wardowski [23], Altun et al. [24], Bianchini [25], and Nadler [5] to the context of m_v^b-metric spaces. These results contribute significantly to the study of set-valued fixed points and their applications in Fredholm integral inclusions.

This paper is organized as follows: In Section 2, we present essential definitions, notations, and foundational concepts related to m_v^b-metrics and set-valued mappings, forming the basis for subsequent developments. In Section 3, building on Hausdorff's ideas, we define the Hausdorff m_v^b-metric and explore its properties. We extend classical contractions such as those by Banach, Sehgal, Wardowski, Altun et al., Bianchini, and Nadler to the set-valued context within m_v^b-metric spaces. Theoretical results are validated through illustrative examples. Section 4 applies the developed fixed point results to demonstrate the existence of solutions to Fredholm integral inclusions, highlighting the practical utility of the theoretical framework. In Section 5, we summarize the main contributions of the paper, highlighting the successful generalization of classical contractions to the m_v^b-metric setting and emphasizing their applicability in mathematical and engineering problems, particularly those involving integral inclusions.

2. Preliminaries

We now recall Nadler's [5] famous contraction theorem for set-valued mapping.

Theorem 1. *[5] Let $(\mathcal{Y}, \rho)$ be a complete metric space and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ be a set-valued function,* *so that*

$$\mathcal{H}(\mathcal{T}u, \mathcal{T}w) \leq \xi\rho(u, w), \forall u, w \in \mathcal{Y},\ 0 \leq \xi < 1,$$

where $\mathcal{H}$ is the Hausdorff metric induced by ρ (see [10]). Then, $\mathcal{T}$ has a fixed point in $\mathcal{Y}$, that is, there exists $w \in \mathcal{Y}$ so that $w \in \mathcal{T}w$.

We have the notations

$$m^b_{v_{u,w}} = \min\{m^b_v(u, u), m^b_v(w, w)\},$$

and

$$\mathcal{M}^b_{v_{u,w}} = \max\{m^b_v(u, u), m^b_v(w, w)\},$$

which are used by Joshi et al. in [17, 26] to generalize an $\mathcal{M}_v$−metric space [16], and many others (see [13–15] and so on) by introducing m^b_v−metric space.

Definition 1. *[17] A pair $(\mathcal{Y}, m^b_v)$ consisting of a non-empty set $\mathcal{Y}$ and a mapping $m^b_v : \mathcal{Y} \times \mathcal{Y} \longrightarrow [0, \infty)$ is called an m^b_v−metric space if*
(m^b_v1) $m^b_v(u, w) = m^b_v(u, u) = m^b_v(w, w) \Leftrightarrow u = w$,
(m^b_v2) $m^b_{v_{u,w}} \leq m^b_v(u, w)$,
(m^b_v3) $m^b_v(u, w) = m^b_v(w, u)$,
(m^b_v4) $$\left(m^b_v(u, w) - m^b_{v_{u,w}}\right) \leq s\left[\left(m^b_v(u, \chi_1) - m^b_{v_{u,\chi_1}}\right) + \left(m^b_v(\chi_1, \chi_2) - m^b_{v_{\chi_1,\chi_2}}\right) + \cdots + \left(m^b_v(\chi_v, w) - m^b_{v_{\chi_v,w}}\right)\right] - \sum_{j=1}^{v} m^b_v(\chi_j, \chi_j),\ s \geq 1,$$
for all distinct $u, w, \chi_j \in \mathcal{Y},\ 0 \leq j \leq v$, and for some particular choice of $v \in \mathbb{N}$.

Definition 2. *In a m^b_v−metric space $(\mathcal{Y}, m^b_v)$, a sequence $\{w_m\}$ is*

(a) m^b_v-convergent to $w \in \mathcal{Y}$ if and only if $\lim_{m\to\infty}\left(m^b_v(w_m, w) - m^b_{v_{w_m,w}}\right) = 0$.

(b) m^b_v-Cauchy if and only if $\lim_{m,p\to\infty}\left(m^b_v(w_m, w_p) - m^b_{v_{w_m,w_p}}\right)$ and $\lim_{m,p\to\infty}\left(\mathcal{M}^b_{v_{w_m,w_p}} - m^b_{v_{w_m,w_p}}\right)$ exist finitely.

Again an m^b_v−metric space $(\mathcal{Y}, m^b_v)$ is m^b_v−complete if each m^b_v−Cauchy sequence in $(\mathcal{Y}, m^b_v)$ is m^b_v−convergent in $(\mathcal{Y}, m^b_v)$. The terms convergent (converges) and m^b_v-convergent (m^b_v-converges) in the context of m^b_v−metric spaces is the same throughout the paper.

In 2015, in a complete metric space, Altun et al. [24] utilized the collection $\mathcal{F}$ of strictly increasing mappings $\mathcal{F} : \mathbb{R}^+ \longrightarrow \mathbb{R}$ so that

$$\lim_{n\to\infty} r_n = 0 \Leftrightarrow \lim_{n\to\infty} \mathcal{F}(r_n) = -\infty, \forall\{r_n\} \subset \mathbb{R}^+,$$

and

$$\exists\, k \in (0, 1)\ :\ \lim_{r\to 0^+} r^k \mathcal{F}(r) = 0$$

introduced by Wardowski [23] to prove some set-valued fixed point results in a different way.

Theorem 2. *[24] Let $(\mathcal{Y}, \rho)$ be a complete metric space and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ be a set-valued function, so that*

$$\tau + \mathcal{F}(\mathcal{H}(\mathcal{T}u, \mathcal{T}w)) \leq \mathcal{F}(\rho(u, w)), \forall u, w \in \mathcal{Y},$$

where $\tau > 0$, $\mathcal{F} \in \mathcal{F}$, and $\mathcal{F}(\inf \mathcal{A}) = \inf \mathcal{F}(\mathcal{A}), \forall \mathcal{A} \subseteq \mathbb{R}^+$ with $\inf \mathcal{A} > 0$. Then $\mathcal{T}$ has a fixed point in $\mathcal{Y}$.

