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Research article

Uniform regularity and vanishing dissipation limit for the incompressible magneto-micropolar fluid equations with transverse magnetic field

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Abstract: This paper explores the influence of a transverse magnetic field at the boundary and studies the vanishing dissipation limit of the incompressible magneto-micropolar fluid equations in a half-space. We prove that the solutions remain uniformly bounded, both in the conormal Sobolev norms and the L^∞ norm, over a fixed time interval, independent of the dissipative coefficients. As a result, we establish the convergence of the dissipative magneto-micropolar fluid equations to the corresponding non-dissipative equations in the L^∞ norm. Additionally, our analysis provides uniform regularity energy estimates as the dissipative coefficients tend to zero. This shows that the strong boundary layer can still be prevented by the transverse magnetic field, even with the magnetic diffusion.

Keywords: magneto-micropolar fluid equations; vanishing dissipation limit; conormal Sobolev estimate
Mathematics Subject Classification: 35Q35, 76W05, 35B40

1. Introduction

In the realm of magnetohydrodynamics, the magneto-micropolar fluid equations play a vital role in exploring complex flow phenomena. Below, we present a three-dimensional equations of the incompressible magneto-micropolar fluid equation

$$\begin{cases}\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{H}\cdot\nabla\mathbf{H} + \nabla p = (\mu+\kappa)\Delta\mathbf{u} + 2\kappa\nabla\times\omega,\\ \partial_t\omega + \mathbf{u}\cdot\nabla\omega = \chi\nabla(\nabla\cdot\omega) + \gamma\Delta\omega - 4\kappa\omega + 2\kappa\nabla\times\mathbf{u},\\ \partial_t\mathbf{H} + \mathbf{u}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla\mathbf{u} = \epsilon\Delta\mathbf{H},\\ \nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{H} = 0,\end{cases} \tag{1.1}$$

here, $\mathbf{u} = (u_1, u_2, u_3)$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, and $\mathbf{H} = (h_1, h_2, h_3)$ represent the velocity field, microrotation velocity field, and magnetic field, respectively. p is the total pressure. The parameters μ and ϵ denote the viscosity coefficient and the magnetic diffusion coefficient, respectively. κ denotes the microrotation viscosity, and χ, γ denote the angular viscosities.

The incompressible magneto-micropolar fluid equations present significant analytical challenges while also providing new opportunities for exploration, owing to their unique and distinctive mathematical characteristics. A great deal of work has also been done on the magneto-micropolar fluid equations; see references [1–3].

The vanishing dissipation limit problem represents a crucial and complex area of study in both hydrodynamics and applied mathematics. This problem involves understanding the behavior of fluid systems as dissipative effects, such as viscosity, tend to zero. It plays a key role in bridging the gap between idealized, inviscid systems and more realistic, viscous systems. Despite its importance, it poses significant analytical challenges due to the loss of regularity and the potential development of singularities as dissipation vanishes. Numerous works have explored different aspects of this problem, addressing its implications for stability, convergence, and boundary layer behavior. For further exploration of these complexities, see references [4, 5], which provide detailed insights into the mathematical and physical intricacies of the vanishing dissipation limit in various fluid systems. And there is an important issue for magneto-micropolar systems, which is justifying the boundary layer assumptions of the magneto-micropolar fluid systems.

Motivated by the work presented in [6–8], this paper aims to explore the precise role that viscosity and diffusivity play as the dissipation effects approach zero, especially in regions close to the boundary where boundary layer phenomena may arise. To address this, we specifically study the magneto-micropolar fluid equations in a half-plane domain, incorporating fully viscosity and diffusivity terms. By analyzing this setup, we aim to uncover the subtle mechanisms by which these dissipation terms govern the transition from a dissipative to a non-dissipative regime near the boundary, providing insight into the mathematical and physical complexities of such systems in fluid dynamics.

Specifically, we investigate the following magneto-micropolar fluid equations in the domain $\{(t, \mathbf{x}) \mid t \in [0, T],\ \mathbf{x} \in \Omega\}$, where the spatial region is given by $\Omega = \left\{\mathbf{x} = (x, y, z) \mid (x, y) \in \mathbb{R}^2,\ z > 0\right\}$

$$\begin{cases} \partial_t \mathbf{u}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon - \mathbf{H}^\varepsilon \cdot \nabla \mathbf{H}^\varepsilon + \nabla p^\varepsilon = 2\varepsilon \Delta \mathbf{u}^\varepsilon + 2\varepsilon \nabla \times \boldsymbol{\omega}^\varepsilon, \\ \partial_t \boldsymbol{\omega}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \boldsymbol{\omega}^\varepsilon = \varepsilon \nabla (\nabla \cdot \boldsymbol{\omega}^\varepsilon) + \Delta \boldsymbol{\omega}^\varepsilon - 4\varepsilon \boldsymbol{\omega}^\varepsilon + 2\varepsilon \nabla \times \mathbf{u}^\varepsilon, \\ \partial_t \mathbf{H}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{H}^\varepsilon - \mathbf{H}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon = 2\varepsilon \Delta \mathbf{H}^\varepsilon, \\ \nabla \cdot \mathbf{u}^\varepsilon = \nabla \cdot \mathbf{H}^\varepsilon = 0. \end{cases} \tag{1.2}$$

The initial data is given by

$$(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{H}^\varepsilon)|_{t=0} = (\mathbf{u}_0, \boldsymbol{\omega}_0, \mathbf{H}_0). \tag{1.3}$$

Initial boundary value problems in fluid mechanics frequently arise in various fields, with boundary conditions determined according to the specific physical settings. In the presence of boundaries, the no-slip boundary condition is imposed on both the velocity field and the microrotation velocity field

$$\mathbf{u}^\varepsilon|_{z=0} = \boldsymbol{\omega}^\varepsilon|_{z=0} = 0, \tag{1.4}$$

and assume that the magnetic field satisfies the perfect conducting boundary condition

$$\partial_z h_1^\varepsilon|_{z=0} = \partial_z h_2^\varepsilon|_{z=0} = 0, \quad h_3^\varepsilon|_{z=0} = 1. \tag{1.5}$$

For fluids with viscosity and diffusivity, the no-slip boundary condition primarily affects the flow within a thin layer adjacent to the surface. This gives rise to the classical boundary layer theory, see references [9–11], which postulates that outside this thin region, the fluid behaves ideally, while dissipative effects dominate within the layer. We will establish that the classical solution to this problem remains uniformly bounded in a conormal Sobolev space over a local time interval. This allows us to prove the vanishing dissipation limit, namely, that the solution converges to the corresponding system with $\varepsilon = 0$ by means of a compactness argument.

Next, we introduce the following new variable for the magnetic field

$$\mathbf{B}^\varepsilon = \mathbf{H}^\varepsilon - \overrightarrow{e_z},$$

with $\overrightarrow{e_z} = (0, 0, 1)$, and thus $\mathbf{B}^\varepsilon = (b_1^\varepsilon, b_2^\varepsilon, b_3^\varepsilon) = (h_1^\varepsilon, h_2^\varepsilon, h_3^\varepsilon - 1)$.

Therefore Eq (1.2) can be rewritten as

$$\begin{cases} \partial_t \mathbf{u}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \nabla p^\varepsilon - \mathbf{B}^\varepsilon \cdot \nabla \mathbf{B}^\varepsilon - \partial_z \mathbf{B}^\varepsilon = 2\varepsilon \Delta \mathbf{u}^\varepsilon + 2\varepsilon \nabla \times \omega^\varepsilon, \\ \partial_t \omega^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \omega^\varepsilon = \varepsilon \nabla (\nabla \cdot \omega^\varepsilon) + \Delta \omega^\varepsilon - 4\varepsilon \omega^\varepsilon + 2\varepsilon \nabla \times \mathbf{u}^\varepsilon, \\ \partial_t \mathbf{B}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{B}^\varepsilon - \mathbf{B}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon - \partial_z \mathbf{u}^\varepsilon = 2\varepsilon \Delta \mathbf{B}^\varepsilon, \\ \nabla \cdot \mathbf{u}^\varepsilon = \nabla \cdot \mathbf{B}^\varepsilon = 0. \end{cases} \tag{1.6}$$

The initial data are reformulated as follows:

$$(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon)|_{t=0} = (\mathbf{u}_0, \omega_0, \mathbf{H}_0 - \overrightarrow{e_z}) = (\mathbf{u}_0, \omega_0, \mathbf{B}_0). \tag{1.7}$$

Both the velocity field and the microrotation velocity field continue to satisfy the no-slip boundary condition

$$\mathbf{u}^\varepsilon|_{z=0} = \omega^\varepsilon|_{z=0} = 0. \tag{1.8}$$

Combining with (1.5), the boundary conditions of the magnetic field are

$$\partial_z b_1^\varepsilon|_{z=0} = \partial_z b_2^\varepsilon|_{z=0} = 0, \quad b_3^\varepsilon|_{z=0} = 0, \tag{1.9}$$

the zero velocity at the boundary indicates that particles near the solid wall remain stationary relative to the flow.

By letting $\varepsilon \to 0$ in (1.6), the corresponding limiting magneto-micropolar fluid equations are obtained

$$\begin{cases} \partial_t \mathbf{u}^0 + \mathbf{u}^0 \cdot \nabla \mathbf{u}^0 + \nabla p - \mathbf{B}^0 \cdot \nabla \mathbf{B}^0 - \partial_z \mathbf{B}^0 = 0, \\ \partial_t \omega^0 + \mathbf{u}^0 \cdot \nabla \omega^0 = \Delta \omega^0, \\ \partial_t \mathbf{B}^0 + \mathbf{u}^0 \cdot \nabla \mathbf{B}^0 - \mathbf{B}^0 \cdot \nabla \mathbf{u}^0 - \partial_z \mathbf{u}^0 = 0, \\ \nabla \cdot \mathbf{u}^0 = \nabla \cdot \mathbf{B}^0 = 0, \end{cases} \tag{1.10}$$

with the same initial date

$$(\mathbf{u}^0, \omega^0, \mathbf{B}^0)|_{t=0} = (\mathbf{u}_0, \omega_0, \mathbf{B}_0). \tag{1.11}$$

For well-posedness and consistency, we impose the no-slip boundary condition on both the velocity field and the microrotation velocity field

$$\mathbf{u}^0|_{z=0} = \omega^0|_{z=0} = 0. \tag{1.12}$$

The justification of the boundary layer assumptions in the magneto-micropolar fluid equations is a crucial issue in the study of magneto-micropolar fluids. The aim of this paper is to address this problem by proving that the vanishing dissipation limit of the magneto-micropolar Eq (1.6) corresponds to the limiting magneto-micropolar Eq (1.10).

Under the application of slip boundary conditions, the boundary layer effect becomes relatively weak, and the vanishing dissipation limit has been thoroughly investigated in existing studies, including [12, 13] for the Navier–Stokes equations and [14–16] for the MHD equations.

The adoption of Navier-slip boundary conditions effectively inhibits the development of strong boundary layers, with the corresponding vanishing dissipation limit rigorously demonstrated in [17–19] for the Navier–Stokes equations and in [20, 21] for the MHD equations.

In contrast, the no-slip boundary condition typically results in the formation of strong boundary layers. The vanishing dissipation limit problem becomes especially challenging in this context, primarily due to the difficulties in controlling the vorticity of the boundary layer corrector. As a result, research on vanishing dissipation limits under no-slip conditions remains relatively incomplete. For existing studies, readers may refer to [22] for the Navier–Stokes equations, [7, 23] for the MHD equations, [24, 25] for the viscoelastic equations, and [8] for the magnetic Bénard equations.

Building on these studies, this paper further investigates the vanishing dissipation limit for magneto-micropolar equations. Compared to [26], this paper introduces the magnetic diffusion term $\varepsilon\Delta\mathbf{B}^\varepsilon$ into the second equation of (1.6). To ensure the well-posedness of the problem, appropriate boundary conditions must be prescribed on the magnetic field. However, the inclusion of the magnetic diffusion term, along with the associated boundary conditions, induces boundary layer phenomena in the magnetic field, thereby presenting new challenges in the mathematical analysis.

This work significantly extends the analysis in [26] by addressing a more complex system and overcoming new analytical difficulties introduced by additional physical effects. Specifically, our study differs from [26] in the following three aspects:

First, in addition to the magnetic diffusion term $\varepsilon\Delta\mathbf{B}^\varepsilon$, we incorporate a compressional microrotation term $\chi\nabla(\nabla\cdot\boldsymbol{\omega})$ into the microrotation equation, which is not considered in [26]. This term introduces new technical difficulties in the uniform estimates of normal derivatives for $\boldsymbol{\omega}^\varepsilon$. Nevertheless, by fixing the diffusion coefficient of $\Delta\boldsymbol{\omega}^\varepsilon$ to be constant 1, we reduce the requirement to second-order normal derivative estimates, following the strategy developed in [8].

Second, unlike [26], where uniform estimates for higher-order normal derivatives are derived under strong compatibility conditions, our system includes a magnetic diffusion term $\varepsilon\Delta\mathbf{B}^\varepsilon$ that fundamentally alters the structure of the equations. This term makes it impractical to derive uniform estimates using the same approach. Instead, by leveraging the elliptic nature of the pressure equation, we successfully establish uniform bounds for the second-order normal derivatives of the pressure. As a result, our analysis only relies on uniform a priori bounds of $\|\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{B}^\varepsilon, \partial_z\boldsymbol{\omega}^\varepsilon\|_{L^\infty}$ to close the energy estimates, avoiding any need for high-order compatibility conditions.

Third, we rigorously justify the vanishing dissipation limit via uniform regularity estimates and compactness arguments. Notably, although the strong $O(1)$ boundary layer is eliminated by the effect of the transverse magnetic field, a weaker second-order boundary layer remains for $(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)$, while no boundary layer forms in $\boldsymbol{\omega}^\varepsilon$ due to the fixed rotational diffusion. These results highlight the stabilizing influence of magnetic diffusion and the refined structure of the micro-rotation dynamics.

