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Research article

On one time-optimal control problem for a parabolic equation with involution in a bounded domain

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Abstract: In this paper, we study the problem of time-optimal control for the parabolic equation with involution in a multidimensional parallelepiped domain. A generalized solution to the initial boundary value problem is found, and the control problem is reduced to a first-order Volterra integral equation. To prove the existence and uniqueness of the solution to this integral equation, necessary estimates are obtained for its kernel. The existence of a solution to the integral equation, i.e., the admissibility of the control function, is proven, and an optimal estimate of the minimum time required to heat the domain to a certain average temperature is found.

Keywords: parabolic equation; Volterra integral equation; admissible control; minimal time; eigenfunction; involution
Mathematics Subject Classification: 35K05, 35K15

1. Introduction

Control problems are of constant interest in physics and engineering due to their relevance. Recent research has extensively investigated optimal and time-efficient control strategies for parabolic equations. This paper studies the boundary control problem for parabolic equations with involution in a multidimensional domain. Our main goal is to determine the allowable control to achieve a given temperature distribution in the minimum time and to find the optimal estimate of the minimum time to reach this given average temperature.

A control problem associated with parabolic equations was studied by Friedman [1]. A great deal of developments in the controllability theory of the linear second-order parabolic equation were initiated by Fattorini and Russell [2, 3]. Control problems for the infinite-dimensional case were studied by

Egorov [4], who generalized Pontryagin's maximum principle to a class of equations in Banach space, and the proof of a bang-bang principle was shown in the particular conditions. The time-varying bang-bang property of time-optimal controls for the heat equation and its applications is studied in [5].

Initial investigations into the time optimal control problem associated with a heat conduction equation in a bounded $n-$dimensional domain were studied in [6, 7], and the minimal time estimate for achieving a given average temperature was found. A mathematical model of thermal control processes with integral constraints on the control function was analyzed in [8].

The control problem for the heat transfer equation in a three-dimensional spatial domain was studied in [9], and the admissibility of the control function was proven by geometric bounding. The optimal time problem for heating a non-homogeneous rod to a given average temperature was addressed in [10]. The necessary control function for heating the rod to an average temperature is also found using the Laplace transform method, and the admissibility of the control is proven. Reference [11] investigates boundary control problems in a two-dimensional domain governed by a fourth-order parabolic equation.

In [12], optimal time control problems governed by semilinear parabolic equations with Dirichlet boundary control in the presence of a target state constraint are considered, and a new Hamiltonian functional is defined to specify optimality conditions for the terminal time T. The problem of the existence of time-optimal control for some semilinear parabolic differential equations with distributed control in the subdomain is studied in [13]. Control problems for pseudoparabolic equations in a one-dimensional domain are discussed in detail in [14, 15].

In recent years, there has also been a growing interest in the study of mixed problems for parabolic-type equations involving involution. Inverse problems for equations of parabolic type with involution are studied in work [16, 17]. In [18], a boundary value problem for the heat equation associated with involution in a one-dimensional domain is studied. Many boundary value problems for parabolic-type equations with involution were studied in works [19, 20]. Boundary problems for fourth-order parabolic equations with involution are studied in work [21].

The solution of some inverse problems for the nonlocal analogue of the fourth-order parabolic equation when the domain is a multidimensional parallelepiped was studied in [22]. The inverse problem for a fractional-order parabolic equation involving a nonlocal biharmonic operator in a two-dimensional domain is studied in detail in [23]. In [24], the class of inverse problems for the heat equation with involution is considered using four different boundary conditions, namely Dirichlet, Neumann, periodic, and periodic boundary conditions. In this work, theorems on the existence and uniqueness of solutions are presented and proven.

Boundary value control problems for the heat transfer equation involving involution in one-and two-dimensional domains were studied in [25, 26], and the control function required to heat a thin plate to a given average temperature was found using the Laplace transform method.

In this paper, we study the boundary control problem for the heat transfer equation, the main goal of which is to determine the existence of admissible control and to find an estimate of the minimum time required to heat the sphere to a given average temperature. The analysis is carried out using the separation of variables method and reduces the control problem to a Volterra integral equation of the first kind (Section 3). It is known that it can be difficult to prove the existence of a solution to a Volterra integral equation of the first kind. In Section 4, we obtain some estimates necessary to evaluate the kernel of the integral equation. In Section 5, we obtain important estimates for the kernel of the integral equation and prove that the solution of the integral equation exists and is unique. In Section 6, we obtain an estimate of the minimum time required to heat the domain to an average temperature. In Section 7, the minimum time estimates are evaluated using numerical methods for given parameter values, providing a concrete illustration of the time-optimal control.

2. Statement of problem

Let $\Omega \subset \mathbb{R}^n$ be an open rectangular parallelepiped defined by

$$\Omega := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 < x_i < p_i \text{ for all } i = \overline{1, n}\},$$

where $p_i > 0$ are given edge lengths, and let $\partial\Omega$ denote its topological boundary.

Consider the mappings $S_i : \mathbb{R}^n \to \mathbb{R}^n$ of the type $S_i x = (x_1, \dots, x_{i-1}, p_i - x_i, x_{i+1}, \dots, x_n)$. Obviously, the mappings S_i are involutions, i.e., $S_i^2 = I$, where I is the identity mapping. Let us consider all possible products of mappings S_i, i.e., $S_{ji} = S_j S_j$, or $S_{jik} = S_j S_j S_k, \dots$. The total number of such mappings, taking into account the identity mapping $S_0 x = x$, is equal to 2^n. To number such mappings, we will use the binary number system, namely, if $0 \leq j < 2^n$ in the binary number system, the representation $j \equiv (j_n \dots j_1)_2 = j_1 + 2j_2 + \dots + 2^{n-1} j_n$, where j_k takes one of the values 0 or 1. Therefore, introducing the vector $j = (j_1, \dots, j_n)$, mappings of the type $S_j \equiv S_1^{j_1} \dots S_n^{j_n}$ corresponding to the index j can be considered. Using these mappings, the operator is introduced

$$L_n U(x) = \sum_{j=0}^{2^n - 1} a_j \, \Delta U(S_j x),$$

where $a_0, \dots, a_{2^n-1}$ are a set of real numbers, and Δ is a Laplace operator. We will call operator L_n a non-local analogue of the Laplace operator.