Let $(\mathcal{Y}, m_v^b)$ be an m_v^b−metric space for $s \geq 1$. Let us denote the collection of all subsets of $\mathcal{Y}$ by $\mathcal{P}(\mathcal{Y})$, the collection of all non-empty, closed, and bounded subsets of $\mathcal{Y}$ by $\mathcal{CB}(\mathcal{Y})$, and the family of all non-empty compact subsets of $\mathcal{Y}$ by $\mathcal{K}(\mathcal{Y})$ (for the definitions, see [17]). It is clear that $\mathcal{K}(\mathcal{Y}) \subseteq \mathcal{CB}(\mathcal{Y})$, and if we define the function $\mathcal{H} : \mathcal{CB}(\mathcal{Y}) \times \mathcal{CB}(\mathcal{Y}) \longrightarrow \mathbb{R}$ by

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) = \max \left\{ \sup_{u \in \mathcal{A}} \delta(u, \mathcal{B}), \sup_{w \in \mathcal{B}} \delta(w, \mathcal{A}) \right\}, \forall \mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathcal{Y}),$$

where $\delta(u, \mathcal{B}) = \inf\{m_v^b(u, w), w \in \mathcal{B}\}$. Then $\mathcal{H}$ forms a metric on $\mathcal{CB}(\mathcal{Y})$. We will call $\mathcal{H}$ as Hausdorff m_v^b−metric induced by the metric m_v^b.

Also, from the definitions of $\mathcal{H}$ and δ, it is clear that

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) = \mathcal{H}(\mathcal{B}, \mathcal{A}), \forall \mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathcal{Y}),$$

$$\mathcal{H}(\mathcal{A}, \mathcal{A}) = 0, \forall \mathcal{A} \in \mathcal{CB}(\mathcal{Y}),$$

and

$$\delta(u, \mathcal{A}) = 0 \Rightarrow u \in \mathcal{A}, \forall u \in \mathcal{Y},\ \mathcal{A} \in \mathcal{CB}(\mathcal{Y}).$$

3. Fixed point results for set-valued contractions

Take into consideration a few key ideas from set-valued fixed point theory that will aid in our analysis of the current study. We need the following generalized notations

$$m^b_{v_{\mathcal{T}u,\mathcal{T}w}} = \min\{\mathcal{H}(\mathcal{T}u, \mathcal{T}u), \mathcal{H}(\mathcal{T}w, \mathcal{T}w)\}, m^b_{v_{u,\mathcal{T}w}} = \min\{\delta(u, \mathcal{T}u), \delta(w, \mathcal{T}w)\},$$

$$\mathcal{M}^b_{v_{\mathcal{T}u,\mathcal{T}w}} = \max\{\mathcal{H}(\mathcal{T}u, \mathcal{T}u), \mathcal{H}(\mathcal{T}w, \mathcal{T}w)\} \text{ and } \mathcal{M}^b_{v_{u,\mathcal{T}w}} = \max\{\delta(u, \mathcal{T}u), \delta(w, \mathcal{T}w)\}$$

in the set-valued sense, where $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ is a set-valued function.

Enumerating from 1 to 5, we present the subsequent lemmas within the framework of the $\mathcal{M}_v^b$−metric space.

Lemma 1. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b−metric space for $s \geq 1$ and $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathcal{Y})$. Then for any $u \in \mathcal{A}$ we have,*

$$\delta(u, \mathcal{B}) \leq m_v^b(u, w), \forall w \in \mathcal{B}.$$

Lemma 2. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b−metric space and $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathcal{Y})$. Then we have,*

$$\delta(u, \mathcal{B}) \leq \mathcal{H}(\mathcal{A}, \mathcal{B}), \forall u \in \mathcal{A}.$$

Lemma 3. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b−metric space and $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathcal{Y})$. Then we have,*

$$m_v^b(u, w) \leq \mathcal{H}(\mathcal{A}, \mathcal{B}), \forall u \in \mathcal{A}, w \in \mathcal{B}.$$

Lemma 4. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b−metric space for $s \geq 1$ and $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathcal{Y})$. Then, for any $\zeta > 0$ and $u \in \mathcal{A}$, there exists $w \in \mathcal{B}$ with $m_v^b(w, w) \leq \zeta$, so that*

$$m_v^b(u, w) \leq \mathcal{H}(\mathcal{A}, \mathcal{B}) + \zeta.$$

Lemma 5. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b–metric space for $s \geq 1$ and $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathcal{Y})$. Then, for any $\zeta > 0$ and $u \in \mathcal{A}$, there exists $w \in \mathcal{B}$ with $m_v^b(w, w) \leq \zeta$, so that*

$$m_v^b(u, w) \leq \zeta \mathcal{H}(\mathcal{A}, \mathcal{B}).$$

Definition 3. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b–metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ so that*

$$\mathcal{H}(\mathcal{T}u, \mathcal{T}w) \leq \xi m_v^b(u, w), \forall u, w \in \mathcal{Y} \text{ and for some constant } \xi.$$

Then $\mathcal{T}$ is known to be a set-valued Lipschitz function with Lipschitz constant ξ. Again, $\mathcal{T}$ is known to be a set-valued contraction if $\xi < 1$.

Following Nadler [5], we have the subsequent definition of a fixed point.

Definition 4. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b–metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$. A point $w \in \mathcal{Y}$ is a fixed point of the set-valued function $\mathcal{T}$, if $w \in \mathcal{T}w$. The set of all fixed points of $\mathcal{T}$ is defined by $\mathcal{T}_{\mathcal{F}ix} = \{w \in \mathcal{Y} : w \in \mathcal{T}w\}$.*

We now prove a lemma.

Lemma 6. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b–metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ so that*

$$\mathcal{H}(\mathcal{T}u, \mathcal{T}w) \leq \xi m_v^b(u, w), \forall u, w \in \mathcal{Y} \text{ and for some constant } 0 < \xi < \frac{1}{s}.$$

If a sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\mathcal{Y}$ with $w_{n+1} \in \mathcal{T}w_n$ converges to $w \in \mathcal{Y}$, then $\{\mathcal{T}w_n\}_{n\in\mathbb{N}}$ converges to $\mathcal{T}w$.

Proof. Since $m^b_{v_{\mathcal{T}w_n,\mathcal{T}w}} \leq \mathcal{H}(\mathcal{T}w_n, \mathcal{T}w)$ and suppose $\mathcal{H}(\mathcal{T}w_n, \mathcal{T}w) = 0$, then

$$\lim_{n\to\infty} \left(\mathcal{H}(\mathcal{T}w_n, \mathcal{T}w) - m^b_{v_{\mathcal{T}w_n,\mathcal{T}w}}\right) = 0 = \lim_{n\to\infty} \left(\mathcal{M}^b_{v_{\mathcal{T}w_n,\mathcal{T}w}} - m^b_{v_{\mathcal{T}w_n,\mathcal{T}w}}\right).$$

Consequently, $\{\mathcal{T}w_n\}_{n\in\mathbb{N}}$ converges to $\mathcal{T}w$.

Suppose $\mathcal{H}(\mathcal{T}w_n, \mathcal{T}w) \neq 0$, then by the inequality condition $\mathcal{H}(\mathcal{T}w_n, \mathcal{T}w) \leq \xi m_v^b(w_n, w)$.