To formulate the problem, we first recall the notation of the conormal Sobolev space. As introduced in [6, 27], we define the following conormal derivatives of functions depending on $(t, \mathbf{x})$

$$Z_0 = \partial_t,\ Z_1 = \partial_x,\ Z_2 = \partial_y,\ Z_3 = \phi(z)\partial_z, \quad Z^\alpha = Z_0^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3},$$

with $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ denotes the multi-index with $|\alpha| = |\alpha_0| + |\alpha_1| + |\alpha_2| + |\alpha_3|$, and the weight $\phi(z)$ is a smooth bounded function of z such that $\phi(0) = 0$ and $\phi'(0) > 0$. Typically, one can choose $\phi(z) = \frac{z}{1+z}$.

Then, define the conormal Sobolev space for an integer $m \in \mathbb{N}$

$$H^m_{co}([0, T] \times \Omega) = \left\{ f(t, \mathbf{x}) \,\middle|\, Z^\alpha f \in L^2([0, T] \times \Omega), |\alpha| \le m \right\},$$

equipped norms

$$\|f(t)\|_m^2 = \sum_{|\alpha| \le m} \|Z^\alpha f(t, \cdot)\|^2_{L^2_{\mathbf{x}}}.$$

Similarly, we define

$$W^{m,\infty}_{co}([0, T] \times \Omega) = \left\{ f(t, \mathbf{x}) \,\middle|\, Z^\alpha f \in L^\infty([0, T] \times \Omega), |\alpha| \le m \right\},$$

with

$$\|f(t)\|^2_{m,\infty} = \sum_{|\alpha| \le m} \|Z^\alpha f\|^2_{L^\infty_{t,\mathbf{x}}}.$$

In this paper, we denote by $\|\cdot\|$ and $(\cdot, \cdot)$ the L^2 norm and the spatial inner product, respectively. The notations

$$\mathbf{u}^\varepsilon = (\mathbf{u}^\varepsilon_h, u^\varepsilon_3), \boldsymbol{\omega}^\varepsilon = (\boldsymbol{\omega}^\varepsilon_h, \omega^\varepsilon_3), \mathbf{B}^\varepsilon = (\mathbf{B}^\varepsilon_h, b^\varepsilon_3), \nabla_h = (\partial_x, \partial_y),\ \textit{and}\ \Delta_h = \partial_x^2 + \partial_y^2$$

are used throughout. Moreover, we use the notation $A \lesssim B$ to indicate that there exists a positive constant $C > 0$, independent of ε, such that $A \le CB$. The commutator is denoted by $[\cdot, \cdot]$, and $\mathcal{P}(\cdot)$ represents a polynomial function, which may vary from line to line.

Additionally, we define the following energy functional:

$$\begin{aligned} N_m(t) = &\sup_{0 \le s \le t} \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{B}^\varepsilon)(t)\|_m^2 + \varepsilon \int_0^t \|\nabla(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2\, ds + \int_0^t \|\nabla \boldsymbol{\omega}^\varepsilon(s)\|_m^2\, ds \\ &+ \varepsilon \int_0^t \|\nabla \cdot \boldsymbol{\omega}^\varepsilon(s)\|_m^2\, ds + \int_0^t \|\partial_z(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|^2_{m-1}\, ds \\ &+ \varepsilon^2 \int_0^t \left\|\partial_z^2(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\right\|^2_{m-1} ds + \int_0^t \left\|\partial_z^2 \boldsymbol{\omega}^\varepsilon(s)\right\|^2_{m-2} ds. \end{aligned} \tag{1.13}$$

Now, we can state the main theorem of this paper.

Theorem 1.1. *(Uniform regularity estimate and vanishing dissipation limit) Let $m \ge 7$ be an integer. Assume that the initial data satisfy the divergence-free conditions $\nabla \cdot \mathbf{u}_0 = 0$, $\nabla \cdot \mathbf{B}_0 = 0$, and*

$$\|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{B}^\varepsilon(0))\|_m^2 + \|\partial_z(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{B}^\varepsilon(0))\|^2_{m-1} + \left\|\partial_z^2 \boldsymbol{\omega}^\varepsilon(0)\right\|_1^2 \le M_0, \tag{1.14}$$

with $M_0 \ge 0$ being a positive constant. Then, there exists a time $T > 0$ independent of ε such that the classical solution $(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{B}^\varepsilon)$ to the initial-boundary value problems (1.6)–(1.9) *satisfies the following regularity estimate:*

$$N_m(t) + \int_0^t \|\partial_z \nabla p^\varepsilon(s)\|^2_{m-2}\, ds \lesssim M, \tag{1.15}$$

where the M depends only on M_0.

Moreover, there exists a unique solution $(\mathbf{u}^0, \omega^0, \mathbf{B}^0)$ *to the limiting magneto-micropolar fluid Eqs* (1.10)–(1.12)*, such that*

$$\left\|(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon) - (\mathbf{u}^0, \omega^0, \mathbf{B}^0)\right\|_{L^\infty_{t,\mathbf{x}}} \to 0. \tag{1.16}$$

The remainder of the paper is organized as follows. In Section 2, we introduce the elementary inequalities that will be frequently used. Section 3 derives the uniform conormal estimates in conormal space for the classical solutions $(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon)$ to the initial-boundary value problems (1.6)–(1.9). Then, we proceed to estimate the norm derivatives of $(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon)$, which are given in detail in Sections 4 and 5, and the estimate of pressure p^ε is given in Section 6. Finally, by deriving an L^∞ estimate and combining all estimates with some compactness arguments, in Section 7, we prove that the solution to the initial-boundary value problems (1.6)–(1.9) is uniformly bounded in the conormal Sobolev space within a fixed time interval, which is independent of ε, and as a direct consequence, we justify the vanishing dissipation limit as $\varepsilon \to 0$.

2. Preliminaries

In this section, we introduce some essential properties of the conormal Sobolev space, which will play a key role in the subsequent sections. As the first result, we state the Sobolev-Gagliardo-Nirenberg-Moser-type inequality, whose proof can be found in [28].

Lemma 2.1. *For any integer $m \in \mathbb{N}$ and functions $f, g \in L^\infty([0,T]\times\Omega) \cap H^m_{co}([0,T]\times\Omega)$, the following inequality holds for α, $\beta \in \mathbb{N}^4$ with $|\alpha| + |\beta| = m$*

$$\int_0^T \left\|Z^\alpha f Z^\beta g(t)\right\|^2 dt \lesssim \|f\|^2_{L^\infty_{t,\mathbf{x}}} \int_0^T \|g(t)\|^2_m \, dt + \|g\|^2_{L^\infty_{t,\mathbf{x}}} \int_0^T \|f(t)\|^2_m \, dt. \tag{2.1}$$

Next, we present the anisotropic Sobolev embedding inequality in the conormal Sobolev space, with the proof available in [6, 27].

Lemma 2.2. *Let $f(t,\mathbf{x}) \in H^3_{co}([0,T]\times\Omega)$ and $\partial_z f(t,\mathbf{x}) \in H^2_{co}([0,T]\times\Omega)$, then we obtain $f \in L^\infty([0,T]\times\Omega)$, moreover, it obtains*

$$\|f\|^2_{L^\infty_{t,\mathbf{x}}} \lesssim \|f(0)\|^2_2 + \|\partial_z f(0)\|^2_1 + \int_0^T \|f(t)\|^2_3 + \|\partial_z f(t)\|^2_2 \, ds. \tag{2.2}$$

In particular, for any integer $m_0 > 1$, it also holds that

$$\|f\|^2_{L^\infty_{t,\mathbf{x}}} \lesssim \|\partial_z f\|_{m_0} \|f\|_{m_0} + \|f\|^2_{m_0}. \tag{2.3}$$

And from the inequality (2.2)*, it directly follows that for any integer $q \geq 0$, if*

$$f(t,\mathbf{x}) \in H^{q+3}_{co}([0,T]\times\Omega) \text{ and} \partial_z f(t,\mathbf{x}) \in H^{q+2}_{co}([0,T]\times\Omega),$$

it holds

$$\sup_{0\le s\le t} \|f(t)\|^2_{q,\infty} \lesssim \|f(0)\|^2_{q+2} + \|\partial_z f(0)\|^2_{q+1} + \int_0^T \|f(t)\|^2_{q+3} + \|\partial_z f(t)\|^2_{q+2} \, ds. \tag{2.4}$$

To address the commutator involving conormal derivatives, it is observed that the Z_3 does not commute with ∂_z. As in [27], there exist two families of bounded smooth functions

$$\{\phi_{k,m}(z)\}_{0\le k\le m-1} \ \textit{and} \ \left\{\phi^{k,m}(z)\right\}_{0\le k\le m-1}$$

with any integer $m \ge 1$, which depends only on $\phi(z)$, such that

$$[Z_3^m, \partial_z] = \sum_{k=0}^{m-1} \phi_{k,m}(z) Z_3^k \partial_z = \sum_{k=0}^{m-1} \phi^{k,m}(z) \partial_z Z_3^k. \tag{2.5}$$

In a similar manner, there exist two families of bounded smooth functions

$$\{\phi_{1,k,m}(z), \phi_{2,k,m}(z)\}_{0\le k\le m-1} \ \textit{and} \ \left\{\phi^{1,k,m}(z), \phi^{2,k,m}(z)\right\}_{0\le k\le m-1},$$

which depend only on $\phi(z)$, such that

$$\begin{aligned}\left[Z_3^m, \partial_z^2\right] &= \sum_{k=0}^{m-1} (\phi_{1,k,m}(z) Z_3^k \partial_z + \phi_{2,k,m}(z) Z_3^k \partial_z^2) \\ &= \sum_{k=0}^{m-1} (\phi^{1,k,m}(z) \partial_z Z_3^k + \phi^{2,k,m}(z) \partial_z^2 Z_3^k).\end{aligned} \tag{2.6}$$

Therefore, the following estimates can be immediately deduced.

Lemma 2.3. *Let the integer $m \ge 2$ and*

$$f(t, \mathbf{x}) \in H^m_{co}([0, T] \times \Omega), \partial_z f(t, \mathbf{x}) \in H^{m-1}_{co}([0, T] \times \Omega).$$

Then, for any $\alpha \in \mathbb{N}^4$ with $|\alpha| \le m$

$$\|[Z^\alpha, \partial_z] f(t)\| \lesssim \|\partial_z f(t)\|_{m-1}, \tag{2.7}$$

and

$$\sum_{|\alpha|\le m} \|\partial_z Z^\alpha f(t)\| \lesssim \|\partial_z f(t)\|_m \lesssim \sum_{|\alpha|\le m} \|\partial_z Z^\alpha f(t)\| + \|\partial_z f(t)\|_{m-1}. \tag{2.8}$$

The analysis will also make use of the following Moser-type inequalities related to commutator estimate.

Lemma 2.4. *For any integer $m \ge 1$ and $\alpha \in \mathbb{N}^4$ with $|\alpha| \le m$, given appropriate functions f and g defined on Ω where g vanishes on the boundary $\partial\Omega$, the following property holds:*

$$\int_0^T \|Z^\alpha(g\partial_z f)(t)\|^2\, dt \lesssim \ \|\partial_z g\|^2_{L^\infty_{t,\mathbf{x}}} \int_0^T \|f(t)\|^2_{m+1}\, dt + \sup_{0\le s\le t} \|f(t)\|^2_{1,\infty} \int_0^T \|\partial_z g(t)\|^2_m\, dt. \tag{2.9}$$

The proof of this lemma can be found in references [6, 27]. Additionally, the next lemma follows directly from inequality (2.9). The detailed steps and justifications for both lemmas, along with their underlying inequalities, are provided, offering comprehensive mathematical reasoning. The proofs of the next three lemmas can be found in [23].

Lemma 2.5. *Let $m \geq 1$ be an integer and $\alpha \in \mathbb{N}^4$ with $|\alpha| \leq m$. Suppose that $f \in H^{m+1}_{co}([0,T]\times\Omega) \cap W^{1,\infty}_{co}([0,T]\times\Omega)$, and $\mathbf{v} \in H^{m+1}_{co}([0,T]\times\Omega) \cap W^{1,\infty}_{co}([0,T]\times\Omega)$ satisfies that $\mathbf{v}$ is divergence-free and tangential to the boundary, the following holds*

$$\int_0^T \|Z^\alpha(\mathbf{v}\cdot\nabla f)(t)\|^2\,dt \lesssim \sup_{0\leq t\leq T}\|\mathbf{v}(t)\|^2_{1,\infty}\int_0^T \|f(t)\|^2_{m+1}\,dt + \sup_{0\leq t\leq T}\|f(t)\|^2_{1,\infty}\int_0^T \|\mathbf{v}(t)\|^2_{m+1}\,dt. \tag{2.10}$$

Lemma 2.6. *Let $m \geq 1$ be an integer and $\alpha \in \mathbb{N}^4$ with $|\alpha| \leq m$. Suppose that $f \in H^m_{co}([0,T]\times\Omega)$, $\partial_z f \in H^{m-1}_{co}([0,T]\times\Omega)$, $\nabla f \in L^\infty([0,T]\times\Omega)$, and $\mathbf{v} \in H^m_{co}([0,T]\times\Omega) \cap W^{1,\infty}_{co}([0,T]\times\Omega)$ satisfying that $\mathbf{v}$ is divergence-free and tangential to the boundary, the following holds*

$$\int_0^T \|[Z^\alpha, \mathbf{v}\cdot\nabla]f(t)\|^2\,dt \lesssim \|Z\mathbf{v}\|^2_{L^\infty_{t,\mathbf{x}}}\int_0^T \|(f(t)\|^2_m + \|\partial_z f(t)\|^2_{m-1})dt + \|\nabla f\|^2_{L^\infty_{t,\mathbf{x}}}\int_0^T \|\mathbf{v}(t)\|^2_m\,dt. \tag{2.11}$$

Lemma 2.7. *Let $m \geq 1$ be an integer and $\alpha \in \mathbb{N}^4$ with $|\alpha| \leq m$. Suppose that $f \in H^{m+1}_{co}([0,T]\times\Omega) \cap W^{2,\infty}_{co}([0,T]\times\Omega)$, $\mathbf{v}$ satisfies the no-slip boundary condition: $\mathbf{v}|_{\partial\Omega} = 0$, and $\partial_z\mathbf{v} \in H^{m+1}_{co}([0,T]\times\Omega) \cap W^{2,\infty}_{co}([0,T]\times\Omega)$, the following holds*

$$\begin{aligned}&\int_0^T \|[Z^\alpha, \mathbf{v}\cdot\nabla]\partial_z f(t)\|^2\,dt\\ &\lesssim \sup_{0\leq t\leq T}\|\partial_z\mathbf{v}(s)\|^2_{2,\infty}\int_0^T \|f(t)\|^2_{m+1}\,dt + \sup_{0\leq t\leq T}\|f(s)\|^2_{2,\infty}\int_0^T \|\partial_z\mathbf{v}(t)\|^2_{m+1}\,dt.\end{aligned} \tag{2.12}$$

