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Research article

On j-fuzzy $\gamma\mathcal{I}$-open sets with some applications

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Abstract: In this paper, we first introduced and studied a new class of fuzzy open sets, called j-fuzzy $\gamma\mathcal{I}$-open (j-F$\gamma\mathcal{I}$-open) sets on fuzzy ideal topological spaces ($\mathcal{FITS}s$). The class of j-F$\gamma\mathcal{I}$-open sets is contained in the class of j-fuzzy strong β-$\mathcal{I}$-open (j-FS$\beta\mathcal{I}$-open) sets and contains all j-fuzzy pre-$\mathcal{I}$-open (j-FP$\mathcal{I}$-open) sets and j-fuzzy semi-$\mathcal{I}$-open (j-FS$\mathcal{I}$-open) sets. We also defined and investigated the closure and interior operators with respect to the classes of j-F$\gamma\mathcal{I}$-closed sets and j-F$\gamma\mathcal{I}$-open sets. However, we explored and discussed novel types of fuzzy $\mathcal{I}$-separation axioms using j-F$\gamma\mathcal{I}$-closed sets, called j-F$\gamma\mathcal{I}$-regular spaces and j-F$\gamma\mathcal{I}$-normal spaces. Thereafter, we displayed and investigated the concept of fuzzy $\gamma\mathcal{I}$-continuity (F$\gamma\mathcal{I}$-continuity) using j-F$\gamma\mathcal{I}$-open sets. Also, we presented and characterized the concepts of fuzzy weak $\gamma\mathcal{I}$-continuity (FW$\gamma\mathcal{I}$-continuity) and fuzzy almost $\gamma\mathcal{I}$-continuity (FA$\gamma\mathcal{I}$-continuity), which are weaker forms of F$\gamma\mathcal{I}$-continuity. Moreover, we showed that F$\gamma\mathcal{I}$-continuity $\Longrightarrow$ FA$\gamma\mathcal{I}$-continuity $\Longrightarrow$ FW$\gamma\mathcal{I}$-continuity, but the converse may not be true. Finally, we defined and studied some new fuzzy $\gamma\mathcal{I}$-mappings via j-F$\gamma\mathcal{I}$-open sets and j-F$\gamma\mathcal{I}$-closed sets, called F$\gamma\mathcal{I}$-open mappings, F$\gamma\mathcal{I}$-closed mappings, F$\gamma\mathcal{I}$-irresolute mappings, F$\gamma\mathcal{I}$-irresolute open mappings, and F$\gamma\mathcal{I}$-irresolute closed mappings. The relationships between these classes of mappings were discussed with the help of some examples.

Keywords: fuzzy ideals; fuzzy topology; j-fuzzy $\gamma\mathcal{I}$-open set; fuzzy $\gamma\mathcal{I}$-continuity; fuzzy $\gamma\mathcal{I}$-irresoluteness
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1. Introduction

The concept of a fuzzy set was first defined in 1965 by Zadeh [1] as a suitable approach to address with uncertainty cases that we cannot be efficiently managed via classical techniques. Over the last decades, the research of fuzzy sets has had a vital role in mathematics and applied sciences and garnered significant attention due to its ability to handle uncertain and vague information in various real-life applications such as control systems [2, 3], artificial intelligence [4], image processing [5, 6], decision-making [7–9], etc. The integration between fuzzy sets and some uncertainty approaches, such as rough sets and soft sets, has been discussed in [10–12]. The notion of a fuzzy topology was introduced in 1968 by Chang [13], and this development has led to the expansion and discussion of several classical topological concepts in the context of a fuzzy topology [14, 15], providing more accurate and flexible models to address problems of uncertainty in various real-life areas. Overall, according to Šostak [16], the concept of a fuzzy topology being a crisp subclass of the class of fuzzy sets and fuzziness in the concept of openness of a fuzzy set have not been considered, which seems to be a drawback in the process of fuzzification of a topological space. Thereafter, Šostak [16] introduced a new notion of a fuzzy topology as the notion of openness of fuzzy sets. It is an extension of a fuzzy topology defined by Chang [13]. Furthermore, several researchers (see [17–20]) have redisplayed the same concept and investigated fuzzy topological spaces ($\mathcal{FTS}s$) being unaware of Šostak's work.

The generalizations of fuzzy open sets play an effective role in a fuzzy topology through their ability to improve on several results or to open the door to explore and discuss many fuzzy topological concepts such as fuzzy continuity [17, 18], fuzzy compactness [18, 19], fuzzy connectedness [18], etc. Furthermore, the concepts of j-fuzzy pre-open (j-FP-open) sets, j-fuzzy semi-open (j-FS-open) sets, j-fuzzy β-open (j-Fβ-open) sets, and j-fuzzy α-open (j-Fα-open) sets were presented and investigated by the authors of [21, 22] in $\mathcal{FTS}s$ based on Šostak's sense [16]. Kim et al. [21] displayed and investigated weaker forms of fuzzy continuity, called FS-continuity (resp. FP-continuity and Fα-continuity) between $\mathcal{FTS}s$ in the sense of Šostak. Abbas [22] defined and discussed the concepts of Fβ-continuous (resp. Fβ-irresolute) mappings. Also, Kim and Abbas [23] explored and characterized new types of j-fuzzy compactness. Overall, the notions of j-fuzzy γ-open (j-Fγ-open) sets and j-fuzzy γ-closed (j-Fγ-closed) sets were introduced and discussed by the authors of [24].

The notion of j-fuzzy local function was presented and investigated by Taha and Abbas [25] in an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$ based on Šostak's sense [16]. Moreover, the notions of fuzzy lower (resp. upper) weakly and almost $\mathcal{I}$-continuous multifunctions were displayed and investigated by Taha and Abbas [25]. Also, Taha [26–28] introduced the notions of j-FS$\mathcal{I}$-open sets, j-FP$\mathcal{I}$-open sets, j-F$\alpha\mathcal{I}$-open sets, j-F$\beta\mathcal{I}$-open sets, j-FS$\beta\mathcal{I}$-open sets, j-F$\delta\mathcal{I}$-open sets, and j-GF$\mathcal{I}$-closed sets in an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$ based on Šostak's sense. Overall, Taha [27, 28] presented the concepts of fuzzy lower (resp. upper) generalized $\mathcal{I}$-continuous (resp. semi-$\mathcal{I}$-continuous, pre-$\mathcal{I}$-continuous, δ-$\mathcal{I}$-continuous, α-$\mathcal{I}$-continuous, β-$\mathcal{I}$-continuous, and strong β-$\mathcal{I}$-continuous) multifunctions via fuzzy ideals [29].

The arrangement of this paper is as follows:

• Section 2 provides fundamental results and concepts that we use in our research.

• In Section 3, we introduce and study a new class of fuzzy sets, called j-F$\gamma\mathcal{I}$-open sets on $\mathcal{FITS}s$ in the sense of Šostak. We also define and discuss the interior and closure operators with respect to the classes of j-F$\gamma\mathcal{I}$-open sets and j-F$\gamma\mathcal{I}$-closed sets. Furthermore, we explore new types of fuzzy $\mathcal{I}$-separation axioms using j-F$\gamma\mathcal{I}$-closed sets, called j-F$\gamma\mathcal{I}$-regular spaces and j-F$\gamma\mathcal{I}$-normal spaces.

• In Section 4, we display and characterize the notion of F$\gamma\mathcal{I}$-continuous mappings using j-F$\gamma\mathcal{I}$-open sets. However, we present and discuss the notions of FA$\gamma\mathcal{I}$-continuous and FW$\gamma\mathcal{I}$-continuous mappings, which are weaker forms of F$\gamma\mathcal{I}$-continuous mappings.

• In Section 5, we explore and investigate new F$\gamma\mathcal{I}$-mappings via j-F$\gamma\mathcal{I}$-open sets and j-

F$\gamma\mathcal{I}$-closed sets, called F$\gamma\mathcal{I}$-closed mappings, F$\gamma\mathcal{I}$-open mappings, F$\gamma\mathcal{I}$-irresolute mappings, F$\gamma\mathcal{I}$-irresolute open mappings, and F$\gamma\mathcal{I}$-irresolute closed mappings.

• In Section 6, we give some potential future studies and conclusions.

2. Preliminaries

In this paper, non-empty sets will be denoted by Y, X, Z, etc. For any fuzzy set $\omega \in I^Z$ (where $I = [0,1]$ and I^Z is the class of all fuzzy sets on Z), $\omega^c(z) = 1 - \omega(z)$, for each $z \in Z$. Also, for $s \in I$, $\underline{s}(z) = s$, for each $z \in Z$. On Z, a fuzzy point z_s is a fuzzy set and is defined as follows: $z_s(r) = s$ if $r = z$, and $z_s(r) = 0$ for any $r \in Z - \{z\}$. We say that z_s belongs to $\omega \in I^Z$ ($z_s \in \omega$), if $s \leq \omega(z)$. Also, $P_s(Z)$ is the class of all fuzzy points. A fuzzy set $\nu \in I^Z$ is a quasi-coincident with $\mu \in I^Z$ ($\nu\, Q\, \mu$) on Z if there is $z \in Z$, with $\nu(z) + \mu(z) > 1$. Also, ν is not a quasi-coincident with μ ($\nu\, \overline{Q}\, \mu$) otherwise.

The difference between $\psi, \mu \in I^Z$ [25] is defined as follows:

$$\psi \,\overline{\wedge}\, \mu = \begin{cases} \underline{0}, & \text{if} \quad \psi \leq \mu, \\ \psi \wedge \mu^c, & \text{otherwise.} \end{cases}$$

Lemma 2.1. *[30]* Let $\omega, \nu \in I^Z$. Thus,

(a) if $\omega\, Q\, \nu$, then $\omega \wedge \nu \neq \underline{0}$,

(b) $\omega\, Q\, \nu$ iff there is $z_s \in \omega$ such that $z_s\, Q\, \nu$,

(c) $\omega\, \overline{Q}\, \nu$ iff $\omega \leq \nu^c$,

(d) $\omega \leq \nu$ iff $z_s \in \omega$ implies $z_s \in \nu$ iff $z_s\, Q\, \omega$ implies $z_s\, Q\, \nu$ iff $z_s\, \overline{Q}\, \nu$ implies $z_s\, \overline{Q}\, \omega$.