Now, if $m_v^b(w, w) \leq m_v^b(w_n, w_n)$, then $\lim_{n\to\infty} m_v^b(w_n, w_n) = 0$ implies $\lim_{n\to\infty} m_v^b(w, w) = 0$ and the convergence of $\{w_n\}_{n\in\mathbb{N}}$ to w implies $\lim_{n\to\infty} \left(m_v^b(w_n, w) - m^b_{v_{w_n,w}}\right) = 0$. Which further implies $\lim_{n\to\infty} m_v^b(w_n, w) = 0$. So, $\lim_{n\to\infty} \mathcal{H}(\mathcal{T}w_n, \mathcal{T}w) = 0$, and consequently, $\{\mathcal{T}w_n\}_{n\in\mathbb{N}}$ converges to $\mathcal{T}w$.

Again, if $m_v^b(w, w) \geq m_v^b(w_n, w_n)$ and we know $\lim_{n\to\infty} m_v^b(w_n, w_n) = 0$, the convergence of $\{w_n\}_{n\in\mathbb{N}}$ to w i.e., $\lim_{n\to\infty} \left(m_v^b(w_n, w) - m^b_{v_{w_n,w}}\right) = 0$ implies $\lim_{n\to\infty} m_v^b(w_n, w) = 0$. Consequently, as above, $\{\mathcal{T}w_n\}_{n\in\mathbb{N}}$, converges to $\mathcal{T}w$. □

Now we prove the following set-valued theorem in the context of m_v^b–metric space, which is analogous to the set-valued variant of Banach [1] for $\xi < 1$.

Theorem 3. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b–complete metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ be a set-valued Lipschitz function with Lipschitz constant ξ satisfying $0 < \xi < \frac{1}{s}$. Then, $\mathcal{T}$ has a fixed point in $\mathcal{Y}$.*

Proof. Since $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ is a set-valued Lipschitz function with Lipschitz constant $0 < \xi < \frac{1}{s}$, then

$$\mathcal{H}(\mathcal{T}u, \mathcal{T}w) \leq \xi m_v^b(u, w), \forall u, w \in \mathcal{Y}. \tag{3.1}$$

Let $w_0 \in \mathcal{Y}$ be arbitrary, so that $w_0 \notin \mathcal{T}w_0$, otherwise w_0 will become a fixed point. Also, let $w_1 \in \mathcal{T}w_0$ with $m_v^b(w_1, w_1) \leq \xi$. Then for $\mathcal{T}w_0, \mathcal{T}w_1 \in \mathcal{CB}(\mathcal{Y})$, $w_1 \in \mathcal{T}w_0$ and $0 < \xi < \frac{1}{s}$, by Lemma 4, there is $w_2 \in \mathcal{T}w_1$ with $m_v^b(w_2, w_2) \leq \xi^2$ so that

$$m_v^b(w_1, w_2) \leq \mathcal{H}(\mathcal{T}w_0, \mathcal{T}w_1) + \xi.$$

Utilizing the contraction condition (3.1), we have

$$\begin{aligned} m_v^b(w_1, w_2) &\leq \mathcal{H}(\mathcal{T}w_0, \mathcal{T}w_1) + \xi \\ &\leq \xi m_v^b(w_0, w_1) + \xi. \end{aligned} \tag{3.2}$$

Again for $\mathcal{T}w_1, \mathcal{T}w_2 \in \mathcal{CB}(\mathcal{Y})$, $0 < \xi^2 < \frac{1}{s^2}$ and $w_2 \in \mathcal{T}w_1$, by Lemma 4, there is $w_3 \in \mathcal{T}w_2$ with $m_v^b(w_3, w_3) \leq \xi^3$, so that

$$m_v^b(w_2, w_3) \leq \mathcal{H}(\mathcal{T}w_1, \mathcal{T}w_2) + \xi^2.$$

Utilizing the contraction condition (3.1) and inequality (3.2), we have

$$\begin{aligned} m_v^b(w_2, w_3) &\leq \mathcal{H}(\mathcal{T}w_1, \mathcal{T}w_2) + \xi^2 \\ &\leq \xi m_v^b(w_1, w_2) + \xi^2 \\ &\leq \xi^2 m_v^b(w_0, w_1) + \xi^2 + \xi^2 \\ &\leq \xi^2 m_v^b(w_0, w_1) + 2\xi^2. \end{aligned} \tag{3.3}$$

Similarly proceeding, we find a sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\mathcal{Y}$ so that $0 < \xi^n < \frac{1}{s^n}$, $w_{n+1} \in \mathcal{T}w_n$ with $m_v^b(w_n, w_n) \leq \xi^n$ and

$$m_v^b(w_n, w_{n+1}) \leq \xi^n m_v^b(w_0, w_1) + n\xi^n, \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

Now, to prove $\{w_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, let $n, p \in \mathbb{N}$ so that $n + p \geq \cdots \geq n + v \geq n$. We have two cases:

Case 1. p is odd and $p = 2q + 1$, for some $q \in \mathbb{N}$

$m_v^b(w_n, w_{n+p})$

$$\begin{aligned} &= m_v^b(w_n, w_{n+2q+1}) \\ &\leq s\Big[m_v^b(w_n, w_{n+1}) + m_v^b(w_{n+1}, w_{n+2}) + m_v^b(w_{n+2}, w_{n+3}) + \cdots + m_v^b(w_{n+v-1}, w_{n+v}) \\ &\quad + m_v^b(w_{n+v}, w_{n+2q+1})\Big] - \Big[m_v^b(w_{n+1}, w_{n+1}) + m_v^b(w_{n+2}, w_{n+2}) + \cdots + m_v^b(w_{n+v}, w_{n+v})\Big] \\ &\leq s\Big[(\xi^n + \xi^{n+1} + \xi^{n+2} + \cdots + \xi^{n+v-1}) m_v^b(w_0, w_1) + \Big(n\xi^n + (n+1)\xi^{n+1} + (n+2)\xi^{n+2} \\ &\quad + \cdots + (n+v-1)\xi^{n+v-1}\Big) + m_v^b(w_{n+v}, w_{n+2q+1})\Big] - \Big[(\xi^{n+1} + \xi^{n+2} + \cdots + \xi^{n+v})\Big] \\ &= s\xi^n \frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + s\sum_{j=n}^{n+v-1} j\xi^j + s m_v^b(w_{n+v}, w_{n+2q+1}) - \xi^{n+1}\frac{1-\xi^v}{1-\xi} \\ &\leq s\xi^n \frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + s\sum_{j=n}^{n+v-1} j\xi^j - \xi^{n+1}\frac{1-\xi^v}{1-\xi} + s^2\Big[m_v^b(w_{n+v}, w_{n+v+1}) \end{aligned}$$

