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Research article

Sufficient conditions for global boundedness in inertial chemotaxis systems with logistic growth

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Abstract: This paper investigates a chemotaxis model described by the following system of partial differential equations:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\mathbf{w}) + \gamma(u - u^\alpha), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \mathbf{w}_t = \Delta \mathbf{w} - \mathbf{w} + \chi \nabla v, & x \in \Omega,\ t > 0, \\ \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} v = 0, \quad \mathbf{w} = 0, & x \in \partial\Omega, \end{cases}$$

posed on a smooth, bounded domain $\Omega \subset \mathbb{R}^n$, where $\chi > 0$, $\gamma \geq 0$, and $\alpha > 1$ are parameters. By employing L^p-estimates and a carefully constructed bootstrap iteration method, we analyze the intricate mathematical relationships governing the system. Our findings demonstrate the existence of globally bounded solutions for $n \geq 4$, contingent upon the conditions that $\gamma > 0$ and $\alpha > \frac{3n+6}{n+8}$. These findings significantly refine the known parameter constraints for global solution existence. In particular, our work improves upon previous studies—such as those by Zhang et al. (2019), Dong and Peng (2021), and Mu and Tao (2022)—which typically imposed the stricter condition $\alpha > \frac{1}{2} + \frac{n}{4}$. Our analysis thus provides sharper criteria for α, broadening the understanding of solution behavior in chemotaxis systems.

Keywords: chemotaxis; acceleration; logistic growth; global boundedness; threshold dynamics
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1. Introduction

The mathematical formalism of random diffusion, classically represented by Brownian motion, offers a fundamental approach for characterizing stochastic particle dynamics in spatial systems. As a cornerstone methodology, this framework finds extensive application spanning physical, chemical, and biological sciences, with particular relevance to population genetics [7], ecological dispersal [3], cancer dynamics [4], and tissue regeneration processes [24]. Nevertheless, the classical diffusion paradigm exhibits inherent limitations in modeling sophisticated biological phenomena, especially those involving oriented motility or anomalous transport characteristics. These constraints have motivated the development of enhanced theoretical frameworks that incorporate guidance mechanisms, notably through chemotactic [16] and preytactic models [13, 14], which mathematically formalize organism navigation along chemical concentration gradients.

A well-known example of such directed movement is chemotaxis, where cells or organisms move along chemical gradients. To model this collective behavior, Keller and Segel [16] developed a generalized version of their classical chemotaxis system, formulated as follows: Biological cells or organisms exhibit directed movement in response to a chemical gradient, a phenomenon referred to as chemotaxis. To characterize the collective movement of cells, Keller and Segel [16] propose a generalized version of the classical Keller-Segel model, which is articulated as follows

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $\tau \in [0, 1]$. The mathematical analysis of the Keller-Segel model and its generalizations has constituted a major research direction in partial differential equations. Since its inception, a substantial portion of the theoretical work has concentrated on proving global existence theorems, with important advances documented in [11, 23, 27].

Traditional models typically assume organism velocity is proportional to stimulus gradients. However, this framework overlooks key movement dynamics observed in nature, particularly in schooling fish [8, 20], swarms of flying insects [21], and flea-beetles [15]. For these systems, empirical evidence suggests that acceleration (the rate of velocity change), rather than velocity itself, correlates with gradient intensity.

Motivated by these considerations, Tao and Wang [25] incorporated an acceleration mechanism into the chemotaxis framework, leading to an improved version of system (1.1) as follows:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \mathbf{w}), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ \tau \mathbf{w}_t = \varepsilon \Delta \mathbf{w} - \mathbf{w} + \chi \nabla v, & x \in \Omega, t > 0 \end{cases} \tag{1.2}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, where parameters $\chi > 0$ and $\varepsilon > 0$. Here, $u(x, t)$ and $v(x, t)$ denote the density of a particular species and the concentration of the signal it emits, respectively. The vector function $\mathbf{w}(x, t) := (w_1, w_2, \cdots, w_n)(x, t)$ represents the velocity of the species u. Notably, when $\varepsilon = 0$ (equilibrium case), the velocity $\mathbf{w}$ becomes proportional to the concentration gradient ($\mathbf{w} = k \nabla v$). Substituting this into the first equation recovers the classical chemotaxis model (1.1). The authors analyzed (1.2) with $\tau = 0$, subject to the boundary conditions

$$\partial_{\mathbf{n}} u = \partial_{\mathbf{n}} v = \mathbf{w} \cdot \mathbf{n} = 0, \quad \partial_{\mathbf{n}} \mathbf{w} \times \mathbf{n} = 0, \qquad x \in \partial\Omega, \tag{1.3}$$

where $\mathbf{n}$ denotes the outward normal vector on $\partial\Omega$ and $\partial_{\mathbf{n}} \mathbf{w} := (\partial_{\mathbf{n}} w_1, \partial_{\mathbf{n}} w_2, \cdots, \partial_{\mathbf{n}} w_n)$. They further noted that system (1.2) with $\tau = 0$ and boundary condition (1.3) precludes the blow-up and allows for the existence of globally bounded solutions when $n \leq 3$ under reasonable initial conditions. This stands in contrast to the classical chemotaxis models, which may exhibit blow-up in three or two dimensions with a critical mass.

Extending prior investigations on solution regularity, Mu and Tao [18] examined the fully parabolic system (1.2) to determine whether it prevents finite-time blow-up. Their analysis focused on the modified system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\mathbf{w}) + f(u), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ \mathbf{w}_t = \Delta \mathbf{w} - \mathbf{w} + \chi \nabla v, & x \in \Omega, t > 0, \\ \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} v = 0, \ \mathbf{w} = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where the growth term takes the form $f(u) = \gamma(u - u^\alpha)$. Under the parameter conditions

$$(H_1) : n \leq 3, \gamma \geq 0, \alpha > 1, \text{ or } n \geq 4, \gamma > 0, \alpha > \frac{1}{2} + \frac{n}{4},$$

they established the existence of global bounded solutions for appropriate initial data. The present work aims to refine these parameter constraints, particularly by relaxing the requirement on α, which was previously considered rather restrictive for solution existence.

