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Research article

Spectral analysis and integral representations of the tempered fractional Riesz derivative

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Abstract: In this work, we develop the harmonic analysis associated with the second-order differential operator $\mathscr{L}_\gamma = -\frac{d^2}{dx^2} - 2\gamma\frac{d}{dx} - \gamma^2$. Fractional powers of $\mathscr{L}_\gamma$ are defined via spectral representation, and a singular integral representation is provided. Furthermore, we establish the equivalence between the fractional powers of $\mathscr{L}_\gamma$ and a tempered Riesz derivative.

Keywords: tempered fractional derivative; Fourier transform
Mathematics Subject Classification: 26A33, 42B10

1. Introduction

Tempered fractional calculus is a generalization of classical fractional calculus. In tempered fractional calculus, the fractional derivatives and integrals are modified by multiplying them with an exponential factor, extending the standard concept of fractional calculus [2, 7, 10, 15, 22, 25, 26, 28].

For $\alpha > 0$ and $\lambda > 0$, the left and right fractional tempered derivatives are defined as [23, 27]:

$$D_+^{\alpha,\lambda} f(x) := e^{-\lambda x} D_+^\alpha(e^{\lambda x} f(x)), \quad D_-^{\alpha,\lambda} f(x) := e^{\lambda x} D_-^\alpha(e^{-\lambda x} f(x)),$$

where D_+^α and D_-^α are the classical fractional derivatives defined as follows:

- Right-sided Riemann-Liouville fractional derivative:

$$D_+^\alpha f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^{+\infty} \frac{f(y)}{(y-x)^{\alpha-m+1}}\, dy.$$

- Left-sided Riemann-Liouville fractional derivative:

$$D_-^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{\alpha-m+1}}\, dy.$$

Here, $m = \lceil \alpha \rceil$ represents the smallest integer greater than or equal to α.

The tempered Riesz derivative, denoted by $D_0^{\alpha,\lambda}$, is defined as a linear combination of $D_+^{\alpha,\lambda}$ and $D_-^{\alpha,\lambda}$, as given by [24]:

$$D_0^{\alpha,\lambda} f(x) = \frac{D_+^{\alpha,\lambda} f(x) + D_-^{\alpha,\lambda} f(x)}{2\cos\left(\frac{\alpha\pi}{2}\right)}, \quad \alpha \neq 1, 3, 5, \ldots .$$

This operator extends classical fractional calculus by introducing the tempering parameter λ, which controls exponential decay or growth, thereby broadening the scope of applications.

These formulations, initially developed for stochastic processes [1], have been applied in various branches of physics, including geophysics, statistical physics, plasma physics, and astrophysics [12, 14]. In addition to their role in physics, tempered fractional derivatives are also used in finance to model price fluctuations with semi-heavy tails [14]. In diffusion processes, the concept of a "truncated Lévy flight" was introduced to ensure finite variance [20]. Koponen [17] later demonstrated that the "tempered Lévy flight" achieves the same goal more effectively, offering additional advantages. This approach has been extended to finance, where tempered Lévy processes are used to model asset returns [8, 9, 18].

Tempered versions of fractional derivatives have also been developed in the Fourier-transform framework, including the tempered Liouville derivative [27, p. 335, Eq (18.72)] and the regularized Caputo derivative [23]. For more recent advances, see [13, 29] and the references therein.

These studies particularly emphasize the significance of the tempered Riesz derivative as a fundamental operator in many applications. Despite its recognized utility, there remains a notable scarcity of theoretical results regarding this operator. More critically, a fundamental question remains unanswered: Is the tempered Riesz derivative genuinely a fractional derivative in the conventional sense?

In this work, we investigate whether the tempered Riesz derivative can be regarded as a true fractional derivative in the conventional sense. To this end, we consider a second-order differential operator with constant coefficients, given by

$$\mathscr{L}_\gamma u := -\frac{d^2u}{dx^2} - 2\gamma\frac{du}{dx} - \gamma^2 u. \tag{1.1}$$

This unbounded operator admits a self-adjoint extension with a simple spectrum covering the entire interval $[0, \infty)$.

Our approach is based on the harmonic analysis associated with $\mathscr{L}_\gamma$. Specifically, we introduce a Fourier transform $\mathscr{F}_\gamma$ adapted to $\mathscr{L}_\gamma$, which we refer to as the *tempered Fourier-cosine transform*. Additionally, we define a new generalized translation operator.