$$
\begin{aligned}
&+m_v^b(w_{n+v+1}, w_{n+v+2}) + \cdots + m_v^b(w_{n+2v-1}, w_{n+2v}) + m_v^b(w_{n+2v}, w_{n+2q+1})\Big] \\
&-s\Big[m_v^b(w_{n+v+1}, w_{n+v+1}) + m_v^b(w_{n+v+2}, w_{n+v+2}) + \cdots + m_v^b(w_{n+2v}, w_{n+2v})\Big] \\
\leq\ & s\xi^n \frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + \sum_{j=n}^{n+v-1} js^j\xi^j - \xi^{n+1}\frac{1-\xi^v}{1-\xi} + s^2\Big[(\xi^{n+v} + \xi^{n+v+1} + \xi^{n+v+2} + \cdots \\
&+\xi^{n+2v-1})m_v^b(w_0, w_1) + \sum_{j=n+v}^{n+2v-1} j\xi^j + m_v^b(w_{n+2v}, w_{n+2q+1})\Big] \\
&-s\Big[(\xi^{n+v+1} + \xi^{n+v+2} + \cdots + \xi^{n+2v})\Big] \\
=\ & s\xi^n \frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + \sum_{j=n}^{n+v-1} js^j\xi^j - \xi^{n+1}\frac{1-\xi^v}{1-\xi} + s^2\Big[\xi^{n+v}\frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) \\
&+ \sum_{j=n+v}^{n+2v-1} j\xi^j + m_v^b(w_{n+2v}, w_{n+2q+1})\Big] - s\Big[\xi^{n+v+1}\frac{1-\xi^v}{1-\xi}\Big] \\
\leq\ & (s\xi^n + s^2\xi^{n+v})\frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + \sum_{j=n+1}^{n+2v-1} js^j\xi^j - (\xi^{n+1} + s\xi^{n+v+1})\frac{1-\xi^v}{1-\xi} \\
&+s^2 m_v^b(w_{n+2v}, w_{n+2q+1}) \\
\leq\ & (s\xi^n + s^2\xi^{n+v} + s^3\xi^{n+2v})\frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + \sum_{j=n+1}^{n+3v-1} js^j\xi^j - (\xi^{n+1} + s\xi^{n+v+1} \\
&+s^2\xi^{n+2v+1})\frac{1-\xi^v}{1-\xi} + s^3 m_v^b(w_{n+3v}, w_{n+2q+1}) \\
\leq\ & (s\xi^n + s^2\xi^{n+v} + s^3\xi^{n+2v} + \cdots + s^{\frac{2q}{v}}\xi^{n+2q-v})\frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + \sum_{j=n+1}^{n+2q-1} js^j\xi^j \\
&-(\xi^{n+1} + s\xi^{n+v+1} + s^2\xi^{n+2v+1} + \cdots + s^{\frac{2q}{v}-1}\xi^{n+2q-v+1})\frac{1-\xi^v}{1-\xi} \\
&+s^{\frac{2q}{v}} m_v^b(w_{n+2q}, w_{n+2q+1}) \\
\leq\ & (s\xi^n + s^2\xi^{n+v} + s^3\xi^{n+2v} + \cdots + s^{\frac{2q}{v}}\xi^{n+2q-v})\frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + \sum_{j=n+1}^{n+2q} js^j\xi^j \\
&-(\xi^{n+1} + s\xi^{n+v+1} + s^2\xi^{n+2v+1} + \cdots + s^{\frac{2q}{v}-1}\xi^{n+2q-v+1})\frac{1-\xi^v}{1-\xi} \\
&+s^{\frac{2q}{v}}\xi^{n+2q} m_v^b(w_0, w_1).
\end{aligned}
$$

Case 2. p is even and $p = 2q$, for some $q \in \mathbb{N}$
$m_v^b(w_n, w_{n+p})$

$$
\begin{aligned}
=\ & m_v^b(w_n, w_{n+2q}) \\
\leq\ & s\Big[m_v^b(w_n, w_{n+1}) + m_v^b(w_{n+1}, w_{n+2}) + m_v^b(w_{n+2}, w_{n+3}) + \cdots + m_v^b(w_{n+v-1}, w_{n+v})
\end{aligned}
$$

$$
\begin{aligned}
&+m_v^b(w_{n+v}, w_{n+2q})\Big] - \Big[m_v^b(w_{n+1}, w_{n+1}) + m_v^b(w_{n+2}, w_{n+2}) + \cdots + m_v^b(w_{n+v}, w_{n+v})\Big] \\
\leq\ & s\Big[(\xi^n + \xi^{n+1} + \xi^{n+2} + \cdots + \xi^{n+v-1})m_v^b(w_0, w_1) + \Big(n\xi^n + (n+1)\xi^{n+1} + (n+2)\xi^{n+2} \\
&+ \cdots + (n+v-1)\xi^{n+v-1}\Big) + m_v^b(w_{n+v}, w_{n+2q})\Big] - \Big[(\xi^{n+1} + \xi^{n+2} + \cdots + \xi^{n+v})\Big] \\
=\ & s\xi^n \frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + s\sum_{j=n}^{n+v-1} j\xi^j + s m_v^b(w_{n+v}, w_{n+2q}) - \xi^{n+1}\frac{1-\xi^v}{1-\xi} \\
\leq\ & s\xi^n \frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + s\sum_{j=n}^{n+v-1} j\xi^j - \xi^{n+1}\frac{1-\xi^v}{1-\xi} + s^2\Big[m_v^b(w_{n+v}, w_{n+v+1}) \\
&+m_v^b(w_{n+v+1}, w_{n+v+2}) + \cdots + m_v^b(w_{n+2v-1}, w_{n+2v}) + m_v^b(w_{n+2v}, w_{n+2q})\Big] \\
&-s\Big[m_v^b(w_{n+v+1}, w_{n+v+1}) + m_v^b(w_{n+v+2}, w_{n+v+2}) + \cdots + m_v^b(w_{n+2v}, w_{n+2v})\Big] \\
\leq\ & s\xi^n \frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + \sum_{j=n}^{n+v-1} js^j\xi^j - \xi^{n+1}\frac{1-\xi^v}{1-\xi} + s^2\Big[(\xi^{n+v} + \xi^{n+v+1} + \xi^{n+v+2} + \cdots \\
&+\xi^{n+2v-1})m_v^b(w_0, w_1) + \sum_{j=n+v}^{n+2v-1} j\xi^j + m_v^b(w_{n+2v}, w_{n+2q})\Big] \\
&-s\Big[(\xi^{n+v+1} + \xi^{n+v+2} + \cdots + \xi^{n+2v})\Big] \\
=\ & s\xi^n \frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + \sum_{j=n}^{n+v-1} js^j\xi^j - \xi^{n+1}\frac{1-\xi^v}{1-\xi} + s^2\Big[\xi^{n+v}\frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) \\
&+\sum_{j=n+v}^{n+2v-1} j\xi^j + m_v^b(w_{n+2v}, w_{n+2q})\Big] - s\Big[\xi^{n+v+1}\frac{1-\xi^v}{1-\xi}\Big] \\
\leq\ & (s\xi^n + s^2\xi^{n+v})\frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + \sum_{j=n+1}^{n+2v-1} js^j\xi^j - (\xi^{n+1} + s\xi^{n+v+1})\frac{1-\xi^v}{1-\xi} \\
&+s^2 m_v^b(w_{n+2v}, w_{n+2q}) \\
\leq\ & (s\xi^n + s^2\xi^{n+v} + s^3\xi^{n+2v})\frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + \sum_{j=n+1}^{n+3v-1} js^j\xi^j - (\xi^{n+1} + s\xi^{n+v+1} \\
&+s^2\xi^{n+2v+1})\frac{1-\xi^v}{1-\xi} + s^3 m_v^b(w_{n+3v}, w_{n+2q}) \\
\leq\ & (s\xi^n + s^2\xi^{n+v} + s^3\xi^{n+2v} + \cdots + s^{\frac{2q}{v}-1}\xi^{n+2q-v-1})\frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + \sum_{j=n+1}^{n+2q-v-1} js^j\xi^j \\
&-(\xi^{n+1} + s\xi^{n+v+1} + s^2\xi^{n+2v+1} + \cdots + s^{\frac{2q}{v}-1}\xi^{n+2q-v})\frac{1-\xi^v}{1-\xi} \\
&+s^{\frac{2q}{v}} m_v^b(w_{n+2q-1}, w_{n+2q}) \\
\leq\ & (s\xi^n + s^2\xi^{n+v} + s^3\xi^{n+2v} + \cdots + s^{\frac{2q}{v}-1}\xi^{n+2q-v-1})\frac{1-\xi^v}{1-\xi} m_v^b(w_0, w_1) + \sum_{j=n+1}^{n+2q-v} js^j\xi^j
\end{aligned}
$$