In our paper, we posit that the initial data $(u, v, \mathbf{w})(x, 0) := (u_0, v_0, \mathbf{w}_0)(x)$ conforms to

$$0 \leq u_0 \in C^0(\overline{\Omega}), 0 \leq v_0 \in W^{2,\infty}(\Omega), \mathbf{w}_0 \in W^{2,\infty}(\Omega, \mathbb{R}^n) \text{ and } u_0, v_0 \not\equiv 0. \tag{1.5}$$

The principal findings of this paper are as follows.

Theorem 1.1. *Suppose that* (1.5) *holds with*

$$(H_1') : n \geq 4, \gamma > 0, \alpha > \frac{3n+6}{n+8}.$$

It follows that the system described by (1.4) *admits a unique classical solution denoted as* $(u, v, \mathbf{w})$ *satisfying*

$$u, v \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \ \ \mathbf{w} \in C^0(\overline{\Omega} \times [0, \infty); \mathbb{R}^n) \cap C^{2,1}(\overline{\Omega} \times (0, \infty); \mathbb{R}^n).$$

and maintains the property that $u, v > 0$ *within* $\Omega \times (0, \infty)$. *Furthermore, the solution exhibits uniform boundedness over time, as expressed by*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|\mathbf{w}(\cdot, t)\|_{W^{1,\infty}(\Omega)} < C, \tag{1.6}$$

where $C > 0$ *is a constant that is independent of* t.

Remark 1.1. *The existence analysis in [18] establishes that under appropriate initial conditions satisfying hypothesis* (H_1)*, the nonlinear parabolic system* (1.4) *admits globally bounded classical solutions. Of particular significance, Theorem 1.1 reveals a dimension-dependent regularity threshold: specifically, for spatial dimensions* $n = 2$ *or* 3*, the exponent* α *must satisfy the supercritical condition* $\alpha > 1$ *(or alternatively, the degeneracy condition* $\gamma = 0$ *must hold). In contrast, for higher-dimensional cases with* $n \geq 4$*, this regularity criterion can be relaxed to the fractional requirement* $\alpha > \frac{3n+6}{n+8}$*, which originates from sharp Sobolev embedding estimates and optimal decay rates in the nonlinear reaction terms.*

Remark 1.2. *Let us consider the case where $\tau = 1$ and examine a specific scenario in which a scalar potential function $\phi \in C^1(\Omega)$. $\mathbf{w}$ is defined as a conservative vector field represented by $\mathbf{w} = \chi\nabla\phi$. Subsequently, we establish a connection between system* (1.2) *and the following chemotaxis model that incorporates indirect signal production*

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla\phi) + f(u), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ \phi_t = \Delta\phi - \phi + v, & x \in \Omega, t > 0. \end{cases} \tag{1.7}$$

If $\gamma = 0$, it has been shown by [9] that the critical dimension for the system (1.7) *under homogeneous Neumann or mixed boundary conditions is $n = 4$. Specifically, blow-up phenomena are absent when the initial condition satisfies $\int_\Omega u_0 < \frac{8\pi^2}{\chi}$, with radial symmetry being a requisite for Neumann boundary conditions. Additionally, the global existence of smooth solutions to a quasilinear variant of this system was investigated in [6]. For dimensions $n \geq 2$, with $\gamma > 0$ and $\alpha > 1$, Zhang et al. [28] established that the system* (1.7) *with homogeneous Neumann boundary conditions yields a global bounded solution in $L^\infty(\Omega)$, which converges to the steady state* $(1, 1, 1)$. *In relation to system* (1.4)*, Theorem 1.1 relaxes the restriction on α to $\alpha > 1$ (or $\gamma = 0$) for dimensions $n = 2, 3$, while for $n \geq 4$, the restriction can be further relaxed to $\alpha > \frac{3n+6}{n+8}$.*

In addition, when $\tau = 1$, Lv and Wang [17] examined a system characterized by signal-dependent motility. In this case, the term $\chi\nabla\cdot(u\nabla\phi)$ in (1.7) was substituted with $\Delta(\gamma(v)u)$. They demonstrated the existence of a unique global bounded classical solution under the condition $\alpha > \max\{\frac{1}{2} + \frac{n}{4}, 1\}$. Additionally, when $\alpha > \frac{1}{2} + \frac{n}{4}$, Dong and Peng [5] established the global boundedness of the solution for system (1.7) with rotational sensitivity. This modification implies that $\chi\nabla\cdot(u\nabla\phi)$ in (1.7) is replaced by $\nabla\cdot(S(x, u, v, w)\nabla v)$, with the condition $f(u) \leq k - \mu u^\alpha$.

Based on the aforementioned conclusion, we propose that our methodology may facilitate a relaxation of the constraints imposed on α.

Remark 1.3. *The preceding critical analysis reveals a fundamental constraint pattern in the existing literature: Prevailing results universally require the nonlinearity exponent to satisfy the Sobolev-critical threshold $\alpha > \frac{1}{2} + \frac{n}{4}$. Through comparative asymptotics for $n \geq 4$, rigorous analysis of the competing conditions*

$$\frac{3n+6}{n+8} \quad vs. \quad \frac{1}{2} + \frac{n}{4}$$

yields crucial dimensional asymptotics. The limit analysis

$$\lim_{n\to\infty} \frac{3n+6}{n+8} = 3, \quad \lim_{n\to\infty}\left(\frac{1}{2} + \frac{n}{4}\right) = +\infty$$

(as visualized in Figure 1) demonstrates the asymptotic decoupling phenomenon—specifically, the former condition exhibits saturation at $\alpha = 3$ while the latter becomes asymptotically unbounded. This divergence mechanism permits significant relaxation of dimensional constraints for $\alpha \geq 3$ through nonlinearity saturation effects.

