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Research article

Caputo fractional curvature of curves in the Lorentzian plane

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Abstract: We present a new concept of fractional curvature invariant for regular curves in the Lorentz plane by generalizing the Caputo-fractional curvature from Euclidean geometry to the pseudo-Riemannian setting. Our construction projects the integer-order derivative of the Caputo vector of fractional-order derivatives onto the Lorentzian normal direction, yielding a curvature measure that naturally distinguishes timelike and spacelike curves. Explicit formulas for representative model curves are derived, and we illustrate how the Lorentzian metric signature fundamentally changes fractional curvature behavior. This framework extends fractional-order geometric analysis into relativity, providing new tools for studying memory effects and nonlocal dynamics along curves in relativistic contexts.

Keywords: fractional derivative; Caputo fractional derivative; Lorentz plane; timelike and spacelike curves
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1. Introduction

Fractional calculus extends the notions of differentiation and integration to arbitrary (non-integer) orders, offering a unified framework that naturally incorporates memory and hereditary properties into mathematical models. Over the past few decades, it has proved indispensable for describing anomalous diffusion, power-law relaxation in viscoelastic materials, and a wealth of nonlocal and complex dynamical behaviors across physics, engineering, and biology. By permitting operators of fractional order, one gains access to tools that capture long-range temporal correlations and spatial heterogeneity, thereby transcending the limitations of classical integer-order methods.

Among the various definitions of fractional derivatives, the classical Riemann–Liouville operator, constructed as a convolution-type integral transform, has been particularly influential. Oldham and Spanier systematically developed its rich mathematical structure and wide range of applications in their monograph, which remains a cornerstone reference in the field [32]. However, despite its theoretical appeal, the Riemann–Liouville formulation can complicate the imposition of physically meaningful initial and boundary conditions in differential equations, motivating the search for alternative definitions that better align with experimental and engineering requirements.

To address these limitations, Caputo introduced the modified fractional derivative in 1967 [16]. The Caputo derivative allows standard initial conditions identical in form to integer-order equations, while preserving the essential memory effects characteristic of fractional-order dynamics. Originally proposed in a geophysical context, Caputo's definition accurately models seismic wave attenuation with frequency-independent quality factors. Caputo further demonstrated applications to linear viscoelasticity in his 1971 monograph, where his fractional derivative captured both elastic and dissipative behaviors [17].

By the late 20th century, fractional calculus had matured significantly as a mathematical discipline. Podlubny provided a comprehensive treatment of both Riemann–Liouville and Caputo derivatives, presenting a unified analytical and numerical framework [33]. Complementarily, the detailed trilogy by Samko et al. rigorously explored fractional integrals and derivatives, highlighting their theoretical foundations and wide-ranging applications [36]. Further important contributions were compiled under Hilfer's editorship, addressing fractional calculus applications in physics [22]. More recent advances, such as those presented by Mainardi, underscore the significance of Caputo derivative in modeling wave phenomena and stress relaxation in viscoelastic materials [29].

Li [23] investigated exact optical solutions of the nonlinear Kodama equation involving the M-truncated derivative using the extended (G'/G)-expansion method. In another recent study, Li and Hussain [24] provided qualitative analysis and traveling wave solutions for a generalized (3+1)-dimensional nonlinear Konopelchenko–Dubrovsky–Kaup–Kupershmidt system. Recent studies have addressed the synchronization and stability of fractional-order neural and diffusion systems using advanced control techniques. For instance, projective synchronization for uncertain fractional reaction-diffusion systems was achieved via adaptive sliding mode control under a finite-time scheme [25]. Mittag-Leffler's synchronization of delayed fractional memristor neural networks has been established through adaptive control strategies [26]. Furthermore, the global Mittag-Leffler stability of delayed fractional-coupled reaction-diffusion systems on weakly connected networks has been investigated [15]. Sliding mode control methods have also been employed to achieve global Mittag-Leffler's synchronization in delayed fractional Cohen-Grossberg neural networks [27]. Recent advances have provided deeper insights into the geometric and physical meaning of fractional derivatives, including their relation to function classes and dimensional interpretation in complex systems [3,30]. Recent studies have highlighted the relevance of fractal geometry in characterizing the mechanical behavior of recycled aggregate concretes and porous media. For instance, a fractal-based framework for evaluating the mechanical performance of recycled aggregate concrete was introduced in [20], while the impact of fractal dimensions on the strength of porous concrete was investigated in [21].

Fractional calculus applications span diverse fields. Euler–Lagrange equations for fractional variational problems were comprehensively studied by Agrawal [1], while generalized Hamilton's principles involving fractional derivatives were analyzed by Atanackovic et al. [2]. Differential geometry incorporating fractional calculus has been explored by Aydın et al. in the Euclidean context, with further studies extending into affine geometry [5, 7]. In a recent contribution, Aydın and Kaya [6] introduced fractional equiaffine curvatures for curves in three-dimensional affine space. Moreover, Aydın [4] examined the effect of local fractional derivatives on the Riemann curvature tensor. Fundamental contributions to fractional-order state equations in viscoelasticity and control theory were made by Bagley and Torvik [10–13]. Additionally, fractional-order chaotic dynamics and feedback control systems were investigated by several researchers [9,18,19]. Fractional generalizations of gradient and Hamiltonian systems were proposed by Tarasov [37, 38], and foundational linear infinitesimal operators were studied by Volterra and Hostinsky [39]. Furthermore, fractional dynamical systems' geometric structures in non-Riemannian spaces were explored by Yajima and Nagahama [40].

Motivated by these theoretical developments, we extend the concept of fractional curvature to the Lorentzian geometric setting. The Lorentz plane $\mathbb{L}^2$, endowed with the metric admits timelike, spacelike, and lightlike curves, whose intrinsic geometry differs significantly from the Euclidean case. Some foundational studies are the Lorentzian geometry establishment of the Frenet–Serret apparatus for nondegenerate Lorentzian curves [35] and O'Neill's systematic development of semi-Riemannian submanifold theory [31]. These works provide the essential background for defining curvature, torsion, and normal vectors in Lorentzian geometry.

Building upon these Lorentzian foundations, we introduce the Caputo fractional curvature for regular timelike or spacelike plane curves $\gamma : I \to \mathbb{L}^2$. Specifically, this fractional curvature is obtained by projecting the integer-order derivative of the Caputo vector of fractional-order derivatives onto the Lorentzian normal direction. We demonstrate that this new curvature definition is invariant under Lorentzian isometries (boosts and reflections), confirming its intrinsic nature within $\mathbb{L}^2$ geometry. This construction generalizes the fractional curvature concept recently proposed by Franco Rubio López and Obidio Rubio [34], adapting their Euclidean projection method to accommodate the Lorentzian metric signature, thereby yielding novel fractional invariants in pseudo-Riemannian geometry.

