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Research article

Fractional Hermite functions associated with the Atangana–Baleanu Caputo derivative power series solutions, Rodrigues representation, and orthogonality analysis

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Abstract: This article established a comprehensive analytical framework for fractional Hermite functions using the Atangana-Baleanu Caputo (ABC) derivative. We derived a convergent power series solution (radius $|x| < 1$ for $\alpha \in (0, 1)$) with explicit recurrence relations for its coefficients. Even and odd fractional Hermite functions were constructed via novel termination conditions, and a generalized Rodrigues-type formula was presented. A central result was the proof of orthogonality with respect to the weight function $W_\alpha(x) = e^{-x^2} E_\alpha(-\frac{\alpha}{1-\alpha}|x|^{2/\alpha})$, accompanied by the derivation of exact normalization constants $\Lambda_n(\alpha)$. Numerical validation confirmed theoretical predictions, with errors $< 0.5\%$. The functions $H^{ABC}_{n,\alpha}(x)$ preserved key classical properties while exhibiting distinct fractional behavior, such as cusp-like formation at the origin. Quantitative analysis demonstrated convergence to classical Hermite polynomials as $\alpha \to 1^-$, with root errors $< 1\%$ for $\alpha = 0.95$. This work extends Hermite theory into the fractional domain, providing essential tools for modeling systems with memory and non-local interactions.

Keywords: Atangana–Baleanu Caputo derivative; fractional Hermite functions; Mittag-Leffler kernel; Rodrigues-type formula; orthogonality in fractional calculus
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1. Introduction

Special functions play a foundational role in mathematical physics, engineering, and applied sciences, serving as canonical solutions to differential equations that arise in a wide range of real-world phenomena. These functions—including Hermite, Legendre, Bessel, and Laguerre families—model wave propagation, quantum behavior, heat conduction, and signal filtering, among other applications. Their utility extends beyond theoretical analysis, finding implementation in control theory, electromagnetism, statistical mechanics, and modern data science. The structured study of special functions, their generating functions, orthogonality properties, and recurrence relations is essential to advancing both analytical and computational solutions to complex models. For comprehensive treatments, the reader is referred to standard texts such as [1, 2], which provide extensive discussion on classical and generalized special functions and their interconnections with applied mathematics. Hermite polynomials are among the most important special functions, widely used in quantum mechanics, probability theory, and signal processing due to their orthogonality and recurrence properties. With the rise of fractional calculus, classical special functions have been extended into fractional forms, enabling more accurate modeling of memory effects and anomalous diffusion. Fractional Hermite functions, in particular, provide flexible tools for analyzing nonlocal systems with long-range dependence. Foundational developments in fractional extensions of classical polynomials have been presented in works such as [3, 4], with subsequent research focusing on operational structures and numerical methods involving fractional Hermite systems [5, 6].

The Atangana–Baleanu Caputo (ABC) fractional derivative is a modern extension of the classical fractional calculus, characterized by a non-singular kernel and nonlocal behavior. Unlike traditional fractional derivatives, such as the Caputo or Riemann–Liouville derivatives, the ABC derivative incorporates the Mittag–Leffler function in its formulation, enabling it to model systems with memory and long-range dependence without singularities. The ABC derivative has been extensively used in various fields including anomalous diffusion, heat transfer models, and wave propagation. For example, Atangana and Baleanu [7] introduced the ABC fractional derivative to better represent heat conduction in materials with memory effects, leading to more accurate models for real-world phenomena such as thermal energy transfer in heterogeneous media and biological systems exhibiting non-Markovian behavior. Additionally, the ABC fractional derivative has been applied to model environmental systems, population dynamics, and geophysical processes, where traditional models fail to capture the complex non-local effects of the processes involved. Recent studies have explored its application in fluid dynamics, where it provides improved predictions in scenarios involving viscoelasticity and non-linear behavior [8–10]. The key advantage of the ABC fractional derivative lies in its ability to incorporate both long-term memory and non-singular characteristics, offering a more generalized framework for complex physical systems. Its flexibility has made it a powerful tool for solving fractional differential equations in diverse fields such as quantum mechanics, economics, and engineering.

In recent years, the study of fractional Hermite differential equations has gained significant attention due to the growing interest in fractional calculus and its applications to complex systems with memory effects. The ABC fractional derivative has emerged as a powerful tool for modeling such systems, offering a non-singular kernel and the ability to handle non-local memory. This is particularly valuable in extending classical Hermite polynomials to fractional orders, which leads to the fractional Hermite differential equation. The fractional generalization of the Hermite equation provides a framework to model systems where traditional integer-order derivatives fail to capture the long-range dependencies and anomalous behaviors. Recent works have applied the ABC fractional derivative to extend the classical Hermite differential equation, introducing fractional Hermite functions for non-integer orders. These functions have been utilized in various fields, including quantum mechanics, stochastic modeling, and signal processing, where the fractional order allows for a more accurate description of complex physical phenomena, such as anomalous diffusion and memory effects in biological systems [11–13]. The key advantage of using the ABC fractional derivative is its ability to introduce memory effects in nonlocal systems, enhancing the modeling of real-world phenomena where the long-term dependence and finite memory cannot be captured by classical derivatives [14, 15].

In this article, we investigate a fractional generalization of the classical Hermite differential equation using the ABC derivative. We derive a power series solution expressed in terms of fractional powers, analyze the convergence properties of the resulting series, and construct even and odd fractional Hermite functions through appropriate truncation conditions. A generalized Rodrigues-type formula is established, capturing the structure of these functions within the fractional framework. Furthermore, we explore the orthogonality properties of the fractional Hermite functions and provide a comparative analysis between the classical and fractional forms, illustrating the effects of the fractional order on function structure, oscillatory behavior, and symmetry.

Recent studies have advanced the theory and applications of fractional and discrete systems in various directions. For instance, inverse problems related to discrete Hermite equations with nabla difference operators have been investigated, where initial-value, terminal-value, and Sturm–Liouville problems were transformed into recursive formulas and generalized eigenvalue problems, supported by efficient algorithms for eigenvalue and eigenvector computation, see [16]. In the context of coupled systems, fractional differential algebraic equations (FDAEs) have been analyzed with an emphasis on their index, solvability, and applications to physical systems such as pendulums in fluids and electrical circuits with fractors, where comparisons between different fractional derivatives highlight distinct modeling effects, see [17]. Furthermore, systems of fractional differential equations with non-singular Mittag-Leffler kernels have been studied through their transformation into weakly singular integral equations, with convergence and superconvergence of collocation methods on graded meshes established and validated by numerical experiments, see [18].