In this work, we define the fractional powers of $\mathscr{L}_\gamma$ via the spectral representation:

$$\mathscr{L}_\gamma^{\alpha/2} f = \mathscr{F}_\gamma^{-1}\Big[\xi^\alpha \, (\mathscr{F}_\gamma f)(\xi)\Big]. \tag{1.2}$$

For $\alpha > 0$, the operator $\mathscr{L}_\gamma^{-\alpha/2}$ corresponds to the tempered Riesz fractional integral, denoted by $I_\gamma^\alpha f$. Specifically, for $\alpha > 0$ with $\alpha \neq 1, 3, 5, \ldots$, and for a suitably smooth even function f on $\mathbb{R}$, we show that the tempered Riesz fractional integral is given by:

$$I_\gamma^\alpha f(x) = \frac{1}{\cos\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha)} \int_0^\infty \sigma_x^\gamma f(y)\, e^{\gamma y} y^{\alpha-1}\, dy, \tag{1.3}$$

where the tempered translation operator σ_x^γ is defined as:

$$\sigma_x^\gamma f(y) = \frac{1}{2}\Big(f(x+y) + e^{-2\gamma\min\{x,y\}} f(|x-y|)\Big). \tag{1.4}$$

Furthermore, for $0 < \alpha < 2$, the fractional powers of $\mathscr{L}_\gamma^{\alpha/2}$ have the following pointwise representation:

$$\mathscr{L}_\gamma^{\alpha/2} f = \frac{2^{\alpha+1}\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\left|\Gamma\left(-\frac{\alpha}{2}\right)\right|}\int_0^\infty \frac{e^{-\gamma y} f(x) - \sigma_x^\gamma f(y)}{y^{\alpha+1}} e^{\gamma y} dy.$$

For $\gamma = 0$, we obtain the classical Riesz derivative:

$$\left(-\frac{d^2}{dx^2}\right)^{\alpha/2} f(x) = \frac{2^{\alpha}\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\left|\Gamma\left(-\frac{\alpha}{2}\right)\right|}\int_0^\infty \frac{2f(x) - f(x+y) - f(x-y)}{y^{\alpha+1}} dy.$$

In particular, the fractional operators $\mathscr{L}_\gamma^{\alpha/2}$ and $\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}$ are connected by

$$\mathscr{L}_\gamma^{\alpha/2} f(x) = e^{-\gamma x}\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}(e^{\gamma x} f)(x). \tag{1.5}$$

This formulation establishes a rigorous connection between the tempered Riesz derivative and the spectral properties of $\mathscr{L}_\gamma$, thereby reinforcing its interpretation as a fractional differential operator.

2. Elements of harmonic analysis related to the operator $\mathscr{L}_\gamma$

Let $\gamma \in \mathbb{R}$. We define the following spaces:

- $L_\gamma^p(0,\infty)$: The space $L_\gamma^p(0,\infty)$, for $1 \le p \le \infty$, consists of functions f with the norm:

$$\|f\|_{\gamma,p} = \begin{cases} \left(\int_0^\infty |f(x)|^p e^{p\gamma x}\, dx\right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in[0,\infty)} |f(x)|, & \text{if } p = \infty. \end{cases}$$

- Schwartz Space $\mathcal{S}_e(\mathbb{R})$: The Schwartz space $\mathcal{S}_e(\mathbb{R})$ consists of smooth even functions f that decay rapidly at infinity along with all their derivatives. Specifically, for all $m, n \ge 0$:

$$\sup_{x\in\mathbb{R}}\left|(1+x^2)^m f^{(2n)}(x)\right| < \infty.$$

- $\mathcal{S}_\gamma(\mathbb{R})$: The space $\mathcal{S}_\gamma(\mathbb{R})$ is defined as $e^{-\gamma x}\mathcal{S}_e(\mathbb{R})$. It is equipped with the topology induced by $\mathcal{S}_e(\mathbb{R})$, defined by the semi-norms:

$$\rho_{n,m}(f) = \sup_{x\ge 0}\left(e^{-\gamma x}(1+x^2)^m\left|f^{(2n)}(x)\right|\right).$$

By (1.5), the space $\mathcal{S}_\gamma(\mathbb{R})$ is invariant under the operator $\mathscr{L}_\gamma$.

2.1. Tempered Fourier-cosine integral transform

Consider the differential operator

$$\mathscr{L}_\gamma u := -\frac{d^2 u}{dx^2} - 2\gamma \frac{du}{dx} - \gamma^2 u,$$

This operator can be rewritten as

$$\mathscr{L}_\gamma u = e^{-\gamma x} \, (-\frac{d^2}{dx^2})(e^{\gamma x} u).$$

It immediately follows that the initial value problem

$$\begin{cases} \mathscr{L}_\gamma u(x) = \lambda^2 u(x), \\ u(0) = 1, \quad u'(0) = -\gamma, \end{cases}$$

admits the unique solution

$$\phi_\gamma(x,\lambda) = e^{-\gamma x} \cos(\lambda x).$$

One may view the tempered kernel $\phi_\gamma(x,\lambda)$ as a regularization of $\cos(\lambda x)$.
We define the domain of the operator $\mathscr{L}_\gamma$ by

$$D_{\mathscr{L}_\gamma} = \Big\{u \in L^2_\gamma(0,\infty) : \mathscr{L}_\gamma u \in L^2_\gamma(0,\infty) \quad \text{and} \quad u'(0) + \gamma\, u(0) = 0\Big\},$$

where both u and $\mathscr{L}_\gamma u$ are interpreted in the sense of distributions.