2. Preliminaries

2.1. Geometry of the Lorentzian plane

Let $\mathbb{L}^2$ denote the two-dimensional Lorentzian plane with coordinates (x, y) and metric

$$\langle (x_1, y_1), (x_2, y_2) \rangle_L = x_1 x_2 \; - \; y_1 y_2.$$

A regular curve $\gamma\colon I \to \mathbb{L}^2$, parametrized by proper arc-length s, satisfies

$$\langle \gamma'(s), \gamma'(s) \rangle_L = \varepsilon \in \{+1, -1\},$$

where $\varepsilon = +1$ for spacelike and $\varepsilon = -1$ for timelike curves [35]. Moreover, if

$$\langle \gamma'(s), \gamma'(s) \rangle_L = 0,$$

then γ is called a null (lightlike) curve. The unit tangent vector is then

$$\mathcal{T}(s) = \gamma'(s),$$

normalized so that $\langle \mathcal{T}, \mathcal{T} \rangle_L = \varepsilon$. The Lorentzian normal $\mathcal{N}(s)$ is defined uniquely (up to orientation) by

$$\langle \mathcal{N}, \mathcal{N} \rangle_L = -\varepsilon, \qquad \langle \mathcal{T}, \mathcal{N} \rangle_L = 0,$$

and the pair $\{\mathcal{T}, \mathcal{N}\}$ forms a pseudo-orthonormal Frenet frame along γ [31].

The curvature $\kappa(s)$ of γ in the Lorentz plane measures the rate of change of the frame and is given by the Frenet–Serret relations

$$\mathcal{T}'(s) = \kappa(s)\,\mathcal{N}(s),$$
$$\mathcal{N}'(s) = \kappa(s)\,\mathcal{T}(s).$$

Equivalently, one computes

$$\kappa(s) = \varepsilon\,\langle \mathcal{T}'(s), \mathcal{N}(s) \rangle_L,$$

which incorporates the metric signature via ε and yields an intrinsic invariant under Lorentzian isometries [31, 35].

Throughout this work, the curve γ is assumed to be non-null; that is, for $\epsilon = \pm 1$, it is regarded as a spacelike or timelike curve.

2.2. *Definition of the Caputo fractional-order derivative*

Definition 1. *Consider $f\colon [a,b] \to \mathbb{R}$ as a function with a continuous derivative. The Caputo fractional λ-order derivative is defined by*

$$\mathcal{D}^{\lambda,C}_{0+} f(t) = \frac{1}{\Gamma(1-\lambda)} \int_a^t \frac{f'(u)}{(t-u)^{\lambda}} \mathrm{d}u, \tag{2.1}$$

where $\lambda \in (0,1)$ and $\Gamma(\cdot)$ is the Gamma function [14, 16].

A key property of this operator is its limit to the ordinary derivative as $\lambda \to 1^-$, namely

$$\lim_{\lambda \to 1^-} \mathcal{D}^{\lambda,C}_{0+} f(t) = f'(t), \quad t \in [a,b], \tag{2.2}$$

so that when λ approaches 1, the new derivative effectively reproduces the classical first derivative.

3. Curves parametrized by arc length using the Caputo derivative

Definition 2. *Assume $\gamma(s)$ is an arc length-parametrized curve in $\mathbb{L}^2$. The vector of fractional λ-order derivatives of γ at s is given by*

$$\mathcal{D}^{\lambda,C}_{0+} \gamma(s) = \left(\mathcal{D}^{\lambda,C}_{0+} \gamma_1(s),\ \mathcal{D}^{\lambda,C}_{0+} \gamma_2(s) \right). \tag{3.1}$$

Thus, we can write

$$\mathcal{D}^{\lambda,C}_{0+} \gamma(s) = \xi_\lambda(s)\,\mathcal{T}(s) + \eta_\lambda(s)\,\mathcal{N}(s), \tag{3.2}$$

for all $s \in [a,b]$.

In what follows, the curve

$$\gamma(s) = (\gamma_1(s),\ \gamma_2(s))$$

will be simply denoted by $\gamma(s)$. Similarly, the curve

$$\gamma(t) = (\gamma_1(t),\ \gamma_2(t))$$

will be simply denoted by $\gamma(t)$.

Definition 3. *Assume $\gamma(s)$ is an arc length-parametrized curve in $\mathbb{L}^2$, and $\mathcal{D}_{0+}^{\lambda,C}\gamma(s)$ is the vector of fractional λ-order derivatives of the curve γ at the point $\gamma(s)$. Then, the Caputo fractional curvature is defined by*

$$\kappa_\lambda(s) = -\epsilon[\xi_\lambda(s)\kappa(s) + \frac{d}{ds}\eta_\lambda(s)], s \in [a, b] \tag{3.3}$$

where $\kappa(s)$ indicates the curvature of integer type for the curve γ evaluated at s.

Theorem 1. *Let $\gamma(s)$ be a unit-speed parametrized curve in $\mathbb{L}^2$. Then,*

$$\text{Proj}_{\mathcal{N}(s)}\frac{d}{ds}\left(\mathcal{D}_{0+}^{\lambda,C}\gamma(s)\right) = \kappa_\lambda(s)\,\mathcal{N}(s), \tag{3.4}$$

$$\begin{aligned}\kappa_\lambda(s) &= -\epsilon\langle \text{Proj}_{\mathcal{N}(s)}\frac{d}{ds}\left(\mathcal{D}_{0+}^{\lambda,C}\gamma(s)\right), \mathcal{N}(s)\rangle_L \\ &= -\epsilon\langle\frac{d}{ds}\left(\mathcal{D}_{0+}^{\lambda,C}\gamma(s)\right), \mathcal{N}(s)\rangle_L.\end{aligned} \tag{3.5}$$

Proof. If we take the integer derivative of (3.2), we obtain

$$\frac{d}{ds}\left(\mathcal{D}_{0+}^{\lambda,C}\gamma(s)\right) = \frac{d}{ds}(\xi_\lambda(s))\mathcal{T}(s) + \xi_\lambda(s)\mathcal{T}'(s) + \frac{d}{ds}(\eta_\lambda(s)\mathcal{N}(s) + \eta_\lambda(s)\mathcal{N}'(s)). \tag{3.6}$$