Consequently, Theorem 1.1's structural hypothesis (H_1') *admits an optimized reformulation through bifurcation analysis:*

$$\begin{cases} (H_{11}') : & n \geq 4, \ \gamma > 0, \ \dfrac{3n+6}{n+8} < \alpha < 3, \\ (H_{12}') : & \gamma > 0, \ \alpha \geq 3 \quad (\textit{with dimensional constraints removed}). \end{cases}$$

This parameter dichotomy reflects distinct regularity regimes: The subcritical range (H_{11}') *requires dimensional compensation through* $n \geq 4$*, while the supercritical regime* (H_{12}') *achieves asymptotic tightness independent of spatial dimension.*

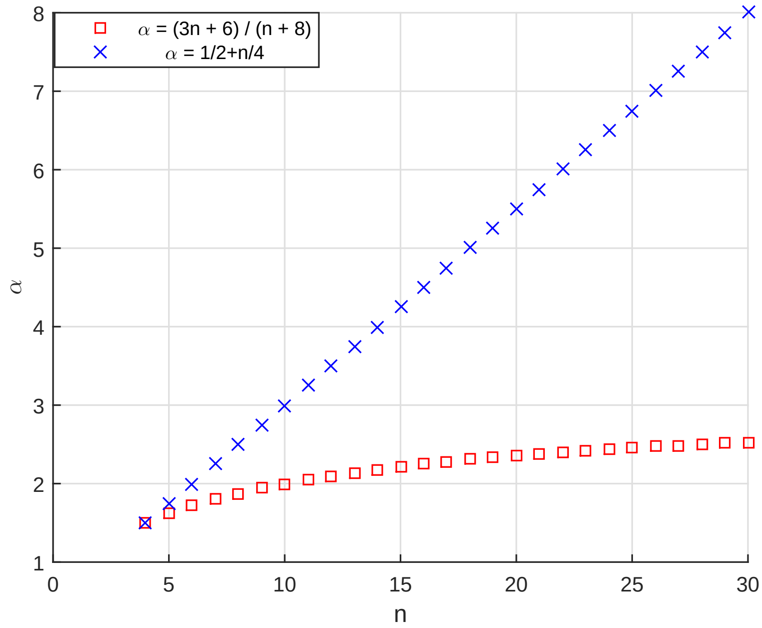

Figure 1. Figure (a) shows that image of $\alpha = \frac{3n+6}{n+8}$ and $\alpha = \frac{1}{2} + \frac{n}{4}$ when $n \geq 4$.

The subsequent sections of this paper are structured as follows: Section 2 presents the well-established result concerning the local existence of solutions to system (1.4), along with the necessary preliminary information. In Section 3, we demonstrate that when $\gamma > 0$, the function u exhibits a property that surpasses mere L^1-boundedness (as detailed in Lemma 3.1). The L^p estimates for both u and v are derived through an elementary energy method, which facilitates the proof of Theorem 1.1.

2. Preliminaries

In this section, we introduce several lemmas that will be utilized in subsequent proofs. To begin, we revisit the established result concerning the local existence of solutions to the system described in Eq (1.4). For comprehensive proof, readers are directed to [18, 25]. This proof relies on conventional arguments associated with the contraction mapping principle. It is important to note that the primary distinction in this case is that $\mathbf{w}$ adheres to a parabolic equation; nevertheless, the homogeneous Dirichlet boundary conditions for $\mathbf{w}$ will ensure adequate regularity in the proof, as discussed in Lemma 2.4 below.

Lemma 2.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary. Assume that the initial data $(u_0, v_0, \mathbf{w}_0)(x)$ satisfy (1.5). Then, system (1.4) admits a unique nonnegative classical solution*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \\ v \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \\ \mathbf{w} \in C^0(\overline{\Omega} \times [0, T_{max}); \mathbb{R}^n) \cap C^{2,1}(\overline{\Omega} \times [0, T_{max}); \mathbb{R}^n), \end{cases} \tag{2.1}$$

where T_{max} denotes the maximal existence time. Moreover, if $T_{max} < \infty$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|\mathbf{w}(\cdot, t)\|_{W^{1,\infty}(\Omega)} \to \infty \quad \text{as } t \nearrow T_{max}.$$

Remark 2.1. *In light of Lemma 2.1, for any $\tau_0 \in (0, T_{max})$ and $\tau_0 \le 1$, there exists $K > 0$ such that*

$$\|u(\cdot, \tau)\|_{L^\infty(\Omega)} + \|v(\cdot, \tau)\|_{W^{1,\infty}(\Omega)} + \|\mathbf{w}(\cdot, \tau)\|_{W^{1,\infty}(\Omega)} \le K \quad \text{for all } \tau \in [0, \tau_0). \tag{2.2}$$

In addressing a specific spatial derivative estimate that incorporates a time-dependent potential function, we employ a modified version of Maximal Sobolev regularity. This approach is informed by the findings presented in [1, Lemmas 2.5 and 3.3] and [2, Lemma 2.3].

Lemma 2.2. *Let $\gamma, q > 1$ as well as $\delta \in (0, 1)$, and assume that $g \in L^\gamma((0, T); L^q(\Omega))$. Consider v as a solution to the following initial boundary value problem:*

$$\begin{cases} v_t - \Delta v + v = g, & x \in \Omega, t \in (0, T), \\ v = 0, & x \in \partial\Omega, \\ v(x, 0) = v_0(x), & x \in \Omega. \end{cases}$$

Then there exists a positive constant C such that if $t_0 \in [0, T)$, $v(\cdot, t_0) \in W^{2,q}(\Omega)$ with $v(\cdot, t_0) = 0$ on $\partial\Omega$, we have

$$\int_{t_0}^T e^{\delta\gamma s} \|\Delta v(\cdot, s)\|^\gamma_{L^q(\Omega)} ds \le C\Big(\int_{t_0}^T e^{\delta\gamma s} \|g(\cdot, s)\|^\gamma_{L^q(\Omega)} ds + e^{\delta\gamma T} \|v(\cdot, t_0)\|^\gamma_{W^{2,q}(\Omega)} \Big), \tag{2.3}$$

where C is a constant that depends on γ, q, and Ω.