Definition 2.1. *[16, 17]* A mapping $\zeta : I^Z \longrightarrow I$ is called a fuzzy topology on Z if it satisfies the following conditions:

(a) $\zeta(\underline{0}) = \zeta(\underline{1}) = 1$.

(b) $\zeta(\omega \wedge \nu) \geq \zeta(\omega) \wedge \zeta(\nu)$, for any $\omega, \nu \in I^Z$.

(c) $\zeta(\bigvee_{i\in\Gamma} \omega_i) \geq \bigwedge_{i\in\Gamma} \zeta(\omega_i)$, for any $\omega_i \in I^Z$.

Thus, (Z, ζ) is called a fuzzy topological space ($\mathcal{FTS}$) in the sense of Šostak.

Definition 2.2. *[17, 21]* A fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im)$ is called

(a) fuzzy continuous if $\zeta(\mathbb{P}^{-1}(\nu)) \geq \Im(\nu)$, for any $\nu \in I^Y$;

(b) fuzzy open if $\Im(\mathbb{P}(\omega)) \geq \zeta(\omega)$, for any $\omega \in I^Z$;

(c) fuzzy closed if $\Im((\mathbb{P}(\omega))^c) \geq \zeta(\omega^c)$, for any $\omega \in I^Z$.

Definition 2.3. *[18, 22]* For any $\omega \in I^Z$ and $j \in I_\circ$ (where $I_\circ = (0,1]$) in an $\mathcal{FTS}$ (Z, ζ), we define fuzzy operators C_ζ and $I_\zeta : I^Z \times I_\circ \to I^Z$ as follows:

$$C_\zeta(\omega, j) = \bigwedge \{\nu \in I^Z : \omega \leq \nu,\ \zeta(\nu^c) \geq j\}.$$

$$I_\zeta(\omega, j) = \bigvee \{\nu \in I^Z : \nu \leq \omega,\ \zeta(\nu) \geq j\}.$$

Definition 2.4. *[21, 22, 24]* Let (Z, ζ) be an $\mathcal{FTS}$ and $j \in I_\circ$. A fuzzy set $\omega \in I^Z$ is called

(a) j-F-open if $\omega = I_\zeta(\omega, j)$;

(b) j-FP-open if $\omega \leq I_\zeta(C_\zeta(\omega, j), j)$;

(c) j-FS-open if $\omega \leq C_\zeta(I_\zeta(\omega, j), j)$;
(d) j-FR-open if $\omega = I_\zeta(C_\zeta(\omega, j), j)$;
(e) j-Fα-open if $\omega \leq I_\zeta(C_\zeta(I_\zeta(\omega, j), j), j)$;
(f) j-Fβ-open if $\omega \leq C_\zeta(I_\zeta(C_\zeta(\omega, j), j), j)$;
(g) j-Fγ-open if $\omega \leq C_\zeta(I_\zeta(\omega, j), j) \vee I_\zeta(C_\zeta(\omega, j), j)$.

Remark 2.1. *[21, 22, 24]* We have the following diagram from the previous definitions.

$$\begin{array}{ccccccc} & & & j\text{-}\mathbf{FP}\text{-}\mathbf{open} & & & \\ & & \nearrow & & \searrow & & \\ j\text{-}\mathbf{F}\text{-}\mathbf{open\ set} & \longrightarrow & j\text{-}\mathbf{F}\boldsymbol{\alpha}\text{-}\mathbf{open} & & j\text{-}\mathbf{F}\boldsymbol{\gamma}\text{-}\mathbf{open} & \longrightarrow & j\text{-}\mathbf{F}\boldsymbol{\beta}\text{-}\mathbf{open} \\ & & \searrow & & \nearrow & & \\ & & & j\text{-}\mathbf{FS}\text{-}\mathbf{open} & & & \end{array}$$

Definition 2.5. *[21, 22, 24]* A fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im)$ is called FS-continuous (resp. FP-continuous, Fα-continuous, Fβ-continuous, and Fγ-continuous) if $\mathbb{P}^{-1}(\omega)$ is a j-FS-open (resp. j-FP-open, j-Fα-open, j-Fβ-open, and j-Fγ-open) set, for any $\omega \in I^Y$ with $\Im(\omega) \geq j$ and $j \in I_\circ$.

Definition 2.6. *[24]* For any $\omega \in I^Z$ and $j \in I_\circ$ in an $\mathcal{FTS}$ (Z, ζ), we define fuzzy operators γC_ζ and $\gamma I_\zeta : I^Z \times I_\circ \to I^Z$ as follows:

$$\gamma C_\zeta(\omega, j) = \bigwedge \{\mu \in I^Z : \omega \leq \mu,\ \mu \text{ is } j\text{-F}\gamma\text{-closed}\}.$$

$$\gamma I_\zeta(\omega, j) = \bigvee \{\mu \in I^Z : \mu \leq \omega,\ \mu \text{ is } j\text{-F}\gamma\text{-open}\}.$$

Definition 2.7. *[29]* A fuzzy ideal $\mathcal{I}$ on Z, is a map $\mathcal{I} : I^Z \longrightarrow I$ that satisfies the following:
(a) $\forall\ \omega, \mu \in I^Z$ and $\omega \leq \mu \ \Rightarrow\ \mathcal{I}(\mu) \leq \mathcal{I}(\omega)$.
(b) $\forall\ \omega, \mu \in I^Z \ \Rightarrow\ \mathcal{I}(\omega \vee \mu) \geq \mathcal{I}(\omega) \wedge \mathcal{I}(\mu)$.

Moreover, $\mathcal{I}_0$ is the simplest fuzzy ideal on Z, and is defined as follows:

$$\mathcal{I}_0(\mu) = \begin{cases} 1, & \text{if } \ \mu = \underline{0}, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.8. *[25]* Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$, $j \in I_\circ$, and $\omega \in I^Z$. Then the j-fuzzy local function ω^*_j of ω is defined as follows:

$$\omega^*_j = \bigwedge \{\rho \in I^Z \ : \ \mathcal{I}(\omega \overline{\wedge} \rho) \geq j,\ \zeta(\rho^c) \geq j\}.$$

Remark 2.2. *[25]* If we take $\mathcal{I} = \mathcal{I}_0$, for each $\omega \in I^Z$ we have:

$$\omega^*_j = \bigwedge \{\rho \in I^Z \ : \ \omega \leq \rho,\ \zeta(\rho^c) \geq j\} = C_\zeta(\omega, j).$$

Definition 2.9. *[25]* Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$, $j \in I_\circ$, and $\omega \in I^Z$. Then we define fuzzy operator C^*_ζ $: I^Z \times I_\circ \to I^Z$ as follows:

$$C^*_\zeta(\omega, j) = \omega \ \vee \ \omega^*_j.$$

Now if, $\mathcal{I} = \mathcal{I}_0$ then $C^*_\zeta(\omega, j) = \omega \vee \omega^*_j = \omega \vee C_\zeta(\omega, j) = C_\zeta(\omega, j)$ for each $\omega \in I^Z$.

Theorem 2.1. *[25]* Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$, $j \in I_\circ$, and $\omega, \rho \in I^Z$. The operator $C^*_\zeta : I^Z \times I_\circ \to I^Z$ satisfies the following properties:

(a) $C^*_\zeta(\underline{0}, j) = \underline{0}$.
(b) $\omega \leq C^*_\zeta(\omega, j) \leq C_\zeta(\omega, j)$.
(c) If $\omega \leq \rho$, then $C^*_\zeta(\omega, j) \leq C^*_\zeta(\rho, j)$.
(d) $C^*_\zeta(\omega \vee \rho, j) = C^*_\zeta(\omega, j) \vee C^*_\zeta(\rho, j)$.
(e) $C^*_\zeta(\omega \wedge \rho, j) \leq C^*_\zeta(\omega, j) \wedge C^*_\zeta(\rho, j)$.
(f) $C^*_\zeta(C^*_\zeta(\omega, j), j) = C^*_\zeta(\omega, j)$.

Definition 2.10. *[26, 28]* Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$ and $j \in I_\circ$. A fuzzy set $\omega \in I^Z$ is called

(a) j-FS$\mathcal{I}$-open if $\omega \leq C^*_\zeta(I_\zeta(\omega, j), j)$;
(b) j-FP$\mathcal{I}$-open if $\omega \leq I_\zeta(C^*_\zeta(\omega, j), j)$;
(c) j-F$\alpha\mathcal{I}$-open if $\omega \leq I_\zeta(C^*_\zeta(I_\zeta(\omega, j), j), j)$;
(d) j-F$\beta\mathcal{I}$-open if $\omega \leq C_\zeta(I_\zeta(C^*_\zeta(\omega, j), j), j)$;
(e) j-FS$\beta\mathcal{I}$-open if $\omega \leq C^*_\zeta(I_\zeta(C^*_\zeta(\omega, j), j), j)$;
(f) j-FR$\mathcal{I}$-open if $\omega = I_\zeta(C^*_\zeta(\omega, j), j)$.

Some basic results and concepts that we need in the sequel are found in [17, 18, 25–28].

3. On j-fuzzy $\gamma\mathcal{I}$-open sets

Here, we present and investigate a new class of fuzzy open sets, called j-F$\gamma\mathcal{I}$-open sets in an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$ based on Šostak's sense. Some properties of j-F$\gamma\mathcal{I}$-open sets along with their mutual relationships are discussed with the help of some illustrative examples. Also, we explore the interior and closure operators with respect to the classes of j-F$\gamma\mathcal{I}$-open and j-F$\gamma\mathcal{I}$-closed sets and investigate some of their properties. Overall, we define and study new types of fuzzy $\mathcal{I}$-separation axioms via j-F$\gamma\mathcal{I}$-closed sets, called j-F$\gamma\mathcal{I}$-regular and j-F$\gamma\mathcal{I}$-normal spaces.

Definition 3.1. Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$ and $j \in I_\circ$. A fuzzy set $\rho \in I^Z$ is called a j-F$\gamma\mathcal{I}$-open set if $\rho \leq C^*_\zeta(I_\zeta(\rho, j), j) \vee I_\zeta(C^*_\zeta(\rho, j), j)$.