$$-(\xi^{n+1} + s\xi^{n+v+1} + s^2\xi^{n+2v+1} + \cdots + s^{\frac{2q}{v}-1}\xi^{n+2q-v})\frac{1-\xi^v}{1-\xi}$$
$$+s^{\frac{2q}{v}}\xi^{n+2q-1}m_v^b(w_0, w_1).$$

Utilizing inequality (3.4) and because the series $\sum_{j=1}^{\infty} js^j\xi^j$ is convergent, both Cases 1 and 2 imply

$$\lim_{n,p\to\infty} m_v^b(w_n, w_{n+p}) < \infty,$$

and

$$\begin{aligned}\mathcal{M}^b_{v_{w_n,w_p}} &= \max\{m_v^b(w_n, w_n), m_v^b(w_p, w_p)\}\\ &\leq \max\{\xi^n, \xi^p\},\end{aligned}$$

$$\begin{aligned}m^b_{v_{w_n,w_p}} &= \min\{m_v^b(w_n, w_n), m_v^b(w_p, w_p)\}\\ &\leq \min\{\xi^n, \xi^p\}\end{aligned}$$

implies

$$\lim_{n,p\to\infty}\left(m_v^b(w_n, w_p) - m^b_{v_{w_n,w_p}}\right) < \infty \text{ and } \lim_{n,p\to\infty}\left(\mathcal{M}^b_{v_{w_n,w_p}} - m^b_{v_{w_n,w_p}}\right) < \infty.$$

Thus, the sequence $\{w_n\}$ is m_v^b–Cauchy in an m_v^b–complete space $\mathcal{Y}$. Then there exists $w \in \mathcal{Y}$ so that

$$\lim_{n\to\infty}\left(m_v^b(w_n, w) - m^b_{v_{w_n,w}}\right) = 0 \text{ and } \lim_{n\to\infty}\left(\mathcal{M}^b_{v_{w_n,w}} - m^b_{v_{w_n,w}}\right) = 0.$$

That is, the sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\mathcal{Y}$ with $w_{n+1} \in \mathcal{T}w_n$ converges to $w \in \mathcal{Y}$, then by Lemma 6 $\{\mathcal{T}w_n\}_{n\in\mathbb{N}}$ converges to $\mathcal{T}w$. That is,

$$\lim_{n\to\infty}\left(\mathcal{H}(\mathcal{T}w_n, \mathcal{T}w) - m^b_{v_{\mathcal{T}w_n,\mathcal{T}w}}\right) = 0$$
$$\Rightarrow \lim_{n\to\infty} \mathcal{H}(\mathcal{T}w_n, \mathcal{T}w) = 0.$$

Again, since $w_{n+1} \in \mathcal{T}w_n$, we have

$$\delta(w_{n+1}, \mathcal{T}w_n) = 0$$
$$\Rightarrow \lim_{n\to\infty} \delta(w_{n+1}, \mathcal{T}w_n) = 0$$
$$\Rightarrow \delta(w, \mathcal{T}w) = 0.$$

That is, $w \in \mathcal{T}w$ and hence w is a fixed point of $\mathcal{T}$. □

Remark 1. *Since set-valued results are the generalization of single-valued results, the Theorem 3 is a generalization of Joshi et al. [26] in a set-valued sense. Also, a generalization of Banach [1] and Nadler [5] in the sense of generalized metric structure m_v^b–metric.*

An alternate statement of Theorem 3 is

Theorem 4. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b-complete metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ be a set-valued function, so that*

$$\mathcal{H}(\mathcal{T}u, \mathcal{T}w) \leq \xi m_v^b(u, w), \forall u, w \in \mathcal{Y}, \ 0 < \xi < \frac{1}{s}.$$

Then $\mathcal{T}$ has a fixed point in $\mathcal{Y}$.

We give an illustrative example that follows Theorem 3.

Example 1. *Let $\mathcal{Y} = [0, 1]$, $s = 2$ and $m_v^b : \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as*

$$m_v^b(u, w) = \left(\frac{|u - w|}{2}\right)^2, \forall u, w \in \mathcal{Y}.$$

Since $m_v^b(u, w) = 0 \Rightarrow u = w$ and $u = w \Rightarrow m_v^b(u, w) = 0$, the mapping m_v^b satisfies $(m_v^b 1)$. Now for distinct u, w, we get $m_v^b(u, w) > 0$. Consequently, m_v^b satisfies $(m_v^b 2)$, as $m_v^b(u, u) = 0$ and $m_v^b(w, w) = 0$.