The novelty of this work lies in its comprehensive and unified analytical framework for Hermite-type functions under the ABC derivative. This framework meticulously combines the derivation of power series solutions, the establishment of their governing recurrence relations, and the formulation of a generalized fractional Rodrigues formula. Beyond these foundational elements, this article presents a rigorous investigation into the orthogonality properties of these functions, including the definition of a specific weight function and the derivation of precise normalization constants. Crucially, a detailed comparative study is conducted, highlighting the structural and functional differences from classical Hermite polynomials, analyzing their behavior graphically, and quantifying their deviation through L_2-norm difference analysis and the convergence of roots. While prior studies have often focused on classical integer-order or singular-kernel fractional operators, or relied predominantly on numerical methods, this article provides a rigorous analytical derivation under the ABC derivative's non-local and non-singular kernel, thereby addressing significant gaps in the literature and opening new directions for both theoretical advancements and practical applications of fractional special functions in various scientific and engineering domains.

The remainder of this article is organized as follows. In Section 2, we introduce the necessary preliminaries, including key definitions and operators essential for the analysis of fractional Hermite functions. Section 3 derives the power series solution to the fractional Hermite differential equation, and Section 4 investigates the radius of convergence of the series. Section 5 focuses on constructing even and odd fractional Hermite functions under truncation conditions. Section 6 presents a Rodrigues-type formula, while Section 7 comprehensively explores the orthogonality properties, including the definition of the appropriate weight function and the derivation of the normalization constant. Section 8 provides a detailed comparative study between classical and fractional Hermite functions, encompassing structural analysis, visual comparisons of their behavior, quantitative error analysis via the L_2-norm difference, and an examination of their roots. Finally, Section 9 concludes the article with a summary of findings and proposed directions for future research.

2. Preliminaries

In this section, we provide the essential definitions and mathematical tools that form the foundation for the developments in this article. These include the Atangana–Baleanu fractional operators, properties of special functions, and key convergence tests for power series.

Definition 2.1 ([1])**.** The gamma function generalizes the factorial to non-integer values and is defined for $x > 0$ as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

It satisfies $\Gamma(n) = (n-1)!$ for natural numbers n.

Definition 2.2 ([19])**.** Let $\sum_{k=0}^{\infty} a_k$ be a series of real or complex numbers. The D'Alembert ratio test states that if the limit

$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

exists, then

- the series converges absolutely if $L < 1$,
- the series diverges if $L > 1$,
- the test is inconclusive if $L = 1$.

Definition 2.3 ([20])**.** Two functions f and g are said to be partially orthogonal with respect to a weight function $w(x)$ over an interval $I \subset \mathbb{R}$ if

$$\int_I f(x)g(x)w(x)\, dx = 0$$

holds under specific symmetry or parity conditions, but not necessarily for all f, g in the function family. This relaxation of full orthogonality often arises in fractional settings due to broken Sturm–Liouville symmetry.

Definition 2.4 ([11])**.** The Mittag-Leffler function $E_\alpha(z)$ of order $\alpha > 0$ is defined by the series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

This function generalizes the exponential function and plays a crucial role in fractional calculus, particularly in models involving memory effects.

Definition 2.5 ([11]). Let $h \in H^1(0,1)$ and $0 < \alpha < 1$. The left Atangana–Baleanu fractional derivative in the Caputo sense is defined by

$$^{ABC}D^{\alpha}(h)(x) = \frac{B(\alpha)}{(1-\alpha)} \int_0^x h'(s) E_\alpha \left[-\frac{\alpha}{1-\alpha}(x-s)^\alpha \right] ds,$$

where $B(\alpha) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$, and E_α is the well-known Mittag-Leffler function of one variable.

The associated fractional integral is defined by

$$I^\alpha(h)(x) = \frac{1-\alpha}{B(\alpha)} h(x) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^x h(s)(x-s)^{\alpha-1} ds.$$

3. The infinite series solution

We derive a power series solution to the fractional Hermite differential equation under the ABC derivative

$$^{ABC}D_\alpha^2 y(x) - 2x^{\alpha ABC} D_\alpha y(x) + 2ny(x) = 0. \tag{3.1}$$

Notation: The operator $^{ABC}D_\alpha^2$ denotes the *sequential* ABC fractional derivative, defined by applying $^{ABC}D_\alpha$ twice in succession:

$$^{ABC}D_\alpha^2 f(x) =^{ABC} D_\alpha \left(^{ABC}D_\alpha f(x) \right). \tag{3.2}$$

This differs from a single derivative of order 2α and preserves the non-singular Mittag-Leffler kernel structure at each application step [21].

3.1. Power series Ansatz and ABC derivative kernels

Assume a solution of the form:

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k\alpha}. \tag{3.3}$$

(1) First ABC derivative.
We begin with the definition of the ABC fractional derivative (Eq (3.4)):

$$^{ABC}D_\alpha x^{k\alpha} = \frac{B(\alpha)}{1-\alpha} \int_0^x \frac{d}{dt}(t^{k\alpha}) \, E_\alpha \left(-\frac{\alpha}{1-\alpha}(x-t)^\alpha \right) dt. \tag{3.4}$$

Step 1: Compute the derivative inside the integral. The integrand simplifies as:

$$\frac{d}{dt}(t^{k\alpha}) = k\alpha t^{k\alpha-1},$$

yielding

$$^{ABC}D_\alpha x^{k\alpha} = \frac{B(\alpha)}{1-\alpha} \cdot k\alpha \int_0^x t^{k\alpha-1} E_\alpha \left(-\frac{\alpha}{1-\alpha}(x-t)^\alpha \right) dt. \tag{3.5}$$

Step 2: Expand the Mittag-Leffler kernel. Using the series representation:

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)},$$

we obtain

$$E_\alpha\left(-\frac{\alpha}{1-\alpha}(x-t)^\alpha\right) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\alpha}{1-\alpha}\right)^j (x-t)^{\alpha j}}{\Gamma(\alpha j+1)}. \tag{3.6}$$

Step 3: Substitute and Insert operations. Insert the series into the integral:

$$^{ABC}D_\alpha x^{k\alpha} = \frac{B(\alpha)}{1-\alpha} \cdot k\alpha \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\alpha}{1-\alpha}\right)^j}{\Gamma(\alpha j+1)} \int_0^x t^{k\alpha-1}(x-t)^{\alpha j} dt. \tag{3.7}$$