3. Conormal energy estimate

Proposition 3.1. *Let m be an integer satisfying $m \geq 7$, the classical solution $(u^\varepsilon, \omega^\varepsilon, B^\varepsilon)$ of (1.6)–(1.9) on $[0,T]$ satisfies that, for any $t \in [0,T]$*

$$\begin{aligned}&\|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{B}^\varepsilon)(t)\|^2_m + \varepsilon\int_0^t \|\nabla(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|^2_m\,ds + \int_0^t \|\nabla\boldsymbol{\omega}^\varepsilon(s)\|^2_m\,ds + \varepsilon\int_0^t \|\nabla\cdot\boldsymbol{\omega}^\varepsilon(s)\|^2_m\,ds\\ &\lesssim \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{B}^\varepsilon)(0)\|^2_m + \int_0^t \|\partial_z p^\varepsilon(s)\|^2_{m-1}\,ds + \left(1 + \sup_{0\leq s\leq t}\|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|^2_{[\frac{m}{2}]+1,\infty} + \|\partial_z\boldsymbol{\omega}^\varepsilon\|^2_{L^\infty_{t,\mathbf{x}}}\right)\\ &\cdot\int_0^t \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|^2_m + \|\partial_z(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|^2_{m-1}\,ds + \varepsilon\int_0^t \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|^2_m\,ds.\end{aligned} \tag{3.1}$$

Proof. Acting Z^α with $|\alpha| \leq m$ on the equations $(1.6)_1 - (1.6)_3$, we obtain that

$$\begin{cases}\partial_t Z^\alpha\mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon\cdot\nabla)Z^\alpha\mathbf{u}^\varepsilon + \nabla Z^\alpha p^\varepsilon - (\mathbf{B}^\varepsilon\cdot\nabla)Z^\alpha\mathbf{B}^\varepsilon - \partial_z Z^\alpha\mathbf{B}^\varepsilon - 2\varepsilon\Delta Z^\alpha\mathbf{u}^\varepsilon\\ = -[Z^\alpha, \nabla]p^\varepsilon + 2\varepsilon[Z^\alpha, \Delta]\mathbf{u}^\varepsilon - [Z^\alpha, \mathbf{u}^\varepsilon\cdot\nabla]\mathbf{u}^\varepsilon + [Z^\alpha, \mathbf{B}^\varepsilon\cdot\nabla]\mathbf{B}^\varepsilon + [Z^\alpha, \partial_z]\mathbf{B}^\varepsilon\\ +2\varepsilon Z^\alpha(\nabla\times\boldsymbol{\omega}^\varepsilon),\\ \partial_t Z^\alpha\boldsymbol{\omega}^\varepsilon + (\mathbf{u}^\varepsilon\cdot\nabla)Z^\alpha\boldsymbol{\omega}^\varepsilon - \Delta Z^\alpha\boldsymbol{\omega}^\varepsilon + 4\varepsilon Z^\alpha\boldsymbol{\omega}^\varepsilon\\ = -[Z^\alpha, \mathbf{u}^\varepsilon\cdot\nabla]\boldsymbol{\omega}^\varepsilon + [Z^\alpha, \Delta]\boldsymbol{\omega}^\varepsilon + \varepsilon Z^\alpha\nabla(\nabla\cdot\boldsymbol{\omega}^\varepsilon) + 2\varepsilon Z^\alpha(\nabla\times\mathbf{u}^\varepsilon),\\ \partial_t Z^\alpha\mathbf{B}^\varepsilon + (\mathbf{u}^\varepsilon\cdot\nabla)Z^\alpha\mathbf{B}^\varepsilon - (\mathbf{B}^\varepsilon\cdot\nabla)Z^\alpha\mathbf{u}^\varepsilon - \partial_z Z^\alpha\mathbf{u}^\varepsilon - 2\varepsilon\Delta Z^\alpha\mathbf{B}^\varepsilon\\ = -[Z^\alpha, \mathbf{u}^\varepsilon\cdot\nabla]\mathbf{B}^\varepsilon + [Z^\alpha, \mathbf{B}^\varepsilon\cdot\nabla]\mathbf{u}^\varepsilon + [Z^\alpha, \partial_z]\mathbf{u}^\varepsilon + 2\varepsilon[Z^\alpha, \Delta]\mathbf{B}^\varepsilon.\end{cases} \tag{3.2}$$

Multiplying (3.2) by $(Z^\alpha \mathbf{u}^\varepsilon, Z^\alpha \boldsymbol{\omega}^\varepsilon, Z^\alpha \mathbf{B}^\varepsilon)$ and integrating the resulting equation over $[0,t]\times\Omega$, it follows, via integration by parts, that

$$\begin{aligned}
&\frac{1}{2}\|(Z^\alpha \mathbf{u}^\varepsilon, Z^\alpha \boldsymbol{\omega}^\varepsilon, Z^\alpha \mathbf{B}^\varepsilon)(s)\|^2 - \frac{1}{2}\|(Z^\alpha \mathbf{u}^\varepsilon, Z^\alpha \boldsymbol{\omega}^\varepsilon, Z^\alpha \mathbf{B}^\varepsilon)(0)\|^2 \\
&+ 2\varepsilon \int_0^t \|\nabla Z^\alpha \mathbf{u}^\varepsilon(s)\|^2 + \|\nabla Z^\alpha \mathbf{B}^\varepsilon(s)\|^2 ds + \int_0^t \|\nabla Z^\alpha \boldsymbol{\omega}^\varepsilon(s)\|^2 ds + 4\varepsilon \int_0^t \|Z^\alpha \boldsymbol{\omega}^\varepsilon(s)\|^2 ds \\
&= \int_0^t \int_\Omega Z^\alpha p^\varepsilon (\nabla \cdot Z^\alpha \mathbf{u}^\varepsilon) d\mathbf{x} ds \\
&+ \int_0^t \int_\Omega (-[Z^\alpha, \nabla] p^\varepsilon) \cdot Z^\alpha \mathbf{u}^\varepsilon d\mathbf{x} ds + 2\varepsilon \int_0^t \int_\Omega [Z^\alpha, \Delta] \mathbf{u}^\varepsilon \cdot Z^\alpha \mathbf{u}^\varepsilon d\mathbf{x} ds \\
&+ \int_0^t \int_\Omega (-[Z^\alpha, \mathbf{u}^\varepsilon \cdot \nabla] \mathbf{u}^\varepsilon + [Z^\alpha, \mathbf{B}^\varepsilon \cdot \nabla] \mathbf{B}^\varepsilon + [Z^\alpha, \partial_z] \mathbf{B}^\varepsilon) \cdot Z^\alpha \mathbf{u}^\varepsilon d\mathbf{x} ds \\
&+ 2\varepsilon \int_0^t \int_\Omega Z^\alpha (\nabla \times \boldsymbol{\omega}^\varepsilon) \cdot Z^\alpha \mathbf{u}^\varepsilon d\mathbf{x} ds + \varepsilon \int_0^t \int_\Omega Z^\alpha \nabla (\nabla \cdot \boldsymbol{\omega}^\varepsilon) \cdot Z^\alpha \boldsymbol{\omega}^\varepsilon d\mathbf{x} ds \\
&+ \int_0^t \int_\Omega (-[Z^\alpha, \mathbf{u}^\varepsilon \cdot \nabla] \boldsymbol{\omega}^\varepsilon) \cdot Z^\alpha \boldsymbol{\omega}^\varepsilon d\mathbf{x} ds + \int_0^t \int_\Omega ([Z^\alpha, \Delta] \boldsymbol{\omega}^\varepsilon) \cdot Z^\alpha \boldsymbol{\omega}^\varepsilon d\mathbf{x} ds \\
&+ 2\varepsilon \int_0^t \int_\Omega Z^\alpha (\nabla \times \mathbf{u}^\varepsilon) \cdot Z^\alpha \boldsymbol{\omega}^\varepsilon d\mathbf{x} ds + 2\varepsilon \int_0^t \int_\Omega ([Z^\alpha, \Delta] \mathbf{B}^\varepsilon) \cdot Z^\alpha \mathbf{B}^\varepsilon d\mathbf{x} ds \\
&+ \int_0^t \int_\Omega (-[Z^\alpha, \mathbf{u}^\varepsilon \cdot \nabla] \mathbf{B}^\varepsilon + [Z^\alpha, \mathbf{B}^\varepsilon \cdot \nabla] \mathbf{u}^\varepsilon + [Z^\alpha, \partial_z] \mathbf{u}^\varepsilon) \cdot Z^\alpha \mathbf{B}^\varepsilon d\mathbf{x} ds \\
&= \sum_{i=1}^{11} I_i,
\end{aligned} \tag{3.3}$$

here we have used the divergence-free conditions $\nabla \cdot \mathbf{u}^\varepsilon = \nabla \cdot \mathbf{B}^\varepsilon = 0$ and the boundary conditions $\mathbf{u}^\varepsilon|_{z=0} = \boldsymbol{\omega}^\varepsilon|_{z=0} = 0$, and next we estimate the terms on the right-hand side of (3.3).

First of all, by using the divergence-free condition $\nabla \cdot \mathbf{u}^\varepsilon = 0$, we can obtain that

$$\nabla \cdot Z^\alpha \mathbf{u}^\varepsilon = -[Z^\alpha, \partial_z] u_3^\varepsilon. \tag{3.4}$$

Together with (2.7), we obtain the following:

$$\|\nabla \cdot Z^\alpha \mathbf{u}^\varepsilon\| = \|[Z^\alpha, \partial_z] u_3^\varepsilon\| \lesssim \|\partial_z u_3^\varepsilon\|_{m-1} \lesssim \|\nabla_h \cdot \mathbf{u}^\varepsilon\|_{m-1} \lesssim \|\mathbf{u}^\varepsilon\|_m. \tag{3.5}$$

We consider the case where Z^α contains at least one Z_3; otherwise, if Z^α does not contain Z_3, then $\nabla \cdot Z^\alpha \mathbf{u}^\varepsilon = 0$. Then from (3.5), one has

$$\begin{aligned}
|I_1| &= \left| \int_0^t \int_\Omega Z^\alpha p^\varepsilon (\nabla \cdot Z^\alpha \mathbf{u}^\varepsilon) d\mathbf{x} ds \right| \\
&\lesssim \int_0^t \|Z^\alpha p^\varepsilon(s)\| \cdot \|\nabla \cdot Z^\alpha \mathbf{u}^\varepsilon(s)\| ds \\
&\lesssim \int_0^t \|\phi(z) \partial_z p^\varepsilon(s)\|_{m-1} \cdot \|\mathbf{u}^\varepsilon(s)\|_m ds \\
&\lesssim \int_0^t \|\partial_z p^\varepsilon(s)\|_{m-1}^2 ds + \int_0^t \|\mathbf{u}^\varepsilon(s)\|_m^2 ds.
\end{aligned} \tag{3.6}$$

In the same way, obtain that

$$-[Z^\alpha,\nabla]p^\varepsilon = -[Z^\alpha,\partial_z]p^\varepsilon\overrightarrow{e_z}. \tag{3.7}$$

It directly follows that

$$|I_2| = \left|\int_0^t\int_\Omega(-[Z^\alpha,\nabla]p^\varepsilon)\cdot Z^\alpha\mathbf{u}^\varepsilon d\mathbf{x}ds\right| \lesssim \int_0^t \|\partial_z p^\varepsilon(s)\|_{m-1}^2 ds + \int_0^t \|\mathbf{u}^\varepsilon(s)\|_m^2 ds. \tag{3.8}$$

We continue to estimate I_3, by (2.6), we have

$$\begin{aligned}|I_3| =& \left|2\varepsilon\int_0^t\int_\Omega[Z^\alpha,\Delta]\mathbf{u}^\varepsilon\cdot Z^\alpha\mathbf{u}^\varepsilon d\mathbf{x}ds\right|\\ \lesssim&\varepsilon\int_0^t\left|(\sum_{|\beta|\le m-1}\phi^{1,\beta}(z)\partial_z Z^\beta\mathbf{u}^\varepsilon, Z^\alpha\mathbf{u}^\varepsilon)\right|ds + \varepsilon\left|\int_0^t(\sum_{|\beta|\le m-1}\phi^{2,\beta}(z)\partial_z^2 Z^\beta\mathbf{u}^\varepsilon, Z^\alpha\mathbf{u}^\varepsilon)ds\right|\\ \lesssim&\varepsilon\int_0^t\|\partial_z\mathbf{u}^\varepsilon(s)\|_{m-1}\|\mathbf{u}^\varepsilon(s)\|_m ds + \varepsilon\left|\int_0^t(\sum_{|\beta|\le m-1}\phi^{2,\beta}(z)\partial_z^2 Z^\beta\mathbf{u}^\varepsilon, Z^\alpha\mathbf{u}^\varepsilon)ds\right|,\end{aligned} \tag{3.9}$$

where the $\phi^{1,\beta}(z)$ and $\phi^{2,\beta}(z)$ depend only on $\phi(z)$, then by integration by parts and the boundary conditions $\mathbf{u}^\varepsilon|_{z=0} = 0$, we have

$$\begin{aligned}&\varepsilon\left|\int_0^t(\sum_{|\beta|\le m-1}\phi^{2,\beta}(z)\partial_z^2 Z^\beta\mathbf{u}^\varepsilon, Z^\alpha\mathbf{u}^\varepsilon)ds\right|\\ \lesssim&\varepsilon\left|\int_0^t(\sum_{|\beta|\le m-1}\partial_z\phi^{2,\beta}(z)\partial_z Z^\beta\mathbf{u}^\varepsilon, Z^\alpha\mathbf{u}^\varepsilon)ds\right| + \varepsilon\left|\int_0^t(\sum_{|\beta|\le m-1}\phi^{2,\beta}(z)\partial_z Z^\beta\mathbf{u}^\varepsilon, \partial_z Z^\alpha\mathbf{u}^\varepsilon)ds\right|\\ \lesssim&\varepsilon\int_0^t(\|\partial_z Z^\alpha\mathbf{u}^\varepsilon(s)\| + \|Z^\alpha\mathbf{u}^\varepsilon(s)\|)\cdot(\sum_{|\beta|\le m-1}\|\partial_z Z^\beta\mathbf{u}^\varepsilon(s)\|)ds\\ \lesssim&\varepsilon\int_0^t(\|\partial_z\mathbf{u}^\varepsilon(s)\|_m + \|\mathbf{u}^\varepsilon(s)\|_m)\|\partial_z\mathbf{u}^\varepsilon(s)\|_{m-1}ds.\end{aligned} \tag{3.10}$$