From the definition of the mapping m_v^b it is clear that $m_v^b(u, w) = m_v^b(w, u)$, that is m_v^b satisfies $(m_v^b 3)$. Again, for distinct u, χ, w, we get

$$\begin{aligned} m_v^b(u, w) - m^b_{v_{u,w}} &= \frac{|u - w|^2}{4} - 0 \\ &= \frac{|u - \chi + \chi - w|^2}{4} \\ &\leq \frac{|u - \chi|^2}{4} + \frac{|\chi - w|^2}{4} + 2\frac{|u - \chi|}{2}\frac{|\chi - w|}{2} \\ &\leq 2\left[\frac{|u - \chi|^2}{4} + \frac{|\chi - w|^2}{4}\right] \\ &\leq 2\left[\left(m_v^b(u, \chi) - m^b_{v_{u,\chi}}\right) + \left(m_v^b(\chi, w) - m^b_{v_{\chi,w}}\right)\right] - m_v^b(\chi, \chi). \end{aligned}$$

In a similar procedure, we can prove that m_v^b satisfies $(m_v^b 4)$ for distinct u, χ_j, w, $0 \leq j \leq v$. Hence $(\mathcal{Y}, m_v^b)$, for $s = 2$, is a m_v^b-metric space.

Let $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ be a set-valued map defined as

$$\mathcal{T}w = \begin{cases} \{u \in \mathcal{Y} : u \in [\frac{w}{7}, \frac{w}{5}]\}, & \text{when } w \in \mathcal{Y} \cap \mathbb{Q}, \\ \{u \in \mathcal{Y} : u \in [\frac{w}{4}, \frac{w}{3}]\}, & \text{elsewhere.} \end{cases}$$

Then $\mathcal{T}$ satisfies Theorem 3 for $0 < \xi = \frac{1}{3} < \frac{1}{s} = \frac{1}{2}$ having a fixed point; in fact, $\mathcal{T}\{0\} = \{0\}$.

The following corollary is a generalization of the Sehgal [22] type contraction for set-valued mappings in m_v^b-metric space.

Corollary 1. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b-complete metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ be a set-valued function, so that*

$$\mathcal{H}(\mathcal{T}u, \mathcal{T}w) \leq \xi \max\left\{m_v^b(u, w), \delta(u, \mathcal{T}u), \delta(w, \mathcal{T}w)\right\}, \forall u, w \in \mathcal{Y}, \ 0 < \xi < \frac{1}{s}. \tag{3.5}$$

Then, $\mathcal{T}$ has a fixed point in $\mathcal{Y}$.

Proof. For any sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\mathcal{Y}$ with $w_{n+1} \in \mathcal{T}w_n$, putting $u = w_n$ and $w = w_{n+1}$, from the inequality (3.5), we have

$$\mathcal{H}(\mathcal{T}w_n, \mathcal{T}w_{n+1}) \leq \xi \max\left\{m_v^b(w_n, w_{n+1}), \delta(w_n, \mathcal{T}w_n), \delta(w_{n+1}, \mathcal{T}w_{n+1})\right\}. \tag{3.6}$$

By Lemmas 1 and 2, we have

$$\delta(w_n, \mathcal{T}w_n) \leq m_v^b(w_n, w_{n+1}),$$

and

$$\delta(w_{n+1}, \mathcal{T}w_{n+1}) \leq \mathcal{H}(\mathcal{T}w_n, \mathcal{T}w_{n+1}).$$

So that, the inequality (3.6) becomes

$$\mathcal{H}(\mathcal{T}w_n, \mathcal{T}w_{n+1}) \leq \xi \max\left\{m_v^b(w_n, w_{n+1}), \mathcal{H}(\mathcal{T}w_n, \mathcal{T}w_{n+1})\right\}. \tag{3.7}$$

Suppose $\max\left\{m_v^b(w_n, w_{n+1}), \mathcal{H}(\mathcal{T}w_n, \mathcal{T}w_{n+1})\right\} = \mathcal{H}(\mathcal{T}w_n, \mathcal{T}w_{n+1})$; then from inequality (3.7), we have $\mathcal{H}(\mathcal{T}w_n, \mathcal{T}w_{n+1}) \leq \xi\mathcal{H}(\mathcal{T}w_n, \mathcal{T}w_{n+1}) < \mathcal{H}(\mathcal{T}w_n, \mathcal{T}w_{n+1})$, a contradiction. Consequently, $\max\left\{m_v^b(w_n, w_{n+1}), \mathcal{H}(\mathcal{T}w_n, \mathcal{T}w_{n+1})\right\} = m_v^b(w_n, w_{n+1})$ and the inequality (3.7) becomes

$$\mathcal{H}(\mathcal{T}w_n, \mathcal{T}w_{n+1}) \leq \xi m_v^b(w_n, w_{n+1}), \tag{3.8}$$

which is the same as the inequality of Theorem 3 and so, the proof now becomes similar to the Theorem 3. □

Remark 2. *Since* $m_v^b(u, w) \leq \max\left\{m_v^b(u, w), \delta(u, \mathcal{T}u), \delta(w, \mathcal{T}w)\right\}$*, Corollary 1 is a clear generalization of Theorem 3.*

Now, we state a corollary as a consequence of Corollary 1, which generalizes the Bianchini contraction [25] for set-valued mappings in the context of m_v^b-metric space.

Corollary 2. *Let* $(\mathcal{Y}, m_v^b)$ *be an* m_v^b-*complete metric space for* $s \geq 1$ *and* $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ *be a set-valued function, so that*

$$\mathcal{H}(\mathcal{T}u, \mathcal{T}w) \leq \xi \max\left\{\delta(u, \mathcal{T}u), \delta(w, \mathcal{T}w)\right\}, \forall u, w \in \mathcal{Y},\ 0 < \xi < \frac{1}{s}. \tag{3.9}$$

Then, $\mathcal{T}$ *has a fixed point in* $\mathcal{Y}$.

Proof. The proof is obvious from Corollary 1. □

We next present an example that will distinguish the Corollaries 1 and 2.