It is known that the operator L_n is defined as follows for $n = 2$:

$$L_2 U(x_1, x_2) = a_0 \Delta U(x_1, x_2) + a_1 \Delta U(p_1 - x_1, x_2) + a_2 \Delta U(x_1, p_2 - x_2) + a_3 \Delta U(p_1 - x_1, p_2 - x_2).$$

In this paper, we investigate the parabolic equation with involution

$$u_t(x, t) - L_n u(x, t) = h(x) \, v(t), \quad x \in \Omega, \quad t > 0, \tag{2.1}$$

with boundary condition

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{2.2}$$

and initial condition

$$u(x, 0) = 0, \quad x \in \overline{\Omega}, \tag{2.3}$$

where $h(x)$ is a given function and $v(t)$ is the control function.

Here the involution operator S_i represents spatial reflection with respect to the mid-plane $x_i = p_i/2$ of the rectangular domain. For example, $S_i(x_1, \dots, x_n) = (x_1, \dots, p_i - x_i, \dots, x_n)$ assigns to the point $(x_1, \dots, x_i, \dots, x_n)$ the value of the function at the symmetric point $(x_1, \dots, p_i - x_i, \dots, x_n)$. Physically, this means that the state of the system at a given point is coupled with the state at its mirror-symmetric counterpart, which may occur, for instance, in heat conduction models for media with geometric symmetry.

We say that the control function $v(t) \in L_2(\mathbb{R}_+)$ is *admissible*, if it fulfills the condition $|v(t)| \leq 1$ on the half-line $t \geq 0$.

In the present work we consider the following thermal control problem:

Time optimal problem. *This problem involves finding the minimum value of* $T \geq 0$ *for a given constant* $\Theta > 0$ *so that, for* $t \geq T$*, the solution* $u(x,t)$ *of the initial-boundary value problem* (2.1)–(2.3) *with some admissible control* $v(t)$ *exists and, for some* $T_1 > T$*, satisfies the equation*

$$\int\limits_{\Omega} \rho(x)\, u(x,t)\, dx = \Theta, \quad T \leq t \leq T_1. \tag{2.4}$$

We consider the following spectral problem:

$$L_n w(x) + \lambda\, w(x) = 0, \quad x \in \Omega, \tag{2.5}$$

$$w(x) = 0, \quad x \in \partial\Omega. \tag{2.6}$$

If, $a_0 = 1$, $a_j = 0$ $(j = \overline{1,n})$ then the problem coincides with the spectral problem with the Dirichlet condition for the classical Laplace operator.

Let us use the expression for the eigenvalues and eigenfunctions of the following Dirichlet problem to determine the eigenvalues and eigenfunctions of the main problem (2.5)-(2.6):

$$\Delta\vartheta(x) + \mu\,\vartheta(x) = 0, \quad x \in \Omega, \quad \vartheta(x) = 0, \quad x \in \partial\Omega. \tag{2.7}$$

It is known that the eigenfunctions of problem (2.7) are as follows (see, e.g., [27], p.331)

$$\vartheta_{m_1 \ldots m_n}(x) = C_n \prod_{k=1}^{n} \sin \frac{m_k\, \pi\, x_k}{p_k}, \tag{2.8}$$

where $C_n = 2^{n/2} \prod_{k=1}^{n} \frac{1}{\sqrt{p_k}}$.

System (2.8) is complete and orthonormal in $L_2(\Omega)$. The corresponding eigenvalues are of the form

$$\mu_{m_1 \ldots m_n} = \pi^2 \sum_{k=1}^{n} \frac{m_k^2}{p_k^2}. \tag{2.9}$$

Thus, the eigenfunctions of problem (2.5)-(2.6) are the functions

$$w_{m_1 \ldots m_n}(x) = \vartheta_{m_1 \ldots m_n}(x) = C_n \prod_{k=1}^{n} \sin \frac{m_k \pi x_k}{p_k}, \tag{2.10}$$

and the corresponding eigenvalues have the form

$$\lambda_{m_1 \ldots m_n} = \theta_{m_1 \ldots m_n}\, \pi^2 \sum_{k=1}^{n} \frac{m_k^2}{p_k^2}, \tag{2.11}$$

where

$$\theta_{m_1 \ldots m_n} = \sum_{j=0}^{2^n - 1} a_j\, (-1)^{|j| + j_1 m_1 + j_2 m_2 + \ldots + j_n m_n}. \tag{2.12}$$

Since the system $\vartheta_{m_1...m_n}(x)$ is complete, then problem (2.5)-(2.6) does not have other eigenfunctions when the condition $\theta_{m_1...m_n} \neq 0$ is met. The coefficients a_j are chosen so that $\theta_{m_1...m_n} > 0$ for all $m_1, \dots, m_n \in \mathbb{N}$.

Since the $\vartheta_{m_1...m_n}(x)$ system is complete and $w_{m_1...m_n}(x) = \vartheta_{m_1...m_n}(x)$, the following lemma holds.

Lemma 2.1. (see [28]) *The system of functions $w_{m_1...m_n}(x)$, $m_k \geq 1$, $k = \overline{1,n}$ are orthonormal and complete in space $L_2(\Omega)$.*

Suppose that the Fourier coefficients of a given function $h(x)$ satisfy the following condition:

$$h_{m_1...m_n} \geq 0, \tag{2.13}$$

where

$$h_{m_1...m_n} = \int_0^{p_1} \dots \int_0^{p_n} h(x_1, ..., x_n)\, w_{m_1...m_n}(x_1, ...x_n)\, dx_1...dx_n.$$

The class of functions $\rho(x)$ is defined as

$$\mathcal{A}(\Omega) := \left\{ \rho \in \mathring{W}_2^1(\Omega) \,\middle|\, \rho(x) \geq 0, \quad \frac{\partial}{\partial x_k}\rho(x) \leq 0, \quad \int_\Omega \rho(x)\, dx = 1 \right\},$$

where $k = \overline{1,n}$ and $\mathring{W}_2^1(\Omega)$ denotes the Sobolev space of functions vanishing on the boundary in the trace sense.

Consider the weight function

$$\rho(x) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \rho_{m_1...m_n}\, w_{m_1...m_n}(x), \quad x \in \Omega,$$

where

$$\rho_{m_1...m_n} = \int_\Omega \rho(x)\, w_{m_1...m_n}(x)\, dx. \tag{2.14}$$

Set

$$\Psi_{m_1...m_n} = h_{m_1...m_n}\, \rho_{m_1...m_n}, \qquad m_k \geq 1, \quad k = \overline{1,n}, \tag{2.15}$$

where $h_{m_1...m_n}$ and $\rho_{m_1...m_n}$ are the Fourier coefficients of the functions $h(x)$ and $\rho(x)$, respectively.