Moreover, the operator $\mathscr{L}_\gamma$ can also be expressed as

$$\mathscr{L}_\gamma u(x) = -e^{-2\gamma x} \, \frac{d}{dx}\Big(e^{2\gamma x}\frac{du}{dx}(x)\Big) - \gamma^2 u(x).$$

This alternative representation demonstrates that the operator $(\mathscr{L}_\gamma,\, D_{\mathscr{L}_\gamma})$ is self-adjoint. By employing the spectral decomposition of the second derivative operator $-\frac{d^2}{dx^2}$, one deduces that its spectrum is simple and coincides with $[0,\infty)$. For further details on the spectral analysis of operators of the form

$$-\frac{1}{A(x)} \, \frac{d}{dx}\Big(A(x)\frac{du}{dx}\Big),$$

the reader is referred to [11].

For $f \in L^1_\gamma(0,\infty)$, the tempered Fourier-cosine integral transform is defined by

$$(\mathscr{F}_\gamma f)(\lambda) = \int_0^\infty f(x)\, \phi_\gamma(x,\lambda)\, d\mu_\gamma(x), \quad \lambda \in \mathbb{R},$$

with the measure given by

$$d\mu_\gamma(x) = e^{2\gamma x}\, dx.$$

In the case $\gamma = 0$, this transform reduces to the classical Fourier-cosine transform:

$$(\mathscr{F}_0 f)(\lambda) = \int_0^\infty f(x) \cos(\lambda x)\, dx, \quad \lambda \in [0,\infty). \tag{2.1}$$

This generalization incorporates an exponential tempering factor, which improves the transform's effectiveness for functions exhibiting specific decay properties—such as those encountered in tempered fractional calculus.

Furthermore, note that the mapping

$$(M_\gamma f)(x) := e^{\gamma x} f(x)$$

defines an isometry from $L^p_\gamma(0,\infty)$ onto $L^p(0,\infty)$, with the inverse operator given by $M_\gamma^{-1} = M_{-\gamma}$. Consequently, the tempered Fourier-cosine transform can be expressed in terms of the classical Fourier-cosine transform as

$$\mathscr{F}_\gamma = \mathscr{F}_0 \circ M_\gamma. \tag{2.2}$$

With this relationship in mind, and by using the well-known properties of the classical Fourier-cosine transform, we can deduce the following results.

Theorem 2.1. *Let $\gamma \in \mathbb{R}$. The tempered Fourier-cosine transform satisfies the following properties:*

(i) If $f \in L^1_\gamma(0,\infty)$, then $\mathscr{F}_\gamma f \in \mathscr{C}_e(\mathbb{R})$, meaning that $\mathscr{F}_\gamma f$ is an even, continuous function that vanishes at infinity.

(ii) The tempered Fourier-cosine transform is a topological isomorphism of $\mathcal{S}_\gamma(\mathbb{R})$.

(iii) If $f \in L^1_\gamma(0,\infty)$ and $\mathscr{F}_\gamma f \in L^1(0,\infty)$, then the inversion formula holds:

$$f(x) = \frac{2}{\pi}\int_0^\infty \mathscr{F}_\gamma f(\lambda)\,\phi_\gamma(x,\lambda)\,d\lambda, \quad \text{for almost every } x \in [0,\infty).$$

(iv) For all $f \in L^1_\gamma([0,\infty))$ and $g \in L^1([0,\infty))$, the duality relation is satisfied:

$$\frac{2}{\pi}\int_0^\infty (\mathscr{F}_\gamma f)(\lambda)\,g(\lambda)\,d\lambda = \int_0^\infty f(x)\,(\mathscr{F}_\gamma^{-1} g)(x)\,d\mu_\gamma(x),$$

where the inverse transform $\mathscr{F}_\gamma^{-1}$ is given by:

$$(\mathscr{F}_\gamma^{-1} g)(x) = \frac{2}{\pi}\int_0^\infty g(\lambda)\phi_\gamma(x,\lambda)\,d\lambda, \quad x \in [0,\infty).$$

(v) For every $f \in L^1_\gamma(0,\infty) \cap L^2_\gamma(0,\infty)$, the Plancherel formula holds:

$$\frac{2}{\pi}\int_0^\infty |(\mathscr{F}_\gamma f)(\lambda)|^2\,d\lambda = \int_0^\infty |f(x)|^2\,d\mu_\gamma(x).$$

The tempered Fourier-cosine transform extends uniquely as an isometry from $L^2_\gamma(0,\infty)$ onto the space of square-integrable functions on $[0,\infty)$ with respect to the measure $\frac{2}{\pi}d\lambda$. The inversion formula holds in the L^2-sense:

$$f(x) = \frac{2}{\pi}\int_0^\infty \mathscr{F}_\gamma f(\lambda)\,\phi_\gamma(x,\lambda)\,d\lambda.$$

(vi) Define the domain:

$$D_\gamma = \left\{ f \in L^2_\gamma([0,\infty)) \;:\; \int_0^\infty \lambda^4 |\mathscr{F}_\gamma f(\lambda)|^2\,d\lambda < \infty \right\}.$$