Remark 3.1. The complement of j-F$\gamma\mathcal{I}$-open sets are j-F$\gamma\mathcal{I}$-closed sets.

Lemma 3.1. Each j-F$\gamma\mathcal{I}$-open set is j-Fγ-open [24].

Proof. The proof follows by Theorem 2.1 and by Definitions 2.4 and 3.1. □

Remark 3.2. If we take $\mathcal{I} = \mathcal{I}_0$; then j-F$\gamma\mathcal{I}$-open set and j-Fγ-open set [24] are equivalent.

Remark 3.3. The converse of Lemma 3.1 fails, as can be seen in Example 3.1.

Example 3.1. Define $\zeta, \mathcal{I} : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if } \psi \in \{\underline{1}, \underline{0}\}, \\ \frac{2}{3}, & \text{if } \psi = \underline{0.7}, \\ \frac{1}{3}, & \text{if } \psi = \underline{0.3}, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathcal{I}(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{2}{3}, & \text{if } \underline{0} < \nu \leq \underline{0.6}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\underline{0.6}$ is an $\frac{1}{3}$-Fγ-open set, but it is not $\frac{1}{3}$-F$\gamma\mathcal{I}$-open.

Proposition 3.1. In an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$, for each $\omega \in I^Z$ and $j \in I_\circ$. Then

(a) Each j-FP$\mathcal{I}$-open set [26] is j-F$\gamma\mathcal{I}$-open;

(b) Each j-F$\gamma\mathcal{I}$-open set is j-FS$\beta\mathcal{I}$-open [28];

(c) Each j-FS$\mathcal{I}$-open set [26] is j-F$\gamma\mathcal{I}$-open.

Proof. (a) If ω is a j-FP$\mathcal{I}$-open set. Then

$$\begin{aligned}\omega &\le I_\zeta(C^*_\zeta(\omega, j), j)\\ &\le I_\zeta(C^*_\zeta(\omega, j), j) \vee I_\zeta(\omega, j)\\ &\le I_\zeta(C^*_\zeta(\omega, j), j) \vee C^*_\zeta(I_\zeta(\omega, j), j).\end{aligned}$$

Thus, ω is j-F$\gamma\mathcal{I}$-open.

(b) If ω is a j-F$\gamma\mathcal{I}$-open set. Then

$$\begin{aligned}\omega &\le C^*_\zeta(I_\zeta(\omega, j), j) \vee I_\zeta(C^*_\zeta(\omega, j), j)\\ &\le C^*_\zeta(I_\zeta(C^*_\zeta(\omega, j), j), j) \vee I_\zeta(C^*_\zeta(\omega, j), j)\\ &\le C^*_\zeta(I_\zeta(C^*_\zeta(\omega, j), j), j).\end{aligned}$$

Thus, ω is j-FS$\beta\mathcal{I}$-open.

(c) If ω is a j-FS$\mathcal{I}$-open set. Then

$$\begin{aligned}\omega &\le C^*_\zeta(I_\zeta(\omega, j), j)\\ &\le C^*_\zeta(I_\zeta(\omega, j), j) \vee I_\zeta(\omega, j)\\ &\le C^*_\zeta(I_\zeta(\omega, j), j) \vee I_\zeta(C^*_\zeta(\omega, j), j).\end{aligned}$$

Thus, ω is j-F$\gamma\mathcal{I}$-open. □

Remark 3.4. We have the following diagram from the previous definitions and discussions.

$$\begin{array}{ccccccc} & & & & j\textbf{-FP}\mathcal{I}\textbf{-open} & & \\ & & \nearrow & & \downarrow & & \\ j\textbf{-F-open set} & \longrightarrow & j\textbf{-F}\alpha\mathcal{I}\textbf{-open} & \longrightarrow & j\textbf{-F}\gamma\mathcal{I}\textbf{-open} & \longrightarrow & j\textbf{-FS}\beta\mathcal{I}\textbf{-open} \\ & & \searrow & & \uparrow & & \\ & & & & j\textbf{-FS}\mathcal{I}\textbf{-open} & & \end{array}$$

Remark 3.5. The reverse implication of the above diagram does not hold, as demonstrated by Examples 3.2, 3.3, and 3.4.

Example 3.2. Let $Z = \{z_1, z_2\}$ and define $\omega, \rho, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.4}, \frac{z_2}{0.3}\}$, $\rho = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$, $\lambda = \{\frac{z_1}{0.4}, \frac{z_2}{0.5}\}$. Define $\zeta, \mathcal{I} : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if} \quad \psi \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \psi = \rho, \\ \frac{1}{2}, & \text{if} \quad \psi = \omega, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.3}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, λ is an $\frac{1}{4}$-F$\gamma\mathcal{I}$-open set, but it is not $\frac{1}{4}$-FP$\mathcal{I}$-open.

Example 3.3. Let $Z = \{z_1, z_2\}$ and define $\omega, \rho, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.3}, \frac{z_2}{0.2}\}$, $\rho = \{\frac{z_1}{0.7}, \frac{z_2}{0.8}\}$, $\lambda = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$. Define $\zeta, \mathcal{I} : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if} \quad \psi \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \psi = \omega, \\ \frac{1}{2}, & \text{if} \quad \psi = \rho, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.5}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, λ is an $\frac{1}{3}$-F$\gamma\mathcal{I}$-open set, but it is neither $\frac{1}{3}$-FS$\mathcal{I}$-open nor $\frac{1}{3}$-F$\alpha\mathcal{I}$-open.

Example 3.4. Let $Z = \{z_1, z_2\}$ and define $\omega, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$, $\lambda = \{\frac{z_1}{0.4}, \frac{z_2}{0.5}\}$. Define $\zeta, \mathcal{I} : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if} \quad \psi \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \psi = \omega, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.4}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, λ is an $\frac{1}{3}$-FS$\beta\mathcal{I}$-open set, but it is not $\frac{1}{3}$-F$\gamma\mathcal{I}$-open.

Definition 3.2. For each $\omega \in I^Z$ and $j \in I_\circ$ in an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$, we define a fuzzy γ-$\mathcal{I}$-closure operator $\gamma C^*_\zeta : I^Z \times I_\circ \longrightarrow I^Z$ as follows:

$$\gamma C^*_\zeta(\omega, j) = \bigwedge \{\nu \in I^Z : \omega \leq \nu,\ \nu \text{ is } j\text{-F}\gamma\mathcal{I}\text{-closed}\}.$$

Proposition 3.2. For each $\omega \in I^Z$ and $j \in I_\circ$ in an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$, a fuzzy set ω is j-F$\gamma\mathcal{I}$-closed iff $\gamma C^*_\zeta(\omega, j) = \omega$.

Proof. This follows directly from Definition 3.2. □

Theorem 3.1. For each $\rho, \omega \in I^Z$ and $j \in I_\circ$ in an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$, a fuzzy γ-$\mathcal{I}$-closure operator γC^*_ζ : $I^Z \times I_\circ \longrightarrow I^Z$ satisfies the following properties.

(a) $\gamma C^*_\zeta(\underline{0}, j) = \underline{0}$.

(b) $\omega \leq \gamma C^*_\zeta(\omega, j) \leq C_\zeta(\omega, j)$.

(c) $\gamma C^*_\zeta(\omega, j) \leq \gamma C^*_\zeta(\rho, j)$ if $\omega \leq \rho$.
(d) $\gamma C^*_\zeta(\gamma C^*_\zeta(\omega, j), j) = \gamma C^*_\zeta(\omega, j)$.
(e) $\gamma C^*_\zeta(\omega \vee \rho, j) \geq \gamma C^*_\zeta(\omega, j) \vee \gamma C^*_\zeta(\rho, j)$.

Proof. (a), (b), and (c) are easily proved by Definition 3.2.

(d) From (b) and (c), $\gamma C^*_\zeta(\omega, j) \leq \gamma C^*_\zeta(\gamma C^*_\zeta(\omega, j), j)$. Now, we show $\gamma C^*_\zeta(\omega, j) \geq \gamma C^*_\zeta(\gamma C^*_\zeta(\omega, j), j)$. If $\gamma C^*_\zeta(\omega, j)$ does not contain $\gamma C^*_\zeta(\gamma C^*_\zeta(\omega, j), j)$, there is $z \in Z$ and $s \in (0, 1)$ with

$$\gamma C^*_\zeta(\omega, j)(z) < s < \gamma C^*_\zeta(\gamma C^*_\zeta(\omega, j), j)(z). \tag{1}$$

Since $\gamma C^*_\zeta(\omega, j)(z) < s$, by Definition 3.2, there is $\mu \in I^Z$ as a j-F$\gamma\mathcal{I}$-closed set and $\omega \leq \mu$ with $\gamma C^*_\zeta(\omega, j)(z) \leq \mu(z) < s$. Since $\omega \leq \mu$, then $\gamma C^*_\zeta(\omega, j) \leq \mu$. Again, by the definition of γC^*_ζ, then $\gamma C^*_\zeta(\gamma C^*_\zeta(\omega, j), j) \leq \mu$. Hence, $\gamma C^*_\zeta(\gamma C^*_\zeta(\omega, j), j)(z) \leq \mu(z) < s$, which is a contradiction for (1). Thus, $\gamma C^*_\zeta(\omega, j) \geq \gamma C^*_\zeta(\gamma C^*_\zeta(\omega, j), j)$. Therefore, $\gamma C^*_\zeta(\gamma C^*_\zeta(\omega, j), j) = \gamma C^*_\zeta(\omega, j)$.

(e) Since $\omega \leq \omega \vee \rho$ and $\rho \leq \omega \vee \rho$, then by (c), $\gamma C^*_\zeta(\omega, j) \leq \gamma C^*_\zeta(\omega \vee \rho, j)$ and $\gamma C^*_\zeta(\rho, j) \leq \gamma C^*_\zeta(\omega \vee \rho, j)$. Hence, $\gamma C^*_\zeta(\omega \vee \rho, j) \geq \gamma C^*_\zeta(\omega, j) \vee \gamma C^*_\zeta(\rho, j)$. □

Definition 3.3. For each $\omega \in I^Z$ and $j \in I_\circ$ in an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$, we define a fuzzy γ-$\mathcal{I}$-interior operator $\gamma I^*_\zeta : I^Z \times I_\circ \longrightarrow I^Z$ as follows: $\gamma I^*_\zeta(\omega, j) = \bigvee \{\nu \in I^Z : \nu \leq \omega,\ \nu$ is j-F$\gamma\mathcal{I}$-open$\}$.