Set

$$\lambda_1 := \lambda_{1...1}, \quad \Psi_1 := \Psi_{1...1}.$$

Theorem 2.1. *Let us assume that*

$$0 < \Theta < \frac{\Psi_1}{\lambda_1}.$$

Set

$$T_0 = -\frac{1}{\lambda_1} \ln\!\left(1 - \frac{\Theta\, \lambda_1}{\Psi_1}\right).$$

Then a solution T_{min} of the Time optimal problem exists, and the estimate $T_{min} \leq T_0$ is valid.

3. Main integral equation

For an arbitrary Banach space B and $T > 0$, we denote by $C([0, T] \to B)$ the Banach space of all continuous mappings $u\colon [0, T] \to B$ endowed with the norm

$$\|u\|_T = \sup_{t\in[0,T]} \|u(t)\|_B.$$

By $\mathring{W}_2^1(\Omega)$, we denote the subspace of the Sobolev space $W_2^1(\Omega)$ consisting of functions with zero trace on $\partial\Omega$. Note that due to the closure $\mathring{W}_2^1(\Omega)$ the sum of a series of functions from $\mathring{W}_2^1(\Omega)$, converging in metric $W_2^1(\Omega)$ also belongs to $\mathring{W}_2^1(\Omega)$.

Definition. *Let $h \in L_2(\Omega)$ and $\nu \in L_2([0, T])$ be given. A function $u(x, t)$ is called a generalized solution of problem* (2.1)–(2.3) *if:*

(1) $u \in C([0, T] \to \mathring{W}_2^1(\Omega))$;
(2) $u(x, 0) = 0$ *holds for a.e.* $x \in \Omega$;
(3) *For every test function* $\chi \in \mathring{W}_2^1(\Omega)$ *and* $t \in [0, T]$,

$$\int_\Omega u_t(x,t)\chi(x)dx + \sum_{j=0}^{2^n-1} a_j \int_\Omega \sum_{i=1}^{n} (-1)^{j_i} \frac{\partial}{\partial x_i} u(S_1^{j_1} \dots S_i^{j_i} \dots S_n^{j_n} x, t) \frac{\partial}{\partial x_i}\chi(x)\, dx = \nu(t) \int_\Omega h(x)\chi(x)\, dx.$$

The class $C([0, T] \to \mathring{W}_2^1(\Omega))$ is a subset of the class $W_2^{1,0}(\Omega_T)$, which was taken into consideration in monograph [29] for defining a solution to the problem of homogeneous boundary conditions (refer to the corresponding uniqueness theorem in Chapter III, Theorem 3.2, pp. 173–176), where $\Omega_T = \Omega \times (0, T)$. Accordingly, the generalized solution mentioned above is likewise a generalized solution in the sense of [29]. However, a solution from the class $C([0, T] \to \mathring{W}_2^1(\Omega))$ continually relies on $t \in [0, T]$ in the metric $L_2(\Omega)$, in contrast to a solution from the class $W_2^{1,0}(\Omega_T)$, which is guaranteed to have a trace for practically everywhere $t \in [0, T]$.

Lemma 3.1. *Let $h \in L_2(\Omega)$ and $\nu \in L_2(\mathbb{R}_+)$. Assume that $\theta_{m_1\dots m_n} > 0$ for all $m_k \geq 1$, $k = \overline{1, n}$. Then the function*

$$u(x,t) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \left(h_{m_1\dots m_n} \int_0^t \nu(s) e^{-\lambda_{m_1\dots m_n}(t-s)} ds \right) w_{m_1\dots m_n}(x), \tag{3.1}$$

is the unique solution of the problem (2.1)–(2.3) *in the class* $C([0, T] \to \mathring{W}_2^1(\Omega))$*, where $h_{m_1\dots m_n}$ are the Fourier coefficients of the function $h(x)$, and $\lambda_{m_1\dots m_n} = \theta_{m_1\dots m_n} \mu_{m_1\dots m_n} > 0$.*

Proof. By using the suggested Fourier series, we will demonstrate that the function $u(x, t)$ is a member of the class $C([0, T] \to \mathring{W}_2^1(\Omega))$. This function's gradient, measured with regard to $x \in \Omega$, may be shown to depend continuously on $t \in [0, T]$ on the space $L_2(\Omega)$ norm. According to Parseval's equality, the norm of this gradient is equal to

$$\begin{aligned} \|\nabla u(\cdot, t)\|^2_{L_2(\Omega)} &= \int_\Omega |\nabla u(x,t)|^2 dx \\ &= \int_\Omega \left| \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} h_{m_1\dots m_n} \left(\int_0^t \nu(s) e^{-\lambda_{m_1\dots m_n}(t-s)} ds \right) \nabla w_{m_1\dots m_n}(x) \right|^2 dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \left| h_{m_1\dots m_n} \int_0^t \nu(s) e^{-\lambda_{m_1\dots m_n}(t-s)} ds \right|^2 \int_\Omega |\nabla w_{m_1\dots m_n}(x)|^2 dx \\
&= \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \left| h_{m_1\dots m_n} \int_0^t \nu(s) e^{-\lambda_{m_1\dots m_n}(t-s)} ds \right|^2 \mu_{m_1\dots m_n} \\
&\le \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} |h_{m_1\dots m_n}|^2 \mu_{m_1\dots m_n} \left(\int_0^t |\nu(s)| e^{-\lambda_{m_1\dots m_n}(t-s)} ds \right)^2 \\
&\le \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} |h_{m_1\dots m_n}|^2 \mu_{m_1\dots m_n} \left(\int_0^t e^{-2\lambda_{m_1\dots m_n}(t-s)} ds \right) \left(\int_0^t |\nu(s)|^2 ds \right) \\
&= \|\nu\|^2_{L_2(0,T)} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} |h_{m_1\dots m_n}|^2 \mu_{m_1\dots m_n} \left(\frac{1 - e^{-2\lambda_{m_1\dots m_n} t}}{2\lambda_{m_1\dots m_n}} \right) \\
&= \|\nu\|^2_{L_2(0,T)} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} |h_{m_1\dots m_n}|^2 \left(\frac{1 - e^{-2\lambda_{m_1\dots m_n} t}}{2\,\theta_{m_1\dots m_n}} \right) \\
&\le C\, \|\nu\|^2_{L_2(0,T)} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} |h_{m_1\dots m_n}|^2 \\
&= C \|\nu\|^2_{L_2(0,T)} \|h\|^2_{L_2(\Omega)},
\end{aligned}
$$

where $C > 0$ is a constant.