For every $f \in D_\gamma$, we have:

$$\mathscr{F}_\gamma(\mathscr{L}_\gamma f)(\lambda) = \lambda^2 \mathscr{F}_\gamma f(\lambda).$$

2.2. Tempered translation operators

Definition 2.2. *Let $\gamma \in \mathbb{R}$. The tempered translation operators σ_x^γ, for $x \in [0,\infty)$, associated with the operator $\mathscr{L}_\gamma$, are defined for continuous functions on $[0,\infty)$ by:*

$$\sigma_x^\gamma f(y) = \frac{1}{2}\Big(f(x+y) + e^{-2\gamma \min\{x,y\}} f(|x-y|)\Big). \tag{2.3}$$

The operators σ_x^γ for $x \geq 0$ satisfy the following properties:

(i) For all $f \in \mathscr{C}([0,\infty))$, we have:

$$\sigma_0^\gamma f(y) = f(y), \quad \text{and} \quad \sigma_x^\gamma f(y) = \sigma_y^\gamma f(x).$$

(ii) The product formula holds:

$$\sigma_x^\gamma \phi_\gamma(y,\lambda) = \phi_\gamma(x,\lambda)\phi_\gamma(y,\lambda), \quad \forall \lambda \in \mathbb{C}.$$

(iii) If $f \in \mathscr{C}([0,\infty))$ satisfies $0 \leq f \leq 1$, then:

$$0 \leq \sigma_x^\gamma f \leq 1.$$

Proposition 2.3. *(i) Let $f \in L_\gamma^p(0,\infty)$ with $1 \leq p \leq \infty$. Then, for all $x \in [0,\infty)$, the operator $\sigma_x^\gamma f$ is well-defined as an element of $L_\gamma^p(0,\infty)$, and it satisfies the following norm estimate:*

$$\|\sigma_x^\gamma f\|_{p,\gamma} \leq \|f\|_{p,\gamma}.$$

(ii) Let $1 \leq p,q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_\gamma^p(0,\infty)$ and $g \in L_\gamma^q(0,\infty)$, then the following Parseval's identity holds:

$$\int_0^\infty \sigma_x^\gamma f(y) g(y)\, d\mu_\gamma(y) = \int_0^\infty f(y)\sigma_x^\gamma g(y)\, d\mu_\gamma(y).$$

(iii) For $f \in L_\gamma^2[0,\infty)$, we have

$$(\mathscr{F}_\gamma \sigma_x^\gamma f)(\lambda) = \phi_\gamma(x,\lambda)\mathscr{F}_\gamma f(\lambda).$$

Proof. The proof of (*i*) is straightforward, so we focus on (*ii*). The proof of (*iii*) follows from (*ii*). To prove (*ii*), we expand the definition of the translation operator σ_x^γ:

$$\int_0^\infty \sigma_x^\gamma f(y) g(y)\, d\mu_\gamma(y) = \int_0^\infty \frac{1}{2}\Big(f(x+y) + e^{-2\gamma \min(x,y)} f(|x-y|)\Big) g(y)\, d\mu_\gamma(y).$$

Splitting the integral into separate terms, we have:

$$\begin{aligned}\int_0^\infty \sigma_x^\gamma f(y) g(y)\, d\mu_\gamma(y) = {} & \frac{1}{2}\int_0^\infty f(x+y) g(y)\, d\mu_\gamma(y) \\ & + \frac{1}{2}\int_0^x f(x-y) e^{-2\gamma y} g(y)\, d\mu_\gamma(y) \\ & + \frac{1}{2}\int_x^\infty f(y-x) e^{-2\gamma x} g(y)\, d\mu_\gamma(y).\end{aligned}$$

Next, we make appropriate changes of variables in each term, and we recognize that the right-hand side matches the original form:

$$\int_0^\infty f(y)\sigma_x^\gamma g(y)\,d\mu_\gamma(y).$$

Thus, we conclude that the duality relation holds:

$$\int_0^\infty \sigma_x^\gamma f(y)g(y)\,d\mu_\gamma(y) = \int_0^\infty f(y)\sigma_x^\gamma g(y)\,d\mu_\gamma(y).$$

This completes the proof. □

3. Tempered Riesz fractional calculus

3.1. Tempered Riesz fractional integral

In this section, we present a formulation of the tempered Riesz fractional integral and establish its essential properties. We begin by recalling the classical Riesz fractional integral, often referred to as the Riesz potential.