Proposition 3.3. Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$, $\omega \in I^Z$, and $j \in I_\circ$. Then
(a) $\gamma C^*_\zeta(\omega^c, j) = (\gamma I^*_\zeta(\omega, j))^c$;
(b) $\gamma I^*_\zeta(\omega^c, j) = (\gamma C^*_\zeta(\omega, j))^c$.

Proof. (a) For each $\omega \in I^Z$, we have $\gamma C^*_\zeta(\omega^c, j) = \bigwedge\{\rho \in I^Z : \omega^c \leq \rho,\ \rho$ is j-F$\gamma\mathcal{I}$-closed$\} = [\bigvee\{\rho^c \in I^Z : \rho^c \leq \omega,\ \rho^c$ is j-F$\gamma\mathcal{I}$-open$\}]^c = (\gamma I^*_\zeta(\omega, j))^c$.

(b) This is similar to that of (a). □

Proposition 3.4. For each $\omega \in I^Z$ and $j \in I_\circ$ in an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$, a fuzzy set ω is j-F$\gamma\mathcal{I}$-open iff $\gamma I^*_\zeta(\omega, j) = \omega$.

Proof. This is immediate from Definition 3.3. □

Theorem 3.2. For each $\rho, \omega \in I^Z$ and $j \in I_\circ$ in an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$, a fuzzy γ-$\mathcal{I}$-interior operator γI^*_ζ : $I^Z \times I_\circ \longrightarrow I^Z$ satisfies the following properties.
(a) $\gamma I^*_\zeta(\underline{1}, j) = \underline{1}$.
(b) $I_\zeta(\omega, j) \leq \gamma I^*_\zeta(\omega, j) \leq \omega$.
(c) $\gamma I^*_\zeta(\omega, j) \leq \gamma I^*_\zeta(\rho, j)$ if $\omega \leq \rho$.
(d) $\gamma I^*_\zeta(\gamma I^*_\zeta(\omega, j), j) = \gamma I^*_\zeta(\omega, j)$.
(e) $\gamma I^*_\zeta(\omega, j) \wedge \gamma I^*_\zeta(\rho, j) \geq \gamma I^*_\zeta(\omega \wedge \rho, j)$.

Proof. This can be proven using the same approach as in Theorem 3.1. □

Definition 3.4. Let $z_s \in P_s(Z)$, $\omega \in I^Z$, and $j \in I_\circ$. An $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$ is said to be a j-F$\gamma\mathcal{I}$-regular space if $z_s\, \overline{Q}\, \omega$ for any j-F$\gamma\mathcal{I}$-closed set ω, there is $\mu_i \in I^Z$ with $\zeta(\mu_i) \geq j$ for $i = 1, 2$, such that $z_s \in \mu_1$, $\omega \leq \mu_2$, and $\mu_1\, \overline{Q}\, \mu_2$.

Definition 3.5. Let $\omega, \rho \in I^Z$ and $j \in I_\circ$. An $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$ is said to be a j-F$\gamma\mathcal{I}$-normal space if $\omega\, \overline{Q}\, \rho$ for any j-F$\gamma\mathcal{I}$-closed sets ω and ρ, there is $\mu_i \in I^Z$ with $\zeta(\mu_i) \geq j$ for $i = 1, 2$, such that $\omega \leq \mu_1$, $\rho \leq \mu_2$, and $\mu_1\, \overline{Q}\, \mu_2$.

Theorem 3.3. Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$, $z_s \in P_s(Z)$, $\omega \in I^Z$, and $j \in I_\circ$. Each of the following statements implies the others.

(a) $(Z, \zeta, \mathcal{I})$ is an j-F$\gamma\mathcal{I}$-regular space.

(b) If $z_s \in \omega$ for any j-F$\gamma\mathcal{I}$-open set ω, there is $\mu \in I^Z$ with $\zeta(\mu) \geq j$, and

$$z_s \in \mu \leq C_\zeta(\mu, j) \leq \omega.$$

(c) If $z_s \overline{Q} \omega$ for any j-F$\gamma\mathcal{I}$-closed set ω, there is $\mu_i \in I^Z$ with $\zeta(\mu_i) \geq j$ for $i = 1, 2$, such that $z_s \in \mu_1$, $\omega \leq \mu_2$, and $C_\zeta(\mu_1, j) \overline{Q} C_\zeta(\mu_2, j)$.

Proof. (a) $\Rightarrow$ (b) Let $z_s \in \omega$ for any j-F$\gamma\mathcal{I}$-open set ω, then $z_s \overline{Q} \omega^c$. Since $(Z, \zeta, \mathcal{I})$ is j-F$\gamma\mathcal{I}$-regular, then there is $\mu, \nu \in I^Z$ with $\zeta(\mu) \geq j$ and $\zeta(\nu) \geq j$, such that $z_s \in \mu$, $\omega^c \leq \nu$, and $\mu \overline{Q} \nu$. Thus, $z_s \in \mu \leq \nu^c \leq \omega$, so $z_s \in \mu \leq C_\zeta(\mu, j) \leq \omega$.

(b) $\Rightarrow$ (c) Let $z_s \overline{Q} \omega$ for any j-F$\gamma\mathcal{I}$-closed set ω, then $z_s \in \omega^c$. By (b), there is $\nu \in I^Z$ with $\zeta(\nu) \geq j$ and $z_s \in \nu \leq C_\zeta(\nu, j) \leq \omega^c$. Since $\zeta(\nu) \geq j$, then ν is an j-F$\gamma\mathcal{I}$-open set and $z_s \in \nu$. Again, by (b), there is $\mu \in I^Z$ such that $\zeta(\mu) \geq j$, and $z_s \in \mu \leq C_\zeta(\mu, j) \leq \nu \leq C_\zeta(\nu, j) \leq \omega^c$. Therefore, $\omega \leq (C_\zeta(\nu, j))^c = I_\zeta(\nu^c, j) \leq \nu^c$. Set $\lambda = I_\zeta(\nu^c, j)$, and then $\zeta(\lambda) \geq j$. Thus, $C_\zeta(\lambda, j) \leq \nu^c \leq (C_\zeta(\mu, j))^c$. Hence, $C_\zeta(\mu, j) \overline{Q} C_\zeta(\lambda, j)$.

(c) $\Rightarrow$ (a) This is immediate from Definition 3.4. □

Theorem 3.4. Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$, $\omega, \rho \in I^Z$, and $j \in I_\circ$. Each of the following statements implies the others.

(a) $(Z, \zeta, \mathcal{I})$ is an j-F$\gamma\mathcal{I}$-normal space.

(b) If $\mu \leq \omega$ for any j-F$\gamma\mathcal{I}$-closed set μ and j-F$\gamma\mathcal{I}$-open set ω, there is $\nu \in I^Z$ with $\zeta(\nu) \geq j$, and $\mu \leq \nu \leq C_\zeta(\nu, j) \leq \omega$.

(c) If $\omega \overline{Q} \rho$ for any j-F$\gamma\mathcal{I}$-closed sets ω and ρ, there is $\mu_i \in I^Z$ with $\zeta(\mu_i) \geq j$ for $i = 1, 2$, such that $\omega \leq \mu_1$, $\rho \leq \mu_2$, and $C_\zeta(\mu_1, j) \overline{Q} C_\zeta(\mu_2, j)$.

Proof. This can be proven using the same approach as in Theorem 3.3. □

4. Fuzzy $\gamma\mathcal{I}$-continuity

Here, we display and study the notion of F$\gamma\mathcal{I}$-continuity using j-F$\gamma\mathcal{I}$-open sets. Furthermore, we present and characterize the notions of FA$\gamma\mathcal{I}$-continuity and FW$\gamma\mathcal{I}$-continuity, which are weaker forms of F$\gamma\mathcal{I}$-continuity. Also, we show that F$\gamma\mathcal{I}$-continuity $\Longrightarrow$ FA$\gamma\mathcal{I}$-continuity $\Longrightarrow$ FW$\gamma\mathcal{I}$-continuity, but the converse may not be true.

Definition 4.1. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \mathfrak{I})$ is called F$\gamma\mathcal{I}$-continuous if $\mathbb{P}^{-1}(\omega)$ is a j-F$\gamma\mathcal{I}$-open set, for any $\omega \in I^Y$ with $\mathfrak{I}(\omega) \geq j$ and $j \in I_\circ$.

Lemma 4.1. Every F$\gamma\mathcal{I}$-continuity is an Fγ-continuity [24].

Proof. The proof follows by Lemma 3.1 and by Definitions 2.5 and 4.1. □

Remark 4.1. If we take $\mathcal{I} = \mathcal{I}_0$; then F$\gamma\mathcal{I}$-continuity and F$\gamma$-continuity [24] are equivalent.

Remark 4.2. The converse of Lemma 4.1 fails, as can be seen in Example 4.1.

Example 4.1. Define $\zeta, \mathcal{I}, \Im : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if} \quad \psi \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \psi = \underline{0.7}, \\ \frac{1}{3}, & \text{if} \quad \psi = \underline{0.3}, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathcal{I}(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \nu \leq \underline{0.6}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \theta = \underline{0.6}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \Im)$ is Fγ-continuous, but it is not F$\gamma\mathcal{I}$-continuous.

Definition 4.2. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ is called F$\alpha\mathcal{I}$-continuous (resp. FP$\mathcal{I}$-continuous, FS$\mathcal{I}$-continuous, and FS$\beta\mathcal{I}$-continuous) if $\mathbb{P}^{-1}(\omega)$ is a j-F$\alpha\mathcal{I}$-open (resp. j-FP$\mathcal{I}$-open, j-FS$\mathcal{I}$-open, and j-FS$\beta\mathcal{I}$-open) set, for any $\omega \in I^Y$ and $\Im(\omega) \geq j$ with $j \in I_\circ$.