Proof. For given $t_0 \in (0, T)$, we observe that $v(\cdot, t_0) = 0$ on the boundary $\partial\Omega$. We define $d := \min\{\frac{1}{4}(T - t_0), 1\}$ and let $\chi \in C_0^\infty([0, \infty))$ denote a smooth cut-off function that satisfies

$$\begin{cases} \chi(t) = 1, & t = 0, \\ \chi(t) \le 1, & 0 < t < d, \\ \chi(t) = 0, & t \ge d, \\ |\chi'(t)| \le \frac{2}{d}, & t \in [0, \infty). \end{cases} \tag{2.4}$$

Let us define the function $w(x,t)$ as follows:

$$w(x,t) := e^{\delta t}v(x,t+t_0) - \chi(t)v(x,t_0) \ \text{ for } (x,t) \in \Omega \times [0, T-t_0),$$

where $\delta \in (0,1)$. It can be readily verified that w solves the following equation:

$$\begin{cases} w_t(x,t) = (\Delta - \delta)w(x,t) + e^{\delta t}g(x,t+t_0) + f(x,t), & (x,t) \in \Omega \times [0,T-t_0), \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T-t_0), \\ w(x,0) = 0, & x \in \Omega, \end{cases} \tag{2.5}$$

where $f(x,t) := \chi(t)\Delta v(x,t_0) - \chi'(t)v(x,t_0) - \delta\chi(t)v(x,t_0)$ in $\Omega \times [0,T-t_0)$. An application of the maximal Sobolev regularity result from [10] implies

$$\begin{aligned} & \int_0^{T-t_0} \|\Delta w(\cdot,t)\|_{L^q(\Omega)}^{\gamma} dt \\ \leq \ & C_1 \int_0^{T-t_0} \|e^{\delta t}g(\cdot,t+t_0)\|_{L^q(\Omega)}^{\gamma} dt \\ & + C_1 \int_0^{T-t_0} \|\chi(t)\Delta v(\cdot,t_0) - \chi'(t)v(\cdot,t_0) - \delta\chi(t)v(\cdot,t_0)\|_{L^q(\Omega)}^{\gamma} dt \\ \leq \ & C_1 \int_0^{T-t_0} \|e^{\delta t}g(\cdot,t+t_0)\|_{L^q(\Omega)}^{\gamma} dt + C_2\|v(\cdot,t_0)\|_{W^{2,q}(\Omega)}^{\gamma}. \end{aligned}$$

Given that $e^{\delta t}\Delta v(x,t+t_0) = \Delta w(x,t) + \chi(t)\Delta v(x,t_0)$, we can arrive at

$$\begin{aligned} & \int_0^{T-t_0} e^{\delta\gamma t}\|\Delta v(\cdot,t+t_0)\|_{L^q(\Omega)}^{\gamma} dt \\ \leq \ & C_3 \int_0^{T-t_0} \|\Delta w(\cdot,t)\|_{L^q(\Omega)}^{\gamma} dt + C_3 \int_0^{T-t_0} \|\chi(t)\Delta v(\cdot,t_0)\|_{L^q(\Omega)}^{\gamma} dt \\ \leq \ & C_4 \int_0^{T-t_0} e^{\delta\gamma t}\|g(\cdot,t+t_0)\|_{L^q(\Omega)}^{\gamma} dt + C_5\|v(\cdot,t_0)\|_{W^{2,q}(\Omega)}^{\gamma}. \end{aligned}$$

By substituting $t = T - s$, we obtain

$$\begin{aligned} & \int_{t_0}^{T} e^{-\delta\gamma(T-s)}\|\Delta v(\cdot,s)\|_{L^q(\Omega)}^{\gamma} ds \\ \leq \ & C_4 \int_{t_0}^{T} e^{-\delta\gamma(T-s)}\|g(\cdot,s)\|_{L^q(\Omega)}^{\gamma} ds + C_5\|v(\cdot,t_0)\|_{W^{2,q}(\Omega)}^{\gamma} \\ \leq \ & C\Big(\int_{t_0}^{T} e^{-\delta\gamma(T-s)}\|g(\cdot,s)\|_{L^q(\Omega)}^{\gamma} ds + \|v(\cdot,t_0)\|_{W^{2,q}(\Omega)}^{\gamma}\Big). \end{aligned} \tag{2.6}$$

Multiplying the above inequality by $e^{\delta\gamma T}$ yields

$$\int_{t_0}^{T} e^{\delta\gamma s}\|\Delta v(\cdot,s)\|_{L^q(\Omega)}^{\gamma} ds \leq C(\int_{t_0}^{T} e^{\delta\gamma s}\|g(\cdot,s)\|_{L^q(\Omega)}^{\gamma} ds + e^{\delta\gamma T}\|v(\cdot,t_0)\|_{W^{2,q}(\Omega)}^{\gamma}).$$

Thus, the proof is complete. □

In this paper, in addition to applying the well-known smooth properties of the Neumann heat semigroup, we also utilize the L^p-L^q estimates of the Dirichlet heat semigroup. First, Lemma 2.3 provides an estimate for the Neumann heat semigroup [26, Lemma 1.3].

Lemma 2.3. *Let $e^{t\Delta}$ be the Neumann heat semigroup in $\Omega \subset \mathbb{R}^n (n \geq 1)$, and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under the Neumann boundary condition. If $p, q \in [1, \infty]$, then for any $v \in L^q(\Omega)$, it holds that with $\int_\Omega v = 0$,*

$$\|e^{t\Delta} v\|_{L^p(\Omega)} \leq C(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}) e^{-\lambda_1 t} \|v\|_{L^q(\Omega)} \quad \text{for all } t > 0.$$

The next lemma provides crucial estimates for the Dirichlet heat semigroup ([19, Lemma 2.4 (i)], which are utilized in [22, Propositions 48.4*, 48.5, and 48.7*], along with a similar approach to the proof of [26, Lemma 1.3]).

Lemma 2.4. *[19] Let $e^{t\Delta}$ be the Dirichlet heat semigroup in $\Omega \subset \mathbb{R}^n (n \geq 1)$, and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under the Dirichlet boundary condition. If $1 \leq q \leq p \leq \infty$, then for any $v \in L^q(\Omega)$, it holds that*

$$\|e^{t\Delta} v\|_{L^p(\Omega)} \leq C(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}) e^{-\lambda_1 t} \|v\|_{L^q(\Omega)} \quad \text{for all } t > 0,$$

and

$$\|\nabla e^{t\Delta} v\|_{L^p(\Omega)} \leq C(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}) e^{-\lambda_1 t} \|v\|_{L^q(\Omega)} \quad \text{for all } t > 0.$$

3. Global existence and boundedness

Our primary objective is to establish the existence of classical solutions and derive uniform bounds for systems (1.4) and (1.5). Let $(u, v, \mathbf{w})$ denote the classical solution of (1.4) and (1.5) with maximal existence time $T_{max} \in (0, \infty]$. The following fundamental properties will be crucial for our analysis.