Consequently, we get

$$
\|\nabla u(\cdot, t)\|^2_{L_2(\Omega)} \le C \|\nu\|^2_{L_2(0,T)} \|h\|^2_{L_2(\Omega)}.
$$

The function $u(x, t)$ is a generalized solution in the sense of the integral identity (3.5) of monograph [29] that follows from Parseval's equality. □

We can write using the integral condition (2.4) and the solution (3.1)

$$
\begin{aligned}
\phi(t) &= \int_\Omega \rho(x)\, u(x, t)\, dx \\
&= \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} h_{m_1\dots m_n} \left(\int_0^t e^{-\lambda_{m_1\dots m_n}(t-s)}\, \nu(s)\, ds \right) \int_\Omega \rho(x)\, w_{m_1\dots m_n}(x)\, dx \\
&= \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} h_{m_1\dots m_n}\, \rho_{m_1\dots m_n} \int_0^t e^{-\lambda_{m_1\dots m_n}(t-s)}\, \nu(s)\, ds \\
&= \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \Psi_{m_1\dots m_n} \int_0^t e^{-\lambda_{m_1\dots m_n}(t-s)}\, \nu(s)\, ds,
\end{aligned}
$$

where the given function $\phi(t) = \Theta$ for $t \in [T, T_1]$, and

$$
\Psi_{m_1\dots m_n} = h_{m_1\dots m_n}\, \rho_{m_1\dots m_n}, \quad m_k \ge 1, \quad k = \overline{1, n}.
$$

Let's introduce this function:

$$K(t) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \Psi_{m_1 \dots m_n} \, e^{-\lambda_{m_1 \dots m_n} t} \qquad t > 0. \tag{3.2}$$

Thus, we have the following Volterra integral equation of the first kind to determine the control function:

$$\int_0^t K(t-s)\, \nu(s)\, ds = \phi(t), \qquad t > 0, \tag{3.3}$$

where $\phi(t) = \Theta$ for $t \in [T, T_1]$.

Our approach remains valid even when Θ is time-dependent, i.e., $\Theta = \Theta(t)$. This generalization permits the examination of (3.3) for any sufficiently smooth function ϕ.

Differentiating (3.3) yields the following Volterra integral equation of the second kind:

$$K(0)\, \nu(t) + \int_0^t K'(t-s)\, \nu(s)\, ds = \phi'(t), \qquad t > 0. \tag{3.4}$$

We will show in the following sections that $K(0)$ is nonzero and bounded. We establish that for an arbitrary non-negative function $\phi(t)$, the solution $\nu(t)$ of (3.4) remains non-negative. Consequently, this solution exhibits monotonic dependence on the right-hand side of (3.4). This property enables us to derive essential estimates for the control function ν.

4. Main estimates

In this section, we will consider the estimates needed to evaluate the kernel of an integral equation.

Lemma 4.1. (see [30]) *If the function $\psi(x_k)$ satisfies the following conditions:*

$$\psi(x_k) \geq 0, \qquad \frac{d\psi}{dx_k} \leq 0, \qquad \text{for all} \quad x_k \in [0, \infty),$$

then the following inequality holds:

$$\int_0^{m_k \pi} \psi(x_k)\, \sin x_k \, dx_k \geq 0, \quad k = \overline{1, n}. \tag{4.1}$$

Lemma 4.2. *In the event that the function $P(x_1, \dots, x_n)$ meets the following requirements:*

$$P(x_1, \dots, x_n) \geq 0, \qquad \frac{\partial P}{\partial x_k} \leq 0, \qquad \text{for all} \quad x \in [0, \infty)^n, \qquad k = \overline{1, n},$$

then the inequality that follows is true:

$$\int_0^{m_1 \pi} \int_0^{m_2 \pi} \dots \int_0^{m_n \pi} P(x_1, \dots, x_n) \sin x_1 \dots \sin x_n \, dx_1 \dots dx_n \geq 0. \tag{4.2}$$

Proof. Using mathematical induction, we progress with the number of variables n.

Step 1. When $n = 1$, the result follows directly from Lemma 4.1. Thus we have

$$\int_0^{m_1\pi} P(x_1)\sin x_1\, dx_1 \geq 0.$$

Step 2. Assume the statement holds for $n-1$ variables. For the n-dimensional case, we can write

$$I_m = \int_0^{m_n\pi}\left(\int_0^{m_1\pi}\cdots\int_0^{m_{n-1}\pi} P(x_1,\ldots,x_n)\sin x_1\cdots\sin x_{n-1}\, dx_1\cdots dx_{n-1}\right)\sin x_n\, dx_n.$$

We define the inner $(n-1)$-dimensional integral as follows:

$$F(x_n) := \int_0^{m_1\pi}\cdots\int_0^{m_{n-1}\pi} P(x_1,\ldots,x_n)\sin x_1\cdots\sin x_{n-1}\, dx_1\cdots dx_{n-1}.$$

We know that the function $F(x_n)$ is non-negative and decreasing. That is,

$$F(x_n) \geq 0,$$

and

$$\frac{dF}{dx_n} = \int_0^{m_1\pi}\cdots\int_0^{m_{n-1}\pi} \frac{\partial P}{\partial x_n}\sin x_1\cdots\sin x_{n-1}\, dx_1\cdots dx_{n-1} \leq 0.$$

By Lemma 4.1, we get

$$I_m = \int_0^{m_n\pi} F(x_n)\sin x_n\, dx_n \geq 0.$$

This completes the induction and proves the lemma. □

Corollary. *Assume that $\rho \in \mathcal{A}(\Omega)$. Then the following inequality holds:*

$$\rho_{m_1\ldots m_n} \geq 0, \quad m_k \geq 1, \quad k = \overline{1,n}.$$

The proof of this follows directly from Lemma 4.2.

Lemma 4.3. *Let $\rho \in \mathcal{A}(\Omega)$. Then the following estimate holds:*

$$|\rho_{m_1\ldots m_n}| \leq C\lambda_{m_1\ldots m_n}^{-1/2}\,\|\nabla\rho\|_{L_2(\Omega)},$$

where $\rho_{m_1\ldots m_n} = (\rho, w_{m_1\ldots m_n})$, and $C > 0$ is a constant.