Considering the equations (Frenet frame for $\mathbb{L}^2$) in (3.6), we get

$$\begin{aligned}\frac{d}{ds}\left(\mathcal{D}_{0+}^{\lambda,C}\gamma(s)\right) &= \frac{d}{ds}(\xi_\lambda(s))\mathcal{T}(s) + \xi_\lambda(s)\kappa(s)\mathcal{N}(s) \\ &+ \frac{d}{ds}(\eta_\lambda(s))\mathcal{N}(s) + \eta_\lambda(s)\kappa(s)\mathcal{T}(s) \\ &= \left[\frac{d}{ds}(\xi_\lambda(s)) + \eta_\lambda(s)\kappa(s)\right]\mathcal{T}(s) \\ &+ \left[\frac{d}{ds}(\eta_\lambda(s)) + \xi_\lambda(s)\kappa(s)\right]\mathcal{N}(s).\end{aligned} \tag{3.7}$$

Then, the orthogonal projection is given by

$$\text{Proj}_{\mathcal{N}(s)}\frac{d}{ds}\left(\mathcal{D}_{0+}^{\lambda,C}\gamma(s)\right) = \kappa_\lambda(s)\,\mathcal{N}(s),$$

$$\begin{aligned}\kappa_\lambda(s) &= -\epsilon\langle \text{Proj}_{\mathcal{N}(s)}\frac{d}{ds}\left(\mathcal{D}_{0+}^{\lambda,C}\gamma(s)\right), \mathcal{N}(s)\rangle_L \\ &= -\epsilon\langle\frac{d}{ds}\left(\mathcal{D}_{0+}^{\lambda,C}\gamma(s)\right), \mathcal{N}(s)\rangle_L.\end{aligned}$$

From (3.7), we define the function $\bar{\kappa}_\lambda$ by

$$\bar{\kappa}_\lambda = \epsilon[\frac{d}{ds}(\xi_\lambda(s)) + \eta_\lambda(s)\kappa(s)], \quad s \in [a, b]. \tag{3.8}$$

□

Theorem 2. *Let* $\gamma\colon [a,b] \subset \mathbb{R} \to \mathbb{L}^2$ *be a regular curve with unit-speed parametrization. Then*

$$\lim_{\lambda\to 1} \xi_\lambda(s) = 1, \tag{3.9}$$

$$\lim_{\lambda\to 1} \eta_\lambda(s) = 0, \tag{3.10}$$

where $s \in [a,b]$.

Proof. From (2.2), for all $t \in [a,b]$

$$\lim_{\lambda\to 1} \mathcal{D}^{\lambda,C}_{0+} f(t) = f'(t),$$

$$\lim_{\lambda\to 1} \mathcal{D}^{\lambda,C}_{0+} \gamma(s) = \gamma'(s) = \mathcal{T}(s), \tag{3.11}$$

and by Eq (3.2), we have Eqs (3.9) and (3.10). □

Theorem 3. *Take* $\gamma\colon [a,b] \subset \mathbb{R} \to \mathbb{L}^2$ *to be a regular, unit-speed curve. Then,*

$$\xi_\lambda(s) = {\gamma_1}'(s)\mathcal{D}^{\lambda,C}_{0+}\gamma_1(s) - {\gamma_2}'(s)\mathcal{D}^{\lambda,C}_{0+}\gamma_2(s), \tag{3.12}$$

$$\eta_\lambda(s) = {\gamma_1}'(s)\mathcal{D}^{\lambda,C}_{0+}\gamma_2(s) - {\gamma_2}'(s)\mathcal{D}^{\lambda,C}_{0+}\gamma_1(s). \tag{3.13}$$

Proof. If we consider Eqs (3.1) and (3.2), we obtain

$$\begin{aligned}
\mathcal{D}^{\lambda,C}_{0+}\gamma(s) &= \left(\mathcal{D}^{\lambda,C}_{0+}\gamma_1(s),\ \mathcal{D}^{\lambda,C}_{0+}\gamma_2(s)\right) \\
&= \xi_\lambda(s)\mathcal{T}(s) + \eta_\lambda(s)\mathcal{N}(s) \\
&= \xi_\lambda(s)({\gamma_1}'(s),\ {\gamma_2}'(s)) + \eta_\lambda(s)({\gamma_2}'(s),\ {\gamma_1}'(s)) \\
&= \left(\xi_\lambda(s){\gamma_1}'(s) + \eta_\lambda(s){\gamma_2}'(s),\ \xi_\lambda(s){\gamma_2}'(s) + \eta_\lambda(s){\gamma_1}'(s)\right).
\end{aligned} \tag{3.14}$$

Consequently, we get

$$\begin{aligned}
\mathcal{D}^{\lambda,C}_{0+}\gamma_1(s) &= \xi_\lambda(s)\,{\gamma_1}'(s) + \eta_\lambda(s)\,{\gamma_2}'(s), \\
\mathcal{D}^{\lambda,C}_{0+}\gamma_2(s) &= \xi_\lambda(s)\,{\gamma_2}'(s) + \eta_\lambda(s)\,{\gamma_1}'(s).
\end{aligned} \tag{3.15}$$

Then, the coefficient determinant of the system (3.15) is non-zero, that is

$$\begin{vmatrix} {\gamma_1}'(s) & {\gamma_2}'(s) \\ {\gamma_2}'(s) & {\gamma_1}'(s) \end{vmatrix} = ({\gamma_1}'(s))^2 - ({\gamma_2}'(s))^2 = -\epsilon \neq 0, \quad s \in [a,b].$$

If we solve system (3.15), we get (3.12) and (3.13). □

Theorem 4. *Suppose* $\gamma(s)$ *is a unit-speed curve in the Lorentzian plane* $\mathbb{L}^2$*. Then, for* $\forall\ s \in [a,b]$

$$\lim_{\lambda\to 1} \kappa_\lambda(s) = \kappa(s). \tag{3.16}$$

Proof. From Eq (3.3), we get

$$\kappa_\lambda(s) = -\epsilon\left(\xi_\lambda(s)\kappa(s) + \frac{d}{ds}\eta_\lambda(s)\right), \quad s \in [a,b]$$

and by Eqs (3.9) and (3.10), we obtain

$$\lim_{\lambda\to 1}\kappa_\lambda(s) = -\epsilon\kappa(s).$$

This concludes the proof. □

Theorem 5. *Assume* $\gamma\colon [a,b] \subset \mathbb{R} \to \mathbb{L}^2$ *is regularly parametrized by arc length. Then,*

$$\kappa_\lambda(s) = -\epsilon[-\gamma_2{}'(s)\frac{d}{ds}\left(\mathcal{D}^{\lambda,C}_{0+}\gamma_1(s)\right) + \gamma_1{}'(s)\frac{d}{ds}\left(\mathcal{D}^{\lambda,C}_{0+}\gamma_2(s)\right)]. \tag{3.17}$$