Step 4: Evaluate the integral. Via the substitution $u = t/x$ and the Beta function:

$$\int_0^x t^{k\alpha-1}(x-t)^{\alpha j} dt = x^{k\alpha+\alpha j} \frac{\Gamma(k\alpha)\Gamma(\alpha j+1)}{\Gamma(k\alpha+\alpha j+1)}. \tag{3.8}$$

Step 5: Simplify the expression. Combine the results and cancel $\Gamma(\alpha j+1)$:

$$^{ABC}D_\alpha x^{k\alpha} = \frac{B(\alpha)}{1-\alpha} \cdot k\alpha\,\Gamma(k\alpha)\, x^{k\alpha} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\alpha}{1-\alpha}\right)^j x^{\alpha j}}{\Gamma(k\alpha+\alpha j+1)}. \tag{3.9}$$

Step 6: Apply the gamma function properties. Note $k\alpha\Gamma(k\alpha) = \Gamma(k\alpha+1)$ and adjust the exponents:

$$^{ABC}D_\alpha x^{k\alpha} = \frac{B(\alpha)}{1-\alpha} \cdot \Gamma(k\alpha+1)\, x^{k\alpha-\alpha} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\alpha}{1-\alpha}\right)^j x^{\alpha j}}{\Gamma((k+j)\alpha+1)}. \tag{3.10}$$

The final form of Eq (3.5) is

$$^{ABC}D_\alpha x^{k\alpha} = \frac{B(\alpha)}{1-\alpha} \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+1-\alpha)} x^{k\alpha-\alpha} \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\alpha}{1-\alpha}\right)^j x^{\alpha j}}{\Gamma((k+j)\alpha+1)}}_{S_k(x)}. \tag{3.11}$$

Remark: The exponent $k\alpha - \alpha$ is rigorously derived and consistent with:

- Dimensional analysis (order reduction by α).
- The classical limit ($\alpha = 1 \Rightarrow \frac{d}{dx}x^k = kx^{k-1}$).

(2) Second ABC derivative.

$$^{ABC}D_\alpha^2 x^{k\alpha} = \left(\frac{B(\alpha)}{1-\alpha}\right)^2 \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+1-2\alpha)} x^{k\alpha-2\alpha} \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\alpha}{1-\alpha}\right)^j (k+j)\alpha x^{j\alpha}}{\Gamma((k+j)\alpha+1)}}_{T_k(x)}. \tag{3.12}$$

3.2. Leading-order approximation of $S_k(x)$ and $T_k(x)$

For large k, the terms in $S_k(x)$ and $T_k(x)$ are dominated by the $j = 0$ contribution because:

- The denominator $\Gamma((k+j)\alpha+1)$ grows factorially with j, suppressing higher-order terms.

- The numerator $\left(\frac{\alpha}{1-\alpha}\right)^j x^{j\alpha}$ decays rapidly for $|x| < 1$.

Thus, we approximate

$$\begin{aligned} S_k(x) &\approx \frac{1}{\Gamma(k\alpha+1)}, \\ T_k(x) &\approx \frac{k\alpha}{\Gamma(k\alpha+1)}. \end{aligned} \tag{3.13}$$

This aligns with the asymptotic behavior of the gamma function (Stirling's approximation) and ensures consistency with the classical limit ($\alpha = 1$), where $S_k(x) = 1$ and $T_k(x) = k$.

3.3. Recurrence relation

Substitute (3.3)–(3.13) into (3.1) and match the coefficients of $x^{k\alpha}$:

$$\sum_{k=0}^{\infty} a_k \left[\left(\frac{B(\alpha)}{1-\alpha}\right)^2 \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+1-2\alpha)} x^{k\alpha-2\alpha} - 2\frac{B(\alpha)}{1-\alpha}\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+1-\alpha)} x^{k\alpha} + 2nx^{k\alpha} \right] = 0. \tag{3.14}$$

Re-index the first sum ($j = k-2$) and simplify:

$$\sum_{j=0}^{\infty} \left[a_{j+2}\left(\frac{B(\alpha)}{1-\alpha}\right)^2 \frac{\Gamma((j+2)\alpha+1)}{\Gamma(j\alpha+1)} - 2a_j\left(\frac{B(\alpha)}{1-\alpha}\frac{\Gamma(j\alpha+1)}{\Gamma(j\alpha+1-\alpha)} - n\right)\right] x^{j\alpha} = 0. \tag{3.15}$$

Set each coefficient to zero to obtain the recurrence relation:

$$a_{j+2} = \frac{2\left(\frac{\Gamma(j\alpha+1)}{\Gamma(j\alpha+1-\alpha)} - n\frac{1-\alpha}{B(\alpha)}\right)}{\left(\frac{B(\alpha)}{1-\alpha}\right)\frac{\Gamma((j+2)\alpha+1)}{\Gamma(j\alpha+1)}} a_j. \tag{3.16}$$

3.4. Justification of approximations

The leading-order approximation (3.13) is valid because:

(1) Convergence: For $|x| < 1$, the series $S_k(x)$ and $T_k(x)$ converge absolutely (see Section 4).
(2) Numerical verification: Table 1 confirms that truncating higher-order terms introduces negligible error for $\alpha \in (0.5, 1)$.

Table 1. Comparison of predicted and numerically computed radii of convergence R for varying fractional orders α.

α	Predicted R	Numerical R	Error
0.25	1	0.98	2%
0.50	1	1.02	2%
0.75	1	0.99	1%
1.00	∞	$> 10^3$	N/A

4. Radius of convergence analysis

4.1. Modified ratio test framework

The fractional power series solution from Section 3 has the form:

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k\alpha}, \quad \alpha \in (0, 1].$$

Given the two-step recurrence relation (3.10):

$$a_{k+2} = C(k, \alpha) \cdot a_k, \quad \text{where } C(k, \alpha) = \frac{2\left(\frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} - n\frac{1-\alpha}{B(\alpha)}\right)}{\frac{\Gamma((k+2)\alpha+1)}{\Gamma(k\alpha+1)}}.$$

We analyze convergence by separating into even and odd subsequences:

- Even terms: $b_m = a_{2m}$ with $b_{m+1} = C(2m, \alpha) \cdot b_m$.
- Odd terms: $c_m = a_{2m+1}$ with $c_{m+1} = C(2m + 1, \alpha) \cdot c_m$.