The Riesz fractional integral of order $\alpha > 0$ is defined by [19, 27]

$$I^\alpha f(x) := \frac{1}{2\cos\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha)} \int_{\mathbb{R}} f(x-y)\,|y|^{\alpha-1}\,dy. \tag{3.1}$$

Its Fourier transform satisfies

$$\mathscr{F}(I^\alpha f)(\xi) = |\xi|^{-\alpha}\,\mathscr{F}f(\xi), \quad \xi \in \mathbb{R}, \tag{3.2}$$

where the Fourier transform $\mathscr{F}$ is defined by

$$(\mathscr{F}f)(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)\,dx, \quad \xi \in \mathbb{R}.$$

Note that if f is an even function on $\mathbb{R}$, then $(I^\alpha f)(x)$ is also an even function. In this case, the representation (3.1) can be rewritten as

$$I^\alpha f(x) = \frac{1}{\cos\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha)} \int_0^\infty \sigma_x^0 f(y)\,y^{\alpha-1}\,dy, \tag{3.3}$$

where the translation operator σ_x^0 is defined by

$$\sigma_x^0 f(y) = \frac{1}{2}\Big[f(x+y) + f(|x-y|)\Big].$$

Moreover, for even functions the Fourier transform $\mathscr{F}$ coincides with the Fourier-cosine transform $\mathscr{F}_0$, so that (3.2) may be equivalently expressed as

$$(\mathscr{F}_0\,I^\alpha f)(\xi) = |\xi|^{-\alpha}\,(\mathscr{F}_0 f)(\xi),$$

with $\mathscr{F}_0$ defined as in (2.1).

Definition 3.1. *For a suitable even function f on $\mathbb{R}$, the tempered Riesz fractional integral is defined by*

$$I^\alpha_\gamma f(x) := e^{-\gamma x}\, I^\alpha\Big(e^{\gamma x} f(x)\Big). \tag{3.4}$$

This definition is valid for any positive α except when α is an odd integer.

We note that a sufficient condition for the integrals in (3.4) to converge is that

$$f(x) = O(e^{-\gamma x} x^{-\alpha-\epsilon}), \quad \epsilon > 0, \quad x \to \infty. \tag{3.5}$$

Theorem 3.2. *Let $\alpha > 0$ with $\alpha \neq 1, 3, 5, \ldots$. Then, for every even continuous function on $\mathbb{R}$ satisfying the condition* (3.5)*, the tempered Riesz fractional integral is given by*

$$I^\alpha_\gamma f(x) = \frac{1}{\cos\!\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha)} \int_0^\infty \sigma^\gamma_x f(y)\, e^{\gamma y}\, y^{\alpha-1}\, dy, \tag{3.6}$$

where $\sigma^\gamma_x f(y)$ is the tempered translation operator defined in (2.3).

Proof. Using the representation (3.3) for I^α, we have for $x > 0$

$$I^\alpha_\gamma f(x) = \frac{e^{-\gamma x}}{\cos\!\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha)} \int_0^\infty \sigma^0_x\Big(e^{\gamma(\cdot)} f\Big)(y)\, y^{\alpha-1}\, dy.$$

Since the (classical) translation operator σ^0_x acts via

$$\sigma^0_x\Big(e^{\gamma(\cdot)} f\Big)(y) = \frac{1}{2}\Big[e^{\gamma(x+y)} f(x+y) + e^{\gamma|x-y|} f(|x-y|)\Big],$$

we can write

$$I^\alpha_\gamma f(x) = \frac{e^{-\gamma x}}{2\cos\!\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha)} \int_0^\infty \Big[e^{\gamma(x+y)} f(x+y) + e^{\gamma|x-y|} f(|x-y|)\Big]\, y^{\alpha-1}\, dy.$$

Noting that

$$\min\{x, y\} = \frac{1}{2}\Big(x + y - |x-y|\Big)$$

one verifies that the above expression is equivalent to

$$I^\alpha_\gamma f(x) = \frac{1}{\cos\!\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha)} \int_0^\infty \sigma^\gamma_x f(y)\, e^{\gamma y}\, y^{\alpha-1}\, dy,$$

which is precisely the representation stated in (3.6). □

Lemma 3.3. *Let $0 < \alpha < 1$ and let f be an even function in $L^1_\gamma([0,\infty))$. Then, for every $\xi > 0$,*

$$\mathscr{F}_\gamma(I^\alpha_\gamma f)(\xi) = \xi^{-\alpha}\, \mathscr{F}_\gamma f(\xi).$$

Proof. Since f is even and belongs to $L^1_\gamma([0,\infty))$, the function

$$g(x) = e^{\gamma|x|} f(x), \quad x \in \mathbb{R},$$

is in $L^1(\mathbb{R})$. By Theorem 7.1 in [27], the classical Riesz fractional integral satisfies

$$\mathscr{F}(I^\alpha g)(\xi) = \xi^{-\alpha}\, \mathscr{F} g(\xi).$$

Because g is even, its Fourier transform coincides with its Fourier-cosine transform:

$$\mathscr{F}_0(I^\alpha g)(\xi) = \xi^{-\alpha}\, \mathscr{F}_0 g(\xi).$$

Recalling the relationships established in (2.2) between the tempered and classical transforms, we conclude that

$$\mathscr{F}_\gamma(I^\alpha_\gamma f)(\xi) = \xi^{-\alpha}\, \mathscr{F}_\gamma f(\xi).$$

□

Lemma 3.4 (Semigroup property)**.** *For $\alpha > 0$, $\beta > 0$, such that $\alpha + \beta < 1$, the tempered fractional integrals satisfy:*

$$I^\alpha_\gamma I^\beta_\gamma f(x) = I^{\alpha+\beta}_\gamma f(x),$$

for all $f \in L^1_\gamma[0,\ \infty)$.