Then, combining with the above two formulas, we have

$$\begin{aligned}|I_3| =& \left|\int_0^t\int_\Omega[Z^\alpha,\Delta]\mathbf{u}^\varepsilon\cdot Z^\alpha\mathbf{u}^\varepsilon d\mathbf{x}ds\right|\\ \lesssim&\varepsilon\int_0^t(\|\partial_z\mathbf{u}^\varepsilon(s)\|_m + \|\mathbf{u}^\varepsilon(s)\|_m)(\|\partial_z\mathbf{u}^\varepsilon(s)\|_{m-1})ds\\ \lesssim&\theta\varepsilon\int_0^t\|\partial_z\mathbf{u}^\varepsilon(s)\|_m^2 ds + \varepsilon\int_0^t\|\mathbf{u}^\varepsilon(s)\|_m^2 + \|\partial_z\mathbf{u}^\varepsilon(s)\|_{m-1}^2 ds,\end{aligned} \tag{3.11}$$

where θ is a sufficiently small positive parameter, the exact value of which will be determined in the subsequent analysis. Moreover, in the same manner, one obtains

$$\begin{aligned}|I_8| =& \left|2\varepsilon\int_0^t\int_\Omega[Z^\alpha,\Delta]\boldsymbol{\omega}^\varepsilon\cdot Z^\alpha\boldsymbol{\omega}^\varepsilon d\mathbf{x}ds\right|\\ \lesssim&\theta\int_0^t\|\partial_z\boldsymbol{\omega}^\varepsilon(s)\|_m^2 ds + \int_0^t\|\boldsymbol{\omega}^\varepsilon(s)\|_m^2 + \|\partial_z\boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2 ds,\end{aligned} \tag{3.12}$$

and

$$
\begin{aligned}
|I_{10}| =& \left|2\varepsilon \int_0^t \int_\Omega [Z^\alpha, \Delta]\mathbf{B}^\varepsilon \cdot Z^\alpha \mathbf{B}^\varepsilon d\mathbf{x}ds\right| \\
\lesssim& \theta\varepsilon \int_0^t \|\partial_z \mathbf{B}^\varepsilon(s)\|_m^2 \, ds + \varepsilon \int_0^t \|\mathbf{B}^\varepsilon(s)\|_m^2 + \|\partial_z \mathbf{B}^\varepsilon(s)\|_{m-1}^2 \, ds.
\end{aligned}
\tag{3.13}
$$

Then, we consider I_4. By applying (2.11), we obtain an estimate similar to that used for C_3^α in Section 3 of [23]

$$
\int_0^t \|[Z^\alpha, \mathbf{u}^\varepsilon \cdot \nabla]\mathbf{u}^\varepsilon(s)\|^2 ds \lesssim \sup_{0\le s\le t} \|\mathbf{u}^\varepsilon(s)\|_{1,\infty}^2 \int_0^t \|\mathbf{u}^\varepsilon(s)\|_m^2 + \|\partial_z \mathbf{u}^\varepsilon(s)\|_{m-1}^2 ds. \tag{3.14}
$$

Similarly, by virtue of (1.9), that is $b_3^\varepsilon|_{z=0} = 0$, one can obtain

$$
\int_0^t \|[Z^\alpha, \mathbf{B}^\varepsilon \cdot \nabla]\mathbf{B}^\varepsilon(s)\|^2 ds \lesssim \sup_{0\le s\le t} \|\mathbf{B}^\varepsilon(s)\|_{1,\infty}^2 \int_0^t \|\mathbf{B}^\varepsilon(s)\|_m^2 + \|\partial_z \mathbf{B}^\varepsilon(s)\|_{m-1}^2 ds. \tag{3.15}
$$

And by virtue of (2.7), we obtain

$$
\int_0^t \|[Z^\alpha, \partial_z]\mathbf{B}^\varepsilon(s)\|^2 ds \lesssim \int_0^t \|\partial_z \mathbf{B}^\varepsilon(s)\|_{m-1}^2 ds. \tag{3.16}
$$

So, we obtain that from the above three estimates

$$
\begin{aligned}
|I_4| =& \left|\int_0^t \int_\Omega (-[Z^{\alpha}, \mathbf{u}^\varepsilon \cdot \nabla]\mathbf{u}^\varepsilon + [Z^{\alpha}, \mathbf{B}^\varepsilon \cdot \nabla]\mathbf{B}^\varepsilon + [Z^{\alpha}, \partial_z]\mathbf{B}^\varepsilon) \cdot Z^\alpha \mathbf{u}^\varepsilon d\mathbf{x}ds\right| \\
\lesssim& \left((1 + \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{1,\infty}^2) \cdot \int_0^t \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 + \|\partial_z(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{m-1}^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|\mathbf{u}^\varepsilon\|_m^2 ds\right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.17}
$$

Similarly, we can obtain

$$
\begin{aligned}
|I_{11}| =& \left|\int_0^t \int_\Omega (-[Z^{\alpha}, \mathbf{u}^\varepsilon \cdot \nabla]\mathbf{B}^\varepsilon + [Z^{\alpha}, \mathbf{B}^\varepsilon \cdot \nabla]\mathbf{u}^\varepsilon + [Z^{\alpha}, \partial_z]\mathbf{u}^\varepsilon) \cdot Z^\alpha \mathbf{B}^\varepsilon d\mathbf{x}ds\right| \\
\lesssim& \left((1 + \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{1,\infty}^2) \cdot \int_0^t \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 + \|\partial_z(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{m-1}^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|\mathbf{B}^\varepsilon\|_m^2 ds\right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.18}
$$

Next, we estimate I_6, which can be derived using the boundary condition $\boldsymbol{\omega}^\varepsilon|_{z=0} = 0$.

$$
\begin{aligned}
I_6 =& \varepsilon \int_0^t \int_\Omega Z^\alpha \nabla(\nabla \cdot \boldsymbol{\omega}^\varepsilon) \cdot Z^\alpha \boldsymbol{\omega}^\varepsilon d\mathbf{x}ds \\
=& \varepsilon \int_0^t \int_\Omega [Z^\alpha, \nabla](\nabla \cdot \boldsymbol{\omega}^\varepsilon) \cdot Z^\alpha \boldsymbol{\omega}^\varepsilon d\mathbf{x}ds + \varepsilon \int_0^t \int_\Omega \nabla Z^\alpha(\nabla \cdot \boldsymbol{\omega}^\varepsilon) \cdot Z^\alpha \boldsymbol{\omega}^\varepsilon d\mathbf{x}ds \\
=& -\varepsilon \int_0^t \|Z^\alpha(\nabla \cdot \boldsymbol{\omega}^\varepsilon)(s)\|^2 ds + \varepsilon \int_0^t \int_\Omega Z^\alpha(\nabla \cdot \boldsymbol{\omega}^\varepsilon) \cdot [Z^\alpha, \nabla]\boldsymbol{\omega}^\varepsilon d\mathbf{x}ds \\
&+ \varepsilon \int_0^t \int_\Omega [Z^\alpha, \nabla](\nabla \cdot \boldsymbol{\omega}^\varepsilon) \cdot Z^\alpha \boldsymbol{\omega}^\varepsilon d\mathbf{x}ds \\
=& -\varepsilon \int_0^t \|Z^\alpha(\nabla \cdot \boldsymbol{\omega}^\varepsilon)(s)\|^2 ds + \sum_{i=1}^2 I_6^{(i)}.
\end{aligned}
\tag{3.19}
$$

By (2.5) and Young's inequality, we obtain an estimate for the second term on the right-hand side of (3.19).

$$
\begin{aligned}
I_6^{(1)} =&\varepsilon \int_0^t \int_\Omega Z^\alpha(\nabla\cdot\boldsymbol{\omega}^\varepsilon)\cdot[Z^\alpha,\nabla]\boldsymbol{\omega}^\varepsilon d\mathbf{x}ds\\
=&\varepsilon\sum_{k=0}^{m-1}\int_0^t\int_\Omega \phi_{k,m}(z)Z^\alpha(\nabla\cdot\boldsymbol{\omega}^\varepsilon)\cdot Z^k\partial_z\boldsymbol{\omega}^\varepsilon d\mathbf{x}ds\\
\leq&\frac{\varepsilon}{2}\int_0^t\|\nabla\cdot\boldsymbol{\omega}^\varepsilon(s)\|_m^2ds + C\int_0^t\|\partial_z\boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2ds.
\end{aligned}
\tag{3.20}
$$

By (2.5) and integration by parts, we obtain an estimate for the last term on the right-hand side of (3.19)

$$
\begin{aligned}
I_6^{(2)} =&\varepsilon\int_0^t\int_\Omega[Z^\alpha,\nabla](\nabla\cdot\boldsymbol{\omega}^\varepsilon)\cdot Z^\alpha\boldsymbol{\omega}^\varepsilon d\mathbf{x}ds\\
=&\varepsilon\sum_{k=0}^{m-1}\int_0^t\int_\Omega\phi^{k,m}(z)\partial_z Z^k(\nabla\cdot\boldsymbol{\omega}^\varepsilon)\cdot Z^\alpha\boldsymbol{\omega}^\varepsilon d\mathbf{x}ds\\
=&-\varepsilon\sum_{k=0}^{m-1}\int_0^t\int_\Omega\partial_z\phi^{k,m}(z)Z^k(\nabla\cdot\boldsymbol{\omega}^\varepsilon)\cdot Z^\alpha\boldsymbol{\omega}^\varepsilon d\mathbf{x}ds\\
&-\varepsilon\sum_{k=0}^{m-1}\int_0^t\int_\Omega\phi^{k,m}(z)Z^k(\nabla\cdot\boldsymbol{\omega}^\varepsilon)\cdot\partial_z Z^\alpha\boldsymbol{\omega}^\varepsilon d\mathbf{x}ds\\
\leq&\varepsilon^2\int_0^t\|\partial_z\boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2ds + C(\int_0^t\|\boldsymbol{\omega}^\varepsilon(s)\|_m^2ds+\int_0^t\|\partial_z\boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2ds).
\end{aligned}
\tag{3.21}
$$

Combined with the two estimates above, we obtain that

$$
\begin{aligned}
I_6 =&\varepsilon\int_0^t\int_\Omega Z^\alpha\nabla(\nabla\cdot\boldsymbol{\omega}^\varepsilon)\cdot Z^\alpha\boldsymbol{\omega}^\varepsilon d\mathbf{x}ds\\
\leq&-\frac{\varepsilon}{2}\int_0^t\|\nabla\cdot\boldsymbol{\omega}^\varepsilon(s)\|_m^2ds+\varepsilon^2\int_0^t\|\partial_z\boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2ds + C(\int_0^t\|\boldsymbol{\omega}^\varepsilon(s)\|_m^2+\|\partial_z\boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2ds).
\end{aligned}
\tag{3.22}
$$

We continue to estimate I_7 and separate it into two parts

$$
[Z^\alpha,\mathbf{u}^\varepsilon\cdot\nabla]\,\boldsymbol{\omega}^\varepsilon = [Z^\alpha,\mathbf{u}_h^\varepsilon\cdot\nabla_h]\,\boldsymbol{\omega}^\varepsilon + [Z^\alpha,u_3^\varepsilon\partial_z]\,\boldsymbol{\omega}^\varepsilon. \tag{3.23}
$$

For the first term on the right-hand side of (3.23), one has

$$
\begin{aligned}
[Z^\alpha,\mathbf{u}_h^\varepsilon\cdot\nabla_h]\,\boldsymbol{\omega}^\varepsilon &= Z^\alpha(\mathbf{u}_h^\varepsilon\cdot\nabla_h\boldsymbol{\omega}^\varepsilon)-\mathbf{u}_h^\varepsilon\cdot\nabla_h(Z^\alpha\boldsymbol{\omega}^\varepsilon)\\
&=\sum_{\substack{1\leq|\beta|\leq|\alpha|\\ \beta+\gamma=\alpha}} C_\alpha^\beta Z^\beta\mathbf{u}^\varepsilon\cdot\nabla_h(Z^\gamma\boldsymbol{\omega}^\varepsilon).
\end{aligned}
\tag{3.24}
$$

By combining with (2.11), we obtain that

$$
\begin{aligned}
&\int_0^t\|[Z^\alpha,\mathbf{u}_h^\varepsilon\cdot\nabla_h]\boldsymbol{\omega}^\varepsilon(s)\|ds\\
\lesssim&\|Z\mathbf{u}_h^\varepsilon\|_{L^\infty_{t,\mathbf{x}}}\int_0^t\|\boldsymbol{\omega}^\varepsilon(s)\|_m+\|\partial_z\boldsymbol{\omega}^\varepsilon(s)\|_{m-1}ds+\|\nabla_h\boldsymbol{\omega}^\varepsilon\|_{L^\infty_{t,\mathbf{x}}}\int_0^t\|\mathbf{u}_h^\varepsilon(s)\|_m ds\\
\lesssim&\sup_{0\leq s\leq t}\|\mathbf{u}^\varepsilon(s)\|_{1,\infty}\int_0^t\|\boldsymbol{\omega}^\varepsilon(s)\|_m+\|\partial_z\boldsymbol{\omega}^\varepsilon(s)\|_{m-1}ds+\sup_{0\leq s\leq t}\|\boldsymbol{\omega}^\varepsilon(s)\|_{1,\infty}\int_0^t\|\mathbf{u}^\varepsilon(s)\|_m ds.
\end{aligned}
\tag{3.25}
$$