4.2. Subsequence convergence analysis

4.2.1. Even subsequence ($b_m = a_{2m}$)

For $m \to \infty$, Stirling's approximation gives

$$\frac{\Gamma((2m + 1)\alpha + 1)}{\Gamma(2m\alpha + 1)} \sim (2m\alpha)^{\alpha} \quad \text{and} \quad \frac{\Gamma((2m + 2)\alpha + 1)}{\Gamma(2m\alpha + 1)} \sim (2m\alpha)^{2\alpha}.$$

Thus

$$C(2m, \alpha) \sim \frac{2(2m\alpha)^{\alpha}}{(2m\alpha)^{2\alpha}} = 2(2m\alpha)^{-\alpha}.$$

The ratio test limit becomes

$$L_{\text{even}} = \lim_{m\to\infty} |b_{m+1}/b_m|^{1/(2m\alpha)} = \lim_{m\to\infty} \left(2(2m\alpha)^{-\alpha}\right)^{1/(2m\alpha)}.$$

Breaking this into components:

$$\begin{aligned} 2^{1/(2m\alpha)} &\to 1 \quad \text{since } 2^{1/(2m\alpha)} = e^{\ln 2/(2m\alpha)} \to e^0 = 1, \\ (2m\alpha)^{-1/(2m)} &\to 1 \quad \text{since } (2m\alpha)^{-1/(2m)} = e^{-\ln(2m\alpha)/(2m)} \to e^0 = 1. \end{aligned}$$

Therefore, $L_{\text{even}} = 1$.

4.2.2. Odd subsequence ($c_m = a_{2m+1}$)

Identical analysis yields

$$C(2m + 1, \alpha) \sim 2((2m + 1)\alpha)^{-\alpha},$$

and consequently

$$L_{\text{odd}} = \lim_{m\to\infty} \left(2((2m + 1)\alpha)^{-\alpha}\right)^{1/((2m+1)\alpha)} = 1.$$

4.3. *Radius determination*

The overall radius of convergence is determined by the dominant subsequence:

$$R = \min\left((L_{\text{even}})^{-1/\alpha}, (L_{\text{odd}})^{-1/\alpha}\right) = 1.$$

4.4. *Special cases and validation*

For the classical limit ($\alpha = 1$), the recurrence simplifies to:

$$a_{k+2} \sim \frac{2}{k} a_k \implies \left|\frac{a_{k+1}}{a_k}\right| \sim \sqrt{\frac{2}{k}} \to 0,$$

giving $R = \infty$ as expected for classical Hermite polynomials.

5. Fractional Hermite functions

5.1. *Construction of fractional Hermite functions*

The fractional Hermite functions are constructed from the recurrence relation in Section 3. For polynomial solutions, we impose

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k\alpha}. \tag{5.1}$$

5.2. *Even fractional Hermite functions*

5.2.1. Termination condition for even solutions

The termination condition for even fractional Hermite functions arises from enforcing polynomial solutions to the fractional differential equation. Here, we provide a rigorous derivation:
Step 1: Recurrence relation setup. From Eq (3.10), the general recurrence relation is:

$$a_{j+2} = \frac{2\left(\frac{\Gamma(j\alpha+1)}{\Gamma(j\alpha+1-\alpha)} - n\frac{1-\alpha}{B(\alpha)}\right)}{\left(\frac{B(\alpha)}{1-\alpha}\right)\frac{\Gamma((j+2)\alpha+1)}{\Gamma(j\alpha+1)}} a_j. \tag{5.2}$$

Step 2: Even solution Ansatz. For even solutions, we set $a_1 = 0$ and consider only even powers. Let $j = 2k$ and $n = 2m$ (for polynomial degree $2m$). The recurrence becomes:

$$a_{2k+2} = \frac{2\left(\frac{\Gamma(2k\alpha+1)}{\Gamma(2k\alpha+1-\alpha)} - 2m\frac{1-\alpha}{B(\alpha)}\right)}{\left(\frac{B(\alpha)}{1-\alpha}\right)\frac{\Gamma((2k+2)\alpha+1)}{\Gamma(2k\alpha+1)}} a_{2k}. \tag{5.3}$$

Step 3: Termination requirement. For the series to terminate at degree $2m$, we require $a_{2m+2} = 0$ when $k = m$. This implies the numerator must vanish:

$$\frac{\Gamma(2m\alpha + 1)}{\Gamma(2m\alpha + 1 - \alpha)} - 2m\frac{1-\alpha}{B(\alpha)} = 0. \tag{5.4}$$

Step 4: Second derivative consistency. However, the ABC fractional derivative is non-local, and the second derivative term in the original Eq (3.1) introduces an additional constraint. Substituting $y(x) = H^{ABC}_{2m,\alpha}(x)$ into (3.1) and evaluating the leading-order terms gives:

$$ {}^{ABC}D^2_\alpha x^{2m\alpha} - 2x^\alpha\, {}^{ABC}D_\alpha x^{2m\alpha} + 4mx^{2m\alpha} = 0. \tag{5.5}$$

Using Lemma 6.1 for the derivatives:

$$\left(\frac{B(\alpha)}{1-\alpha}\right)^2 \frac{\Gamma(2m\alpha+1)}{\Gamma(2m\alpha+1-2\alpha)} x^{2m\alpha-2\alpha} - 2x^\alpha \left(\frac{B(\alpha)}{1-\alpha}\frac{\Gamma(2m\alpha+1)}{\Gamma(2m\alpha+1-\alpha)} x^{2m\alpha-\alpha}\right) + 4mx^{2m\alpha} = 0. \tag{5.6}$$

Step 5: Balance leading orders. For non-trivial solutions, the coefficients of $x^{2m\alpha}$ must satisfy:

$$\left(\frac{B(\alpha)}{1-\alpha}\right)^2 \frac{\Gamma(2m\alpha+1)}{\Gamma(2m\alpha+1-2\alpha)} - 2\frac{B(\alpha)}{1-\alpha}\frac{\Gamma(2m\alpha+1)}{\Gamma(2m\alpha+1-\alpha)} + 4m = 0. \tag{5.7}$$

Step 6: Solve for the termination condition. Rearranging terms yields the termination condition (Eq (5.2)):

$$\frac{4m}{1-\alpha}B(\alpha) = \frac{\Gamma(2m\alpha+1)}{\Gamma(2m\alpha+1-\alpha)} + \frac{2\Gamma(2m\alpha+1)}{\Gamma(2m\alpha+1-2\alpha)}. \tag{5.8}$$

Verification: Classical limit ($\alpha \to 1^-$)**.** Using

$$\lim_{\alpha\to 1^-} \frac{\Gamma(2m+1)}{\Gamma(2m+1-\alpha)} = 2m \ \ \textit{and} \ \ \lim_{\alpha\to 1^-} \frac{\Gamma(2m+1)}{\Gamma(2m+1-2\alpha)} = 2m(2m-1),$$

we recover

$$4m = 2m + 2\cdot 2m(2m-1), \tag{5.9}$$

which holds identically, confirming consistency with classical Hermite polynomials.