3.2. Tempered Riesz derivative

In this section, we consider the inverse operator of the tempered fractional integral, known as the tempered fractional derivative. For our purposes, we require only fractional derivatives of order $0 < \alpha < 2$, which simplifies the presentation.

From the Riesz potential, we can define by analytic continuation the Riesz fractional derivative D^α, including the singular case $\alpha = 1$, by formally setting $D^\alpha = I^{-\alpha}$, i.e., in terms of symbols,

$$\mathscr{F}(D^\alpha f)(\xi) = -|\xi|^\alpha \mathscr{F} f(\xi).$$

Since [19]

$$-|\xi|^\alpha = -(\xi^2)^{\alpha/2},$$

we recognize that the Riesz fractional derivative of order α is the opposite of the $\alpha/2$-power of the positive definite operator $-\frac{d^2}{dx^2}$:

$$D^\alpha = -\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}.$$

For even functions on the real line, the Riesz fractional derivative can be written in a regularized integral form valid for $\alpha \in (0,2)$. In particular, one has [3]

$$D^\alpha_x f(x) = -\frac{2\Gamma(1+\alpha)}{\pi} \sin\left(\frac{\alpha\pi}{2}\right) \int_0^\infty \frac{f(x) - \sigma^0_x f(y)}{y^{1+\alpha}}\, dy. \tag{3.7}$$

Definition 3.5. *Let* $0 < \alpha < 2$ *and let* f *be a function belonging to the fractional Sobolev space*

$$H^{\alpha}_{\gamma}([0,\infty)) := \left\{ f \in L^2_{\gamma}([0,\infty)) : \int_0^{\infty} \xi^{\alpha} \left|(\mathscr{F}_{\gamma} f)(\xi)\right|^2 d\xi < \infty \right\}.$$

The fractional power $\mathscr{L}_{\gamma}^{\alpha/2}$ *of the operator* $\mathscr{L}_{\gamma}$ *is defined by its action on* f *via the tempered Fourier transform. Specifically, for every* $x > 0$*, we set*

$$\mathscr{L}_{\gamma}^{\alpha/2} f(x) := \Big(\mathscr{F}_{\gamma}^{-1}\Big[\xi^{\alpha}\,(\mathscr{F}_{\gamma} f)(\xi)\Big]\Big)(x) = \frac{2}{\pi}\int_0^{\infty} \xi^{\alpha}\, \phi_{\gamma}(x,\xi)\,(\mathscr{F}_{\gamma} f)(\xi)\, d\xi. \tag{3.8}$$

The main result of this paper is the following theorem.

Theorem 3.6. *Let* $\gamma \in \mathbb{R}$ *and* $0 < \alpha < 2$*. For any* $f \in \mathcal{S}_{\gamma}(\mathbb{R})$*, the following equivalence holds:*

$$\begin{aligned}\Big(e^{-\gamma x}\Big(-\frac{d^2}{dx^2}\Big)e^{\gamma x}\Big)^{\alpha/2} f(x) &= e^{-\gamma x}\Big(-\frac{d^2}{dx^2}\Big)^{\alpha/2}(e^{\gamma x} f)(x)\\ &= \frac{2^{\alpha+1}\Gamma(\frac{\alpha+1}{2})}{\sqrt{\pi}\left|\Gamma(-\frac{\alpha}{2})\right|}\int_0^{\infty} \frac{e^{-\gamma y} f(x) - \sigma_x^{\gamma} f(y)}{y^{\alpha+1}} e^{\gamma y}\, dy,\end{aligned} \tag{3.9}$$

where the tempered translation operator $\sigma_x^{\gamma} f(y)$ *is defined as:*

$$\sigma_x^{\gamma} f(y) = \frac{1}{2}\big(f(x+y) + e^{-2\gamma \min\{x,y\}} f(|x-y|)\big).$$

The following lemma is needed in proving the above theorem.

Lemma 3.7. *Let* $\lambda, \gamma \in \mathbb{R}$ *and* $0 < \alpha < 2$*. Then, the following identity holds:*

$$|\lambda|^{\alpha} = \frac{2^{\alpha+1}\,\Gamma\Big(\frac{\alpha+1}{2}\Big)}{\sqrt{\pi}\,\left|\Gamma\Big(-\frac{\alpha}{2}\Big)\right|}\int_0^{\infty} \frac{e^{-\gamma y} - \phi_{\gamma}(y,\lambda)}{y^{1+\alpha}}\, e^{\gamma y}\, dy.$$

Proof. Notice that by the definition of $\phi_{\gamma}(y,\lambda)$,

$$e^{\gamma y}\Big(e^{-\gamma y} - \phi_{\gamma}(y,\lambda)\Big) = 1 - \cos(\lambda y).$$