Lemma 3.1. *Under the assumptions in Theorem 1.1, then there exist positive constants C and*

$$M := \begin{cases} \|u_0\|_{L^1(\Omega)}, & \gamma = 0, \\ \max\{\|u_0\|_{L^1(\Omega)}, |\Omega|\}, & \gamma > 0, \end{cases}$$

such that if $\gamma = 0$, then

$$\int_\Omega u = \int_\Omega u_0 \leq M \qquad \text{for all } t \in (0, T_{max}), \tag{3.1}$$

and if $\gamma > 0$, then

$$\int_\Omega u \leq \max\left\{ \int_\Omega u_0, |\Omega| \right\} \leq M \qquad \text{for all } t \in (0, T_{max}). \tag{3.2}$$

Moreover, we have

$$\int_\Omega v \leq M \qquad \text{for all } t \in (0, T_{max}). \tag{3.3}$$

In addition,

$$\int_t^{t+\tau} \int_\Omega u^\alpha \leq C \qquad \text{for all } t \in (0, T_{max} - \tau), \tag{3.4}$$

where $\tau := \min\{1, \frac{1}{2}T_{max}\}$.

Proof. Integrating the first equation of (1.4) yields

$$\frac{d}{dt}\int_\Omega u = \gamma\int_\Omega u - \gamma\int_\Omega u^\alpha \quad \text{for all } t\in(0,T_{max}). \tag{3.5}$$

By Hölder inequality $\|u\|_{L^1(\Omega)} \le |\Omega|^{1-\frac{1}{\alpha}}\|u\|_{L^\alpha(\Omega)}$, we have

$$\frac{d}{dt}\|u\|_{L^1(\Omega)} \le \gamma\|u\|^{\alpha}_{L^1(\Omega)}(\|u\|^{1-\alpha}_{L^1(\Omega)} - |\Omega|^{1-\alpha}) \qquad \text{for all } t\in(0,T_{max}), \tag{3.6}$$

which implies (3.2). Then, integrating the second equation of (1.4), yields

$$\frac{d}{dt}\int_\Omega v = -\int_\Omega v + \int_\Omega u \le \int_\Omega u \le M \qquad \text{for all } t\in(0,T_{max}).$$

Moreover, let $\tau := \min\{1, \frac{1}{2}T_{max}\}$, then integrating (3.5) over $(t, t+\tau)$ in time and using (3.2), it can deduce that there exists a positive constant C such that

$$\int_t^{t+\tau}\int_\Omega u^\alpha(\cdot,s)ds = \int_t^{t+\tau}\int_\Omega u(\cdot,s)ds + \frac{1}{\gamma}\Big(\int_\Omega u(\cdot,t) - \int_\Omega u(\cdot,t+\tau)\Big) \le C$$

for $t\in(0,T_{max}-\tau)$. □

In the following, we will establish an iterative step that depends on a series of a priori estimates to develop the main component of our result. To this end, we will first present the following lemma.

Lemma 3.2. *Let $(u, v, \mathbf{w})$ be the solution of (1.4) with the initial data satisfying (1.5), and suppose the hypothesis of Theorem 1.1 holds. Then, for any $p > 1$, one can find a positive constant C satisfying*

$$\int_\Omega u^p \le C \qquad \text{for all } t\in(0,T_{max}), \tag{3.7}$$

and

$$\int_\Omega v^p \le C \qquad \text{for all } t\in(0,T_{max}). \tag{3.8}$$

Proof. For arbitrary $p > p_0$, multiplying the first equation of (1.4) by u^{p-1} and subsequently integrating over Ω yields

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\int_\Omega u^{p-2}|\nabla u|^2\\
=\ & -\int_\Omega u^{p-1}\nabla\cdot(u\mathbf{w}) + \gamma\int_\Omega u^p - \gamma\int_\Omega u^{p+\alpha-1}\\
=\ & (p-1)\int_\Omega u^{p-1}\mathbf{w}\cdot\nabla u - \gamma\int_\Omega u^{p+\alpha-1} + \gamma\int_\Omega u^p\\
\le\ & \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{2}\int_\Omega u^p|\mathbf{w}|^2 - \gamma\int_\Omega u^{p+\alpha-1} + \gamma\int_\Omega u^p
\end{aligned} \tag{3.9}$$

for all $t\in(0,T_{max})$. Moreover, according to Young's inequality,

$$\frac{p-1}{2}\int_\Omega u^p|\mathbf{w}|^2 \le C_1\int_\Omega |\mathbf{w}|^{\frac{2(p+\alpha-1)}{\alpha-1}} + \frac{\gamma}{2}\int_\Omega u^{p+\alpha-1}, \tag{3.10}$$

and

$$\gamma\int_\Omega u^p \le \frac{\gamma}{4}\int_\Omega u^{p+\alpha-1} + C_2. \tag{3.11}$$

Combining (3.9) with (3.10) as well as (3.11), we have

$$\frac{d}{dt}\int_\Omega u^p \le C_1 p\int_\Omega |\mathbf{w}|^{\frac{2(p+\alpha-1)}{\alpha-1}} - \frac{\gamma p}{4}\int_\Omega u^{p+\alpha-1} + C_2 p, \qquad t\in(0,T_{max}). \tag{3.12}$$

Furthermore, for any $\delta_1 \in (0,1)$ and $\gamma_1 > 1$, by adding $\delta_1\gamma_1\int_\Omega u^p$ to both sides of (3.12), it can be deduced that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^p + \delta_1\gamma_1\int_\Omega u^p \\ \le\ & C_1 p\int_\Omega |\mathbf{w}|^{\frac{2(p+\alpha-1)}{\alpha-1}} - \frac{\gamma p}{4}\int_\Omega u^{p+\alpha-1} + \delta_1\gamma_1\int_\Omega u^p + C_2 p \end{aligned} \tag{3.13}$$

for all $t\in(0,T_{max})$. Next, we estimate $\delta_1\gamma_1\int_\Omega u^p$ by Young's inequality, and it reads that

$$\delta_1\gamma_1\int_\Omega u^p \le \frac{\gamma p}{8}\int_\Omega u^{p+\alpha-1} + C_3 \quad \text{for all } t\in(0,T_{max}). \tag{3.14}$$

Therefore, it follows from (3.13) and (3.14) that for all $t\in(0,T_{max})$, we have

$$\frac{d}{dt}\int_\Omega u^p + \delta_1\gamma_1\int_\Omega u^p \le -\frac{\gamma p}{8}\int_\Omega u^{p+\alpha-1} + C_1 p\int_\Omega |\mathbf{w}|^{\frac{2(p+\alpha-1)}{\alpha-1}} + C_4 \tag{3.15}$$

with $C_4 = C_2 p + C_3 > 0$.