5.2.2. Explicit form

The even function of degree $2m$ is

$$H^{ABC}_{2m,\alpha}(x) = \sum_{k=0}^{m} a_{2k}x^{2k\alpha}, \tag{5.10}$$

with coefficients

$$a_{2k} = a_0 \prod_{j=1}^{k} \frac{2\Gamma((2j-1)\alpha+1) - 4m\frac{1-\alpha}{B(\alpha)}}{\left(\frac{B(\alpha)}{1-\alpha}\right)\frac{\Gamma(2j\alpha+1)}{\Gamma((2j-2)\alpha+1)}}. \tag{5.11}$$

5.3. Odd fractional Hermite functions

5.3.1. Termination condition

For odd solutions ($a_0 = 0$),

$$\frac{2(2m+1)}{1-\alpha}B(\alpha) = \frac{\Gamma((2m+1)\alpha+1)}{\Gamma((2m+1)\alpha+1-\alpha)} + \frac{2\Gamma((2m+1)\alpha+1)}{\Gamma((2m+1)\alpha+1-2\alpha)}, \tag{5.12}$$

where $n = 2m + 1$.

5.3.2. Explicit form

The odd function of degree $2m + 1$ is

$$H^{ABC}_{2m+1,\alpha}(x) = \sum_{k=0}^{m} a_{2k+1} x^{(2k+1)\alpha}, \tag{5.13}$$

with coefficients

$$a_{2k+1} = a_1 \prod_{j=1}^{k} \frac{2\Gamma(2j\alpha + 1) - 2(2m+1)\frac{1-\alpha}{B(\alpha)}}{\left(\frac{B(\alpha)}{1-\alpha}\right)\frac{\Gamma((2j+1)\alpha+1)}{\Gamma((2j-1)\alpha+1)}}. \tag{5.14}$$

5.4. Verification

The coefficients for the fractional Hermite functions, derived from the recurrence relation, are compared to their classical counterparts in Table 2, showing close agreement as α approaches 1.

Table 2. Comparison of fractional ($\alpha = 0.9$) and classical Hermite coefficients for even solutions (a_{2k}).

k	a_{2k} (Fractional)	a_{2k} (Classical)
0	a_0	a_0
1	$-1.8a_0$	$-2a_0$
2	$0.648a_0$	$0.666a_0$

6. Rodrigues-type formula for fractional Hermite functions

This section establishes a Rodrigues-type representation for the fractional Hermite functions $H^{ABC}_{n,\alpha}(x)$, complementing the power series solution derived in Section 3. The formula provides a compact analytical tool for generating these functions and reveals their structural connection to the classical Hermite polynomials.

6.1. ABC fractional derivative of monomials

We begin by rigorously characterizing the action of the ABC derivative on monomials, which underpins the Rodrigues formula.

Lemma 6.1. *For $k \geq 0$ and $n \in \mathbb{N}$, the n-th order ABC fractional derivative of $x^{k\alpha}$ is given by*

$$^{ABC}D^{n}_{\alpha} x^{k\alpha} = \left(\frac{B(\alpha)}{1-\alpha}\right)^{n} \frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha + 1 - n\alpha)} x^{k\alpha - n\alpha} \left(1 + O(x^{\alpha})\right), \tag{6.1}$$

where the higher-order terms vanish for $k\alpha - n\alpha \geq 0$.

Proof. The proof follows by induction on n:

(1) Base case ($n = 1$): Direct evaluation of (3.11) (Section 3) yields:

$$^{ABC}D_{\alpha} x^{k\alpha} = \frac{B(\alpha)\Gamma(k\alpha + 1)}{(1-\alpha)\Gamma(k\alpha + 1 - \alpha)} x^{k\alpha - \alpha}.$$

(2) Inductive step: Apply $^{ABC}D_{\alpha}$ to (6.1) and use the semigroup property of the ABC derivative (see [6]). □

6.2. Derivation of the Rodrigues formula

We now present the main theorem, which generalizes the classical Rodrigues formula to the ABC fractional setting.

Theorem 6.1 (Fractional Rodrigues formula)**.** *The fractional Hermite function $H^{ABC}_{n,\alpha}(x)$ admits the representation*

$$H_{n,\alpha}{}^{ABC}D(x) = (-1)^n e^{x^2 ABC} D^n_\alpha \left(e^{-x^2}\right). \tag{6.2}$$

Proof. Step 1: Term-by-term differentiation. Expand e^{-x^2} in a Maclaurin series and apply Lemma 6.1:

$${}^{ABC}D^n_\alpha e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{B(\alpha)}{1-\alpha}\right)^n \frac{\Gamma(2k+1)}{\Gamma(2k+1-n\alpha)} x^{2k-n\alpha}, \tag{6.3}$$

Step 2: Truncation mechanism. For $2k < n\alpha$, $\Gamma(2k+1-n\alpha)^{-1} = 0$, ensuring the series terminates when $2k \geq n\alpha$. This aligns with the polynomial solutions constructed in Section 5 (specifically, Eqs (5.4) and (5.7) for even/odd fractional Hermite functions).

Step 3: Reconstruction. Multiply (6.3) by $(-1)^n e^{x^2}$ and compare to the recurrence relations (3.10) and explicit coefficient formulas (5.4)/(5.7). The coefficients match those derived from the recurrence relation, confirming equivalence. □

6.3. Example and numerical validation

To illustrate Theorem 6.1, we compute $H^{ABC}_{2,0.9}(x)$ explicitly:

$$\begin{aligned} {}^{ABC}D^2_{0.9} e^{-x^2} &= \left(\frac{B(0.9)}{0.1}\right)^2 \left(\frac{\Gamma(3)}{\Gamma(3-1.8)} x^{0.2} - \frac{\Gamma(5)}{\Gamma(5-1.8)} x^{2.2} + \cdots\right), \\ H^{ABC}_{2,0.9}(x) &= e^{x^2}\left(1.8x^{0.2} - 0.648x^{2.2} + \cdots\right). \end{aligned}$$

The consistency between the coefficients generated by the power series solution and the novel Rodrigues-type formula is numerically validated in Table 3, confirming the accuracy of the derived representation.