Proof. If we consider Eq (3.1), we have

$$\frac{d}{ds}\left(\mathcal{D}^{\lambda,C}_{0+}\gamma(s)\right) = \left(\frac{d}{ds}\left(\mathcal{D}^{\lambda,C}_{0+}\gamma_1(s)\right),\ \frac{d}{ds}\left(\mathcal{D}^{\lambda,C}_{0+}\gamma_2(s)\right)\right).$$

From (3.5), we know that

$$\begin{aligned}\kappa_\lambda(s) &= -\epsilon\langle\frac{d}{ds}\left(\mathcal{D}^{\lambda,C}_{0+}\gamma(s)\right), \mathcal{N}(s)\rangle_L \\ &= -\epsilon\langle\frac{d}{ds}\left(\mathcal{D}^{\lambda,C}_{0+}\gamma(s)\right), (\gamma_2{}'(s), \gamma_1{}'(s))\rangle_L.\end{aligned}$$

Thus, we get

$$\kappa_\lambda(s) = -\epsilon\left[-\gamma_2{}'(s)\frac{d}{ds}\left(\mathcal{D}^{\lambda,C}_{0+}\gamma_1(s)\right) + \gamma_1{}'(s)\frac{d}{ds}\left(\mathcal{D}^{\lambda,C}_{0+}\gamma_2(s)\right)\right].$$

□

Theorem 6. *Assume* $\gamma\colon [a,b] \subset \mathbb{R} \to \mathbb{L}^2$ *is regularly parametrized by arc length. Then,*

$$\left\|\frac{d}{ds}\left(\mathcal{D}^{\lambda,C}_{0+}\gamma(s)\right)\right\|^2 = \epsilon[(\bar{\kappa}_\lambda)^2 - (\kappa_\lambda)^2]. \tag{3.18}$$

Proof. If we consider Eqs (3.3), (3.7), and (3.8), we have Eq (3.18), which completes the poof. □

Theorem 7. *Assume* $\gamma\colon [a,b] \subset \mathbb{R} \to \mathbb{L}^2$ *is regularly parametrized by arc length. Then,*

$$\frac{d}{ds}\left(\left\|\mathcal{D}^{\lambda,C}_{0+}\gamma(s)\right\|^2\right) = 2\epsilon[\xi_\lambda(s)\bar{\kappa}_\lambda(s) + \eta_\lambda(s)\kappa_\lambda(s)]. \tag{3.19}$$

Proof. If we multiply (3.3) by $\eta_\lambda(s)$ and (3.12) by $\xi_\lambda(s)$ and add these equations, we get

$$\eta_\lambda(s)\kappa_\lambda(s) = -\epsilon[\xi_\lambda(s)\kappa_\lambda(s) + \frac{d}{ds}\eta_\lambda(s)]\eta_\lambda(s),$$

$$\xi_\lambda(s)\bar{\kappa}_\lambda(s) = \epsilon[\frac{d}{ds}(\xi_\lambda(s)) + \eta_\lambda(s)\kappa(s)]\xi_\lambda(s).$$

From the last equations, we obtain the following equality:

$$\begin{aligned}\xi_\lambda(s)\bar{\kappa}_\lambda(s) + \eta_\lambda(s)\kappa_\lambda(s) &= \epsilon\left[\xi_\lambda(s)\frac{d}{ds}\xi_\lambda(s) - \eta_\lambda(s)\frac{d}{ds}\eta_\lambda(s)\right] \\ &= \frac{1}{2}\epsilon\frac{d}{ds}\left[(\xi_\lambda(s))^2 - (\eta_\lambda(s))^2\right] \\ &= \frac{1}{2}\epsilon\frac{d}{ds}\left(\left\|\mathcal{D}_{0+}^{\lambda,C}\gamma(s)\right\|^2\right).\end{aligned}$$

Thus, we obtain

$$\frac{d}{ds}\left(\left\|\mathcal{D}_{0+}^{\lambda,C}\gamma(s)\right\|^2\right) = 2\epsilon\left[\xi_\lambda(s)\bar{\kappa}_\lambda(s) + \eta_\lambda(s)\kappa_\lambda(s)\right],$$

which completes the proof. □

Theorem 8. *Assume $\gamma\colon [a,b] \subset \mathbb{R} \to \mathbb{L}^2$ is regularly parametrized by arc length. Accordingly, one can choose $f, g\colon [a,b] \to \mathbb{R}$ satisfying*

$$f(s)\kappa_\lambda(s) = \kappa(s), \tag{3.20}$$

$$g(s)\bar{\kappa}_\lambda(s) = -\kappa(s), \tag{3.21}$$

where $s \in [a,b]$.

Proof. If we consider Eqs (3.3), (3.7), and (3.8), we have

$$\frac{d}{ds}(\mathcal{D}_{0+}^{\lambda,C}(s)) = \epsilon[\bar{\kappa}_\lambda(s)\mathcal{T}(s) - \kappa_\lambda(s)\mathcal{N}(s)]$$

for all $s \in [a,b]$.

Then, $\mathrm{Proj}_{\mathcal{N}(s)}(\frac{d}{ds}(\mathcal{D}_{0+}^{\lambda,C}(s)))$ is parallel to $\mathcal{T}'(s)$, and $\kappa_\lambda(s)\,\mathcal{N}(s)$ is parallel to $\mathcal{T}'(s)$. Then, there is a function $f\colon [a,b] \to \mathbb{R}$ such that

$$f(s)\,\kappa_\lambda(s) = \kappa(s) \quad \forall\, s \in [a,b].$$

Similarly, then $\mathrm{Proj}_{\mathcal{T}(s)}(\frac{d}{ds}(\mathcal{D}_{0+}^{\lambda,C}(s)))$ is parallel to $\mathcal{N}'(s)$, and $\kappa_\lambda(s)\,\mathcal{T}(s)$ is parallel to $\mathcal{N}'(s)$. Then, there is a function $g\colon [a,b] \to \mathbb{R}$ such that

$$g(s)\,\kappa_\lambda(s) = -\kappa(s) \quad \forall\, s \in [a,b].$$

Therefore, the proof is concluded. □

Corollary 1. *Assume $\gamma\colon [a,b] \subset \mathbb{R} \to \mathbb{L}^2$ is a regular curve parametrized by its arc length. Hence, one can find mappings $f, g\colon [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ satisfying*

$$\kappa(s) = \frac{1}{2}\left[f(s)\kappa_\lambda(s) - g(s)\bar{\kappa}_\lambda(s)\right], \tag{3.22}$$

$$f(s)\kappa_\lambda(s) + g(s)\bar{\kappa}_\lambda(s) = 0, \tag{3.23}$$

where $s \in [a,b]$ and $\lambda \in [0,1]$

Proof. The proof is trivial from (3.20) and (3.21). □

Theorem 9. *Assume $\gamma\colon [a,b] \subset \mathbb{R} \to \mathbb{L}^2$ is a regular curve parametrized by its arc length. Hence,*

$$\kappa_\lambda(s) = 0 \iff \kappa(s) = 0 \quad \forall\, s \in [a,b]. \tag{3.24}$$