In what follows, we will focus on estimating $\|\mathbf{w}\|^{\frac{2(p+\alpha-1)}{\alpha-1}}_{L^{\frac{2(p+\alpha-1)}{\alpha-1}}(\Omega)}$ based on the theories of partial differential equations. At first, similarly as in [12, Lemma 4.1], by employing L^p-L^q estimates of the Neumann heat semigroup and the Hölder inequality, as well as (3.3) and (3.4), we conclude that

$$\begin{aligned} \|\nabla v\|_{L^q(\Omega)} \le\ & C_5(t-\eta_0)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}e^{-\lambda_1(t-\eta_0)}\|v\|_{L^1(\Omega)} \\ & + \int_{\eta_0}^t [1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\alpha}-\frac{1}{q})}]e^{-\lambda_1(t-\eta_0)}\|u(\cdot,s)\|_{L^\alpha(\Omega)}ds \\ \le\ & C_6 + C_7\Big(\int_{\eta_0}^t \|u(\cdot,s)\|^\alpha_{L^\alpha(\Omega)}ds\Big)^{\frac{1}{\alpha}}\Big(\int_{\eta_0}^t [1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\alpha}-\frac{1}{q})}]^{\frac{\alpha}{\alpha-1}}e^{-\lambda_1(t-s)\frac{\alpha}{\alpha-1}}ds\Big)^{\frac{\alpha-1}{\alpha}} \\ \le\ & C_8 \qquad \text{for all } t\in(\eta_0,T_{max}) \text{ and } 1\le q<\tfrac{n\alpha}{(n+2-\alpha)_+}, \end{aligned} \tag{3.16}$$

where $\eta_0 = \frac{\tau_0}{2}$ and τ_0 is the same as (2.2). Subsequently, it can be concluded from the third equation of (1.4) that for all $s_1\in[1,\frac{n\alpha}{(n+2-2\alpha)_+})$ and $t\in(\eta_0,T_{max})$,

$$\|\mathbf{w}(\cdot,t)\|_{W^{1,s_1}(\Omega)} \le C_9(\|\nabla v(\cdot,t)\|_{L^q(\Omega)}+1) \le C_{10}. \tag{3.17}$$

In light of the Sobolev embedding theorem, we derive that for any $s_2\in[1,\frac{n\alpha}{(n+2-3\alpha)_+})$,

$$\|\mathbf{w}(\cdot,t)\|_{L^{s_2}(\Omega)} \le C_{11}. \tag{3.18}$$

When $n \geq 4$, on the other hand, in light of $\alpha > \frac{3n+6}{n+8}$, we can choose $k_0 \in (1, \frac{n\alpha}{(n+2-3\alpha)_+})$ and $k_1 \in (1, \frac{n\alpha}{(n+2-2\alpha)_+})$, which are sufficiently close to $\frac{n\alpha}{(n+2-3\alpha)_+}$ and $\frac{n\alpha}{(n+2-2\alpha)_+}$, respectively, yielding

$$\begin{cases} \dfrac{2(p+\alpha-1)}{\alpha-1} \cdot \dfrac{\frac{1}{k_0} - \frac{\alpha-1}{2(p+\alpha-1)}}{\frac{1}{k_0} + \frac{2}{n} - \frac{1}{\gamma_1}} < \gamma_1, \\ \gamma_1 \cdot \dfrac{\frac{1}{k_1} + \frac{1}{n} - \frac{1}{\gamma_1}}{\frac{1}{k_1} + \frac{2}{n} - \frac{1}{m}} < m, \\ m < p + \alpha - 1. \end{cases} \tag{3.19}$$

Without any loss of generality, we may assume that $\gamma_1 \leq p + \alpha - 1$. This assumption is justified because, in the case where $\gamma_1 > p + \alpha - 1$, an analogous argument can be employed by symmetrically interchanging the roles of γ_1 and $p+\alpha-1$. Consequently, this does not restrict the generality of the proof and ensures the validity of the result in all cases. Utilizing the well-established Gagliardo-Nirenberg inequality, in conjunction with Eq (3.18), leads to

$$\begin{aligned} C_1 p \|\mathbf{w}\|^{\gamma_1}_{L^{\gamma_1}(\Omega)} &\leq C_{13} \|\Delta \mathbf{w}\|^{\gamma_1 \theta}_{L^{\gamma_1}(\Omega)} \|\mathbf{w}\|^{\gamma_1(1-\theta)}_{L^{k_0}(\Omega)} + C_{14} \|\mathbf{w}\|^{\gamma_1}_{L^{k_0}(\Omega)} \\ &\leq C_{15}(\|\Delta \mathbf{w}\|^{\gamma_1 \theta}_{L^{\gamma_1}(\Omega)} + 1) \\ &\leq \tfrac{1}{2} \|\Delta \mathbf{w}\|^{\gamma_1}_{L^{\gamma_1}(\Omega)} + C_{16} \quad \text{for all } t \in (0, T_{max}) \end{aligned} \tag{3.20}$$

with $\theta = \frac{\frac{1}{k_0} - \frac{1}{\gamma_1}}{\frac{1}{k_0} + \frac{2}{n} - \frac{1}{\gamma_1}}$. Furthermore, by applying Lemma 2.2 and (3.16), we can assert the existence of a positive constant C_{γ_1} such that for the above η_0, if $\mathbf{w}(\cdot, \eta_0) \in W^{2,\gamma_1}(\Omega), t \in (\eta_0, T_{max})$ with $\mathbf{w}(\cdot, \eta_0) = 0$ on $\partial\Omega$, then

$$\begin{aligned} & \int_{\eta_0}^t e^{\delta_1 \gamma_1 s} \|\Delta \mathbf{w}(\cdot, s)\|^{\gamma_1}_{L^{\gamma_1}(\Omega)} ds \\ \leq\ & C_{\gamma_1} \Big(\int_{\eta_0}^t e^{\delta_1 \gamma_1 s} \|\nabla v(\cdot, s)\|^{\gamma_1}_{L^{\gamma_1}(\Omega)} ds + e^{\delta_1 \gamma_1 t} \|\mathbf{w}(\cdot, \eta_0)\|^{\gamma_1}_{W^{2,\gamma_1}(\Omega)} \Big) \\ \leq\ & C_{\gamma_1} \int_{\eta_0}^t e^{\delta_1 \gamma_1 s} \|\nabla v(\cdot, s)\|^{\gamma_1}_{L^{\gamma_1}(\Omega)} ds + C_{17}. \end{aligned} \tag{3.21}$$