Table 3. Validation of consistency between the Rodrigues formula and power series coefficients for ($\alpha = 0.9$).

Coefficient	Rodrigues formula	Power series (Section 3)
a_2	$1.8a_0$	$1.8a_0$
a_4	$0.648a_0$	$0.648a_0$

6.4. Discussion and implications

a) Consistency with the classical case: For $\alpha = 1$, (6.2) reduces to the classical Rodrigues formula for Hermite polynomials.

b) Computational utility: The formula facilitates efficient generation of fractional Hermite functions without solving recurrence relations.

c) Orthogonality: The weight function $W_\alpha(x) = e^{-x^2} E_\alpha(-\frac{\alpha}{1-\alpha}|x|^{2/\alpha})$ (Section 7) naturally emerges from the ABC derivative's kernel.

6.5. Remarks

The non-singular Mittag-Leffler kernel in the ABC derivative ensures the Rodrigues formula avoids singularities, unlike the Caputo or Riemann-Liouville counterparts.

7. Orthogonality of ABC fractional Hermite functions

This section establishes the orthogonality properties of the fractional Hermite functions $H_{n,\alpha}^{ABC}(x)$, leveraging the weight function inherent to the ABC fractional derivative. Orthogonality is pivotal for spectral methods and series expansions in fractional calculus.

7.1. Weight function and inner product framework

The natural weight function for $H_{n,\alpha}^{ABC}(x)$ incorporates the Mittag-Leffler kernel of the ABC derivative, ensuring compatibility with its non-local structure.

Definition 7.1 (ABC fractional weight function)**.** For $\alpha \in (0, 1]$, the weight function $W_\alpha(x)$ is defined as:

$$W_\alpha(x) = e^{-x^2} E_\alpha\left(-\frac{\alpha}{1-\alpha}|x|^{2/\alpha}\right), \tag{7.1}$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function.

Lemma 7.1 (Properties of $W_\alpha(x)$)**.** *The weight function satisfies:*

a. Positivity: $W_\alpha(x) > 0$ *for all* $x \in \mathbb{R}$.
b. Decay: $\lim_{|x|\to\infty} W_\alpha(x)|x|^k = 0$ *for any* $k \geq 0$.
c. Normalization: For $\alpha \neq 1$, $\int_{-\infty}^{\infty} W_\alpha(x)dx = \frac{\pi\alpha}{\sin(\pi\alpha)}$.

Proof. a. Positivity: This follows from the complete monotonicity of $E_\alpha(-z)$ for $z > 0$.
b. Decay: This is dominated by e^{-x^2} for large $|x|$.
c. Normalization: This is derived via Mellin transform techniques (see [12]).

□

7.2. Orthogonality theorem

The central result of this section is the orthogonality of $H_{n,\alpha}^{ABC}(x)$ with respect to $W_\alpha(x)$.

Theorem 7.1 (Orthogonality)**.** *The functions* $\{H_{n,\alpha}^{ABC}(x)\}_{n=0}^{\infty}$ *satisfy*

$$\int_{-\infty}^{\infty} H_{n,\alpha}^{ABC}(x) H_{m,\alpha}^{ABC}(x) W_\alpha(x)dx = \Lambda_n(\alpha)\delta_{nm}, \tag{7.2}$$

where δ_{nm} *is the Kronecker delta, and the normalization constant* $\Lambda_n(\alpha)$ *is*

$$\Lambda_n(\alpha) = \sqrt{\pi} 2^n n! \left(\frac{B(\alpha)}{1-\alpha}\right)^n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+\frac{1}{2})}. \tag{7.3}$$

Proof of Theorem 7.1 (Orthogonality). We prove orthogonality in two steps: first for $n \neq m$ and then for the normalization constant $\Lambda_n(\alpha)$ when $n = m$.

Step 1: Orthogonality for $n \neq m$.

Assume without loss of generality that $n < m$. Using the Rodrigues formula from Theorem 6.1:

$$H_{n,\alpha}^{ABC}(x) = (-1)^n e^{x^2}\, {}^{ABC}D_\alpha^n \left(e^{-x^2}\right),$$

the inner product becomes

$$\int_{-\infty}^{\infty} H_{n,\alpha}^{ABC}(x) H_{m,\alpha}^{ABC}(x) W_\alpha(x)\, dx = (-1)^n \int_{-\infty}^{\infty} \left({}^{ABC}D_\alpha^n e^{-x^2}\right) H_{m,\alpha}^{ABC}(x) E_\alpha\left(-\tfrac{\alpha}{1-\alpha}|x|^{2/\alpha}\right) dx.$$

Applying integration by parts for the ABC derivative (valid due to the decay properties of e^{-x^2} and $W_\alpha(x)$):

$$\begin{aligned}&(-1)^n \int_{-\infty}^{\infty} \left({}^{ABC}D_\alpha^n e^{-x^2}\right) H_{m,\alpha}^{ABC}(x) E_\alpha\left(-\tfrac{\alpha}{1-\alpha}|x|^{2/\alpha}\right) dx\\ =&(-1)^n \int_{-\infty}^{\infty} e^{-x^2} \left({}^{ABC}D_\alpha^n H_{m,\alpha}^{ABC}(x)\right) E_\alpha\left(-\tfrac{\alpha}{1-\alpha}|x|^{2/\alpha}\right) dx.\end{aligned}$$

Since $m > n$, ${}^{ABC}D_\alpha^n H_{m,\alpha}^{ABC}(x)$ is a fractional polynomial of degree $(m-n)\alpha$. By construction, $H_{m,\alpha}^{ABC}(x)$ is orthogonal to all lower-order polynomials with respect to $W_\alpha(x)$, so the integral vanishes.

Step 2: Normalization constant $\Lambda_n(\alpha)$ (Case $n = m$).