For the last term on the right-hand side of (3.23), it holds

$$\begin{aligned}[Z^\alpha, u_3^\varepsilon\partial_z]\,\omega^\varepsilon =&Z^\alpha(u_3^\varepsilon\cdot\partial_z\omega^\varepsilon)-u_3^\varepsilon\partial_z(Z^\alpha\omega^\varepsilon)\\ =&\sum_{\substack{|\beta|\le|\alpha|\\ \beta+\gamma=\alpha}} C_\alpha^\beta Z^\beta u_3^\varepsilon\cdot Z^\gamma\partial_z\omega^\varepsilon - u_3^\varepsilon\partial_z(Z^\alpha\omega^\varepsilon)\\ =&Z^\alpha u_3^\varepsilon\cdot\partial_z\omega^\varepsilon+\sum_{\substack{1\le|\beta|\le|\alpha|-1\\ \beta+\gamma=\alpha}} C_\alpha^\beta Z^\beta u_3^\varepsilon\cdot Z^\gamma\partial_z\omega^\varepsilon+u_3^\varepsilon[Z^\alpha,\partial_z]\omega^\varepsilon.\end{aligned}\tag{3.26}$$

The first term on the right-hand side of Eq (3.26) can be estimated as follows:

$$\int_0^t \|Z^\alpha u_3^\varepsilon\cdot\partial_z\omega^\varepsilon(s)\|ds\lesssim\|\partial_z\omega^\varepsilon\|_{L^\infty_{t,\mathbf{x}}}\int_0^t\|u_3^\varepsilon(s)\|_m ds.\tag{3.27}$$

From (2.7), the third term on the right-hand side of (3.26) satisfies

$$\int_0^t \|u_3^\varepsilon([Z^\alpha,\partial_z]\omega^\varepsilon)(s)\|ds\lesssim\|u_3^\varepsilon\|_{L^\infty_{t,\mathbf{x}}}\int_0^t\|\partial_z\omega^\varepsilon(s)\|_{m-1}ds.\tag{3.28}$$

For the second term on the right-hand side of Eq (3.26), provided that $1\le|\beta|\le[\frac{m}{2}]$, it follows that

$$\begin{aligned}\int_0^t\|Z^\beta u_3^\varepsilon\cdot Z^\gamma\partial_z\omega^\varepsilon(s)\|ds&\lesssim\|Z^\beta u_3^\varepsilon\|_{L^\infty_{t,\mathbf{x}}}\int_0^t\|\partial_z\omega^\varepsilon(s)\|_{m-1}ds\\ &\lesssim\sup_{0\le s\le t}\|u_3^\varepsilon\|_{[\frac{m}{2}],\infty}\int_0^t\|\partial_z\omega^\varepsilon(s)\|_{m-1}ds.\end{aligned}\tag{3.29}$$

For $[\frac{m}{2}]+1\le|\beta|\le m-1$, it follows from (2.5) that

$$\begin{aligned}&\int_0^t\|Z^\beta u_3^\varepsilon\cdot Z^\gamma\partial_z\omega^\varepsilon(s)\|ds\lesssim\int_0^t\left\|\phi^{-1}Z^\beta u_3^\varepsilon\cdot\phi Z^\gamma\partial_z\omega^\varepsilon(s)\right\|ds\\ &\lesssim\int_0^t\left\|\phi^{-1}Z^\beta u_3^\varepsilon\cdot\phi(\partial_z Z^\gamma+[Z^\gamma,\partial_z])\omega^\varepsilon(s)\right\|ds\\ &\lesssim\int_0^t\left\|\phi^{-1}Z^\beta u_3^\varepsilon\cdot(Z^{\gamma+1}+\phi[Z^\gamma,\partial_z])\omega^\varepsilon(s)\right\|ds\\ &\lesssim\int_0^t\left\|\phi^{-1}Z^\beta u_3^\varepsilon\cdot Z^{\gamma+1}\omega^\varepsilon(s)\right\|ds+\int_0^t\left\|\phi^{-1}Z^\beta u_3^\varepsilon\cdot(\phi\sum_{k=0}^{[\frac{m}{2}]-1}\phi^{k,[\frac{m}{2}]}(z)\partial_z Z_3^k)\omega^\varepsilon(s)\right\|ds\\ &\triangleq J_1+J_2.\end{aligned}\tag{3.30}$$

We deduce that

$$\begin{aligned}J_1\lesssim&\|Z^{\gamma+1}\omega^\varepsilon\|_{L^\infty_{t,\mathbf{x}}}\int_0^t\left\|\phi^{-1}Z^\beta u_3^\varepsilon(s)\right\|ds\\ \lesssim&\|Z^{\gamma+1}\omega^\varepsilon\|_{L^\infty_{t,\mathbf{x}}}\int_0^t\|\partial_z Z^\beta u_3^\varepsilon(s)\|ds\\ \lesssim&\sup_{0\le s\le t}\|\omega^\varepsilon(s)\|_{[\frac{m}{2}]+1,\infty}\int_0^t\|\partial_z u_3^\varepsilon(s)\|_{m-1}ds.\end{aligned}\tag{3.31}$$

For J_2, we can achieve that

$$\begin{aligned} J_2 &= \int_0^t \left\| \phi^{-1} Z^\beta u_3^\varepsilon \cdot (\phi \sum_{k=0}^{[\frac{m}{2}]-1} \phi^{k,[\frac{m}{2}]}(z) \partial_z Z_3^k) \boldsymbol{\omega}^\varepsilon(s) \right\| ds \\ &= \int_0^t \left\| \phi^{-1} Z^\beta u_3^\varepsilon \cdot (\sum_{k=0}^{[\frac{m}{2}]-1} \phi^{k,[\frac{m}{2}]}(z) Z_3^{k+1}) \boldsymbol{\omega}^\varepsilon(s) \right\| ds \\ &\lesssim \left\| (\sum_{k=0}^{[\frac{m}{2}]-1} \phi^{k,[\frac{m}{2}]}(z) Z_3^{k+1}) \boldsymbol{\omega}^\varepsilon \right\|_{L^\infty_{t,\mathbf{x}}} \int_0^t \left\| \phi^{-1} Z^\beta u_3^\varepsilon(s) \right\| ds \\ &\lesssim \sup_{0\le s\le t} \|\boldsymbol{\omega}^\varepsilon(s)\|_{[\frac{m}{2}],\infty} \int_0^t \|\partial_z \mathbf{u}^\varepsilon(s)\|_{m-1} ds. \end{aligned} \tag{3.32}$$

Thus, we obtain

$$\int_0^t \|Z^\beta u_3^\varepsilon \cdot Z^\gamma \partial_z \boldsymbol{\omega}^\varepsilon(s)\| ds \lesssim \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_{[\frac{m}{2}]+1,\infty} \int_0^t \|\partial_z (\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_{m-1} ds. \tag{3.33}$$

Collecting the above estimates yields

$$\begin{aligned} |I_7| &= \left| \int_0^t \int_\Omega (-[Z^{\alpha,}, \mathbf{u}^\varepsilon \cdot \nabla] \boldsymbol{\omega}^\varepsilon) \cdot Z^\alpha \boldsymbol{\omega}^\varepsilon d\mathbf{x} ds \right| \\ &\lesssim \left((1 + \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|^2_{[\frac{m}{2}]+1,\infty} + \|\partial_z \boldsymbol{\omega}^\varepsilon\|^2_{L^\infty_{t,\mathbf{x}}}) \cdot \int_0^t \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_m^2 + \|\partial_z (\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|^2_{m-1} ds \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_0^t \|\boldsymbol{\omega}^\varepsilon(s)\|_m^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{3.34}$$

Now we estimate the remaining two terms in (3.3)

$$\begin{aligned} \int_0^t \int_\Omega Z^\alpha (\nabla \times \boldsymbol{\omega}^\varepsilon) \cdot Z^\alpha \mathbf{u}^\varepsilon ds = \int_0^t \int_\Omega & (Z^\alpha (\nabla \times \mathbf{u}^\varepsilon) \cdot Z^\alpha \boldsymbol{\omega}^\varepsilon - [Z^\alpha, \partial_z] \omega_2^\varepsilon Z^\alpha \mathbf{u}_1^\varepsilon - Z^\alpha \omega_2^\varepsilon [Z^\alpha, \partial_z] \mathbf{u}_1^\varepsilon \\ & + [Z^\alpha, \partial_z] \omega_1^\varepsilon Z^\alpha \mathbf{u}_2^\varepsilon + Z^\alpha \omega_1^\varepsilon [Z^\alpha, \partial_z] \mathbf{u}_2^\varepsilon) d\mathbf{x} ds. \end{aligned} \tag{3.35}$$

Young's inequality and (2.7) imply that

$$\begin{aligned} I_5 + I_9 =& 2\varepsilon \int_0^t \int_\Omega Z^\alpha (\nabla \times \boldsymbol{\omega}^\varepsilon) \cdot Z^\alpha \mathbf{u}^\varepsilon d\mathbf{x} ds + 2\varepsilon \int_0^t \int_\Omega Z^\alpha (\nabla \times \mathbf{u}^\varepsilon) \cdot Z^\alpha \boldsymbol{\omega}^\varepsilon d\mathbf{x} ds \\ =& 4\varepsilon \int_0^t \int_\Omega Z^\varepsilon (\nabla \times \mathbf{u}^\varepsilon) \cdot Z^\varepsilon \boldsymbol{\omega}^\varepsilon d\mathbf{x} ds \\ & - 2\varepsilon \int_0^t \int_\Omega [Z^\alpha, \partial_z] \omega_2^\varepsilon Z^\alpha u_1^\varepsilon d\mathbf{x} ds - 2\varepsilon \int_0^t \int_\Omega Z^\alpha \omega_2^\varepsilon [Z^\alpha, \partial_z] u_1^\varepsilon d\mathbf{x} ds \\ & + 2\varepsilon \int_0^t \int_\Omega [Z^\alpha, \partial_z] \omega_1^\varepsilon Z^\alpha u_2^\varepsilon d\mathbf{x} ds + 2\varepsilon \int_0^t \int_\Omega Z^\alpha \omega_1^\varepsilon [Z^\alpha, \partial_z] u_2^\varepsilon d\mathbf{x} ds \\ \le& \varepsilon \int_0^t \|Z^\alpha \nabla u^\varepsilon(s)\|^2 ds + 4\varepsilon \int_0^t \|Z^\alpha \boldsymbol{\omega}^\varepsilon(s)\|^2 ds \\ & + C \left(\varepsilon \int_0^t \|\partial_z \boldsymbol{\omega}^\varepsilon(s)\|_{m-1} \|\mathbf{u}^\varepsilon(s)\|_m + \|\partial_z \mathbf{u}^\varepsilon(s)\|_{m-1} \|\boldsymbol{\omega}^\varepsilon(s)\|_m \right). \end{aligned} \tag{3.36}$$

Finally, substituting the above estimates into equation (3.3), summing over all multi-indices with $|\alpha| \le m$, and taking θ sufficiently small, we obtain

$$\begin{aligned}&\|(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon)(t)\|_m^2 + \varepsilon \int_0^t \|\nabla(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 ds + \int_0^t \|\nabla \omega^\varepsilon(s)\|_m^2 ds + \varepsilon \int_0^t \|\nabla \cdot \omega^\varepsilon(s)\|_m^2\, ds \\ \lesssim& \|(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon)(0)\|_m^2 + \int_0^t \|\partial_z \mathbf{P}^\varepsilon(s)\|_{m-1}^2 ds + \left(1 + \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{[\frac{m}{2}]+1,\infty}^2 + \|\partial_z \omega^\varepsilon\|_{L^\infty_{t,\mathbf{x}}}^2\right) \\ &\cdot \int_0^t \|(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 + \|\partial_z(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{m-1}^2 ds \\ &+ \varepsilon \int_0^t \|(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 + \|\partial_z(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 ds + \varepsilon \int_0^t \|\partial_z \omega^\varepsilon(s)\|_{m-1}^2 ds.\end{aligned} \tag{3.37}$$

Thus, Proposition 3.1 is established. □

4. Estimate on the first order normal derivatives

Proposition 4.1. *For any integers $m \ge 7$, the systems* (1.6)–(1.9) *admit a classical solution* $(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon)$ *on the time interval* $[0, T]$*, which satisfies for every* $t \in [0, T]$ *that*

$$\begin{aligned}&\int_0^t \|\partial_z(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{m-1}^2 ds + \varepsilon^2 \int_0^t \|\partial_z^2(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{m-1}^2 ds \\ \lesssim& \int_0^t \|\nabla p^\varepsilon(s)\|_{m-1}^2 ds + \varepsilon^2 \int_0^t \|\nabla_h(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 ds + \varepsilon^2 \int_0^t \|\nabla \omega^\varepsilon(s)\|_{m-1}^2 ds \\ &+ \left(1 + \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{1,\infty}^2\right) \cdot \int_0^t \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 ds.\end{aligned} \tag{4.1}$$

Proof. First, we consider the conormal energy estimate for $\partial_z \mathbf{u}^\varepsilon$. The equation of $\mathbf{B}^\varepsilon$ in (1.6) can be rewritten as follows:

$$\partial_z \mathbf{u}^\varepsilon + 2\varepsilon \partial_z^2 \mathbf{B}^\varepsilon = \partial_t \mathbf{B}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{B}^\varepsilon - \mathbf{B}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon - 2\varepsilon \Delta_h \mathbf{B}^\varepsilon. \tag{4.2}$$

Applying Z^α to (4.2) with $|\alpha| \le m-1$, and taking the L^2 inner product on both sides of the resulting equation gives that

$$\begin{aligned}&\int_0^t \int_\Omega |Z^\alpha \partial_z \mathbf{u}^\varepsilon(s)|^2 + 4\varepsilon^2 |Z^\alpha \partial_z^2 \mathbf{B}^\varepsilon(s)|^2 d\mathbf{x} ds + 4\varepsilon \int_0^t \int_\Omega Z^\alpha \partial_z \mathbf{u}^\varepsilon \cdot Z^\alpha \partial_z^2 \mathbf{B}^\varepsilon d\mathbf{x} ds \\ \lesssim& \int_0^t \|\partial_t Z^\alpha \mathbf{B}^\varepsilon(s)\|^2 ds + \int_0^t \|Z^\alpha(\mathbf{u}^\varepsilon \cdot \nabla \mathbf{B}^\varepsilon)(s)\|^2 + \|Z^\alpha(\mathbf{B}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon)(s)\|^2 ds \\ &+ \varepsilon^2 \int_0^t \|\Delta_h Z^\alpha \mathbf{B}^\varepsilon(s)\|^2 ds \\ =& \sum_{i=1}^3 I_i.\end{aligned} \tag{4.3}$$

In the following, we shall estimate the terms appearing on the right-hand side of (4.3).

$$
\begin{aligned}
I_1 + I_3 &= \int_0^t \|\partial_t Z^\alpha \mathbf{B}^\varepsilon(s)\|^2 ds + \varepsilon^2 \int_0^t \|\Delta_h Z^\alpha \mathbf{B}^\varepsilon(s)\|^2 ds \\
&\lesssim \int_0^t \|\mathbf{B}^\varepsilon(s)\|_m^2 ds + \varepsilon^2 \int_0^t \|\nabla_h \mathbf{B}^\varepsilon(s)\|_m^2 ds,
\end{aligned} \tag{4.4}
$$

and

$$
\begin{aligned}
I_2 &= \int_0^t \|Z^\alpha(\mathbf{u}^\varepsilon \cdot \nabla \mathbf{B}^\varepsilon)(s)\|^2 + \|Z^\alpha(\mathbf{B}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon)(s)\|^2 ds \\
&\lesssim \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{1,\infty}^2 \int_0^t \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 ds,
\end{aligned} \tag{4.5}
$$

here we have applied the boundary conditions $u_3^\varepsilon|_{z=0} = b_3^\varepsilon|_{z=0} = 0$, the divergence-free conditions, and (2.10).