Proof. Using (2.7), we can write

$$
\begin{aligned}
\lambda_{m_1\dots m_n}\,\rho_{m_1\dots m_n} &= \lambda_{m_1\dots m_n}\int\limits_{\Omega}\rho(x)\,w_{m_1\dots m_n}(x)\,dx\\
&= -\int\limits_{\Omega}\rho(x)\,L_n\,w_{m_1\dots m_n}(x)\,dx\\
&= -\sum_{j=0}^{2^n-1} a_j\int_{\Omega}\rho(x)\,\Delta w_{m_1\dots m_n}(S_1^{j_1}\dots S_n^{j_n}x)\,dx\\
&= \sum_{j=0}^{2^n-1} a_j\sum_{i=1}^{n}(-1)^{j_i}\int_{\Omega}\frac{\partial}{\partial x_i}w_{m_1\dots m_n}\left(S_1^{j_1}\dots S_i^{j_i}\dots S_n^{j_n}x\right)\frac{\partial}{\partial x_i}\rho(x)\,dx.
\end{aligned}
$$

Note that

$$
\begin{aligned}
w_{m_1\dots m_n}(S_1^{j_1}\dots S_n^{j_n}x) &= C_n\prod_{k=1}^{n}\sin\frac{m_k\,\pi}{p_k}S_k^{j_k}x_k\\
&= C_n\prod_{k=1}^{n}(-1)^{(m_k+1)j_k}\sin\frac{m_k\,\pi}{p_k}x_k.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\partial}{\partial x_i}w_{m_1\dots m_n}\left(S_1^{j_1}\dots S_n^{j_n}x\right) &= C_n\,(-1)^{(m_i+1)j_i}\,\frac{m_i\,\pi}{p_i}\,\cos\frac{m_i\,\pi}{p_i}x_i\prod_{k=1,k\neq i}^{n}(-1)^{(m_k+1)j_k}\sin\frac{m_k\,\pi}{p_k}x_k\\
&= \left(\prod_{k=1}^{n}(-1)^{(m_k+1)j_k}\right)\frac{\partial}{\partial x_i}w_{m_1\dots m_n}(x)\\
&= C_{i,j}\,\frac{\partial}{\partial x_i}w_{m_1\dots m_n}(x).
\end{aligned}
$$

It is clear that

$$
\begin{aligned}
||\nabla w_{m_1\dots m_n}||^2_{L_2(\Omega)} &= (\nabla w_{m_1\dots m_n},\nabla w_{m_1\dots m_n})\\
&= (w_{m_1\dots m_n},-\Delta w_{m_1\dots m_n})\\
&= \mu_{m_1\dots m_n},
\end{aligned}
$$

where $\mu_{m_1\dots m_n}$ is defined by (2.9).

Therefore,

$$
\begin{aligned}
|\lambda_{m_1\dots m_n}\,\rho_{m_1\dots m_n}| &\le \left(\sum_{j=0}^{2^n-1}|a_j|\right)||\nabla\rho||_{L_2(\Omega)}\,||\nabla w_{m_1\dots m_n}||_{L_2(\Omega)}\\
&= \left(\sum_{j=0}^{2^n-1}|a_j|\right)\sqrt{\mu_{m_1\dots m_n}}\,||\nabla\rho||_{L_2(\Omega)}.
\end{aligned}
$$

Using $\lambda_{m_1\dots m_n} = \theta_{m_1\dots m_n}\,\mu_{m_1\dots m_n} > 0$, we get the required estimate

$$
|\rho_{m_1\dots m_n}| \le C\lambda^{-1/2}_{m_1\dots m_n}\,||\nabla\rho||_{L_2(\Omega)}.
$$

□

5. The existence of a solution to an integral equation

In this section, we will consider the existence and uniqueness of a solution to the Volterra integral equation.

Lemma 5.1. *Assume that* $\theta_{m_1...m_n} > 0$ *for all* $m_k \geq 1$, $k = \overline{1, n}$. *Let* $h \in L_2(\Omega)$, *and let* $\rho \in \mathcal{A}(\Omega)$. *Then, the kernel function* $K(t)$ *defined by* (3.2) *is continuous on the half-line* $t \geq 0$.

Proof. Using Lemma 4.2 and condition (2.13), we can say that Ψ_m is non-negative. Therefore

$$K(t) > 0, \quad \text{and} \quad K'(t) < 0, \quad t > 0. \tag{5.1}$$

From Lemma 4.3 and definition (2.15), we have

$$\begin{aligned} \Psi_{m_1...m_n} &= h_{m_1...m_n} \rho_{m_1...m_n} \\ &\leq C\, |h_{m_1...m_n}| \lambda_{m_1...m_n}^{-1/2} \|\nabla \rho\|_{L_2(\Omega)}, \end{aligned}$$

where $\lambda_{m_1...m_n} > 0$ are the eigenvalues of the operator L_n.

We have the estimate

$$\begin{aligned} K(t) &= \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \Psi_{m_1...m_n}\, e^{-\lambda_{m_1...m_n} t} \\ &\leq C\, \|\nabla \rho\|_{L_2(\Omega)} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{h_{m_1...m_n}}{\sqrt{\lambda_{m_1...m_n}}}\, e^{-\lambda_{m_1...m_n} t} \\ &\leq C\, \|\nabla \rho\|_{L_2(\Omega)} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{h_{m_1...m_n}}{\sqrt{\lambda_{m_1...m_n}}}. \end{aligned}$$

□

We can write

$$K(0) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \Psi_{m_1...m_n} > 0,$$

and

$$\int_0^{\infty} K(s)\, ds = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\Psi_{m_1...m_n}}{\lambda_{m_1...m_n}}. \tag{5.2}$$

Moreover, according to (5.1) and (5.2),

$$\begin{aligned} \int_0^{\infty} |K'(s)|\, ds &= -\int_0^{\infty} K'(s)\, ds \\ &= K(0). \end{aligned}$$

Direct computation reveals that for any absolutely continuous function $\phi(t)$ satisfying $\phi(0) = 0$, the following equivalence holds: every bounded solution of (3.3) simultaneously solves (3.4), and conversely, any solution to (3.4) yields a solution to (3.3).

We set

$$A\nu(t) = -\frac{1}{K(0)} \int\limits_0^t K'(t-s)\,\nu(s)\,ds.$$

Then (3.4) takes the form

$$(I - A)\nu(t) = f(t), \tag{5.3}$$

where

$$f(t) = \frac{\phi'(t)}{K(0)}.$$

For any bounded function $f(t)$, the solution to (5.3) exists, is unique, and admits a representation via Neumann series:

$$\nu(t) = \sum_{n=0}^{\infty} A^n f(t), \tag{5.4}$$

where A denotes the corresponding integral operator.