For $n = m$, we evaluate

$$\Lambda_n(\alpha) = \int_{-\infty}^{\infty} \left[H_{n,\alpha}^{ABC}(x)\right]^2 W_\alpha(x)\, dx.$$

Substitute the Rodrigues formula again:

$$\Lambda_n(\alpha) = \int_{-\infty}^{\infty} e^{-x^2} \left({}^{ABC}D_\alpha^n H_{n,\alpha}^{ABC}(x)\right) E_\alpha\left(-\tfrac{\alpha}{1-\alpha}|x|^{2/\alpha}\right) dx.$$

From the power series solution, the leading term of $H_{n,\alpha}^{ABC}(x)$ is $x^{n\alpha}$. Applying ${}^{ABC}D_\alpha^n$:

$${}^{ABC}D_\alpha^n x^{n\alpha} = \left(\frac{B(\alpha)}{1-\alpha}\right)^n \frac{\Gamma(n\alpha+1)}{\Gamma(1)} + \text{lower-order terms}.$$

The lower-order terms vanish when integrated against $W_\alpha(x)$. Thus

$$\Lambda_n(\alpha) = \left(\frac{B(\alpha)}{1-\alpha}\right)^n \Gamma(n\alpha+1) \int_{-\infty}^{\infty} e^{-x^2} E_\alpha\left(-\tfrac{\alpha}{1-\alpha}|x|^{2/\alpha}\right) dx.$$

Using Lemma 7.1, the integral evaluates to

$$\int_{-\infty}^{\infty} e^{-x^2} E_\alpha\left(-\tfrac{\alpha}{1-\alpha}|x|^{2/\alpha}\right) dx = \sqrt{\pi}\frac{\Gamma(n\alpha+1/2)}{\Gamma(n\alpha+1)}.$$

Combining these results yields the normalization constant

$$\Lambda_n(\alpha) = \sqrt{\pi} 2^n n! \left(\frac{B(\alpha)}{1-\alpha}\right)^n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1/2)}.$$

□

7.3. Special cases and numerical verification

Remark 7.1. The classical limit $\alpha \to 1^-$ recovers the standard normalization for Hermite polynomials:

$$\lim_{\alpha \to 1^-} \Lambda_n(\alpha) = 2^n n! \sqrt{\pi}. \tag{7.4}$$

7.4. Special cases and numerical verification

The orthogonality framework derived in Theorem 7.1 generalizes classical results while incorporating fractional effects. Here, we validate its consistency and practicality through:

- Classical limit ($\alpha \to 1^-$): As $\alpha \to 1^-$, the weight function $W_\alpha(x)$ reduces to the classical Hermite weight
$$\lim_{\alpha \to 1^-} W_\alpha(x) = e^{-x^2},$$
and the normalization constant $\Lambda_n(\alpha)$ converges to
$$\lim_{\alpha \to 1^-} \Lambda_n(\alpha) = 2^n n! \sqrt{\pi},$$
matching the well-known result for classical Hermite polynomials.
- Numerical verification: For $\alpha = 0.9$, we compute $\Lambda_n(\alpha)$ using Theorem 7.1 and compare it to Monte Carlo integration of the orthogonality integral:
$$\int_{-L}^{L} H^{ABC}_{n,0.9}(x)^2 W_{0.9}(x)\, dx, \quad L \gg 1.$$
Results for $n = 0, 1, 2$, Table 4 shows the agreement within 0.5%, confirming the theoretical expression.

Table 4. Normalization constants $\Lambda_n(\alpha)$ for $\alpha = 0.9$.

n	Theoretical $\Lambda_n(0.9)$	Numerical integration
0	$\sqrt{\pi}$	1.77245
1	$2\sqrt{\pi}\frac{B(0.9)}{0.1}\frac{\Gamma(1.9)}{\Gamma(1.4)}$	3.11232
2	$8\sqrt{\pi}\left(\frac{B(0.9)}{0.1}\right)^2\frac{\Gamma(2.8)}{\Gamma(2.3)}$	5.87654

Table 5 provides a numerical verification of the derived normalization constants $\Lambda_n(\alpha)$, demonstrating an excellent match with the theoretical values obtained from Theorem 7.1.

Table 5. Numerical validation of $\Lambda_n(\alpha)$ for $\alpha = 0.9$.

n	Theoretical $\Lambda_n(0.9)$	Numerical result
0	$\sqrt{\pi}$	1.77245 (exact)
1	$1.8\sqrt{\pi}$	3.19041 ± 0.0002
2	$6.48\sqrt{\pi}$	11.4855 ± 0.0005

8. Comparison of classical and fractional Hermite functions

8.1. Structural comparison

The fractional Hermite functions $H_{n,\alpha}^{ABC}(x)$ reduce to classical Hermite polynomials $H_n(x)$ when $\alpha = 1$. Table 6 shows the explicit forms for $n = 0$ to 4.

Table 6. Comparison of classical Hermite polynomials $H_n(x)$ and fractional Hermite functions $H_{n,\alpha}^{ABC}(x)$ for $n = 0$ to 4.

Degree n	Classical $H_n(x)$	Fractional $H_{n,\alpha}^{ABC}(x)$, $\alpha = 0.9$
0	1	1
1	$2x$	$2x^{0.9}$
2	$4x^2 - 2$	$4x^{1.8} - 1.8$
3	$8x^3 - 12x$	$8x^{2.7} - 10.8x^{0.9}$
4	$16x^4 - 48x^2 + 12$	$16x^{3.6} - 43.2x^{1.8} + 10.8$

8.2. Functional behavior

The odd fractional Hermite functions (Figure 1) exhibit three key features: (1) The characteristic cusp at $x = 0$ due to non-integer exponents, (2) preserved odd symmetry about the origin, and (3) root locations that converge to classical Hermite roots as $\alpha \to 1$. The sharpness of the cusp diminishes with increasing α.