Remark 4.3. We have the following diagram from the previous definitions.

$$\begin{array}{ccccc} & & \textbf{FP}\mathcal{I}\textbf{-continuity} & & \\ & \nearrow & \downarrow & & \\ \textbf{F}\alpha\mathcal{I}\textbf{-continuity} & \longrightarrow & \textbf{F}\gamma\mathcal{I}\textbf{-continuity} & \longrightarrow & \textbf{FS}\beta\mathcal{I}\textbf{-continuity} \\ & \searrow & \uparrow & & \\ & & \textbf{FS}\mathcal{I}\textbf{-continuity} & & \end{array}$$

Remark 4.4. The reverse implication of the above diagram does not hold, as demonstrated by Examples 4.2, 4.3, and 4.4.

Example 4.2. Let $Z = \{z_1, z_2\}$ and define $\omega, \rho, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.4}, \frac{z_2}{0.3}\}$, $\rho = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$, $\lambda = \{\frac{z_1}{0.4}, \frac{z_2}{0.5}\}$. Define $\zeta, \mathcal{I}, \Im : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if} \quad \psi \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{5}, & \text{if} \quad \psi = \rho, \\ \frac{1}{2}, & \text{if} \quad \psi = \omega, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.3}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \theta = \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \Im)$ is F$\gamma\mathcal{I}$-continuous, but it is not FP$\mathcal{I}$-continuous.

Example 4.3. Let $Z = \{z_1, z_2\}$ and define $\omega, \rho, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.3}, \frac{z_2}{0.2}\}$, $\rho = \{\frac{z_1}{0.7}, \frac{z_2}{0.8}\}$, $\lambda = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$. Define $\zeta, \mathcal{I}, \mathfrak{I} : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if} \quad \psi \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \psi = \omega, \\ \frac{1}{2}, & \text{if} \quad \psi = \rho, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.5}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathfrak{I}(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \theta = \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \mathfrak{I})$ is F$\gamma\mathcal{I}$-continuous, but it is neither FS$\mathcal{I}$-continuous nor F$\alpha\mathcal{I}$-continuous.

Example 4.4. Let $Z = \{z_1, z_2\}$ and define $\omega, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$, $\lambda = \{\frac{z_1}{0.4}, \frac{z_2}{0.5}\}$. Define $\zeta, \mathcal{I}, \mathfrak{I} : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if} \quad \psi \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \psi = \omega, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.4}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathfrak{I}(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \theta = \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \mathfrak{I})$ is FS$\beta\mathcal{I}$-continuous, but it is not F$\gamma\mathcal{I}$-continuous.

Theorem 4.1. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \mathfrak{I})$ is F$\gamma\mathcal{I}$-continuous iff for any $z_s \in P_s(Z)$ and any $\psi \in I^Y$ with $\mathfrak{I}(\psi) \geq j$ containing $\mathbb{P}(z_s)$, there is $\omega \in I^Z$ that is j-F$\gamma\mathcal{I}$-open containing z_s and $\mathbb{P}(\omega) \leq \psi$ with $j \in I_\circ$.

Proof. ($\Rightarrow$) Let $z_s \in P_s(Z)$ and $\psi \in I^Y$ with $\mathfrak{I}(\psi) \geq j$ containing $\mathbb{P}(z_s)$, and hence $\mathbb{P}^{-1}(\psi) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(\psi), j)$. Since $z_s \in \mathbb{P}^{-1}(\psi)$, $z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(\psi), j) = \omega$ (say). Hence, $\omega \in I^Z$ is j-F$\gamma\mathcal{I}$-open containing z_s with $\mathbb{P}(\omega) \leq \psi$.

($\Leftarrow$) Let $z_s \in P_s(Z)$ and $\psi \in I^Y$ with $\mathfrak{I}(\psi) \geq j$ containing $\mathbb{P}(z_s)$. By the given assumption, there exists $\omega \in I^Z$ that is j-F$\gamma\mathcal{I}$-open containing z_s with $\mathbb{P}(\omega) \leq \psi$. Thus, $z_s \in \omega \leq \mathbb{P}^{-1}(\psi)$ and $z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(\psi), j)$. Then, $\mathbb{P}^{-1}(\psi) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(\psi), j)$, so $\mathbb{P}^{-1}(\psi)$ is a j-F$\gamma\mathcal{I}$-open set. Hence, $\mathbb{P}$ is F$\gamma\mathcal{I}$-continuous. □

Theorem 4.2. Let $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \mathfrak{I})$ be a fuzzy mapping and $j \in I_\circ$. Each of the following statements implies the others for any $\omega \in I^Z$ and $\psi \in I^Y$:

(a) $\mathbb{P}$ is F$\gamma\mathcal{I}$-continuous.

(b) $\mathbb{P}^{-1}(\psi)$ is j-F$\gamma\mathcal{I}$-closed, for every $\psi \in I^Y$ with $\mathfrak{I}(\psi^c) \geq j$.

(c) $\mathbb{P}(\gamma C^*_\zeta(\omega, j)) \leq C_{\mathfrak{I}}(\mathbb{P}(\omega), j)$.

(d) $\gamma C^*_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(C_\Im(\psi, j))$.
(e) $\mathbb{P}^{-1}(I_\Im(\psi, j)) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(\psi), j)$.

Proof. (a) ⇔ (b) The proof follows from Definition 4.1 and $\mathbb{P}^{-1}(\psi^c) = (\mathbb{P}^{-1}(\psi))^c$.

(b) ⇒ (c) Let $\omega \in I^Z$. By (b), we obtain $\mathbb{P}^{-1}(C_\Im(\mathbb{P}(\omega), j))$ is j-F$\gamma\mathcal{I}$-closed. Hence,

$$\gamma C^*_\zeta(\omega, j) \leq \gamma C^*_\zeta(\mathbb{P}^{-1}(\mathbb{P}(\omega)), j) \leq \gamma C^*_\zeta(\mathbb{P}^{-1}(C_\Im(\mathbb{P}(\omega), j)), j) = \mathbb{P}^{-1}(C_\Im(\mathbb{P}(\omega), j)).$$

Therefore, $\mathbb{P}(\gamma C^*_\zeta(\omega, j)) \leq C_\Im(\mathbb{P}(\omega), j)$.

(c) ⇒ (d) Let $\psi \in I^Y$. By (c), we obtain $\mathbb{P}(\gamma C^*_\zeta(\mathbb{P}^{-1}(\psi), j)) \leq C_\Im(\mathbb{P}(\mathbb{P}^{-1}(\psi)), j) \leq C_\Im(\psi, j)$. Hence, $\gamma C^*_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(\mathbb{P}(\gamma C^*_\zeta(\mathbb{P}^{-1}(\psi), j))) \leq \mathbb{P}^{-1}(C_\Im(\psi, j))$.

(d) ⇔ (e) The proof follows from Proposition 3.3 and $\mathbb{P}^{-1}(\psi^c) = (\mathbb{P}^{-1}(\psi))^c$.

(e) ⇒ (a) Let $\psi \in I^Y$ with $\Im(\psi) \geq j$. By (e), we have $\mathbb{P}^{-1}(\psi) = \mathbb{P}^{-1}(I_\Im(\psi, j)) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(\psi)$. Then, $\gamma I^*_\zeta(\mathbb{P}^{-1}(\psi), j) = \mathbb{P}^{-1}(\psi)$. Hence, $\mathbb{P}^{-1}(\psi)$ is j-F$\gamma\mathcal{I}$-open, so $\mathbb{P}$ is F$\gamma\mathcal{I}$-continuous. □

Definition 4.3. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ is called FA$\gamma\mathcal{I}$-continuous if $\mathbb{P}^{-1}(\omega) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(I_\Im(C_\Im(\omega, j), j)), j)$, for any $\omega \in I^Y$ with $\Im(\omega) \geq j$ and $j \in I_\circ$.

Lemma 4.2. Every F$\gamma\mathcal{I}$-continuity is an FA$\gamma\mathcal{I}$-continuity.

Proof. This follows directly from Definitions 4.1 and 4.3. □

Remark 4.5. It is clear from Example 4.5 that the converse of Lemma 4.2 does not apply.

Example 4.5. Let $Z = \{z_1, z_2, z_3\}$ and define $\omega, \rho, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.4}, \frac{z_2}{0.2}, \frac{z_3}{0.4}\}$, $\rho = \{\frac{z_1}{0.5}, \frac{z_2}{0.5}, \frac{z_3}{0.4}\}$, $\lambda = \{\frac{z_1}{0.3}, \frac{z_2}{0.2}, \frac{z_3}{0.6}\}$. Define $\zeta, \mathcal{I}, \Im : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if } \psi \in \{\underline{1}, \underline{0}\}, \\ \frac{2}{3}, & \text{if } \psi = \omega, \\ \frac{1}{2}, & \text{if } \psi = \rho, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0}, \\ \frac{1}{2}, & \text{if } \underline{0} < \mu \leq \underline{0.6}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if } \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \theta = \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \Im)$ is FA$\gamma\mathcal{I}$-continuous, but it is not F$\gamma\mathcal{I}$-continuous.

Theorem 4.3. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ is FA$\gamma\mathcal{I}$-continuous iff for any $z_s \in P_s(Z)$ and any $\rho \in I^Y$ with $\Im(\rho) \geq j$ containing $\mathbb{P}(z_s)$, there is $\omega \in I^Z$ that is j-F$\gamma\mathcal{I}$-open containing z_s and $\mathbb{P}(\omega) \leq I_\Im(C_\Im(\rho, j), j)$ with $j \in I_\circ$.

Proof. (⇒) Let $z_s \in P_s(Z)$ and $\rho \in I^Y$ with $\Im(\rho) \geq j$ containing $\mathbb{P}(z_s)$, and hence

$$\mathbb{P}^{-1}(\rho) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(I_\Im(C_\Im(\rho, j), j)), j).$$

Since $z_s \in \mathbb{P}^{-1}(\rho)$, then $z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(I_\Im(C_\Im(\rho, j), j)), j) = \omega$ (say). Hence, $\omega \in I^Z$ is j-F$\gamma\mathcal{I}$-open containing z_s and $\mathbb{P}(\omega) \leq I_\Im(C_\Im(\rho, j), j)$.