Moreover, utilizing the Gagliardo-Nirenberg inequality to estimate the term $C_{\gamma_1} \int_{\eta_0}^t e^{\delta_1 \gamma_1 s} \|\nabla v(\cdot, s)\|^{\gamma_1}_{L^{\gamma_1}(\Omega)} ds$, it deduces that

$$\begin{aligned} & C_{\gamma_1} \int_{\eta_0}^t e^{\delta_1 \gamma_1 s} \|\nabla v(\cdot, s)\|^{\gamma_1}_{L^{\gamma_1}(\Omega)} ds \\ \leq\ & C_{18} \int_{\eta_0}^t e^{\delta_1 \gamma_1 s} \|\Delta v(\cdot, s)\|^{\gamma_1 \theta_1}_{L^m(\Omega)} \|v(\cdot, s)\|^{\gamma_1(1-\theta_1)}_{L^{k_1}(\Omega)} + C_{19} \int_{\eta_0}^t e^{\delta_1 \gamma_1 s} \|v(\cdot, s)\|^{\gamma_1}_{L^{k_1}(\Omega)} \\ \leq\ & C_{20} \int_{\eta_0}^t e^{\delta_1 \gamma_1 s} (\|\Delta v(\cdot, s)\|^{\gamma_1 \theta_1}_{L^m(\Omega)} + 1) \quad \text{for all } t \in (\eta_0, T_{max}) \end{aligned} \tag{3.22}$$

with $\theta_1 = \frac{\frac{1}{k_1} + \frac{1}{n} - \frac{1}{\gamma_1}}{\frac{1}{k_1} + \frac{2}{n} - \frac{1}{m}}$. Consequently, for any $t \in (\eta_0, T_{max})$, by employing the variation-of-constants formula to the aforementioned inequality, together with (3.15) and (3.19)–(3.21), we derive

$$\begin{aligned}\int_\Omega u^p &\le e^{-\delta_1\gamma_1(t-\eta_0)}\int_\Omega u^p(\cdot,\eta_0)+C_1pe^{-\gamma_1 t}\int_{\eta_0}^t e^{\delta_1\gamma_1 s}\int_\Omega |\mathbf{w}|^{\frac{2(p+\alpha-1)}{\alpha-1}}ds\\ &\quad -\frac{\gamma p}{8}e^{-\delta_1\gamma_1 t}\int_{\eta_0}^t e^{\delta_1\gamma_1 s}ds\int_\Omega u^{p+\alpha-1}+C_4\int_{\eta_0}^t e^{-\delta_1\gamma_1(t-s)}ds\\ &\le e^{-\delta_1\gamma_1(t-\eta_0)}\int_\Omega u^p(\cdot,\eta_0)+\frac{1}{2}e^{-\delta_1\gamma_1 t}\int_{\eta_0}^t e^{\delta_1\gamma_1 s}\|\Delta\mathbf{w}(\cdot,s)\|_{L^{\gamma_1}(\Omega)}^{\gamma_1}ds\\ &\quad -\frac{\gamma p}{8}e^{-\delta_1\gamma_1 t}\int_{\eta_0}^t e^{\delta_1\gamma_1 s}ds\int_\Omega u^{p+\alpha-1}+C_4\int_{\eta_0}^t e^{-\delta_1\gamma_1(t-s)}ds\\ &\le e^{-\delta_1\gamma_1(t-\eta_0)}\int_\Omega u^p(\cdot,\eta_0)+\frac{C_{\gamma_1}}{2}e^{-\delta_1\gamma_1 t}\int_{\eta_0}^t e^{\delta_1\gamma_1 s}\|\nabla v(\cdot,s)\|_{L^{\gamma_1}(\Omega)}^{\gamma_1}ds+\frac{C_{17}}{2}\\ &\quad -\frac{\gamma p}{8}e^{-\delta_1\gamma_1 t}\int_{\eta_0}^t e^{\delta_1\gamma_1 s}ds\int_\Omega u^{p+\alpha-1}+C_4\int_{\eta_0}^t e^{-\delta_1\gamma_1(t-s)}ds,\end{aligned}$$

whence returning to (3.19) combined with (3.22), and by means of Young's inequality and the fact that $\gamma_1\theta_1 < m < p+\alpha-1$, it follows that for any $\eta > 0$,

$$\begin{aligned}\int_\Omega u^p &\le C_{21}+e^{-\delta_1\gamma_1 t}\int_{\eta_0}^t e^{\delta_1\gamma_1 s}\Big(\frac{C_{20}}{2}\|\Delta v\|_{L^m(\Omega)}^{\gamma_1\theta_1}-\frac{\gamma p}{8}\int_\Omega u^{p+\alpha-1}+C_{22}\Big)ds\\ &\le C_{21}+e^{-\delta_1\gamma_1 t}\int_{\eta_0}^t e^{\delta_1\gamma_1 s}\Big(\eta\int_\Omega|\Delta v|^{p+\alpha-1}-\frac{\gamma p}{8}\int_\Omega u^{p+\alpha-1}+C_{23}\Big)ds\end{aligned}$$

with $C_{22} = \frac{C_{20}}{2} + C_4$ and some $C_{23} > 0$. Recalling that $\gamma_1 \le p+\alpha-1$, whereby thanks to the Sobolev maximum principle and choosing $\eta > 0$ small enough, the above then yields

$$\begin{aligned}\int_\Omega u^p &\le C_{21}+e^{-\delta_1\gamma_1 t}\int_{\eta_0}^t e^{\delta_1\gamma_1 s}\Big(\frac{\gamma p}{16}\int_\Omega u^{p+\alpha-1}-\frac{\gamma p}{8}\int_\Omega u^{p+\alpha-1}+C_{25}\Big)ds\\ &= C_{21}+e^{-\delta_1\gamma_1 t}\int_{\eta_0}^t e^{\delta_1\gamma_1 s}\Big(-\frac{\gamma p}{16}\int_\Omega u^{p+\alpha-1}+C_{24}\Big)ds\\ &\le C_{25}.\end{aligned}$$