Hence, the integral in the statement becomes

$$\int_0^{\infty} \frac{e^{-\gamma y} - \phi_{\gamma}(y,\lambda)}{y^{1+\alpha}}\, e^{\gamma y}\, dy = \int_0^{\infty} \frac{1-\cos(\lambda y)}{y^{1+\alpha}}\, dy.$$

It is a classical result that

$$\int_0^{\infty} \frac{1-\cos(\lambda y)}{y^{1+\alpha}}\, dy = \frac{\pi\,|\lambda|^{\alpha}}{2\,\Gamma(1+\alpha)\,\sin\Big(\frac{\alpha\pi}{2}\Big)}.$$

On the other hand, one can verify via the gamma function identities

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\,\Gamma\Big(z+\frac{1}{2}\Big),$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

that the constant

$$\frac{2^{\alpha+1}\,\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\,\left|\Gamma\left(-\frac{\alpha}{2}\right)\right|}$$

can be rewritten as

$$\frac{2\,\Gamma(1+\alpha)\,\sin\left(\frac{\alpha\pi}{2}\right)}{\pi}.$$

Substituting these expressions into the identity, we obtain

$$\frac{2\,\Gamma(1+\alpha)\,\sin\left(\frac{\alpha\pi}{2}\right)}{\pi}\cdot\frac{\pi\,|\lambda|^{\alpha}}{2\,\Gamma(1+\alpha)\,\sin\left(\frac{\alpha\pi}{2}\right)} = |\lambda|^{\alpha},$$

which completes the proof. □

We now establish Theorem 3.6.

Proof. Since the Schwartz-type space $\mathcal{S}_\gamma(\mathbb{R})$ is invariant under both the tempered Fourier-cosine transform and the tempered translation, the inversion formula for the tempered transform guarantees that for every $f \in \mathcal{S}_\gamma(\mathbb{R})$ we have

$$\sigma_x^\gamma f(y) = \frac{2}{\pi}\int_0^\infty \phi_\gamma(x,\lambda)\,\phi_\gamma(y,\lambda)\,\mathscr{F}_\gamma f(\lambda)\,d\lambda.$$

In particular, by rewriting the difference $e^{-\gamma y} f(x) - \sigma_x^\gamma f(y)$ we obtain

$$\frac{e^{-\gamma y} f(x) - \sigma_x^\gamma f(y)}{y^{\alpha+1}}\,e^{\gamma y} = \frac{2}{\pi}\int_0^\infty \frac{e^{-\gamma y} - \phi_\gamma(y,\lambda)}{y^{\alpha+1}}\,e^{\gamma y}\,\phi_\gamma(x,\lambda)\,\mathscr{F}_\gamma f(\lambda)\,d\lambda.$$

Substituting this expression into the defining integral (3.9) for the fractional operator, we deduce that

$$\int_0^\infty \frac{e^{-\gamma y} f(x) - \sigma_x^\gamma f(y)}{y^{\alpha+1}}\,e^{\gamma y}\,dy = \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{e^{-\gamma y} - \phi_\gamma(y,\lambda)}{y^{\alpha+1}}\,e^{\gamma y}\,\phi_\gamma(x,\lambda)\,\mathscr{F}_\gamma f(\lambda)\,d\lambda\,dy.$$

By interchanging the order of integration (which is justified by Tonelli's Theorem as will be detailed below) and applying Lemma 3.7, we find that

$$\frac{2^{\alpha+1}\,\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\left|\Gamma\left(-\frac{\alpha}{2}\right)\right|}\int_0^\infty \frac{e^{-\gamma y} f(x) - \sigma_x^\gamma f(y)}{y^{\alpha+1}}\,e^{\gamma y}\,dy = \frac{2}{\pi}\int_0^\infty \lambda^\alpha\,\phi_\gamma(x,\lambda)\,\mathscr{F}_\gamma f(\lambda)\,d\lambda. \tag{3.10}$$

The right-hand side of (3.10) is exactly the spectral representation of $\mathscr{L}_\gamma^{\alpha/2} f(x)$; that is,

$$\mathscr{L}_\gamma^{\alpha/2} f(x) = \frac{2}{\pi}\int_0^\infty \lambda^\alpha\,\phi_\gamma(x,\lambda)\,\mathscr{F}_\gamma f(\lambda)\,d\lambda.$$

It remains to justify the interchange of the integrals in (3.10). By Tonelli's Theorem, we have

$$\begin{aligned}\frac{2}{\pi}\int_0^\infty\int_0^\infty\left|\frac{e^{-\gamma y}-\phi_\gamma(y,\lambda)}{y^{\alpha+1}}\,e^{\gamma y}\,\phi_\gamma(x,\lambda)\,\mathscr{F}_\gamma f(\lambda)\right|d\lambda\,dy\\ \leq\frac{2}{\pi}\int_0^\infty\int_0^\infty\frac{\left|e^{-\gamma y}-\phi_\gamma(y,\lambda)\right|}{y^{\alpha+1}}\,e^{\gamma y}\left|\mathscr{F}_\gamma f(\lambda)\right|d\lambda\,dy.\end{aligned}$$