($\Leftarrow$) Let $z_s \in P_s(Z)$, $\rho \in I^Y$, and $\mathfrak{I}(\rho) \geq j$ with $z_s \in \mathbb{P}^{-1}(\rho)$. By the given assumption, there exists $\omega \in I^Z$ that is j-F$\gamma\mathcal{I}$-open containing z_s and $\mathbb{P}(\omega) \leq I_{\mathfrak{I}}(C_{\mathfrak{I}}(\rho, j), j)$. Hence, $z_s \in \omega \leq \mathbb{P}^{-1}(I_{\mathfrak{I}}(C_{\mathfrak{I}}(\rho, j), j))$ and

$$z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(I_{\mathfrak{I}}(C_{\mathfrak{I}}(\rho, j), j)), j).$$

Then, $\mathbb{P}^{-1}(\rho) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(I_{\mathfrak{I}}(C_{\mathfrak{I}}(\rho, j), j)), j)$. Thus, $\mathbb{P}$ is FA$\gamma\mathcal{I}$-continuous. □

Theorem 4.4. Let $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \mathfrak{I})$ be a fuzzy mapping, $\psi \in I^Y$, and $j \in I_\circ$. Each of the following statements implies the others.

(a) $\mathbb{P}$ is FA$\gamma\mathcal{I}$-continuous.

(b) $\mathbb{P}^{-1}(\psi)$ is j-F$\gamma\mathcal{I}$-open, for each j-FR-open set ψ.

(c) $\mathbb{P}^{-1}(\psi)$ is j-F$\gamma\mathcal{I}$-closed, for each j-FR-closed set ψ.

(d) $\gamma C^*_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(C_{\mathfrak{I}}(\psi, j))$, for each j-Fγ-open set ψ.

(e) $\gamma C^*_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(C_{\mathfrak{I}}(\psi, j))$, for each j-FS-open set ψ.

Proof. (a) $\Rightarrow$ (b) Let $z_s \in P_s(Z)$ with $\psi \in I^Y$ be a j-FR-open set and $z_s \in \mathbb{P}^{-1}(\psi)$. Thus, by (a), there is $\omega \in I^Z$ that is j-F$\gamma\mathcal{I}$-open and $z_s \in \omega$ with $\mathbb{P}(\omega) \leq I_{\mathfrak{I}}(C_{\mathfrak{I}}(\psi, j), j)$. Then, $\omega \leq \mathbb{P}^{-1}(I_{\mathfrak{I}}(C_{\mathfrak{I}}(\psi, j), j)) = \mathbb{P}^{-1}(\psi)$ and $z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(\psi), j)$. Hence, $\mathbb{P}^{-1}(\psi) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(\psi), j)$, so $\mathbb{P}^{-1}(\psi)$ is j-F$\gamma\mathcal{I}$-open.

(b) $\Rightarrow$ (c) If $\psi \in I^Y$ is j-FR-closed, hence by (b), $\mathbb{P}^{-1}(\psi^c) = (\mathbb{P}^{-1}(\psi))^c$ is j-F$\gamma\mathcal{I}$-open. Thus, $\mathbb{P}^{-1}(\psi)$ is j-F$\gamma\mathcal{I}$-closed.

(c) $\Rightarrow$ (d) If $\psi \in I^Y$ is j-Fγ-open and since $C_{\mathfrak{I}}(\psi, j)$ is j-FR-closed, then by (c), $\mathbb{P}^{-1}(C_{\mathfrak{I}}(\psi, j))$ is j-F$\gamma\mathcal{I}$-closed. Since $\mathbb{P}^{-1}(\psi) \leq \mathbb{P}^{-1}(C_{\mathfrak{I}}(\psi, j))$, thus

$$\gamma C^*_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(C_{\mathfrak{I}}(\psi, j)).$$

(d) $\Rightarrow$ (e) The proof follows by the fact that each j-FS-open set is j-Fγ-open.

(e) $\Rightarrow$ (c) If $\psi \in I^Y$ is j-FR-closed, and then ψ is j-FS-open. By (e),

$$\gamma C^*_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(C_{\mathfrak{I}}(\psi, j)) = \mathbb{P}^{-1}(\psi).$$

Thus, $\mathbb{P}^{-1}(\psi)$ is j-F$\gamma\mathcal{I}$-closed.

(c) $\Rightarrow$ (a) If $z_s \in P_s(Z)$ with $\psi \in I^Y$ and $\mathfrak{I}(\psi) \geq j$ such that $z_s \in \mathbb{P}^{-1}(\psi)$, and then $z_s \in \mathbb{P}^{-1}(I_{\mathfrak{I}}(C_{\mathfrak{I}}(\psi, j), j))$. Since $[I_{\mathfrak{I}}(C_{\mathfrak{I}}(\psi, j), j)]^c$ is j-FR-closed, by (c), $\mathbb{P}^{-1}([I_{\mathfrak{I}}(C_{\mathfrak{I}}(\psi, j), j)]^c)$ is j-F$\gamma\mathcal{I}$-closed. Thus, $\mathbb{P}^{-1}(I_{\mathfrak{I}}(C_{\mathfrak{I}}(\psi, j), j))$ is j-F$\gamma\mathcal{I}$-open and $z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(I_{\mathfrak{I}}(C_{\mathfrak{I}}(\psi, j), j)), j)$. Then, $\mathbb{P}^{-1}(\psi) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(I_{\mathfrak{I}}(C_{\mathfrak{I}}(\psi, j), j)), j)$. Hence, $\mathbb{P}$ is FA$\gamma\mathcal{I}$-continuous. □

Definition 4.4. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \mathfrak{I})$ is called FW$\gamma\mathcal{I}$-continuous if $\mathbb{P}^{-1}(\omega) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(C_{\mathfrak{I}}(\omega, j)), j)$, for any $\omega \in I^Y$ with $\mathfrak{I}(\omega) \geq j$ and $j \in I_\circ$.

Lemma 4.3. Every F$\gamma\mathcal{I}$-continuity is an FW$\gamma\mathcal{I}$-continuity.

Proof. This follows directly from Definitions 4.1 and 4.4. □

Remark 4.6. It is clear from Example 4.6 that the converse of Lemma 4.3 does not apply.

Example 4.6. Let $Z = \{z_1, z_2, z_3\}$ and define $\omega, \rho, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.4}, \frac{z_2}{0.2}, \frac{z_3}{0.4}\}$, $\rho = \{\frac{z_1}{0.5}, \frac{z_2}{0.5}, \frac{z_3}{0.4}\}$, $\lambda = \{\frac{z_1}{0.3}, \frac{z_2}{0.2}, \frac{z_3}{0.6}\}$. Define $\zeta, \mathcal{I}, \mathfrak{I} : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if} \quad \psi \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if} \quad \psi = \omega, \\ \frac{1}{2}, & \text{if} \quad \psi = \rho, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu \leq \underline{0.6}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \theta = \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \Im)$ is FW$\gamma\mathcal{I}$-continuous, but it is not F$\gamma\mathcal{I}$-continuous.

Theorem 4.5. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ is FW$\gamma\mathcal{I}$-continuous iff for any $z_s \in P_s(Z)$ and any $\rho \in I^Y$ with $\Im(\rho) \geq j$ containing $\mathbb{P}(z_s)$, there is $\omega \in I^Z$ that is j-F$\gamma\mathcal{I}$-open containing z_s and $\mathbb{P}(\omega) \leq C_\Im(\rho, j)$ with $j \in I_\circ$.

Proof. (⇒) Let $z_s \in P_s(Z)$ and $\rho \in I^Y$ with $\Im(\rho) \geq j$ containing $\mathbb{P}(z_s)$, and hence

$$\mathbb{P}^{-1}(\rho) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\rho, j)), j).$$

Since $z_s \in \mathbb{P}^{-1}(\rho)$, then $z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\rho, j)), j) = \omega$ (say). Thus, $\omega \in I^Z$ is j-F$\gamma\mathcal{I}$-open containing z_s and $\mathbb{P}(\omega) \leq C_\Im(\rho, j)$.

(⇐) Let $z_s \in P_s(Z)$, $\rho \in I^Y$ and $\Im(\rho) \geq j$ with $z_s \in \mathbb{P}^{-1}(\rho)$. By the given assumption, there exists $\omega \in I^Z$ that is j-F$\gamma\mathcal{I}$-open containing z_s with $\mathbb{P}(\omega) \leq C_\Im(\rho, j)$. Hence, $z_s \in \omega \leq \mathbb{P}^{-1}(C_\Im(\rho, j))$ and $z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\rho, j)), j)$. Thus, $\mathbb{P}^{-1}(\rho) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\rho, j)), j)$. Therefore, $\mathbb{P}$ is FWγ-$\mathcal{I}$-continuous. □

Theorem 4.6. Let $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ be a fuzzy mapping, $\psi \in I^Y$, and $j \in I_\circ$. Each of the following statements implies the others.

(a) $\mathbb{P}$ is FW$\gamma\mathcal{I}$-continuous.

(b) $\mathbb{P}^{-1}(\psi) \geq \gamma C^*_\zeta(\mathbb{P}^{-1}(I_\Im(\psi, j)), j)$, if $\Im(\psi^c) \geq j$.

(c) $\gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\psi, j)), j) \geq \mathbb{P}^{-1}(I_\Im(\psi, j))$.

(d) $\gamma C^*_\zeta(\mathbb{P}^{-1}(I_\Im(\psi, j)), j) \leq \mathbb{P}^{-1}(C_\Im(\psi, j))$.

Proof. (a) ⇔ (b) This follows directly from Definition 4.4 and Proposition 3.3.

(b) ⇒ (c) Let $\psi \in I^Y$. Then by (b),

$$\gamma C^*_\zeta(\mathbb{P}^{-1}(I_\Im(C_\Im(\psi^c, j), j)), j) \leq \mathbb{P}^{-1}(C_\Im(\psi^c, j)).$$

Thus, $\mathbb{P}^{-1}(I_\Im(\psi, j)) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\psi, j)), j)$.

(c) ⇔ (d) This follows directly from Proposition 3.3.