Proof. Suppose that $\kappa_\lambda(s) = 0$ for all $s \in [a,b]$, for all $\lambda \in [0,1]$. Hence by (3.20), we write

$$f(s)\kappa_\lambda(s) = \kappa(s) = 0 \quad \text{for all } s \in [a,b].$$

Conversely, assume that $\kappa(s) = 0$ for every $s \in [a,b]$. Applying the Frenet formulas, we obtain

$$\gamma(s) = (x_0 + sv_1,\ y_0 + sv_2 = (\gamma_1(s), \gamma_2(s)),$$

where $v = (v_1, v_2) \in \mathbb{R}^2$, $\|v\| = 1$. Taking the integer-order derivative of the coordinate functions, we obtain

$$\gamma_1{}' = v_1, \quad \gamma_2{}' = v_2, \quad \text{for all } s \in [a,b]. \tag{3.25}$$

We find the Caputo derivative

$$\mathcal{D}^{\lambda,C}_{0+}\gamma_1(s) = \frac{s^{1-\lambda}}{\Gamma(2-\lambda)}v_1, \quad \mathcal{D}^{\lambda,C}_{0+}\gamma_2(s) = \frac{s^{1-\lambda}}{\Gamma(2-\lambda)}v_2. \tag{3.26}$$

If we use Eqs (3.25) and (3.26), we obtain

$$\begin{aligned}\xi_\lambda(s) &= \gamma_1{}'(s)\mathcal{D}^{\lambda,C}_{0+}\gamma_1(s) - \gamma_2{}'(s)\mathcal{D}^{\lambda,C}_{0+}\gamma_2(s) = -\frac{s^{1-\lambda}}{\Gamma(2-\lambda)},\\ \eta_\lambda(s) &= \gamma_1{}'(s)\mathcal{D}^{\lambda,C}_{0+}\gamma_2(s) - \gamma_2{}'(s)\mathcal{D}^{\lambda,C}_{0+}\gamma_1(s) = 0.\end{aligned}$$

Substituting (3.25) and (3.26) into (3.3), $\kappa_\lambda(s)$ vanishes on $[a,b]$ for any $\lambda \in [0,1]$.

□

Theorem 10. *Consider $\gamma(s)$ as a unit-speed parametrized curve in $\mathbb{L}^2$. Let $F\colon \mathbb{L}^2 \to \mathbb{L}^2$ denote an isometric mapping, and define $\Gamma = F \circ \gamma : [a,b] \subset \mathbb{R} \to \mathbb{L}^2$. Hence, we write*

$$\mathcal{D}^{\lambda,C}_{0+}\Gamma(s) = dF_{\gamma(s)}\left(\mathcal{D}^{\lambda,C}_{0+}\gamma(s)\right), \tag{3.27}$$

$$\kappa^\Gamma_\lambda(s) = \kappa^\gamma_\lambda(s). \tag{3.28}$$

Proof. Let $\{\mathcal{T}_\gamma(s), \mathcal{N}_\gamma(s)\}$ and $\{\mathcal{T}_\Gamma(s), \mathcal{N}_\Gamma(s)\}$ be the Frenet–Serret apparatus for γ and Γ, respectively. Given that F is an isometry, we obtain

$$\mathcal{T}_\Gamma(s) = dF_{\gamma(s)}\left(\mathcal{T}_\gamma(s)\right), \tag{3.29}$$

$$\mathcal{N}_\Gamma(s) = dF_{\gamma(s)}\left(\mathcal{N}_\gamma(s)\right), \tag{3.30}$$

$$\kappa_\Gamma(s) = \kappa_\gamma(s), \tag{3.31}$$

where $\kappa_\Gamma(s)$ and $\kappa_\gamma(s)$ indicate the integer-order curvature functions of the curves γ and Γ, respectively. Moreover, we obtain,

$$\mathcal{D}_{0+}^{\lambda,C}\gamma(s) = \xi_\lambda(s)\mathcal{T}_\gamma(s) + \eta_\lambda(s)\mathcal{N}_\gamma(s),$$
$$\mathcal{D}_{0+}^{\lambda,C}\Gamma(s) = \xi_\lambda^1(s)\mathcal{T}_\Gamma(s) + \eta_\lambda^1(s)\mathcal{N}_\Gamma(s).$$

As F is an isometry, it decomposes into a translation T_q followed by an orthogonal transformation C, i.e.,

$$F = T_q \circ C. \tag{3.32}$$

From (3.32), we obtain

$$\begin{aligned}\Gamma(s) &= (T_p \circ \gamma)(s) = (T_p \circ C)(\gamma(s))\\ &= p + C(\gamma(s)) = p + C\left(\gamma_1(s)(\mathbf{e}_1) + \gamma_2(s)(\mathbf{e}_2)\right)\\ &= p + \gamma_1(s)C(\mathbf{e}_1) + \gamma_2(s)C(\mathbf{e}_2).\end{aligned}$$

Hence, the Caputo derivative becomes

$$\begin{aligned}\mathcal{D}_{0+}^{\lambda,C}\Gamma(s) &= \mathcal{D}_{0+}^{\lambda,C}(p) + \left(\mathcal{D}_{0+}^{\lambda,C}\gamma_1(s)\right)\cdot C(\mathbf{e}_1) + \left(\mathcal{D}_{0+}^{\lambda,C}\gamma_2(s)\right)\cdot C(\mathbf{e}_2)\\ &= C\left(\mathcal{D}_{0+}^{\lambda,C}\gamma_1(s)\mathbf{e}_1 + \mathcal{D}_{0+}^{\lambda,C}\gamma_2(s)\mathbf{e}_2\right)\\ &= C\left(\mathcal{D}_{0+}^{\lambda,C}\gamma(s)\right).\end{aligned}$$

Hence, we write

$$\mathcal{D}_{0+}^{\lambda,C}\Gamma(s) = C(\mathcal{D}_{0+}^{\lambda,C}\gamma(s)) = dF_{\gamma(s)}\left(\mathcal{D}_{0+}^{\lambda,C}\gamma(s)\right).$$