Example 2. *Consider* m_v^b-*metric space* $(\mathcal{Y}, m_v^b)$ *for* $s = 1$*, where* $\mathcal{Y} = \{1, 2, 3, \ldots\}$ *and* $m_v^b : \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ *be defined as*

$$m_v^b(u, w) = |u - w| + \max\{u, w\}, \forall u, w \in \mathcal{Y}.$$

Let $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ *be a set-valued map defined as*

$$\mathcal{T}w = \begin{cases} \{1, 2\}, & \text{when } w \in \{1, 2\}, \\ \{w - 2, w - 1\}, & \text{elsewhere.} \end{cases}$$

Then, $\mathcal{T}$ *satisfies Corollary 1 for* $0 < \xi = \frac{1}{2} < \frac{1}{s} = 1$*, whence* $\mathcal{T}$ *satisfies Corollary 2 for* $0 < \xi = \frac{2}{3} < \frac{1}{s} = 1$*, having a fixed point, in fact* $\mathcal{T}\{1, 2\} = \{1, 2\}$.

Remark 3. *From Example 2, it is clear that the convergence of any sequence in the space using Corollary 1 is more rapid than using Corollary 2.*

In an m_v^b−metric space, using Wardowski contraction [23], the corollary stated below will generalize the F−set-valued contraction introduced by Altun et al. [24].

Corollary 3. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b−complete metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ be a set-valued function, so that*

$$\tau + \mathcal{F}(\mathcal{H}(\mathcal{T}u, \mathcal{T}w)) \leq \mathcal{F}(m_v^b(u, w)), \forall u, w \in \mathcal{Y}, \tag{3.10}$$

where $\mathcal{F} \in \mathcal{F}, \tau > 0$. Then $\mathcal{T}$ has a fixed point in $\mathcal{Y}$.

Proof. Starting with some $w_0 \in \mathcal{Y}$, we will obtain a sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\mathcal{Y}$ with $w_{n+1} \in \mathcal{T}w_n$. Since $\mathcal{F}$ is strictly increasing, from the Lemma 5 and utilizing $u = w_{n-1}$ and $w = w_n$ in inequality (3.10), we have

$$\begin{aligned} \mathcal{F}(m_v^b(w_n, w_{n+1})) &\leq \mathcal{F}(\mathcal{H}(\mathcal{T}w_{n-1}, \mathcal{T}w_n)) \\ &\leq \mathcal{F}(m_v^b(w_{n-1}, w_n)) - \tau, \ \forall n \in \mathbb{N}. \end{aligned} \tag{3.11}$$

Recursively, the right side of the inequality (3.11) will give us the following inequality:

$$\mathcal{F}(m_v^b(w_n, w_{n+1})) \leq \mathcal{F}(m_v^b(w_0, w_1)) - n\tau. \tag{3.12}$$

Again $\mathcal{F}$ satisfies the limiting condition, so that

$$\lim_{n\to\infty} \mathcal{F}(m_v^b(w_n, w_{n+1})) = -\infty \Rightarrow \lim_{n\to\infty} m_v^b(w_n, w_{n+1}) = 0,$$

and $\exists\, k \in (0, 1)$ with $\lim_{n\to\infty} (m_v^b(w_n, w_{n+1}))^k \mathcal{F}(m_v^b(w_n, w_{n+1})) = 0$.

Consequently, if we take the limit after multiplying the term $(m_v^b(w_n, w_{n+1}))^k$, both side of the inequality (3.12), we have

$$\begin{gathered} -\lim_{n\to\infty} n\tau(m_v^b(w_n, w_{n+1}))^k \leq 0 \\ \Rightarrow \lim_{n\to\infty} n(m_v^b(w_n, w_{n+1}))^k = 0 \\ \Rightarrow m_v^b(w_n, w_{n+1}) \leq \frac{1}{n^k}, \ \forall n \in \mathbb{N}. \end{gathered}$$

Using the above inequality, we can prove easily the Cauchyness of the sequence $\{w_n\}_{n\in\mathbb{N}}$ and from here the proof follows Theorem 3. □

Remark 4. *Note that Corollary 3 is a generalization of Theorem 3 and Nadler's theorem in [5].*

We now give an example that will satisfy Corollary 3 and conclude the above remark but does not satisfy any of Theorem 3 and Nadler's theorem in [5].

Example 3. *Consider m_v^b−metric space $(\mathcal{Y}, m_v^b)$ for $s = 2$, where $\mathcal{Y} = \{1 - \frac{1}{3^n} : n \in \mathbb{N}\} \cup \{1\}$ and $m_v^b : \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as*

$$m_v^b(u, w) = \left(\frac{|u - w|}{2}\right)^2 + (\max\{u, w\})^2, \forall u, w \in \mathcal{Y}.$$

Let $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ be a set-valued map defined as

$$\mathcal{T}w = \begin{cases} \{1 - \frac{1}{3^n}, 1 - \frac{1}{3^{n+1}}\}, & \text{when } w = 1 - \frac{1}{3^n}, n \in \mathbb{N} \setminus \{1\}, \\ \{1 - \frac{1}{3}, 1\}, & \text{when } w \in \left\{1 - \frac{1}{3}, 1\right\}. \end{cases}$$

Since for $u = 1, w = 1 - \frac{1}{3^2}$ we have $\mathcal{H}(\mathcal{T}u, \mathcal{T}w) = \frac{71}{81} > \frac{289}{324} = m_v^b(u, w)$, $\mathcal{T}$ does not satisfy Theorem 3 and Nadler's theorem in [5] for any $\xi < 1$.

But $\mathcal{T}$ satisfies Corollary 3 for $\mathcal{F}(r) = \ln(r) + r$ and $\tau = 0.015 > 0$, having a fixed point, in fact $\mathcal{T}\{1, \frac{2}{3}\} = \{1, \frac{2}{3}\}$.

Now, we state the corollaries which are a consequence of Corollary 3, and generalize the Corollaries 1 and 2 in the $F-$set-valued sense.

Corollary 4. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b-complete metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ be a set-valued function, so that*

$$\tau + \mathcal{F}(\mathcal{H}(\mathcal{T}u, \mathcal{T}w)) \leq \mathcal{F}(\max\left\{m_v^b(u, w), \delta(u, \mathcal{T}u), \delta(w, \mathcal{T}w)\right\}), \forall u, w \in \mathcal{Y}, \tag{3.13}$$

where $\mathcal{F} \in \mathcal{F}, \tau > 0$. Then $\mathcal{T}$ has a fixed point in $\mathcal{Y}$.

Corollary 5. *Let $(\mathcal{Y}, m_v^b)$ be an m_v^b-complete metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{CB}(\mathcal{Y})$ be a set-valued function, so that*

$$\tau + \mathcal{F}(\mathcal{H}(\mathcal{T}u, \mathcal{T}w)) \leq \mathcal{F}(\max\{\delta(u, \mathcal{T}u), \delta(w, \mathcal{T}w)\}), \forall u, w \in \mathcal{Y}, \tag{3.14}$$

where $\mathcal{F} \in \mathcal{F}, \tau > 0$. Then $\mathcal{T}$ has a fixed point in $\mathcal{Y}$.