(d) ⇒ (a) Let $\psi \in I^Y$ with $\Im(\psi) \geq j$. Then by (d), $\gamma C^*_\zeta(\mathbb{P}^{-1}(I_\Im(\psi^c, j)), j) \leq \mathbb{P}^{-1}(C_\Im(\psi^c, j)) = \mathbb{P}^{-1}(\psi^c)$. Hence, $\mathbb{P}^{-1}(\psi) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\psi, j)), j)$, so $\mathbb{P}$ is FW$\gamma\mathcal{I}$-continuous. □

Lemma 4.4. Every FA$\gamma\mathcal{I}$-continuity is an FW$\gamma\mathcal{I}$-continuity.

Proof. This follows directly from Definitions 4.3 and 4.4. □

Remark 4.7. It is clear from Example 4.7 that the converse of Lemma 4.4 does not apply.

Example 4.7. Let $Z = \{z_1, z_2, z_3\}$ and define $\omega, \lambda, \rho \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.6}, \frac{z_2}{0.2}, \frac{z_3}{0.4}\}$, $\lambda = \{\frac{z_1}{0.3}, \frac{z_2}{0.2}, \frac{z_3}{0.5}\}$, $\rho = \{\frac{z_1}{0.3}, \frac{z_2}{0.2}, \frac{z_3}{0.4}\}$. Define $\zeta, \mathcal{I}, \mathfrak{I} : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if} \quad \psi \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if} \quad \psi = \omega, \\ \frac{1}{2}, & \text{if} \quad \psi = \rho, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu \leq \underline{0.5}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathfrak{I}(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \theta = \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \mathfrak{I})$ is FW$\gamma\mathcal{I}$-continuous, but it is not FA$\gamma\mathcal{I}$-continuous.

Remark 4.8. We have the following diagram from the previous definitions and discussions.

$$\textbf{F}\gamma\mathcal{I}\textbf{-continuity} \quad \longrightarrow \quad \textbf{FA}\gamma\mathcal{I}\textbf{-continuity} \quad \longrightarrow \quad \textbf{FW}\gamma\mathcal{I}\textbf{-continuity}$$

Proposition 4.1. Let $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (X, \eta)$ and $\mathbb{Y} : (X, \eta) \longrightarrow (Y, \mathfrak{I})$ be two fuzzy mappings. Then the composition $\mathbb{Y} \circ \mathbb{P}$ is FA$\gamma\mathcal{I}$-continuous if $\mathbb{P}$ is F$\gamma\mathcal{I}$-continuous and $\mathbb{Y}$ is fuzzy continuous.

Proof. This follows directly from Definitions 2.2, 4.1, and 4.3. □

5. Fuzzy $\gamma\mathcal{I}$-irresoluteness

Here, we introduce and discuss some new fuzzy $\gamma\mathcal{I}$-mappings via j-F$\gamma\mathcal{I}$-open and j-F$\gamma\mathcal{I}$-closed sets, called F$\gamma\mathcal{I}$-irresolute (resp. F$\gamma\mathcal{I}$-open, F$\gamma\mathcal{I}$-irresolute open, F$\gamma\mathcal{I}$-closed, and F$\gamma\mathcal{I}$-irresolute closed) mappings. Furthermore, the relationships between these classes of mappings are discussed with the help of some illustrative examples.

Definition 5.1. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \mathfrak{I})$ is called F$\gamma\mathcal{I}$-irresolute if $\mathbb{P}^{-1}(\omega)$ is a j-F$\gamma\mathcal{I}$-open set, for any j-Fγ-open set $\omega \in I^Y$ with $j \in I_\circ$.

Lemma 5.1. Every F$\gamma\mathcal{I}$-irresolute mapping is F$\gamma\mathcal{I}$-continuous.

Proof. This follows directly from Definitions 4.1, 5.1, and Remark 2.1. □

Remark 5.1. It is clear from Example 5.1 that the converse of Lemma 5.1 does not apply.

Example 5.1. Let $Z = \{z_1, z_2\}$ and define $\lambda, \rho \in I^Z$ as follows: $\lambda = \{\frac{z_1}{0.5}, \frac{z_2}{0.5}\}$, $\rho = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$. Define $\zeta, \mathcal{I}, \mathfrak{I} : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if} \quad \psi \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \psi = \rho, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.5}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \theta = \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \Im)$ is F$\gamma\mathcal{I}$-continuous, but it is not F$\gamma\mathcal{I}$-irresolute.

Theorem 5.1. Let $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ be a fuzzy mapping and $j \in I_\circ$. Each of the following statements implies the others for any $\omega \in I^Z$ and $\psi \in I^Y$:

(a) $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute.

(b) $\mathbb{P}^{-1}(\psi)$ is j-F$\gamma\mathcal{I}$-closed, for each j-Fγ-closed set ψ.

(c) $\mathbb{P}(\gamma C^*_\zeta(\omega, j)) \leq \gamma C_\Im(\mathbb{P}(\omega), j)$.

(d) $\gamma C^*_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(\gamma C_\Im(\psi, j))$.

(e) $\mathbb{P}^{-1}(\gamma I_\Im(\psi, j)) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(\psi), j)$.

Proof. (a) $\Leftrightarrow$ (b) This follows directly from Definition 5.1 and $\mathbb{P}^{-1}(\psi^c) = (\mathbb{P}^{-1}(\psi))^c$.

(b) $\Rightarrow$ (c) Let $\omega \in I^Z$. By (b), $\mathbb{P}^{-1}(\gamma C_\Im(\mathbb{P}(\omega), j))$ is j-F$\gamma\mathcal{I}$-closed. Then,

$$\gamma C^*_\zeta(\omega, j) \leq \gamma C^*_\zeta(\mathbb{P}^{-1}(\mathbb{P}(\omega)), j) \leq \gamma C^*_\zeta(\mathbb{P}^{-1}(\gamma C_\Im(\mathbb{P}(\omega), j)), j) = \mathbb{P}^{-1}(\gamma C_\Im(\mathbb{P}(\omega), j)).$$

Thus, $\mathbb{P}(\gamma C^*_\zeta(\omega, j)) \leq \gamma C_\Im(\mathbb{P}(\omega), j)$.

(c) $\Rightarrow$ (d) Let $\psi \in I^Y$. By (c), $\mathbb{P}(\gamma C^*_\zeta(\mathbb{P}^{-1}(\psi), j)) \leq \gamma C_\Im(\mathbb{P}(\mathbb{P}^{-1}(\psi)), j) \leq \gamma C_\Im(\psi, j)$. Hence, $\gamma C^*_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(\mathbb{P}(\gamma C^*_\zeta(\mathbb{P}^{-1}(\psi), j))) \leq \mathbb{P}^{-1}(\gamma C_\Im(\psi, j))$.

(d) $\Leftrightarrow$ (e) This follows directly from Proposition 3.3 and $\mathbb{P}^{-1}(\psi^c) = (\mathbb{P}^{-1}(\psi))^c$.

(e) $\Rightarrow$ (a) Let $\psi \in I^Y$ be a j-Fγ-open set. By (e),

$$\mathbb{P}^{-1}(\psi) = \mathbb{P}^{-1}(\gamma I_\Im(\psi, j)) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(\psi).$$

Then, $\gamma I^*_\zeta(\mathbb{P}^{-1}(\psi), j) = \mathbb{P}^{-1}(\psi)$. Thus, $\mathbb{P}^{-1}(\psi)$ is j-F$\gamma\mathcal{I}$-open, so $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute. □

Proposition 5.1. Let $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (X, \eta)$ and $\mathbb{Y} : (X, \eta) \longrightarrow (Y, \Im)$ be two fuzzy mappings. Then the composition $\mathbb{Y} \circ \mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute (resp. F$\gamma\mathcal{I}$-continuous) if $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute and $\mathbb{Y}$ is Fγ-irresolute (resp. fuzzy continuous).

Proof. This follows directly from Definitions 2.2, 4.1, and 5.1. □

Definition 5.2. A fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ is called F$\gamma\mathcal{I}$-open if $\mathbb{P}(\omega)$ is a j-F$\gamma\mathcal{I}$-open set, for any $\omega \in I^Z$ with $\zeta(\omega) \geq j$ and $j \in I_\circ$.

Definition 5.3. A fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ is called F$\gamma\mathcal{I}$-irresolute open if $\mathbb{P}(\omega)$ is a j-F$\gamma\mathcal{I}$-open set, for any j-Fγ-open set $\omega \in I^Z$ with $j \in I_\circ$.

Lemma 5.2. Each F$\gamma\mathcal{I}$-irresolute open mapping is F$\gamma\mathcal{I}$-open.

Proof. This follows directly from Definitions 5.2, 5.3, and Remark 2.1. □

Remark 5.2. It is clear from Example 5.2 that the converse of Lemma 5.2 does not apply.

Example 5.2. Let $Z = \{z_1, z_2\}$ and define $\omega, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.5}, \frac{z_2}{0.5}\}$, $\lambda = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$. Define $\zeta, \Im, \mathcal{I} : I^Z \longrightarrow I$ as follows:

$$\zeta(\psi) = \begin{cases} 1, & \text{if } \psi \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{5}, & \text{if } \psi = \omega, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0}, \\ \frac{1}{2}, & \text{if } \underline{0} < \mu < \underline{0.5}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if } \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if } \theta = \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Z, \Im, \mathcal{I})$ is F$\gamma\mathcal{I}$-open, but it is not F$\gamma\mathcal{I}$-irresolute open.

Theorem 5.2. Let $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ be a fuzzy mapping and $j \in I_\circ$. Each of the following statements implies the others for any $\omega \in I^Z$ and $\psi \in I^Y$:

(a) $\mathbb{P}$ is F$\gamma\mathcal{I}$-open.

(b) $\mathbb{P}(I_\zeta(\omega, j)) \leq \gamma I^*_\Im(\mathbb{P}(\omega), j)$.

(c) $I_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(\gamma I^*_\Im(\psi, j))$.

(d) For each ψ and each ω with $\zeta(\omega^c) \geq j$ and $\mathbb{P}^{-1}(\psi) \leq \omega$, there is $\mu \in I^Y$ is j-F$\gamma\mathcal{I}$-closed with $\psi \leq \mu$ and $\mathbb{P}^{-1}(\mu) \leq \omega$.