Also by (3.29) and (3.30), we have

$$\begin{aligned}\xi^1(s)\mathcal{T}_\Gamma(s) + \eta^1(s)\mathcal{N}_\Gamma(s) &= \mathcal{D}_{0+}^{\lambda,C}\Gamma(s)\\ &= dF_{\gamma(s)}(\xi_\lambda(s)\mathcal{T}_\gamma(s) + \eta_\lambda(s)\mathcal{N}_\gamma(s))\\ &= \xi_\lambda(s)dF_{\gamma(s)}(\mathcal{T}_\gamma(s)) + \eta_\lambda(s)dF_{\gamma(s)}(\mathcal{N}_\gamma(s))\\ &= \xi_\lambda(s)\mathcal{T}_\Gamma(s) + \eta_\lambda(s)\mathcal{N}_\Gamma(s).\end{aligned}$$

Thus, we obtain

$$\xi_\lambda^1(s) = \xi_\lambda(s),$$
$$\eta_\lambda^1(s) = \eta_\lambda(s).$$

Therefore, we have

$$\begin{aligned}\kappa_\lambda^\Gamma(s) &= \xi_\lambda^1(s)\kappa_\gamma(s) + \frac{d}{ds}(\eta_\lambda^1(s))\\ &= \xi_\lambda(s)\kappa_\gamma(s) + \frac{d}{ds}\xi(\eta_\lambda(s))\\ &= \kappa_\gamma(s).\end{aligned}$$

□

Thus, the proof is completed.

4. Fractional curvature of Lorentzian curves under arbitrary reparametrization

Suppose $\gamma\colon I \subset \mathbb{R} \to \mathbb{L}^2$ is a regular curve parametrized by an arbitrary parameter t, $\{\mathcal{T}(t), \mathcal{N}(t)\}$ are the tangent and normal vectors (Frenet–Serret frame) of γ at t, where

$$\mathcal{T}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|},$$

$$\mathcal{N}(t) = \frac{(\gamma_2{}'(t), \gamma_1{}'(t))}{\|\gamma'(t)\|},$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Also, we have

$$\mathcal{T}'(t) = \|\gamma'(t)\|\kappa(t)\mathcal{N}(t), \tag{4.1}$$

$$\mathcal{N}'(t) = \|\gamma'(t)\|\kappa(t)\mathcal{T}(t), \tag{4.2}$$

where

$$\kappa(t) = \frac{\epsilon(\gamma_1{}''\gamma_2{}' - \gamma_1{}'\gamma_2{}'')}{\|\gamma'(t)\|^3}. \tag{4.3}$$

Thus, we get

$$\mathcal{D}^{\lambda,C}_{0+}\gamma(t) = \xi_\lambda(t)\mathcal{T}(t) + \eta_\lambda(t)\mathcal{N}(t). \tag{4.4}$$

We know that,

$$\lim_{\lambda\to 1} \mathcal{D}^{\lambda,C}_{0+} f(t) = f'(t) \tag{4.5}$$

for any $t \in [a, b]$. For λ sufficiently near 1, the Caputo fractional derivative closely reproduces the properties of the integer derivative. It follows from (4.5) that we have the next theorem.

Theorem 11. *Suppose $\gamma\colon I \subset \mathbb{R} \to \mathbb{L}^2$ is a regular curve parametrized by an arbitrary parameter t. Then*

$$\lim_{\lambda\to 1} \xi_\lambda(t) = \|\gamma'(t)\|, \tag{4.6}$$

$$\lim_{\lambda\to 1} \eta_\lambda(t) = 0 \tag{4.7}$$

where $t \in I$.

Proof. The proof is trivial.

If we take integer derivative of the Eq (4.4), we obtain

$$\frac{d}{dt}(\mathcal{D}^{\lambda,C}_{0+}\gamma(t)) = \frac{d}{dt}(\xi_\lambda(t))\mathcal{T}(t) + \xi_\lambda(t)\mathcal{T}'(t) + \frac{d}{dt}(\eta_\lambda(t))\mathcal{N}(t) + \eta_\lambda(t)\mathcal{N}'(t). \tag{4.8}$$

Using Eqs (4.1) and (4.2) in (4.8), we get

$$\frac{d}{dt}(\mathcal{D}_{0+}^{\lambda,C}\gamma(t)) = [\frac{d}{dt}(\xi_\lambda(t)) + \eta_\lambda(t)||\gamma'(t)||\kappa(t)]\mathcal{T}(t) + [\xi_\lambda(t)||\gamma'(t)||\kappa(t) + \frac{d}{dt}\eta_\lambda(t)]\mathcal{N}(t). \tag{4.9}$$

From (4.9), we define the function $\kappa_1^\lambda : I \to \mathbb{R}$, by

$$\kappa_1^\lambda(t) = -\epsilon[\xi_\lambda(t)||\gamma'(t)||\kappa(t) + \frac{d}{dt}\eta_\lambda(t)]. \tag{4.10}$$

Thus, we have

$$\kappa_1^\lambda(t) = -\epsilon\langle\frac{d}{dt}(\mathcal{D}_{0+}^{\lambda,C}\gamma(t)), \mathcal{N}(t)\rangle_L$$

for all $t \in I$. □

Theorem 12. *Suppose $\gamma\colon I \subset \mathbb{R} \to \mathbb{L}^2$ is a regular curve parametrized by an arbitrary parameter t. Then*

$$\kappa(t) = -\epsilon \lim_{\lambda\to 1}\frac{\kappa_1^\lambda(t)}{||\gamma'(t)||^2} \tag{4.11}$$

for all $t \in I$.

Proof. From (4.10), we have

$$\kappa_1^\lambda(t) = -\epsilon[\xi_\lambda(t)||\gamma'(t)||\kappa(t) + \frac{d}{dt}\eta_\lambda(t)].$$

Then, finding the limit

$$\lim_{\lambda\to 1}\kappa_1^\lambda(t) = -\epsilon[\lim_{\lambda\to 1}\xi_\lambda(t)]||\gamma'(t)||\kappa(t) + \lim_{\lambda\to 1}\frac{d}{dt}(\eta_\lambda(t))].$$

From (4.6) and (4.7), we obtain

$$\lim_{\lambda\to 1}\kappa_1^\lambda(t) = -\epsilon||\gamma'(t)||^2\kappa(t).$$

Thus,

$$\kappa(t) = -\epsilon \lim_{\lambda\to 1}\frac{\kappa_1^\lambda(t)}{||\gamma'(t)||^2}$$

for all $t \in I$. □

Definition 4. *Consider $\gamma\colon I \subset \mathbb{R} \to \mathbb{L}^2$ as a regular curve parametrized by the arbitrary variable t, $\mathcal{D}_{0+}^{\lambda,C}\gamma(t)$, and the Caputo fractional-order derivative vector of γ at t is*

$$\kappa_\lambda(t) = -\epsilon\frac{\kappa_1^\lambda(t)}{||\gamma'(t)||^2} \tag{4.12}$$

for all $t \in I$.