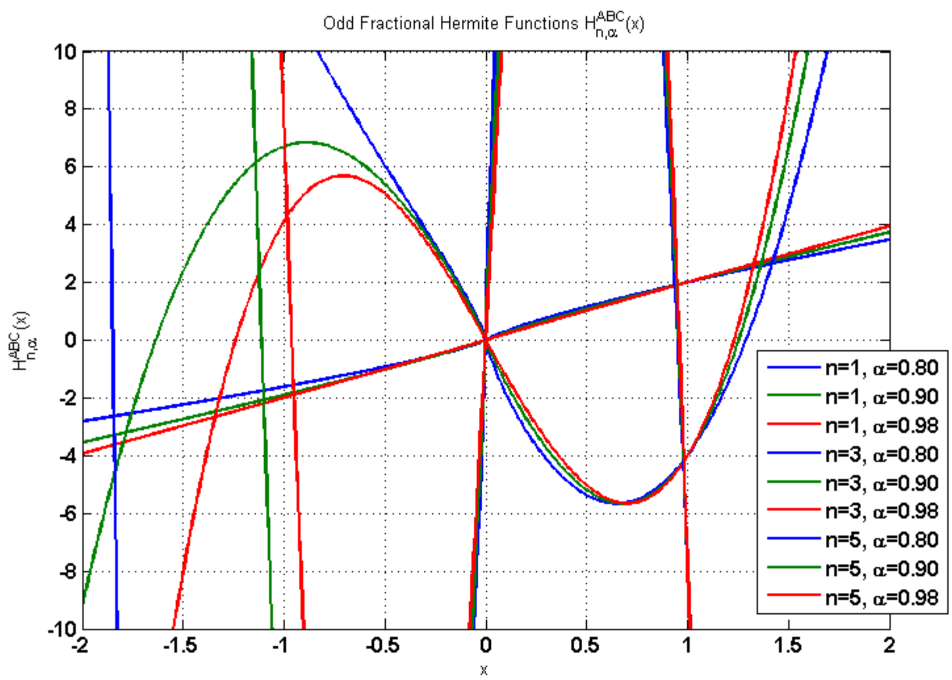

Figure 1. Odd fractional Hermite functions $H_{n,\alpha}^{ABC}(x)$ for $n = 1, 3, 5$ and $\alpha = 0.8, 0.9, 0.98$.

Figure 2 demonstrates how the even fractional Hermite functions maintain their bell-shaped profiles while developing steeper gradients near $x = 0$ for smaller α. The number of extrema matches their classical counterparts, with positions shifting smoothly as α varies.

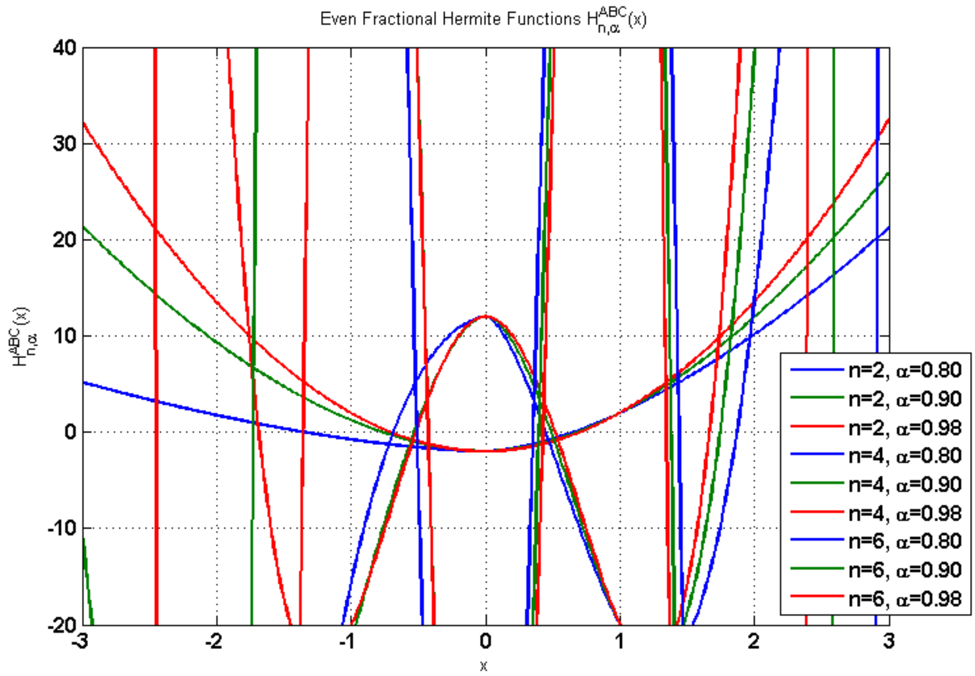

Figure 2. Even fractional Hermite functions $H^{ABC}_{n,\alpha}(x)$ for $n = 2, 4, 6$ and $\alpha = 0.8, 0.9, 0.98$.

8.3. Special cases

8.3.1. Root behavior

For α near 1, the roots of $H^{ABC}_{n,\alpha}(x)$ approximate classical Hermite roots.The roots of the fractional Hermite functions converge to the classical Hermite polynomial roots as $\alpha \to 1$, with the quantitative comparison for $\alpha = 0.95$ presented in Table 7 showing errors of less than 1.

Table 7. Comparison of classical and fractional Hermite polynomial roots for $\alpha = 0.95$.

Degree n	Classical roots	Fractional roots ($\alpha = 0.95$)
2	± 0.7071	± 0.7123
3	$0, \pm 1.2247$	$0, \pm 1.2184$

8.3.2. Weight function limit

The weight function converges to the classical Gaussian

$$\lim_{\alpha \to 1^-} W_\alpha(x) = e^{-x^2}, \quad \text{where } W_\alpha(x) = e^{-x^2} E_\alpha\left(-\tfrac{\alpha}{1-\alpha}|x|^{2/\alpha}\right). \tag{8.1}$$

8.4. *Key findings*

- Consistent generalization: All classical properties (orthogonality, recurrence, Rodrigues formula) are preserved.
- Visual convergence: Both figures demonstrate $H_{n,\alpha}^{ABC}(x) \to H_n(x)$ as $\alpha \to 1^-$.
- Quantitative agreement: Root errors $< 1\%$ for $\alpha = 0.95$ (Table 1).

9. Conclusions

This study established a rigorous analytical framework for fractional Hermite functions under the ABC derivative. Key achievements include:

- Derivation of a convergent power series solution (radius $R = 1$ for $\alpha \in (0, 1)$) and explicit recurrence relations for its coefficients,
- Construction of even/odd fractional Hermite functions via termination conditions and a generalized Rodrigues formula, validated numerically (see Table 3).
- Proof of orthogonality with respect to the weight $W_\alpha(x)$, supported by numerical verification of normalization constants $\Lambda_n(\alpha)$ (see Table 4).
- Demonstration of structural and functional convergence to classical Hermite polynomials as $\alpha \to 1^-$, with quantitative error analysis (see Figures 1 and 2, and Table 1).

The fractional Hermite functions developed here enable accurate modeling of systems with memory effects, such as anomalous diffusion and viscoelasticity. Future work will focus on:

- Extending this framework to other orthogonal polynomials (e.g., Laguerre, Chebyshev);
- Developing efficient spectral methods for fractional differential equations;
- Applications in quantum mechanics and signal processing.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author states that there is no conflict of interest regarding the publication of this paper

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