Hence, for any absolutely continuous function $\phi(t)$ satisfying $\phi' \in L_\infty([0,T])$ with $\phi(0) = 0$, there exists a unique solution to Eq (3.3).

Remark 5.1. *The assumption $\phi' \in L_\infty$ is not restrictive for most practical applications, since it is satisfied when the input data are sufficiently smooth. In particular, in many control and heat transfer problems, ϕ represents a physically measurable quantity with bounded variation. This condition is mainly imposed to guarantee the existence and uniqueness of solutions and to simplify the technical estimates in the analysis.*

Lemma 5.2. *Let $\phi(t)$ be an absolutely continuous function so that $\phi(0) = 0$ and $\phi' \in L_\infty([0,T])$. If the following two-sided inequality*

$$0 \le \phi'(t) \le K(t), \quad t \ge 0, \tag{5.5}$$

is fulfilled, then the solution $\nu(t)$ of (3.3) *exists, is unique, and satisfies the following conditions:*

$$0 \le \nu(t) \le 1, \quad t \ge 0. \tag{5.6}$$

Proof. The left inequality in (5.6) follows directly from the representation formula (5.4). For clarity, we introduce the substitution $\nu_0(t) = 1 - \nu(t)$, which appears repeatedly in the subsequent argument. This allows us to rewrite the integral equation in terms of ν_0, making it transparent that the solution is non-negative and satisfies $\nu(t) \le 1$.

Then, using (3.3), we obtain

$$\int_0^t K(t-s)\left(1 - \nu_0(s)\right) ds = \phi(t).$$

Rearranging provides:

$$\int_0^t K(t-s)\,ds - \int_0^t K(t-s)\nu_0(s)\,ds = \phi(t).$$

Let

$$\phi_0(t) = \int_0^t K(t-s)\,ds - \phi(t).$$

Then
$$\int_0^t K(t-s)\nu_0(s)\,ds = \phi_0(t).$$
Differentiate of function $\phi_0(t)$:
$$\phi_0'(t) = K(t) - \phi'(t) \geq 0.$$
Since $\phi_0(0) = 0$ and $\phi_0'(t) \geq 0$, it follows that $\phi_0(t) \geq 0$ for all $t \geq 0$.
The following is the equation for $\nu_0(t)$:
$$\int_0^t K(t-s)\nu_0(s)\,ds = \phi_0(t).$$
Since $\phi_0(t) \geq 0$ and $K(t-s) \geq 0$, the solution $\nu_0(t)$ must also be non-negative. Thus
$$\nu_0(t) = 1 - \nu(t) \geq 0,$$
and we get the following estimate:
$$\nu(t) \leq 1.$$
□

We introduce the function
$$H(t) = \int_0^t K(t-s)\,ds = \int_0^t K(s)\,ds.$$
It is known that $H(0) = 0$ and $H'(t) = K(t) > 0$. The physical meaning of this function is that $H(t)$ denotes the average temperature in Ω.

Define
$$H^* = \lim_{t\to\infty} H(t) = \int_0^\infty K(s)\,ds.$$
It is known that H^* is finite. Indeed, using (5.2), we may write
$$H^* = \int_0^\infty K(s)\,ds = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\Psi_{m_1\ldots m_n}}{\lambda_{m_1\ldots m_n}} < +\infty.$$
We can say that the average temperature in the domain Ω cannot exceed H^*.

It is clear that if and only if the real number Θ satisfies the condition $0 < \Theta < H^*$ then there exists $T > 0$ so that
$$H(T) = \Theta. \tag{5.7}$$

Lemma 5.3. *Let $T > 0$ be a root of (5.7). Let Θ be a positive integer $0 < \Theta < H^*$. Then there is a measurable real-valued function $\nu(t)$ such that $|\nu(t)| \leq 1$ for all $t \geq 0$ and the following equality holds:*
$$\int_0^t K(t-s)\,\nu(s)\,ds = \Theta, \quad T \leq t \leq T_1. \tag{5.8}$$

Proof. We consider the Volterra integral equation (3.3)

$$\int_0^t K(t-s)\,\nu(s)\,ds = \phi(t), \quad t > 0,$$

where

$$\phi(t) = \begin{cases} H(t), & \text{for } 0 \le t \le T; \\ \Theta, & \text{for } T \le t \le T_1. \end{cases}$$

It is evident that this function is absolutely continuous on the half-line $t > 0$ and

$$\phi'(t) = \begin{cases} K(t), & \text{for } 0 \le t \le T; \\ 0, & \text{for } T \le t \le T_1. \end{cases}$$

Obviously,

$$\phi'(t) \le K(t), \quad t \ge 0.$$

Consequently, by Lemma 5.2, there exists a solution ν to the integral equation (3.3) that satisfies the two-sided inequality (5.6). Moreover, this solution ν identically satisfies the equality (5.8). □

Remark 5.2. *Note that this function $\nu(t)$, as it follows from* (3.4)*, has a jump at the point $t = T$.*

6. Estimate of minimal time

We consider the integral equation (3.3) for the case $t \in [T, T_1]$

$$\int_0^t K(t-s)\,\nu(s)\,ds \;=\; \Theta,$$

where the kernel $K(t)$ is defined by (3.2).

Lemma 6.1. *For the kernel $K(t)$ the following estimate holds:*

$$K(t) \ge \Psi_1\, e^{-\lambda_1 t}, \quad t > 0.$$

The proof of this lemma follows from the fact that the kernel $K(t)$ is positive on the half-line $t \ge 0$

Lemma 6.2. *Let be*

$$0 < \Theta < \frac{\Psi_1}{\lambda_1}.$$

Then the solution T to (5.7) *exists and satisfies the following inequalities holds:*

$$0 < T \le -\frac{1}{\lambda_1} \ln\Big(1 - \frac{\Theta\,\lambda_1}{\Psi_1}\Big).$$

Proof. Using Lemma 6.1, we can write the following inequality:

$$\begin{aligned} H(t) &= \int_0^t K(s)\,ds \\ &\geq \Psi_1 \int_0^t e^{-\lambda_1 s} ds \\ &= \frac{\Psi_1}{\lambda_1}\left(1 - e^{-\lambda_1 t}\right). \end{aligned} \tag{6.1}$$

At $t = T_0$ we consider the Eq (6.1), then we have the equation

$$\frac{\Psi_1}{\lambda_1}\left(1 - e^{-\lambda_1 T_0}\right) = \Theta. \tag{6.2}$$

Then, we have

$$T_0 = -\frac{1}{\lambda_1} \ln\left(1 - \frac{\Theta \lambda_1}{\Psi_1}\right).$$

According to (6.1) and (6.2), we get

$$0 < \Theta \leq H(T_0).$$

Then it is clear that T $(0 < T < T_0)$ exists, which is a solution to Eq (5.7). □

The proof of main Theorem 2.1 follows easily from Lemmas 5.3 and 6.2.