Theorem 13. *Assume $\gamma\colon I \subset \mathbb{R} \to \mathbb{L}^2$ is a regular curve parametrized by an arbitrary t. Then,*

$$\kappa_\lambda(t) = -\epsilon\left[\frac{\xi_\lambda(t)}{\|\gamma'(t)\|}\kappa(t) + \frac{1}{\|\gamma'(t)\|^2}\frac{d}{dt}(\eta_\lambda(t))\right] \tag{4.13}$$

for all $t \in I$.

Proof. Using (4.10) and Definition 4.3 , we obtain

$$\kappa_\lambda(t) = -\epsilon\frac{\kappa_1^\lambda(t)}{\|\gamma'(t)\|^2} = -\epsilon\frac{\xi_\lambda(t)\|\gamma'(t)\|\kappa(t) + \frac{d}{dt}(\eta_\lambda(t))}{\|\gamma'(t)\|^2}$$

for all $t \in I$. Then,

$$\kappa_\lambda(t) = -\epsilon\left[\frac{\xi_\lambda(t)}{\|\gamma'(t)\|}\kappa(t) + \frac{1}{\|\gamma'(t)\|^2}\frac{d}{dt}(\eta_\lambda(t))\right]$$

for all $t \in I$. □

Theorem 14. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{L}^2$ be a regular curve with arbitrary parameter t, and $\mathcal{D}_{0+}^{\lambda,C}\gamma(t) = \xi_\lambda(t)\mathcal{T}(t) + \eta_\lambda(t)\mathcal{N}(t)$ the vector of fractional-order derivatives. Then,*

$$\xi_\lambda(t) = \frac{\left[\gamma_1{}'(t)\mathcal{D}_{0+}^{\lambda,C}\gamma_1(t) + \gamma_2{}'(t)\mathcal{D}_{0+}^{\lambda,C}\gamma_2(t)\right]}{\|\gamma'(t)\|}, \tag{4.14}$$

$$\eta_\lambda(t) = \frac{\left[\gamma_1{}'(t)\mathcal{D}_{0+}^{\lambda,C}\gamma_2(t) - \gamma_2{}'(t)\mathcal{D}_{0+}^{\lambda,C}\gamma_1(t)\right]}{\|\gamma'(t)\|}. \tag{4.15}$$

Proof. We have $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and $\gamma'(t) = (\gamma_1{}'(t), \gamma_2{}'(t))$. Hence, for an arbitrary parameter t, the Frenet–Serret frame is given by $\{\mathcal{T}(t), \mathcal{N}(t)\}$, such that

$$\mathcal{T}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}, \quad \mathcal{N}(t) = \left(\frac{\gamma_2{}'(t)}{\|\gamma'(t)\|}, \frac{\gamma_1{}'(t)}{\|\gamma'(t)\|}\right),$$

and

$$\begin{aligned}\mathcal{D}_{0+}^{\lambda,C}\gamma(t) &= \left(\mathcal{D}_{0+}^{\lambda,C}\gamma_1(t), \mathcal{D}_{0+}^{\lambda,C}\gamma_2(t)\right)\\ &= \xi_\lambda(t)\,\mathcal{T}(t) + \eta_\lambda(t)\,\mathcal{N}(t)\\ &= \xi_\lambda(t)\left(\frac{\gamma_1{}'(t)}{\|\gamma'(t)\|}, \frac{\gamma_2{}'(t)}{\|\gamma'(t)\|}\right) + \eta_\lambda(t)\left(\frac{\gamma_2{}'(t)}{\|\gamma'(t)\|}, \frac{\gamma_1{}'(t)}{\|\gamma'(t)\|}\right).\end{aligned} \tag{4.16}$$

Thus, we get the system

$$\begin{aligned}\gamma_1{}'(t)\xi_\lambda(t) + \gamma_2{}'(t)\eta_\lambda(t) &= \mathcal{D}_{0+}^{\lambda,C}\gamma_1(t)\,\|\gamma'(t)\|,\\ \gamma_2{}'(t)\xi_\lambda(t) + \gamma_1{}'(t)\eta_\lambda(t) &= \mathcal{D}_{0+}^{\lambda,C}\gamma_2(t)\,\|\gamma'(t)\|.\end{aligned} \tag{4.17}$$

The determinant of the coefficient matrix of system (4.17) is:

$$\begin{vmatrix} \gamma_1{}'(t) & \gamma_2{}'(t) \\ \gamma_2{}'(t) & \gamma_1{}'(t) \end{vmatrix} = \|\gamma'(t)\|^2 \neq 0, \quad \forall t \in I.$$

Thus, the system yields a unique solution:

$$\xi_\lambda(t) = \frac{\gamma_1{}'(t)\, \mathcal{D}^{\lambda,C}_{0+}\gamma_1(t) - \gamma_2{}'(t)\, \mathcal{D}^{\lambda,C}_{0+}\gamma_2(t)}{\|\gamma'(t)\|}, \tag{4.18}$$

$$\eta_\lambda(t) = \frac{\gamma_1{}'(t)\, \mathcal{D}^{\lambda,C}_{0+}\gamma_2(t) - \gamma_2{}'(t)\, \mathcal{D}^{\lambda,C}_{0+}\gamma_1(t)}{\|\gamma'(t)\|}. \tag{4.19}$$

□

Thus, the proof is completed.