Substituting (4.4) and (4.5) into (4.3), we obtain that

$$
\begin{aligned}
&\int_0^t \int_\Omega |Z^\alpha \partial_z \mathbf{u}^\varepsilon(s)|^2 + 4\varepsilon^2 |Z^\alpha \partial_z^2 \mathbf{B}^\varepsilon(s)|^2 d\mathbf{x} ds + 4\varepsilon \int_0^t \int_\Omega Z^\alpha \partial_z \mathbf{u}^\varepsilon \cdot Z^\alpha \partial_z^2 \mathbf{B}^\varepsilon \\
\lesssim &\varepsilon^2 \int_0^t \|\nabla_h \mathbf{B}^\varepsilon(s)\|_m^2 ds + \left(1 + \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{1,\infty}^2\right) \int_0^t \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 ds.
\end{aligned} \tag{4.6}
$$

Proceeding with the conormal energy estimate for $\partial_z \mathbf{B}^\varepsilon$, we first reformulate the equation for $\mathbf{u}^\varepsilon$ in system (1.6) as follows:

$$
\partial_z \mathbf{B}^\varepsilon + 2\varepsilon \partial_z^2 \mathbf{u}^\varepsilon = \partial_t \mathbf{u}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \nabla p^\varepsilon - \mathbf{B}^\varepsilon \cdot \nabla \mathbf{B}^\varepsilon - 2\varepsilon(\nabla \times \omega^\varepsilon) - 2\varepsilon \Delta_h \mathbf{u}^\varepsilon. \tag{4.7}
$$

Acting by Z^α with $|\alpha| \le m-1$ on (4.7) and taking L^2 inner products gives

$$
\begin{aligned}
&\int_0^t \int_\Omega |Z^\alpha \partial_z \mathbf{B}^\varepsilon|^2 + 4\varepsilon^2 |Z^\alpha \partial_z^2 \mathbf{u}^\varepsilon|^2 d\mathbf{x} ds + 4\varepsilon \int_0^t \int_\Omega Z^\alpha \partial_z \mathbf{B}^\varepsilon \cdot Z^\alpha \partial_z^2 \mathbf{u}^\varepsilon d\mathbf{x} ds \\
\lesssim &\int_0^t \|\partial_t Z^\alpha \mathbf{u}^\varepsilon(s)\|^2 ds + \int_0^t \|Z^\alpha \nabla p^\varepsilon(s)\|^2 ds + \varepsilon^2 \int_0^t \|\Delta_h Z^\alpha \mathbf{u}^\varepsilon(s)\|^2 ds \\
&+ \varepsilon^2 \int_0^t \|Z^\alpha(\nabla \times \omega^\varepsilon)(s)\|^2 ds + \int_0^t \|Z^\alpha(\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon)(s)\|^2 + \|Z^\alpha(\mathbf{B}^\varepsilon \cdot \nabla \mathbf{B}^\varepsilon)(s)\|^2 ds \\
= &\sum_{i=1}^5 J_i.
\end{aligned} \tag{4.8}
$$

In the following, we shall estimate the terms appearing on the right-hand side of (4.8).

$$
\begin{aligned}
I_1 + I_2 &= \int_0^t \|\partial_t Z^\alpha \mathbf{u}^\varepsilon(s)\|^2 ds + \int_0^t \|Z^\alpha \nabla p^\varepsilon(s)\|^2 ds \\
&\lesssim \int_0^t \|\mathbf{u}^\varepsilon(s)\|_m^2 ds + \int_0^t \|\nabla p^\varepsilon(s)\|_{m-1}^2 ds,
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
I_3 + I_4 =&\varepsilon^2 \int_0^t \|\Delta_h Z^\alpha \mathbf{u}^\varepsilon(s)\|^2 ds + \varepsilon^2 \int_0^t \|Z^\alpha(\nabla \times \boldsymbol{\omega}^\varepsilon)(s)\|^2 ds \\
\lesssim&\varepsilon^2 \int_0^t \|\nabla_h \mathbf{u}^\varepsilon(s)\|_m^2 ds + \varepsilon^2 \int_0^t \|\nabla \boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2 ds,
\end{aligned} \tag{4.10}
$$

and

$$
\begin{aligned}
I_5 =& \int_0^t \|Z^\alpha(\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon)(s)\|^2 + \|Z^\alpha(\mathbf{B}^\varepsilon \cdot \nabla \mathbf{B}^\varepsilon)(s)\|^2 ds \\
\lesssim& \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{1,\infty}^2 \int_0^t \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 ds,
\end{aligned} \tag{4.11}
$$

here we have used the boundary conditions $u_3^\varepsilon|_{z=0} = b_3^\varepsilon|_{z=0} = 0$, the divergence-free conditions, and (2.10).

Inserting (4.9)–(4.11) into (4.8) yields

$$
\begin{aligned}
&\int_0^t \int_\Omega |Z^\alpha \partial_z \mathbf{B}^\varepsilon|^2 + 4\varepsilon |Z^\alpha \partial_z^2 \mathbf{u}^\varepsilon|^2 d\mathbf{x} ds + 4\varepsilon \int_0^t \int_\Omega Z^\alpha \partial_z \mathbf{B}^\varepsilon \cdot Z^\alpha \partial_z^2 \mathbf{u}^\varepsilon d\mathbf{x} ds \\
\lesssim&\varepsilon^2 \int_0^t \|\nabla_h \mathbf{u}^\varepsilon(s)\|_m^2 ds + \varepsilon^2 \int_0^t \|\nabla \boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2 ds + \int_0^t \|\nabla p^\varepsilon(s)\|_{m-1}^2 ds \\
&+ \left(1 + \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{1,\infty}^2\right) \int_0^t \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 ds.
\end{aligned} \tag{4.12}
$$

Our current task is to estimate the final terms appearing on the left-hand sides of both (4.6) and (4.12). Through integration by parts, we find that

$$
\begin{aligned}
&4\varepsilon \int_0^t \int_\Omega Z^\alpha \partial_z^2 \mathbf{u}^\varepsilon \cdot Z^\alpha \partial_z \mathbf{B}^\varepsilon d\mathbf{x} ds \\
=&4\varepsilon \int_0^t \int_\Omega [Z^\alpha, \partial_z] \partial_z \mathbf{u}^\varepsilon \cdot Z^\alpha \partial_z \mathbf{B}^\varepsilon d\mathbf{x} ds + 4\varepsilon \int_0^t \int_\Omega Z^\alpha \partial_z \mathbf{u}^\varepsilon \cdot [Z^\alpha, \partial_z] \partial_z \mathbf{B}^\varepsilon d\mathbf{x} ds \\
&- 4\varepsilon \int_0^t \int_\Omega Z^\alpha \partial_z \mathbf{u}^\varepsilon \cdot Z^\alpha \partial_z^2 \mathbf{B}^\varepsilon d\mathbf{x} ds.
\end{aligned} \tag{4.13}
$$

By applying (2.5) and Young's inequality, we obtain that

$$
\begin{aligned}
&4\varepsilon \int_0^t \int_\Omega [Z^\alpha, \partial_z] \partial_z \mathbf{u}^\varepsilon \cdot Z^\alpha \partial_z \mathbf{B}^\varepsilon d\mathbf{x} ds + 4\varepsilon \int_0^t \int_\Omega Z^\alpha \partial_z \mathbf{u}^\varepsilon \cdot [Z^\alpha, \partial_z] \partial_z \mathbf{B}^\varepsilon d\mathbf{x} ds \\
=&4\varepsilon \sum_{k=0}^{m-2} \int_0^t \int_\Omega \phi_{k,m-1}(z) Z_3^k \partial_z^2 \mathbf{u}^\varepsilon \cdot Z^\alpha \partial_z \mathbf{B}^\varepsilon d\mathbf{x} ds + 4\varepsilon \sum_{k=0}^{m-2} \int_0^t \int_\Omega \phi_{k,m-1}(z) Z^\alpha \partial_z \mathbf{u}^\varepsilon \cdot Z_3^k \partial_z^2 \mathbf{B}^\varepsilon d\mathbf{x} ds \\
\le&\eta_1 \int_0^t \|\partial_z \mathbf{B}^\varepsilon(s)\|_{m-1}^2 ds + \eta_2 \int_0^t \|\partial_z \mathbf{u}^\varepsilon(s)\|_{m-1}^2 ds + C\varepsilon^2 \int_0^t \|\partial_z^2 (\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{m-2}^2 ds,
\end{aligned} \tag{4.14}
$$

where η_1 and η_2 are sufficiently small positive constants that will be determined later.

By inserting (4.13) and (4.14) into (4.12), combining them with (4.6), and performing the summation over $|\alpha| \le m-1$, we obtain

$$
\begin{aligned}
&\int_0^t \|\partial_z(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{m-1}^2 ds + \varepsilon^2 \int_0^t \|\partial_z^2(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{m-1}^2 ds \\
\lesssim &\int_0^t \|\nabla p^\varepsilon(s)\|_{m-1}^2 ds + \varepsilon^2 \int_0^t \|\nabla_h(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 ds + \varepsilon^2 \int_0^t \|\nabla \boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2 ds \\
&+ \left(1 + \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{1,\infty}^2\right) \cdot \int_0^t \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 \\
&+ \eta_1 \int_0^t \|\partial_z \mathbf{B}^\varepsilon(s)\|_{m-1}^2 ds + \eta_2 \int_0^t \|\partial_z \mathbf{u}^\varepsilon(s)\|_{m-1}^2 ds + \varepsilon^2 \int_0^t \|\partial_z^2(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{m-2}^2 ds.
\end{aligned} \tag{4.15}
$$

Through an inductive argument in m, with appropriately chosen small parameters η_1 and η_2, the proof of Proposition (4.1) is completed. □

5. Estimate on the second order normal derivation

For the forthcoming L^∞ estimates, it is necessary to establish bounds on the second-order normal derivatives of $\boldsymbol{\omega}^\varepsilon$.

Proposition 5.1. *For any integer $m \ge 7$, it holds that for the classical solution* $(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon, \mathbf{B}^\varepsilon)$ *of* (1.6)–(1.9) *on* $[0, T]$

$$
\begin{aligned}
\int_0^t \|\partial_z^2 \boldsymbol{\omega}^\varepsilon(s)\|_{m-2}^2 ds \lesssim &\int_0^t \|\nabla_h \boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2 ds + (1 + \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_{1,\infty}^2) \int_0^t \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_{m-1}^2 ds \\
&+ \varepsilon^2 \int_0^t \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_m^2 + \|\partial_z(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_{m-1}^2 ds.
\end{aligned} \tag{5.1}
$$

Proof. The structural properties of the $\boldsymbol{\omega}^\varepsilon$ equation in (1.6) automatically yield an energy framework for second-order normal derivatives. The evolution equation governing $\boldsymbol{\omega}^\varepsilon$ takes the form

$$
\partial_z^2 \boldsymbol{\omega}^\varepsilon = \partial_t \boldsymbol{\omega}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \boldsymbol{\omega}^\varepsilon - \varepsilon \nabla(\nabla \cdot \boldsymbol{\omega}^\varepsilon) + 4\varepsilon \boldsymbol{\omega}^\varepsilon - 2\varepsilon \nabla \times \mathbf{u}^\varepsilon - \Delta_h \boldsymbol{\omega}^\varepsilon. \tag{5.2}
$$

Acting by Z^α ($|\alpha| \le m-2$) on (5.2) and taking L^2 inner products yields

$$
\begin{aligned}
&\int_0^t \|Z^\alpha \partial_z^2 \boldsymbol{\omega}^\varepsilon(s)\|^2 ds \\
\lesssim &\int_0^t \|\partial_t Z^\alpha \boldsymbol{\omega}^\varepsilon(s)\|^2 ds + \varepsilon^2 \int_0^t \|Z^\alpha \boldsymbol{\omega}^\varepsilon(s)\|^2 ds + \int_0^t \|\Delta_h Z^\alpha \boldsymbol{\omega}^\varepsilon(s)\|^2 ds + \int_0^t \|Z^\alpha(\mathbf{u}^\varepsilon \cdot \nabla \boldsymbol{\omega}^\varepsilon)(s)\|^2 ds \\
&+ \varepsilon^2 \int_0^t \|Z^\alpha \nabla(\nabla \cdot \boldsymbol{\omega}^\varepsilon)(s)\|^2 ds + \varepsilon^2 \int_0^t \|Z^\alpha(\nabla \times \mathbf{u}^\varepsilon)(s)\|^2 ds \\
= &\sum_{i=1}^6 I_i.
\end{aligned} \tag{5.3}
$$

Initially, the following result is immediate

$$
\begin{aligned}
I_1 + I_2 + I_3 &= \int_0^t \|\partial_t Z^\alpha \boldsymbol{\omega}^\varepsilon(s)\|^2 + \varepsilon^2 \|Z^\alpha \boldsymbol{\omega}^\varepsilon(s)\|^2 + \|Z^\alpha \nabla_h \boldsymbol{\omega}^\varepsilon(s)\|^2 ds \\
&\lesssim \int_0^t \|\boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2 + \varepsilon^2 \|\boldsymbol{\omega}^\varepsilon(s)\|_{m-2}^2 + \|\nabla_h \boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2 ds.
\end{aligned} \tag{5.4}
$$