Remark 6.1. (Small temperature approximation). *For sufficiently small target temperatures $\Theta > 0$, the minimal time estimate T_0 in Theorem 2.1 admits the first-order approximation:*

$$T_0 \approx \frac{\Theta}{\Psi_1}. \tag{6.3}$$

7. Numerical examples

In this section, we present three illustrative examples of the time-optimal control problem (2.4) in the two-dimensional case $n = 2$. We consider the domain $\Omega = (0, 1)^2$ and set the control target $\Theta = 0.02$, which represents the average temperature over the domain. The functions are chosen as $h(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$ and $\rho(x_1, x_2) = 4(1 - x_1)(1 - x_2)$, ensuring that $h_{1,1} = \rho_{1,1} = 1$, which gives $\Psi_1 = 1$. This setup allows a clear investigation of how the involution coefficient $\theta_{1,1}$ affects the time-optimal control.

Example 1: Strong involution ($\theta_{1,1} > 1$). Here, we choose the coefficients as $(a_0, a_1, a_2, a_3) = (1, 1, 1, 1)$, yielding

$$\theta_{1,1} = \sum_{j=0}^{3} a_j(-1)^{|j|+j_1+j_2} = 2 > 1.$$

The first eigenvalue of the operator is $\lambda_1 = \theta_{1,1}\pi^2(1^2 + 1^2) \approx 39.478$, which corresponds to a time-optimal estimate of

$$T_0 = -\frac{1}{\lambda_1} \ln(1 - \Theta\lambda_1) \approx 0.0395.$$

This result indicates that the strong involution effectively couples symmetric points in the domain, enhancing the overall response and slightly accelerating the heat distribution, reflected in the relatively small T_0 value.

Example 2: Weak involution ($0 < \theta_{1,1} < 1$)**.** Choosing $(a_0, a_1, a_2, a_3) = (1, 0, 0, 0.5)$ gives $\theta_{1,1} = 0.5 < 1$, and the first eigenvalue is $\lambda_1 \approx 9.8696$, leading to $T_0 \approx 0.0104$. In this case, the weaker involution reduces the coupling between symmetric points, slowing the heat propagation. Consequently, the system requires a longer duration to reach the target average temperature, which is reflected in the increased T_0 value.

Example 3: Classical Laplace operator ($\theta_{1,1} = 1$)**.** With coefficients $(a_0, a_1, a_2, a_3) = (1, 0, 0, 0)$, the operator reduces to the classical Laplace operator, so that $\theta_{1,1} = 1$. The first eigenvalue is $\lambda_1 = 2\pi^2 \approx 19.739$, yielding $T_0 \approx 0.0212$. This intermediate value demonstrates the standard diffusion behavior without additional enhancement or suppression due to involution effects.

Collectively, these examples demonstrate the significant influence of the involution coefficient $\theta_{1,1}$ on the time-optimal value T_0: a strong involution increases T_0, a weak involution decreases it, and the classical Laplace operator produces an intermediate outcome. For better visualization and comparison, all three cases are presented in a single 2D plot, highlighting the dependence of the minimum control time on the value of $\theta_{1,1}$.

Remark 7.1. *A comparison of the calculated minimum time values T_0 for the three cases reveals a clear dependence on the parameter θ. In the first case, where $\theta > 1$, T_0 attains the smallest value, indicating that the heat dissipation process concludes more rapidly. In the second case, corresponding to $\theta < 1$, T_0 reaches the largest value, implying that the process is comparatively slower. For the third case, with $\theta = 1$, T_0 takes an intermediate value, lying between those of the first and second cases. These observations highlight the significant influence of the involution coefficient θ on the minimum time required for the heat process in systems with direct involution (see Figure 1).*

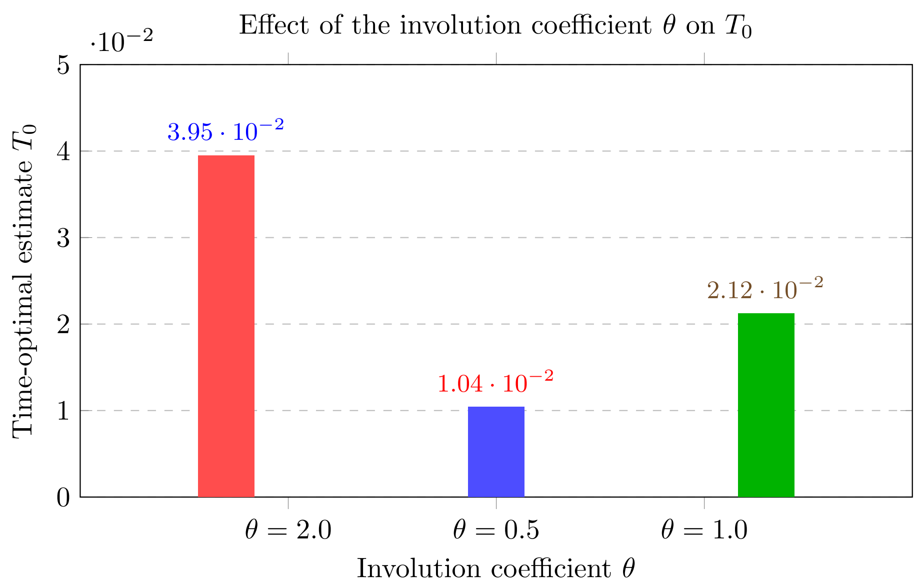

Figure 1. Effect of the involution coefficient θ on the time-optimal value T_0.

8. Conclusions

In this work, we consider the optimal time problem for a parabolic equation with involution in a multidimensional parallelepiped domain. The initial boundary value problem is solved using the separation of variables method. By introducing an additional integral condition, the control problem is reduced to a Volterra integral equation of the first kind. An asymptotic estimate is found for the kernel of the Volterra integral equation. Using this estimate, the existence and uniqueness of a solution to the Volterra integral equation is proved using the Laplace transform method. An optimal estimate is found for the minimum time required to heat the domain to a given average temperature.