5. Applications

Example 5.1. *Let* $\gamma(s) = (\cosh s, \sinh s)$ *be the parametrized curve for* $s \in [0, 2\pi]$. *Its classical curvature is*

$$\kappa(s) = 1.$$

For $0 < \lambda < 1$ and $\epsilon = 1$, the fractional curvature of order λ is denoted $\kappa^\lambda(s)$ and defined via the Caputo-based curvature formula (4.13). Carrying out the necessary computations then yields

$$\kappa^\lambda(s) = \sinh(s) \cdot \sum_{k=0}^{\infty} \frac{(2k-\lambda)\, s^{2k-1-\lambda}}{\Gamma(2k+1-\lambda)} - \cosh(s) \cdot \sum_{k=0}^{\infty} \frac{(2k+1-\lambda)\, s^{2k-\lambda}}{\Gamma(2k+2-\lambda)}.$$

In Figure 1, the function $\kappa^\lambda(s)$ is shown for various choices of λ. Moreover, in Figure 1, one can observe that as λ approaches 1, the graph of $\kappa^\lambda(s)$ converges to the integer-order (classical) curvature $\kappa(s)$

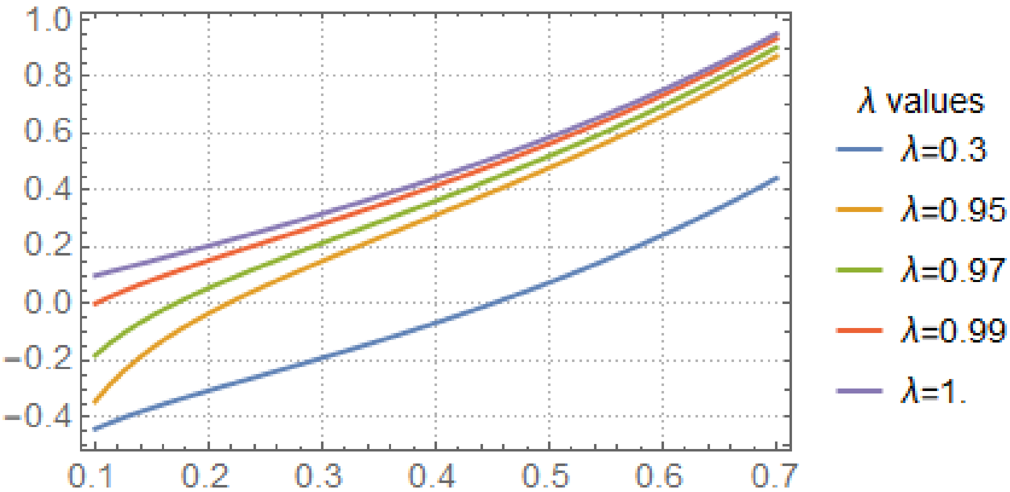

Figure 1. Plot of the fractional curvature $\kappa^\lambda(s)$ across different λ.

The curvatures corresponding to these curves in the Euclidean plane can be found in [34]. In this study, however, the above curvature is computed in the Lorentzian plane, and the corresponding curvature is presented in the graph below.

Example 5.2. *Let*

$$\gamma(t) = (e^{\sqrt{2}t}\cosh t,\ e^{\sqrt{2}t}\sinh t)$$

be the parametrized curve for arbitrary parameter t. Its classical Lorentzian curvature is

$$\kappa(t) = e^{-\sqrt{2}t}.$$

For $0 < \lambda < 1$ and $\epsilon = 1$, the Caputo-based fractional curvature is

$$\kappa^{\lambda}(t) = \frac{e^{-\sqrt{2}t}}{2}\Big[(\sqrt{2}+1)^{\lambda+1}e^{2t} + (\sqrt{2}-1)^{\lambda+1}e^{-2t} + \sqrt{2}((\sqrt{2}-1)(\sqrt{2}+1)^{\lambda} - (\sqrt{2}+1)(\sqrt{2}-1)^{\lambda})\Big].$$

In Figure 2, the function $\kappa^{\lambda}(t)$ is plotted for various choices of λ.

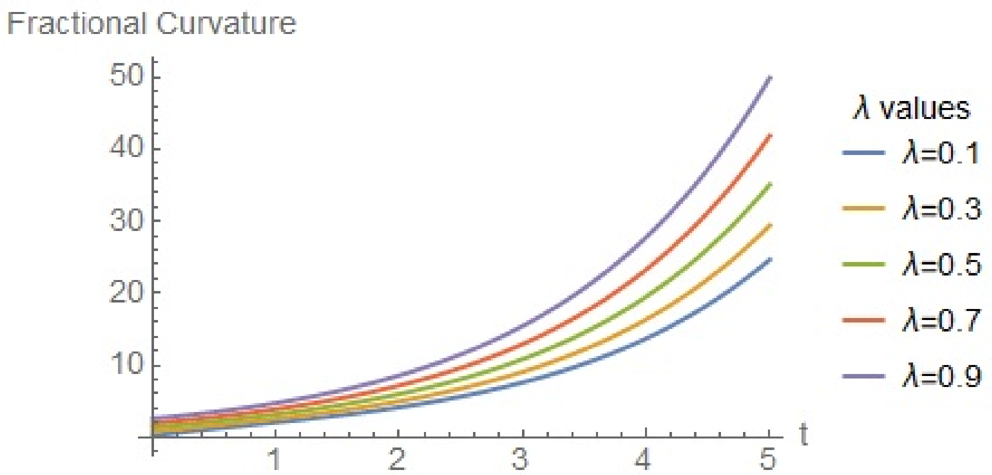

Figure 2. Graph of the fractional curvature $\kappa^{\lambda}(s)$ for different λ values.

6. Physical interpretation of fractional curvature

Fractional curvature can be interpreted as a geometric measure that incorporates nonlocal and memory effects into the description of a curve's behavior. In physical systems, such effects are often encountered in media with anomalous diffusion, viscoelastic materials, or systems governed by fractional dynamics. The inclusion of fractional derivatives allows for a more accurate representation of processes where the classical curvature fails to capture hereditary or path-dependent behaviors.

7. Conclusions

We extend fractional curvature to Lorentzian curves, naturally distinguishing timelike and spacelike trajectories. Explicit expressions illustrate how the pseudo-Riemannian metric signature alters fractional curvature behavior. Our formulation naturally distinguishes timelike and spacelike trajectories and remains invariant under Lorentzian isometries. Explicit expressions for classical model curves illustrate how the pseudo-Riemannian metric signature fundamentally alters fractional curvature behavior compared to the Euclidean setting. By uniting Caputo fractional calculus with Lorentzian geometry, the proposed framework opens new avenues for analyzing nonlocal memory effects and anomalous dynamics along relativistic curves, with potential applications ranging from viscoelastic spacetime models to fractional-order variational principles in general relativity.

Author contributions

Meltem Ogrenmis: writing-original draft, methodology; Handan Oztekin: methodology, writing-original draft; Y. S. Hamed: visualization, methodology, writing-review & editing; Muhammad Bilal Riaz: methodology, conceptualization, funding acquisition, project administration; Muhammad Abbas: visualization, software, writing-review & editing. All authors have read and agreed to publish the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors confirm that they have no relevant financial or non-financial competing interests. All the authors with the consultation of each other completed this research and drafted the manuscript together.

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