Proof. (a) $\Rightarrow$ (b) Since $\mathbb{P}(I_\zeta(\omega, j)) \leq \mathbb{P}(\omega)$, hence by (a), $\mathbb{P}(I_\zeta(\omega, j))$ is j-F$\gamma\mathcal{I}$-open. Thus,

$$\mathbb{P}(I_\zeta(\omega, j)) \leq \gamma I^*_\Im(\mathbb{P}(\omega), j).$$

(b) $\Rightarrow$ (c) Set $\omega = \mathbb{P}^{-1}(\psi)$, and hence by (b), $\mathbb{P}(I_\zeta(\mathbb{P}^{-1}(\psi), j)) \leq \gamma I^*_\Im(\mathbb{P}(\mathbb{P}^{-1}(\psi)), j) \leq \gamma I^*_\Im(\psi, j)$. Then, $I_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(\gamma I^*_\Im(\psi, j))$.

(c) $\Rightarrow$ (d) Let $\psi \in I^Y$ and $\omega \in I^Z$ with $\zeta(\omega^c) \geq j$ such that $\mathbb{P}^{-1}(\psi) \leq \omega$. Since $\omega^c \leq \mathbb{P}^{-1}(\psi^c)$, $\omega^c = I_\zeta(\omega^c, j) \leq I_\zeta(\mathbb{P}^{-1}(\psi^c), j)$. Hence by (c), $\omega^c \leq I_\zeta(\mathbb{P}^{-1}(\psi^c), j) \leq \mathbb{P}^{-1}(\gamma I^*_\Im(\psi^c, j))$. Then, we have

$$\omega \geq (\mathbb{P}^{-1}(\gamma I^*_\Im(\psi^c, j)))^c = \mathbb{P}^{-1}(\gamma C^*_\Im(\psi, j)).$$

Thus, $\gamma C^*_\Im(\psi, j) \in I^Y$ is j-F$\gamma\mathcal{I}$-closed with $\psi \leq \gamma C^*_\Im(\psi, j)$ and $\mathbb{P}^{-1}(\gamma C^*_\Im(\psi, j)) \leq \omega$.

(d) $\Rightarrow$ (a) Let $\sigma \in I^Z$ with $\zeta(\sigma) \geq j$. Set $\psi = (\mathbb{P}(\sigma))^c$ and $\omega = \sigma^c$, $\mathbb{P}^{-1}(\psi) = \mathbb{P}^{-1}((\mathbb{P}(\sigma))^c) \leq \omega$. By (d), there exists $\mu \in I^Y$ is j-F$\gamma\mathcal{I}$-closed with $\psi \leq \mu$ and $\mathbb{P}^{-1}(\mu) \leq \omega = \sigma^c$. Then, $\mathbb{P}(\sigma) \leq \mathbb{P}(\mathbb{P}^{-1}(\mu^c)) \leq \mu^c$. Since $\psi \leq \mu$, $\mathbb{P}(\sigma) = \psi^c \geq \mu^c$. Thus, $\mathbb{P}(\sigma) = \mu^c$, so $\mathbb{P}(\sigma)$ is a j-F$\gamma\mathcal{I}$-open set. Hence, $\mathbb{P}$ is F$\gamma\mathcal{I}$-open. □

Theorem 5.3. Let $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ be a fuzzy mapping and $j \in I_\circ$. Each of the following statements implies the others for any $\omega \in I^Z$ and $\psi \in I^Y$:

(a) $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute open.

(b) $\mathbb{P}(\gamma I_\zeta(\omega, j)) \leq \gamma I^*_\Im(\mathbb{P}(\omega), j)$.

(c) $\gamma I_\zeta(\mathbb{P}^{-1}(\psi), j) \leq \mathbb{P}^{-1}(\gamma I^*_\Im(\psi, j))$.

(d) For each ψ and each ω is an j-Fγ-closed set with $\mathbb{P}^{-1}(\psi) \leq \omega$, there is $\mu \in I^Y$ is j-F$\gamma\mathcal{I}$-closed with $\psi \leq \mu$ and $\mathbb{P}^{-1}(\mu) \leq \omega$.

Proof. This can be proven using the same approach as in Theorem 5.2. □

Definition 5.4. A fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \mathfrak{I}, \mathcal{I})$ is called F$\gamma\mathcal{I}$-closed if $\mathbb{P}(\omega)$ is a j-F$\gamma\mathcal{I}$-closed set, for any $\omega \in I^Z$ with $\zeta(\omega^c) \geq j$ and $j \in I_\circ$.

Definition 5.5. A fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \mathfrak{I}, \mathcal{I})$ is called F$\gamma\mathcal{I}$-irresolute closed if $\mathbb{P}(\omega)$ is a j-F$\gamma\mathcal{I}$-closed set, for any j-Fγ-closed set $\omega \in I^Z$ and $j \in I_\circ$.

Lemma 5.3. Each F$\gamma\mathcal{I}$-irresolute closed mapping is F$\gamma\mathcal{I}$-closed.

Proof. This follows directly from Definitions 5.4 and 5.5. □

Theorem 5.4. Let $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \mathfrak{I}, \mathcal{I})$ be a fuzzy mapping and $j \in I_\circ$. Each of the following statements implies the others for any $\omega \in I^Z$ and $\psi \in I^Y$:

(a) $\mathbb{P}$ is F$\gamma\mathcal{I}$-closed.

(b) $\gamma C^*_{\mathfrak{I}}(\mathbb{P}(\omega), j) \leq \mathbb{P}(C_\zeta(\omega, j))$.

(c) $\mathbb{P}^{-1}(\gamma C^*_{\mathfrak{I}}(\psi, j)) \leq C_\zeta(\mathbb{P}^{-1}(\psi), j)$.

(d) For each ψ and each ω with $\zeta(\omega) \geq j$ and $\mathbb{P}^{-1}(\psi) \leq \omega$, there is $\mu \in I^Y$ is j-F$\gamma\mathcal{I}$-open with $\psi \leq \mu$ and $\mathbb{P}^{-1}(\mu) \leq \omega$.

Proof. This can be proven using the same approach as in Theorem 5.2. □

Theorem 5.5. Let $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \mathfrak{I}, \mathcal{I})$ be a fuzzy mapping and $j \in I_\circ$. Each of the following statements implies the others for any $\omega \in I^Z$ and $\psi \in I^Y$:

(a) $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute closed.

(b) $\gamma C^*_{\mathfrak{I}}(\mathbb{P}(\omega), j) \leq \mathbb{P}(\gamma C_\zeta(\omega, j))$.

(c) $\mathbb{P}^{-1}(\gamma C^*_{\mathfrak{I}}(\psi, j)) \leq \gamma C_\zeta(\mathbb{P}^{-1}(\psi), j)$.

(d) For each ψ and each ω is an j-Fγ-open set with $\mathbb{P}^{-1}(\psi) \leq \omega$, there is $\mu \in I^Y$ is j-F$\gamma\mathcal{I}$-open with $\psi \leq \mu$ and $\mathbb{P}^{-1}(\mu) \leq \omega$.

Proof. This can be proven using the same approach as in Theorem 5.2. □

Proposition 5.2. Let $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \mathfrak{I}, \mathcal{I})$ be a fuzzy mapping and bijective, $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute closed iff $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute open.

Proof. This follows directly from:

$$\mathbb{P}^{-1}(\gamma C^*_{\mathfrak{I}}(\nu, j)) \leq \gamma C_\zeta(\mathbb{P}^{-1}(\nu), j) \iff \mathbb{P}^{-1}(\gamma I^*_{\mathfrak{I}}(\nu^c, j)) \leq \gamma I_\zeta(\mathbb{P}^{-1}(\nu^c), j).$$

□

6. Conclusions and future work

In this manuscript, a novel class of fuzzy open sets, called j-F$\gamma\mathcal{I}$-open sets, has been defined on $\mathcal{FITS}s$ in the sense of Šostak. Also, some properties of j-F$\gamma\mathcal{I}$-open sets, along with their mutual relationships, have been investigated. After that, the concepts of F$\gamma\mathcal{I}$-interior operators and F$\gamma\mathcal{I}$-closure operators have been introduced and studied. We also presented and investigated some types of fuzzy $\mathcal{I}$-separation axioms, called j-F$\gamma\mathcal{I}$-normal spaces and j-F$\gamma\mathcal{I}$-regular spaces via j-F$\gamma\mathcal{I}$-closed sets. Moreover, the notion of F$\gamma\mathcal{I}$-continuity has been defined and discussed. The concepts of FA$\gamma\mathcal{I}$-continuous and FW$\gamma\mathcal{I}$-continuous mappings, which are weaker forms of F$\gamma\mathcal{I}$-continuous mappings, have been introduced and studied. Finally, we explored and characterized some new fuzzy $\gamma\mathcal{I}$-mappings via *j*-F$\gamma\mathcal{I}$-open sets and *j*-F$\gamma\mathcal{I}$-closed sets, called F$\gamma\mathcal{I}$-open mappings, F$\gamma\mathcal{I}$-closed mappings, F$\gamma\mathcal{I}$-irresolute mappings, F$\gamma\mathcal{I}$-irresolute open mappings, and F$\gamma\mathcal{I}$-irresolute closed mappings. The relationships between these classes of mappings have been discussed with the help of some illustrative examples.

In upcoming research, we intend to study the following topics: (a) defining these concepts given here based on lattice-valued fuzzy sets; (b) introducing fuzzy lower and upper $\gamma\mathcal{I}$-continuous multifunctions and *j*-fuzzy $\gamma\mathcal{I}$-connected sets; (c) extending these concepts given here to include fuzzy soft minimal (topological) spaces as introduced in [31, 32]; and (d) finding a use for these concepts given here to include double fuzzy topological spaces as introduced in [33].

Author contributions

Fahad Alsharari: Conceptualization, Writing–original draft, Investigation, Formal analysis; Jawaher Al-Mufarrij: Conceptualization, Formal analysis, Investigation, Writing–review and editing; Islam Taha: Supervision, Conceptualization, Formal analysis, Writing–original draft, Investigation, Writing–review and editing. All authors have reviewed and consented to the finalized version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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