Author contributions

Farrukh Dekhkonov: Conceptualization, Methodology, Writing-original draft, Writing-review & editing. Batirkhan Turmetov: Methodology, Writing-original draft, Writing-review & editing. Both authors have been working together on the mathematical development of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. A. Friedman, Optimal control for parabolic equations, *J. Math. Anal. Appl.*, **18** (1967), 479–491. https://doi.org/10.1016/0022-247X(67)90040-6
2. H. O. Fattorini, D. L. Russell, Exact controllability theorems for linear parabolic equations in one space dimension, *Arch. Rational Mech. Anal.*, **43** (1971), 272–292. https://doi.org/10.1007/BF00250466
3. H. O. Fattorini, Time-optimal control of solutions of operational differential equations, *J. SIAM Control*, **2** (1964), 49–65. https://doi.org/10.1137/0302005
4. Yu. V. Egorov, The optimal control in Banach space (in Russian), *Uspekhi Mat. Nauk*, **18** (1963), 211–213.

5. N. Chen, Y. Wang, D. H. Yang, Time-varying bang-bang property of time optimal controls for heat equation and its applications, *Syst. Control Lett.*, **112** (2018), 18–23. https://doi.org/10.1016/j.sysconle.2017.12.008
6. S. Albeverio, S. A. Alimov, On a time-optimal control problem associated with the heat exchange process, *Appl. Math. Optim.*, **57** (2008), 58–68. https://doi.org/10.1007/s00245-007-9008-7
7. S. Alimov, On the null-controllability of the heat exchange process, *Eurasian Math. J.*, **2** (2011), 5–19.
8. S. Alimov, G. Ibragimov, Time optimal control problem with integral constraint for the heat transfer process, *Eurasian Math. J.*, **15** (2024), 8–22.
9. Z. K. Fayazova, Boundary control of the heat transfer process in the space, *Russian Math.*, **63** (2019), 71–79. https://doi.org/10.3103/S1066369X19120089
10. F. N. Dekhkonov, Boundary control associated with a parabolic equation, *J. Math. Comput. Sci.*, **33** (2024), 146–154. https://doi.org/10.22436/jmcs.033.02.03
11. F. Dekhkonov, W. Li, On the boundary control problem associated with a fourth order parabolic equation in a two-dimensional domain, *Discrete Cont. Dyn. Syst.- S*, **17** (2024), 2478–2488. https://doi.org/10.3934/dcdss.2024004
12. N. Arada, J. P. Raymond, Time optimal problems with Dirichlet boundary conditions, *Discrete Cont. Dyn. Syst.*, **9** (2003), 1549–1570. https://doi.org/10.3934/dcds.2003.9.1549
13. G. Wang, The existence of time optimal control of semilinear parabolic equations, *Syst. Control Lett.*, **53** (2004), 171–175. https://doi.org/10.1016/j.sysconle.2004.04.002
14. F. N. Dekhkonov, On the control problem associated with a pseudo-parabolic type equation in an one-dimensional domain, *Int. J. Appl. Math.*, **37** (2024), 109–118. https://doi.org/10.12732/ijam.v37i1.9
15. F. Dekhkonov, On one boundary control problem for a pseudo-parabolic equation in a two-dimensional domain, *Commun. Anal. Mech.*, **17** (2025), 1–14. https://doi.org/10.3934/cam.2025001
16. B. K. Turmetov, B. J. Kadirkulov, An inverse problem for a parabolic equation with involution, *Lobachevskii J. Math.*, **42** (2021), 3006–3015. https://doi.org/10.1134/S1995080221120350
17. B. K. Turmetov, B. J. Kadirkulov, On the solvability of an initial-boundary value problem for a fractional heat equation with involution, *Lobachevskii J. Math.*, **43** (2022), 249–262. https://doi.org/10.1134/S1995080222040217
18. E. Mussirepova, A. Sarsenbi, A. Sarsenbi, The inverse problem for the heat equation with reflection of the argument and with a complex coefficient, *Bound. Value Probl.*, **2022** (2022), 99. https://doi.org/10.1186/s13661-022-01675-1
19. B. Ahmad, A. Alsaedi, M. Kirane, R. G. Tapdigoglu, An inverse problem for space and time fractional evolution equations with an involution perturbation, *Quaest. Math.*, **40** (2017), 151–160. https://doi.org/10.2989/16073606.2017.1283370
20. A. Kopzhassarova, A. Sarsenbi, Basis properties of eigenfunctions of second-order differential operators with involution, *Abstr. Appl. Anal.*, **2012** (2012), 1–6. https://doi.org/10.1155/2012/576843

21. M. Kirane, A. A. Sarsenbi, Solvability of mixed problems for a fourth-order equation with involution and fractional derivative, *Fractal Fract.*, **7** (2023), 1–12. https://doi.org/10.3390/fractalfract7020131
22. B. Turmetov, V. Karachik, On solvability of some inverse problems for a nonlocal fourth-order parabolic equation with multiple involution, *AIMS Math.*, **9** (2024), 6832–6849. https://doi.org/10.3934/math.2024333
23. M. Muratbekova, B. Kadirkulov, M. Koshanova, B. Turmetov, On solvability of some inverse problems for a fractional parabolic equation with a nonlocal biharmonic operator, *Fractal Fract.*, **7** (2023), 404. https://doi.org/10.3390/fractalfract7050404
24. N. Al-Salti, M. Kirane, B. T. Torebek, On a class of inverse problems for a heat equation with involution perturbation, *Hacet. J. Math. Stat.*, **48** (2016), 669–681.
25. F. N. Dekhkonov, On the control problem for a heat conduction equation with involution in a two-dimensional domain, *Lobachevskii J. Math.*, **46** (2025), 613–623. https://doi.org/10.1134/S199508022560013X
26. F. N. Dekhkonov, Boundary control problem for a parabolic equation with involution, *Eurasian J. Math. Comput. Appl.*, **12** (2024), 22–34. https://doi.org/10.32523/2306-6172-2024-12-3-22-34
27. S. G. Mikhlin, *Linear partial differential equations (in Russian)*, Vysshaya Shkola, Moskva, 1977.
28. B. Turmetov, V. Karachik, On eigenfunctions and eigenvalues of a nonlocal Laplace operator with involution in a parallelepiped, *AIP Conf. Proc.*, **2879** (2023), 040005. https://doi.org/10.1063/5.0175246
29. O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva, *Linear and quasi-linear equations of parabolic type (in Russian)*, Nauka, Moscow, 1967.
30. S. A. Alimov, F. N. Dekhkonov, On a control problem associated with fast heating of a thin rod, *Bull. Natl Univ. Uzb.: Math. Nat. Sci.*, **2** (2019), 1–14.