Recalling Remark 2.1, we conclude that (3.7) holds. In a similar manner, by multiplying the second equation of (1.4) by v^{p-1}, integrating by parts over Ω, we can conclude that for all $t \in (\eta_0, T_{max})$,

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v^p+(p-1)\int_\Omega v^{p-2}|\nabla v|^2 = -\int_\Omega v^p+\int_\Omega uv^{p-1}.$$

Subsequently, applying Young's inequality yields

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega v^p &\le -\int_\Omega v^p+\frac{1}{2p}\int_\Omega v^p+C_{26}\int_\Omega u^p\\ &\le C_{27}-\frac{1}{2p}\int_\Omega v^p,\end{aligned}$$

which implies that for all $t \in (\eta_0, T_{max})$,

$$\int_\Omega v^p \le e^{\frac{1}{2}(t-\eta_0)}\int_\Omega v^p(\cdot,\eta_0)+2pC_{26}(1-e^{\frac{1}{2}(t-\eta_0)}) \le C_{27}. \tag{3.23}$$

According to Remark 2.1, we conclude that (3.8) holds. This completes the proof of Lemma 3.2. □

With Lemmas 3.1 and 3.2 acquired, and by revisiting the second and third equations of (1.4) again, we establish the boundedness of ∇v and $\nabla \mathbf{w}$ in $L^\infty(\Omega)$.

Lemma 3.3. *Presume that Theorem 1.1 holds. Then, there are positive constants K_1 and K_2 such that*

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le K_1, \quad \|\nabla \mathbf{w}(\cdot,t)\|_{L^\infty(\Omega)} \le K_2 \quad \text{for all } t \in (0,T_{max}). \tag{3.24}$$

Proof. By examining the second equation in (1.4), we can apply a related variation-of-constants formula, which indicates that

$$v(\cdot,t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)ds$$

for all $t \in (0,T_{max})$. To achieve this, we utilize the established smoothing property of the Neumann semigroup, in conjunction with Lemma 3.2 and the Hölder inequality. Consequently, we can ascertain the existence of positive constants λ and C_i such that

$$\begin{aligned} \|\nabla v\|_{L^\infty(\Omega)} &\le C_1\|\nabla e^{t(\Delta-1)}v_0\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{(t-s)(\Delta-1)}u\|_{L^\infty(\Omega)}ds \\ &\le C_2\|v_0\|_{W^{1,\infty}(\Omega)} + C_2\int_0^t [1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{2n}-\frac{1}{\infty})}]e^{-\lambda(t-s)}\|u\|_{L^{2n}(\Omega)}ds \\ &\le C_3, \end{aligned} \tag{3.25}$$

where the fact $-\frac{1}{2}-\frac{n}{2}(\frac{1}{2n}-\frac{1}{\infty}) > -1$ and (1.5) are applied. In accordance with the embedding theorem, it can be stated that for any $p \ge 1$, $L^\infty(\Omega) \hookrightarrow L^p(\Omega)$, therefore,

$$\|\nabla v\|_{L^p(\Omega)} = \Big(\int_\Omega |\nabla v|^p\Big)^{\frac{1}{p}} \le \Big(\int_\Omega C_3^p\Big)^{\frac{1}{p}} \le C_3|\Omega|^{\frac{1}{p}} \le C_4. \tag{3.26}$$

In reference to the third equation in (1.4), we can similarly apply a related variation-of-constants formula, leading to the following formulation

$$\mathbf{w}(\cdot,t) = e^{t(\Delta-1)}\mathbf{w}_0 + \chi\int_0^t e^{(t-s)(\Delta-1)}\nabla v(\cdot,s)$$

for all $t \in (0,T_{max})$. By integrating Lemma 3.2 with the application of the Hölder inequality again, we arrive at

$$\begin{aligned} \|\nabla \mathbf{w}\|_{L^\infty(\Omega)} &\le C_5\|\nabla e^{t(\Delta-1)}\mathbf{w}_0\|_{L^\infty(\Omega)} + \chi\int_0^t \|\nabla e^{(t-s)(\Delta-1)}\nabla v\|_{L^\infty(\Omega)}ds \\ &\le C_6\|\mathbf{w}_0\|_{W^{1,\infty}(\Omega)} + C_7\int_0^t [1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{2n}-\frac{1}{\infty})}]e^{-\lambda(t-s)}\|\nabla v\|_{L^{2n}(\Omega)}ds \\ &\le C_7 \end{aligned} \tag{3.27}$$

for all $t \in (0,T_{max})$, where in this derivation, we have utilized the fact $-\frac{1}{2}-\frac{n}{2}(\frac{1}{2n}-\frac{1}{\infty}) > -1$, (1.5), and (3.26) with $p = 2n$. □

Proof of Theorem 1.1. On the basis of the extensibility criterion in Lemma 2.1, we can assert $T_{max} = \infty$ according to (1.6). The assertion follows directly from the implications of Lemmas 3.2 and 3.3, and the proof is completed. □

4. Conclusions

This study examines chemotaxis models incorporating acceleration effects and logistic source terms, formulated within a smooth, bounded domain $\Omega \subset \mathbb{R}^n$. Utilizing L^p-estimates alongside a meticulously developed bootstrap iteration technique, we explore the complex mathematical structures underlying the system. Our results establish the existence of globally bounded solutions for dimensions $n \geq 4$, under the conditions $\gamma > 0$ and $\alpha > \frac{3n+6}{n+8}$. This represents an improvement over prior research, which generally required the more restrictive condition $\alpha > \frac{1}{2} + \frac{n}{4}$. Consequently, the present work refines the parameter thresholds associated with global solution existence. These findings contribute to the advancement of partial differential equation theory by providing more precise criteria for the parameter α and enhancing the understanding of solution dynamics in chemotaxis systems.

Author contributions

Jiashan Zheng provided guidance in the theoretical calculations presented in this paper; Liqiong Pu took charge of drafting the manuscript. All authors participated in the final review of the manuscript and contributed equally to the completion of this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors affirm that they have no conflicting interests to declare.

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