Remark 5. *Since $\mathcal{K}(\mathcal{Y}) \subseteq \mathcal{CB}(\mathcal{Y})$, if we change the co-domain $\mathcal{CB}(\mathcal{Y})$ of the set-valued mapping $\mathcal{T}$ by $\mathcal{K}(\mathcal{Y})$, then the results will still hold.*

4. Application

The mathematical model of Fredholm-type integral inclusions is a generalization of the Fredholm integral equation and plays a significant role in various scientific disciplines where uncertainties, constraints, or differential dependencies exist. These mathematical models are particularly useful in fields where the governing relationships involve set-valued mappings rather than single-valued functions. Overall, Fredholm-type integral inclusions serve as powerful tools in multiple scientific disciplines by effectively handling uncertainty and providing more accurate representations of complex systems. Their applications continue to expand, enhancing predictive modeling and decision-making in various fields.

In order to solve an integral inclusion of the Fredholm type, we use our primary result in this case. Consider m_v^b-metric space $(\mathcal{Y}, m_v^b)$ for $s = 2$, where $\mathcal{Y} = \mathcal{C}([0, 1], \mathbb{R}^2)$ and $m_v^b : \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as

$$m_v^b(u(t), w(t)) = \left(\frac{\sup_{t \in [0,1]} |u(t) - w(t)|}{2} \right)^2, \forall u, w \in \mathcal{Y}, t \in [0, 1],$$

where $\mathcal{C}([0, 1], \mathbb{R}^2)$ is the space of all continuous functions defined from $[0, 1]$ to $\mathbb{R}^2$.

Take into account the Fredholm integral inclusion

$$w(t) \in \psi(t) + \int_0^1 k_w(t, s, w(s))ds, \tag{4.1}$$

so that for all $\mathcal{K}_w : [0, 1]^2 \times \mathbb{R}^2 \longrightarrow \mathcal{K}(\mathcal{Y})$ there is $k_w(t, s, w) \in \mathcal{K}_w(t, s, w)$.

Let us define a set-valued map $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{K}(\mathcal{Y})$ by

$$\mathcal{T}(w(t)) = \left\{ u(t) : u(t) \in \psi(t) + \int_0^1 k_w(t, s, w(s))ds \right\}. \tag{4.2}$$

Theorem 5. *The integral inclusion (4.1) has a solution, if*

1. *there is $w_0 \in \mathcal{Y}$ so that $w_n \in \mathcal{T}w_{n-1}$, $n \in \mathbb{N}$,*
2. *there is a function $\phi : [0, 1] \times [0, 1] \longrightarrow [0, 1]$, which is continuous so that*

$$|k_u(t, s, u(s)) - k_w(t, s, w(s))| \leq \phi(u(s), w(s))|u(s) - w(s)|,$$

and $\phi(u(s), w(s)) \leq \xi < 1, \forall u, w \in \mathcal{Y}, \forall s, t \in [0, 1]$,

3. *$\psi : [0, 1] \longrightarrow \mathbb{R}^2$ and $\mathcal{K}_w : [0, 1]^2 \times \mathbb{R}^2 \longrightarrow \mathcal{K}(\mathcal{Y})$ are continuous.*

Proof. Let $u(t) \in \mathcal{T}(w(t))$ be arbitrary. Then

$$\begin{aligned}
\delta(u(t), \mathcal{T}(w(t))) &\leq m_v^b(u(t), w(t)) \\
&= \left(\frac{\sup_{t \in [0,1]} |u(t) - w(t)|}{2} \right)^2 \\
&= \frac{1}{4} \left(\sup_{t \in [0,1]} \left| \int_0^1 k_u(t, s, u(s))ds - \int_0^1 k_w(t, s, w(s))ds \right| \right)^2 \\
&\leq \frac{1}{4} \left(\sup_{t \in [0,1]} \int_0^1 |k_u(t, s, u(s)) - k_w(t, s, w(s))|ds \right)^2 \\
&\leq \frac{1}{4} \left(\sup_{t \in [0,1]} \int_0^1 \phi(u(s), w(s))|u(s) - w(s)|ds \right)^2, \text{ by condition 2} \\
&\leq \left(\frac{\sup_{t \in [0,1]} |u(t) - w(t)|}{2} \right)^2 \left(\sup_{s \in [0,1]} \int_0^1 \phi(u(s), w(s))ds \right)^2 \\
&\leq m_v^b(u(t), w(t)) \left(\sup_{s \in [0,1]} \int_0^1 \phi(u(s), w(s))ds \right)^2 \\
&< \xi m_v^b(u(t), w(t)),
\end{aligned}$$

using condition 2 and integrating.
In a similar way, one figures out

$$\begin{aligned} \delta(w(t), \mathcal{T}(u(t))) &< \xi m_v^b(w(t), u(t)) \\ &= \xi m_v^b(u(t), w(t)), \end{aligned}$$

consequently, $\mathcal{H}(u(t), \mathcal{T}(w(t))) < \xi m_v^b(u(t), w(t))$.

Thus, we achieve the necessary contraction of Theorem 3 and as a result, $\mathcal{T}$ has a fixed point, that is, a solution of the integral inclusion 4.1. □

Remark 6. *Fredholm-type integral inclusions are widely used in physics (e.g., quantum and statistical mechanics), engineering (e.g., control systems, signal processing), biology (e.g., population dynamics, epidemiology), and economics (e.g., optimization under uncertainty). These models help describe complex phenomena like nonlinear interactions, uncertain parameters, and probabilistic behavior in real-world systems. Applications also span medical imaging, pharmacokinetics, and environmental modeling, making them valuable tools in both theoretical and applied sciences.*

5. Conclusions

This research has explored the realm of set-valued contractions within the context of m_v^b-metric spaces, expanding upon the traditional framework of contraction theory. By introducing and analyzing the Hausdorff m_v^b-metric, we have laid the foundation for the development of set-valued fixed point theorems, significantly advancing this field of study. Our successful generalization of well-known contractions, including Banach [1], Sehgal [22], Wardowski [23], Altun et al. [24], Bianchini [25], and Nadler [5], to the m_v^b-metric space has practical implications and underscores the increasing relevance of set-valued fixed points, shedding light on their critical role in addressing challenges like Fredholm integral inclusions. This research opens doors to further exploration and application of set-valued contractions in diverse scientific and engineering disciplines.

Author contributions

Khairul Habib Alam: Conceptualization, formal analysis, writing original draft preparation, writing review and editing; Yumnam Rohen: Formal analysis, investigation, supervision, writing review and editing; Anita Tomar: Formal analysis, investigation, writing review and editing; Mohammad Sajid: Formal analysis, writing review and editing. All authors have read and agreed to the submitted version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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