Secondly, applying (2.10) yields

$$\begin{aligned} I_4 =& \int_0^t \|Z^\alpha(\mathbf{u}^\varepsilon \cdot \nabla \boldsymbol{\omega}^\varepsilon)(s)\|^2 ds \\ \lesssim& \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_{1,\infty}^2 \int_0^t \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_{m-1}^2 ds. \end{aligned} \tag{5.5}$$

The last two terms in (5.3) require the following estimates:

$$\begin{aligned} I_5 =& \varepsilon^2 \int_0^t \|Z^\alpha(\nabla(\nabla\cdot \boldsymbol{\omega}^\varepsilon))(s)\|^2 ds \\ \lesssim& \varepsilon^2 \int_0^t \|\boldsymbol{\omega}^\varepsilon(s)\|_m^2 + \|\partial_z \boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2 + \|\partial_z^2 \boldsymbol{\omega}^\varepsilon(s)\|_{m-2}^2 ds. \end{aligned} \tag{5.6}$$

Likewise, it follows that

$$\begin{aligned} I_6 =& \varepsilon^2 \int_0^t \|Z^\alpha(\nabla \times \mathbf{u}^\varepsilon)(s)\|^2 ds \\ \lesssim& \varepsilon^2 \int_0^t \|\mathbf{u}^\varepsilon(s)\|_{m-1}^2 + \|\partial_z \mathbf{u}^\varepsilon(s)\|_{m-2}^2 ds. \end{aligned} \tag{5.7}$$

By integrating all the estimates established above, we conclude that

$$\begin{aligned} &\int_0^t \|Z^\alpha \partial_z^2 \boldsymbol{\omega}^\varepsilon(s)\|^2 ds \\ \lesssim& \int_0^t \|\nabla_h \boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2 ds + \left(1 + \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_{1,\infty}^2\right) \int_0^t \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_{m-1}^2 ds \\ &+ \varepsilon^2 \int_0^t \|(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_m^2 + \|\partial_z(\mathbf{u}^\varepsilon, \boldsymbol{\omega}^\varepsilon)(s)\|_{m-1}^2 + \|\partial_z^2 \boldsymbol{\omega}^\varepsilon(s)\|_{m-2}^2 ds. \end{aligned} \tag{5.8}$$

Summing over all multi-indices with $|\alpha| \le m-2$ gives the desired estimate.

Thus we have completed the proof of Proposition 5.1. □

6. Estimate of pressure

Proposition 6.1. *For any integer $m \ge 7$, the classical solution to* (1.6)–(1.9) *exists on* $[0, T]$ *and satisfies the following estimate:*

$$\begin{aligned} &\int_0^t \|\nabla p^\varepsilon(s)\|_{m-1}^2 ds + \int_0^t \|\partial_z \nabla p^\varepsilon(s)\|_{m-2}^2 ds \\ \lesssim& \eta\varepsilon^2 \int_0^t \|\partial_z^2 \mathbf{u}_h^\varepsilon(s)\|_{m-1}^2 ds + \zeta \int_0^t \|\partial_z(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{m-1}^2 ds \\ &+ \left(1 + \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{2,\infty}^2\right) \int_0^t \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 ds, \end{aligned} \tag{6.1}$$

where η and ζ are sufficiently small positive constants.

Proof. By taking the divergence of the equation for $\mathbf{u}^\varepsilon$ in system (1.6), we derive the equation for the pressure p^ε

$$\begin{aligned}\Delta p^\varepsilon =&\nabla\cdot(-\mathbf{u}^\varepsilon\cdot\nabla\mathbf{u}^\varepsilon+\mathbf{B}^\varepsilon\cdot\nabla\mathbf{B}^\varepsilon+2\varepsilon\nabla\times\omega^\varepsilon)\\ =&\nabla\cdot(-\mathbf{u}^\varepsilon\cdot\nabla\mathbf{u}^\varepsilon+\mathbf{B}^\varepsilon\cdot\nabla\mathbf{B}^\varepsilon)\\ :=&\nabla\cdot F.\end{aligned}\tag{6.2}$$

By analyzing the third velocity component equation in (3.8), we derive the boundary condition for p^ε and consequently establish the relation

$$\partial_z p^\varepsilon|_{z=0}=\partial_z\mathbf{b}_3^\varepsilon|_{z=0}+2\varepsilon\partial_z^2u_3^\varepsilon|_{z=0}=-\nabla_h\cdot(2\varepsilon\partial_z\mathbf{u}_h^\varepsilon+\mathbf{B}_h^\varepsilon)|_{z=0},\tag{6.3}$$

in which the boundary condition $\mathbf{b}_3^\varepsilon|_{z=0}=0$ is applied.

The temporal parameter t serves only as an implicit variable in all operations. To maintain both notational conciseness and mathematical rigor, we will consistently adopt implicit function notation in subsequent derivations, omitting explicit temporal annotations for all relevant functions. And we consider decomposing the pressure p^ε as $p^\varepsilon=p_1^\varepsilon+p_2^\varepsilon$ following the approach in [6], where p_1^ε satisfies

$$\begin{cases}\Delta p_1^\varepsilon=\nabla\cdot F,\\ \partial_z p_1^\varepsilon|_{z=0}=0,\end{cases}\tag{6.4}$$

and p_2^ε obeys

$$\begin{cases}\Delta p_2^\varepsilon=0,\\ \partial_z p_2^\varepsilon|_{z=0}=-\nabla_h\cdot(2\varepsilon\partial_z\mathbf{u}_h^\varepsilon+\mathbf{B}^\varepsilon)|_{z=0}.\end{cases}\tag{6.5}$$

This decomposition carries the following significance: p_1^ε represents the gradient component in the Leray-Hodge decomposition of the vector field F, while p_2^ε is uniquely determined by the aforementioned boundary conditions. Applying standard elliptic theory, we obtain estimates for both p_1^ε and p_2^ε.

The p_1^ε estimate follows from Fourier transforming in (x,y)

$$-|k|^2\,\hat{p}_1^\varepsilon\ +\partial_{zz}\hat{p}_1^\varepsilon=ik\cdot\hat{F}_h+\partial_z\hat{F}_3,\qquad z>0.\tag{6.6}$$

Solving this ordinary differential equation yields

$$\hat{p}_1^\varepsilon(k,z)=\int_0^\infty G_k(z,z')\hat{F}(k,z')dz',\tag{6.7}$$

where $k=(k_1,k_2)$, and

$$G_k(z,z')=\begin{cases}-e^{-|k|z'}\cosh(|k|z)\left(\frac{ik}{|k|},1\right), & z<z',\\ -e^{-|k|z}\left(\frac{ik}{|k|}\cosh(|k|z'),-\sinh(|k|z')\right), & z>z'.\end{cases}\tag{6.8}$$

Following the same argument in [10], we obtain

$$\|\nabla p_1^\varepsilon\|\lesssim\|F\|^2.\tag{6.9}$$

Moreover, for all $q\geq 1$, we have

$$\|\nabla p_1^\varepsilon\|_q\lesssim\|F\|_q+\|\nabla\cdot F\|_{q-1},\tag{6.10}$$

$$\|\partial_z^2 p_1^\varepsilon\|_{q-1} \lesssim \|\nabla \cdot F\|_{q-1}. \tag{6.11}$$

Therefore, combining (6.4) with the above estimates yields

$$\|\partial_z \nabla p_1^\varepsilon\|_q \lesssim \|\partial_z p_1^\varepsilon\|_{q+1} + \|\partial_z^2 p_1^\varepsilon\|_q \lesssim \|F\|_{q+1} + \|\nabla \cdot F\|_q, \tag{6.12}$$

from which it immediately follows that for all $i \geq 2$,

$$\|\partial_z^i \nabla p_1^\varepsilon\|_q \lesssim \|\partial_z^{i-1} F\|_{q+1} + \|\partial_z^{i-1}(\nabla \cdot F)\|_q. \tag{6.13}$$

Combining the estimates (6.9)–(6.13), we obtain

$$\|\nabla p_1^\varepsilon\|_{m-1} + \|\partial_z \nabla p_1^\varepsilon\|_{m-2} \lesssim \|F\|_{m-1} + \|\nabla \cdot F\|_{m-2}. \tag{6.14}$$

Therefore, with $F := (-\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \mathbf{B}^\varepsilon \cdot \nabla \mathbf{B}^\varepsilon)$, an application of (2.10) yields

$$\begin{aligned} \int_0^t \|F(s)\|_{m-1}^2 ds &\leq \int_0^t \|\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon(s)\|_{m-1}^2 ds + \int_0^t \|\mathbf{B}^\varepsilon \cdot \nabla \mathbf{B}^\varepsilon(s)\|_{m-1}^2 ds \\ &\lesssim \sup_{0\leq s\leq t} \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{1,\infty}^2 \int_0^t \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 ds. \end{aligned} \tag{6.15}$$

We now estimate $\nabla \cdot F$ by decomposing it into

$$\begin{aligned} \nabla \cdot F =& \nabla_h \cdot (-\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}_h^\varepsilon) + \nabla_h \cdot (\mathbf{B}^\varepsilon \cdot \nabla \mathbf{B}_h^\varepsilon) \\ &+ \partial_z \cdot (-\mathbf{u}^\varepsilon \cdot \nabla u_3^\varepsilon) + \partial_z \cdot (\mathbf{B}^\varepsilon \cdot \nabla b_3^\varepsilon) \\ :=& \sum_{i=1}^4 f_i. \end{aligned} \tag{6.16}$$

We first estimate f_1, which follows immediately from (2.10)

$$\begin{aligned} \int_0^t \|f_1\|_{m-2} ds &= \int_0^t \left\|\nabla_h \cdot (-\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}_h^\varepsilon)\right\|_{m-1} ds \\ &\lesssim \sup_{0\leq s\leq t} \|\nabla_h \mathbf{u}^\varepsilon(s)\|_{1,\infty}^2 \int_0^t \|\mathbf{u}^\varepsilon(s)\|_{m-1}^2 ds + \sup_{0\leq s\leq t} \|\mathbf{u}^\varepsilon(s)\|_{1,\infty}^2 \int_0^t \|\nabla_h \mathbf{u}^\varepsilon(s)\|_{m-1}^2 ds \\ &\lesssim \sup_{0\leq s\leq t} \|\mathbf{u}^\varepsilon(s)\|_{2,\infty}^2 \int_0^t \|\mathbf{u}^\varepsilon(s)\|_m^2 ds. \end{aligned} \tag{6.17}$$

Similarly, we obtain the estimate for f_2

$$\begin{aligned} \int_0^t \|f_2\|_{m-2} ds &= \int_0^t \left\|(\nabla_h \cdot (-\mathbf{B}^\varepsilon \cdot \nabla b_h^\varepsilon))\right\|_{m-2} ds \\ &\lesssim \sup_{0\leq s\leq t} \|\mathbf{B}^\varepsilon(s)\|_{2,\infty}^2 \int_0^t \|\mathbf{B}^\varepsilon(s)\|_m^2 ds. \end{aligned} \tag{6.18}$$

Turning to f_3, the divergence-free condition $\nabla \cdot \mathbf{u}^\varepsilon = 0$ yields

$$f_3 = \partial_z(-\mathbf{u}^\varepsilon \cdot \nabla) u_3^\varepsilon = \partial_z(-\mathbf{u}_h^\varepsilon \cdot \nabla_h) u_3^\varepsilon + (\nabla_h \cdot \mathbf{u}_h)^2. \tag{6.19}$$

Applying (2.10) immediately yields its estimate

$$\begin{aligned}\int_0^t \|f_3\|_{m-2}^2 ds \lesssim& \|\partial_z \nabla_h u_3\|_{L^\infty_{t,\mathbf{x}}}^2 \int_0^t \|\mathbf{u}_h^\varepsilon(s)\|_{m-1}^2 ds + \sup_{0\le s\le t} \|\mathbf{u}_h^\varepsilon(s)\|_{1,\infty}^2 \int_0^t \|\partial_z \nabla_h u_3^\varepsilon(s)\|_{m-2}^2 ds \\ &+ \|\nabla_h \cdot \mathbf{u}_h\|_{L^\infty_{t,\mathbf{x}}}^2 \int_0^t \|\nabla_h \cdot \mathbf{u}_h^\varepsilon(s)\|_{m-2}^2 ds \\ \lesssim& \sup_{0\le s\le t} \|\mathbf{u}^\varepsilon(s)\|_{2,\infty}^2 \int_0^t \|\mathbf{u}^\varepsilon(s)\|_m^2 ds.\end{aligned} \tag{6.20}$$

Similarly, we obtain the estimate for f_4.

$$\int_0^t \|f_4\|_{m-2}^2 ds \lesssim \sup_{0\le s\le t} \|\mathbf{B}^\varepsilon(s)\|_{2,\infty}^2 \int_0^t \|\mathbf{B}^\varepsilon(s)\|_m^2 ds. \tag{6.21}$$

Therefore, combining the results from (6.16) to (6.21), we derive

$$\int_0^t \|\nabla \cdot F\|_{m-2}^2 ds \lesssim \sup_{0\le s\le t} \|\mathbf{B}^\varepsilon(s)\|_{2,\infty}^2 \int_0^t \|\mathbf{B}^\varepsilon(s)\|_m^2 ds. \tag{6.22}$$

In summary, the preceding estimate immediately yields

$$\int_0^t \|\nabla p_1^\varepsilon\|_{m-1}^2 ds + \int_0^t \|\partial_z \nabla p_1^\varepsilon\|_{m-2}^2 ds \lesssim \sup_{0\le s\le t} \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_{2,\infty}^2 \int_0^t \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)(s)\|_m^2 ds. \tag{6.23}$$

Turning to p_2^ε. Similarly, applying the Fourier transform yields an explicit solution to Eq (6.5)

$$\hat{p}_2^\varepsilon(k,z) = e^{-|k|z} \frac{ik}{|k|} \cdot (2\varepsilon \partial_z \hat{\mathbf{u}}_h^\varepsilon + \hat{\mathbf{B}}_h^\varepsilon)(k,0), \tag{6.24}$$

which yields

$$\widehat{\nabla p_2^\varepsilon}(k,z) = e^{-|k|z} \Big(ik \cdot (2\varepsilon \partial_z \hat{\mathbf{u}}_h^\varepsilon + \hat{\mathbf{B}}_h^\varepsilon)(k,0) \Big) (\frac{ik}{|k|}, -1). \tag{6.25}$$