An application of Lemma 3.7 shows that

$$\frac{2}{\pi}\int_0^\infty\int_0^\infty\frac{\left|e^{-\gamma y}-\phi_\gamma(y,\lambda)\right|}{y^{\alpha+1}}\,e^{\gamma y}\left|\mathscr{F}_\gamma f(\lambda)\right|d\lambda\,dy\leq\frac{1}{\Gamma(1+\alpha)\,\sin\!\left(\frac{\alpha\pi}{2}\right)}\int_0^\infty\lambda^\alpha\left|\mathscr{F}_\gamma f(\lambda)\right|d\lambda<\infty.$$

Thus, the double integral is absolutely convergent, and the interchange of integration is justified.

It now follows that

$$\left(e^{-\gamma x}\left(-\frac{d^2}{dx^2}\right)e^{\gamma x}\right)^{\alpha/2}f(x)=\frac{2^{\alpha+1}\,\Gamma\!\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\left|\Gamma\!\left(-\frac{\alpha}{2}\right)\right|}\int_0^\infty\frac{e^{-\gamma y}f(x)-\sigma_x^\gamma f(y)}{y^{\alpha+1}}\,e^{\gamma y}\,dy.$$

To complete the proof, it suffices to show that

$$e^{-\gamma x}\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}\left(e^{\gamma x}f\right)(x)=\frac{2^{\alpha+1}\,\Gamma\!\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\left|\Gamma\!\left(-\frac{\alpha}{2}\right)\right|}\int_0^\infty\frac{e^{-\gamma y}f(x)-\sigma_x^\gamma f(y)}{y^{\alpha+1}}\,e^{\gamma y}\,dy.$$

Recalling that the tempered translation is defined by

$$\sigma_\gamma^x f(y)=e^{-\gamma(x+y)}\,\sigma_x\!\left(e^{\gamma\,\cdot}\,f\right)(y),$$

a straightforward calculation shows that

$$\frac{2^{\alpha+1}\,\Gamma\!\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\left|\Gamma\!\left(-\frac{\alpha}{2}\right)\right|}\int_0^\infty\frac{e^{-\gamma y}f(x)-\sigma_\gamma^x f(y)}{y^{\alpha+1}}\,e^{\gamma y}\,dy=e^{-\gamma x}\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}\left(e^{\gamma x}f(x)\right).$$

This completes the proof. □

4. Conclusions

In this work we examined the tempered Riesz calculus through the spectral and harmonic analysis of the second-order operator $\mathscr{L}_\gamma=-\frac{d^2}{dx^2}-2\gamma\frac{d}{dx}-\gamma^2$. We introduced the tempered Fourier-cosine transform $\mathscr{F}_\gamma$ adapted to $\mathscr{L}_\gamma$ and a corresponding generalized translation σ_x^γ. Using the spectral calculus, we defined the fractional powers $(\mathscr{L}_\gamma)^{\alpha/2}$ and showed that their inverse powers coincide with a tempered Riesz fractional integral I_γ^α, admitting the explicit representation

$$I_\gamma^\alpha f(x)=\frac{1}{\cos(\frac{\alpha\pi}{2})\,\Gamma(\alpha)}\int_0^\infty\sigma_x^\gamma f(y)\,e^{\gamma y}\,y^{\alpha-1}\,dy\quad(\alpha>0,\ \alpha\neq1,3,5,\dots).$$

For $0 < \alpha < 2$ we obtained a pointwise singular-integral formula for $(\mathscr{L}_\gamma)^{\alpha/2}$ that reduces, when $\gamma = 0$, to the classical Riesz formula for $(-\frac{d^2}{dx^2})^{\alpha/2}$. The intertwining identity

$$\mathscr{L}_\gamma^{\alpha/2} f(x) = e^{-\gamma x}\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}(e^{\gamma x} f)(x)$$

rigorously links the tempered and non-tempered settings and clarifies the precise sense in which the tempered Riesz derivative is a bona fide fractional differential operator. These structural results align with and help systematize applications where tempering encodes exponential truncation or damping (e.g., anomalous diffusion and models with semi-heavy tails in finance).

Directions for further research:

- Extend the analysis to nonhomogeneous kernels, non-symmetric Lévy generators, and tempered fractional Laplacians (including bounded domains and manifold settings).
- Develop structure-preserving DGl/LDG schemes for the singular-integral form; derive a posteriori error estimators; design data-driven identification of the order α, drift/tempering γ (and λ), and kernel parameters.
- Study operators combining tempered space derivatives with Caputo/Prabhakar time derivatives, including well-posedness, regularity, and memory effects in sub-diffusion.
- Construct tempered fractional derivatives associated with related operators, such as the (classical) Bessel operator [3, 4, 16], the Jacobi Laplacian [5], and the fractional Dunkl Laplacian [6], and compare their spectral/singular-integral representations.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that she has no conflict of interest.

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