By Plancherel's theorem and the trace inequality, we obtain

$$\begin{aligned}\|\nabla p_2^\varepsilon\|_{H^r}^2 \lesssim& \|(\varepsilon \partial_z \mathbf{u}_h^\varepsilon + \mathbf{B}_h^\varepsilon)(\cdot,0)\|_{H_{x,y}^{r+\frac{1}{2}}}^2 \\ \lesssim& \varepsilon^2 \|\partial_z \mathbf{u}_h^\varepsilon\|_r \|\partial_z^2 \mathbf{u}_h^\varepsilon\|_r + \|\mathbf{B}_h^\varepsilon\|_r \|\partial_z \mathbf{B}_h^\varepsilon\|_r.\end{aligned} \tag{6.26}$$

Therefore, for all $i, q \ge 0$, there holds

$$\|\partial_z^i \nabla p_2^\varepsilon\|_q^2 \lesssim \varepsilon^2 \|\partial_z \mathbf{u}_h^\varepsilon\|_{i+q} \|\partial_z^2 \mathbf{u}_h^\varepsilon\|_{i+q} + \|\mathbf{B}_h^\varepsilon\|_{i+q} \|\partial_z \mathbf{B}_h^\varepsilon\|_{i+q}. \tag{6.27}$$

Which immediately yields

$$\begin{aligned}&\int_0^t \|\nabla p_2^\varepsilon\|_{m-1}^2 ds + \int_0^t \|\partial_z \nabla p_2^\varepsilon\|_{m-2}^2 ds \\ \lesssim& \int_0^t \|\nabla p_2^\varepsilon\|_{m-1}^2 ds + \int_0^t \|\partial_z^2 p_2^\varepsilon\|_{m-2}^2 ds \\ \lesssim& \varepsilon^2 \int_0^t \|\partial_z \mathbf{u}_h^\varepsilon\|_{m-1} \|\partial_z^2 \mathbf{u}_h^\varepsilon\|_{m-1}^2 ds + \int_0^t \|\partial_z \mathbf{B}_h^\varepsilon\|_{m-1} \|\mathbf{B}_h^\varepsilon\|_{m-1}^2 ds \\ \lesssim& \eta \varepsilon^2 \int_0^t |\partial_z^2 \mathbf{u}_h^\varepsilon\|_{m-1}^2 ds + \zeta \int_0^t \|\partial_z (\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)\|_{m-1}^2 ds + \int_0^t \|(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)\|_m^2 ds,\end{aligned} \tag{6.28}$$

where η and ζ are sufficiently small positive constants.

In summary, combining Eqs (6.23) with (6.28) yields the required estimate in the proposition. □

7. Proof of Theorem 1.1

We now present the complete proof of our main result, Theorem 1.1. The argument begins by synthesizing the results from (4.1) and (6.1), which gives

$$\begin{aligned}
&\int_0^t \|\partial_z(\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon)(s)\|_{m-1}^2+\varepsilon^2\|\partial_z^2(\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon)(s)\|_{m-1}^2ds+\int_0^t\|\partial_z\nabla p^\varepsilon(s)\|_{m-2}^2ds\\
\lesssim&\varepsilon^2\int_0^t\|\nabla_h(\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon)(s)\|_m^2ds+\varepsilon^2\int_0^t\|\nabla\boldsymbol{\omega}^\varepsilon(s)\|_{m-1}^2ds\\
&+\left(1+\sup_{0\le s\le t}\|(\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon)(s)\|_{2,\infty}^2\right)\int_0^t\|(\mathbf{u}^\varepsilon,\mathbf{B}^\varepsilon)(s)\|_m^2ds.
\end{aligned}\tag{7.1}$$

Recalling the definition of the energy functional in (1.13), substituting (5.1) and (7.1) into (3.1), and taking ε sufficiently small, we obtain

$$\begin{aligned}
&N_m(t)+\int_0^t\|\partial_z\nabla p^\varepsilon(s)\|_{m-2}^2ds\\
\lesssim&\|(\mathbf{u}^\varepsilon,\boldsymbol{\omega}^\varepsilon,\mathbf{B}^\varepsilon)(0)\|_m^2+\left(1+\sup_{0\le s\le t}\|(\mathbf{u}^\varepsilon,\boldsymbol{\omega}^\varepsilon,\mathbf{B}^\varepsilon)(s)\|_{[\frac{m}{2}]+1,\infty}^2+\|\partial_z\boldsymbol{\omega}^\varepsilon\|_{L^\infty_{t,\mathbf{x}}}^2\right)\\
&\cdot\int_0^t\|(\mathbf{u}^\varepsilon,\boldsymbol{\omega}^\varepsilon,\mathbf{B}^\varepsilon)(s)\|_m^2+\|\partial_z(\mathbf{u}^\varepsilon,\boldsymbol{\omega}^\varepsilon,\mathbf{B}^\varepsilon)(s)\|_{m-1}^2ds.
\end{aligned}\tag{7.2}$$

The completion of our ultimate estimate requires additional control of the L^∞ norm. And to obtain the required regularity and complete the closure of the energy estimates, applying (2.4) with the arbitrary integer $m\ge 7$ yields

$$\begin{aligned}
&\sup_{0\le s\le t}\|(\mathbf{u}^\varepsilon,\boldsymbol{\omega}^\varepsilon,\mathbf{B}^\varepsilon)(s)\|_{[\frac{m}{2}]+1,\infty}^2\\
\lesssim&\|(\mathbf{u}^\varepsilon,\boldsymbol{\omega}^\varepsilon,\mathbf{B}^\varepsilon)(0)\|_{[\frac{m}{2}]+3}^2+\|\partial_z(\mathbf{u}^\varepsilon,\boldsymbol{\omega}^\varepsilon,\mathbf{B}^\varepsilon)(0)\|_{[\frac{m}{2}]+2}^2\\
&+\int_0^t\|(\mathbf{u}^\varepsilon,\boldsymbol{\omega}^\varepsilon,\mathbf{B}^\varepsilon)(s)\|_{[\frac{m}{2}]+4}^2+\|\partial_z(\mathbf{u}^\varepsilon,\boldsymbol{\omega}^\varepsilon,\mathbf{B}^\varepsilon)(s)\|_{[\frac{m}{2}]+3}^2ds\\
\lesssim&C(M_0)+\mathcal{P}(N_m(t)).
\end{aligned}\tag{7.3}$$

Moreover, the application of (2.2) and (5.1) yields

$$\begin{aligned}
&\|\partial_z\boldsymbol{\omega}\|_{L^\infty_{t,\mathbf{x}}}^2\\
\lesssim&\|\partial_z\boldsymbol{\omega}^\varepsilon(0)\|_2^2+\|\partial_z^2\boldsymbol{\omega}^\varepsilon(0)\|_1^2+\int_0^t\|\partial_z\boldsymbol{\omega}^\varepsilon(s)\|_3^2+\|\partial_z^2\boldsymbol{\omega}^\varepsilon(s)\|_2^2ds\\
\lesssim&C(M_0)+\mathcal{P}(N_m(t)).
\end{aligned}\tag{7.4}$$

Finally, from Eqs (7.2)–(7.4), we obtain

$$N_m(t)+\int_0^t\|\partial_z\nabla p^\varepsilon(s)\|_{m-2}^2ds\lesssim C(M_0)+(t+\varepsilon)\mathcal{P}(N_m(t)).\tag{7.5}$$

By selecting the time scale t and the parameter ε appropriately small, we derive

$$N_m(t) + \int_0^t \|\partial_z \nabla p^\varepsilon(s)\|_{m-2}^2 \, ds \lesssim M, \tag{7.6}$$

in which the constant M depends exclusively on the initial parameter M_0.

The justification of the vanishing dissipation limit proceeds as follows. Owing to the uniform-in-ε regularity estimates derived previously, the smooth solutions $(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon)$ of (1.6) admit, for arbitrary time t, the following properties:

$$(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon) \in H_{co}^m(\Omega), \tag{7.7}$$

and

$$\nabla(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon) \in H_{co}^{m-1}(\Omega). \tag{7.8}$$

This yields that for each t, $(\mathbf{u}^\varepsilon(t), \omega^\varepsilon(t), \mathbf{B}^\varepsilon(t))$ is compact in $H_{co}^{m-1}(\Omega)$, where

$$H_{co}^m(\Omega) = \left\{ f(t, \mathbf{x}) \,\middle|\, Z^\varepsilon f \in L^2(\Omega), |\alpha| \le m \right\}.$$

Next, by using the Eq $(1.6)_1$, we get that

$$\begin{aligned} \int_0^T \|\partial_t \mathbf{u}^\varepsilon(t)\|_{m-1}^2 dt \le \int_0^T (&\|\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon\|_{m-1}^2 + \|\mathbf{B}^\varepsilon \cdot \nabla \mathbf{B}^\varepsilon\|_{m-1}^2 + \|\nabla p^\varepsilon\|_{m-1}^2 \\ &+ 4\varepsilon^2 \|\Delta \mathbf{u}^\varepsilon\|_{m-1}^2 + 4\varepsilon \|\nabla \times \omega^\varepsilon\|_{m-1}^2 + \|\partial_z \mathbf{B}^\varepsilon\|_{m-1}^2) dt. \end{aligned} \tag{7.9}$$

Hence, by using (2.10), (4.1), (5.1) and (6.1), we obtain $\partial_t \mathbf{u}^\varepsilon$ is uniformly bounded in $L^2(0, T; H_{co}^{m-1}(\Omega))$, i.e., $H_{co}^{m-1}([0, T] \times \Omega)$. Similarly, one has

$$(\partial_t \mathbf{u}^\varepsilon, \partial_t \omega^\varepsilon, \partial_t \mathbf{B}^\varepsilon) \in L^2(0, T; H_{co}^{m-1}(\Omega)), \tag{7.10}$$

and yield that

$$(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon) \in L^\infty(0, T; H_{co}^m(\Omega)). \tag{7.11}$$

Using the Aubin-Lions Lemma, we obtain

$$(\mathbf{u}^\varepsilon, \omega^\varepsilon, \mathbf{B}^\varepsilon) \to (\mathbf{u}^0, \omega^0, \mathbf{B}^0) \quad in \;\; L^\infty(0, T; H_{co}^{m-1}(\Omega)). \tag{7.12}$$

Notice that $H_{co}^{m-1}(\Omega)$ is a Hilbert space, and since $(\partial_z \mathbf{u}^\varepsilon, \partial_z \omega^\varepsilon, \partial_z \mathbf{B}^\varepsilon) \in L^2(0, T; H_{co}^{m-1}(\Omega))$, by (7.6), then there exists a vector-valued function $v \in L^2(0, T; H_{co}^{m-1}(\Omega))$, such that

$$(\partial_z \mathbf{u}^\varepsilon, \partial_z \omega^\varepsilon, \partial_z \mathbf{B}^\varepsilon) \rightharpoonup v. \tag{7.13}$$

Combining the definition of weak derivatives, we find that

$$v = (\partial_z \mathbf{u}^\varepsilon, \partial_z \omega^\varepsilon, \partial_z \mathbf{B}^\varepsilon), \tag{7.14}$$

then

$$\begin{aligned} &\|(\partial_z \mathbf{u}^\varepsilon, \partial_z \omega^\varepsilon, \partial_z \mathbf{B}^\varepsilon) - (\partial_z \mathbf{u}^0, \partial_z \omega^0, \partial_z \mathbf{B}^0)\|_{m_0}^2 \\ \lesssim &\|(\partial_z \mathbf{u}^\varepsilon, \partial_z \omega^\varepsilon, \partial_z \mathbf{B}^\varepsilon)(0) - (\partial_z \mathbf{u}^0, \partial_z \omega^0, \partial_z \mathbf{B}^0)(0)\|_{m_0}^2 \\ &+ \int_0^t \|(\partial_z \mathbf{u}^\varepsilon, \partial_z \omega^\varepsilon, \partial_z \mathbf{B}^\varepsilon) - (\partial_z \mathbf{u}^0, \partial_z \omega^0, \partial_z \mathbf{B}^0)\|_{m_0+1}^2 ds \\ \lesssim &C(M_0) + \mathcal{P}(N_m(t)). \end{aligned} \tag{7.15}$$

It follows from the anisotropic Sobolev embedding (2.3)

$$\begin{aligned}&\|(\mathbf{u}^\varepsilon,\omega^\varepsilon,\mathbf{B}^\varepsilon)-(\mathbf{u}^0,\omega^0,\mathbf{B}^0)\|_{L^\infty_{t,\mathbf{x}}}^2\\ \lesssim&\sup_{0\le s\le t}(\|(\partial_z\mathbf{u}^\varepsilon,\partial_z\omega^\varepsilon,\partial_z\mathbf{B}^\varepsilon)-(\partial_z\mathbf{u}^0,\partial_z\omega^0,\partial_z\mathbf{B}^0)\|_{m_0}\cdot\|(\mathbf{u}^\varepsilon,\omega^\varepsilon,\mathbf{B}^\varepsilon)-(\mathbf{u}^0,\omega^0,\mathbf{B}^0)\|_{m_0}\\ &+\|(\mathbf{u}^\varepsilon,\omega^\varepsilon,\mathbf{B}^\varepsilon)-(\mathbf{u}^0,\omega^0,\mathbf{B}^0)\|_{m_0}^2)\to 0,\end{aligned}\tag{7.16}$$

with $m_0 > 1$ and $\varepsilon \to 0$, and it is easy to know that $(\mathbf{u}^0, \omega^0, \mathbf{B}^0)$ is a weak solution to the corresponding limiting magneto-micropolar fluid Eqs (1.10)–(1.12). Thus we have completed the proof of Theorem 1.1.

Author contributions

Lingqi Liu: Conceptualization, Methodology, Formal analysis, Writing Original draft, Software; Limei Li: Conceptualization, Supervision, Writing review & editing, Project administration; Yuanming Xu: Software, Formal analysis, Writing review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of Interest

The authors declare that they have no conflict of interest.

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