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Research article

Frequently hypercyclic and Devaney chaotic C_0-semigroups indexed by complex sectors

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Abstract: In this paper, the definitions of frequent hypercyclicity and Devaney chaos for a C_0-semigroup $\{T_t\}_{t\in\Delta}$ indexed by a complex sector Δ are revisited. A general sufficient criterion and necessary conditions for a C_0-semigroup $\{T_t\}_{t\in\Delta}$ to be frequently hypercyclic are established. By adapting the concept of periodic point with a syndetic set of periods, we present clean and effective characterizations of the Devaney chaotic translation semigroup $\{T_t\}_{t\in\Delta}$ on $L^p_\rho(\Delta,\mathbb{K})$, $1 \le p < \infty$, which generalize those of the classical translation semigroup $\{T_t\}_{t\ge 0}$. Moreover, it is shown that Devaney chaos implies frequent hypercyclicity, topological mixing, and distributional chaos for the translation semigroups $\{T_t\}_{t\in\Delta}$ under our revised definitions, and that topological mixing or distributional chaos does not imply Devaney chaos.

Keywords: frequent hypercyclicity; Devaney chaos; complex sector; C_0-semigroups; translation semigroup; periodic point
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1. Introduction

Over the past few decades, the chaotic properties of linear dynamic systems have received considerable attention and have been widely researched, including hypercyclicity [1–3], topological mixing [4–6], Devaney chaos [7–9], frequent hypercyclicity [10–12], and distributional chaos [13–15]. The foundational theory is now well established for classical discrete systems (iterations of an operator T) and continuous systems (C_0-semigroups indexed by $\mathbb{R}_{\ge 0}$). For a comprehensive overview of linear chaos, readers are referred to the monographs [16, 17] and the references therein.

Recently, the focus has expanded to a more complex setting: a C_0-semigroup $\{T_t\}_{t\in\Delta}$ indexed by a complex sector $\Delta := \Delta(\alpha) = \{re^{i\theta} \mid r \geq 0, |\theta| \leq \alpha\}$, where $0 < \alpha \leq \pi/2$. A primary theoretical impetus of this extension was the discovery that dynamics on sectors differ fundamentally from the classical setting. For a classical hypercyclic semigroup $\{T_t\}_{t\geq 0}$, every operator T_t ($t > 0$) is itself hypercyclic. However, this property of inheriting hypercyclicity dramatically fails on complex sectors [18], revealing a richer and more intricate dynamic structure. Moreover, as these semigroups are instances of linear semiflows, the field is closely connected to the broader study of topological dynamics [19–21]. Furthermore, this area is motivated by applications in partial differential equations, as the dynamics of C_0-semigroups can model the solutions to certain partial differential equations (see [22–24]). For instance, many chaotic properties for the translation semigroup—a central focus of this paper—can be mapped to the solutions of certain Lasota-type equations via the conjugation lemma (see [23, 25]).

The foundational work by Conejero and Peris established the first characterizations for translation semigroups on sectors, addressing hypercyclicity and topological mixing in [18] and Devaney chaos on weighted function spaces in [26]. This pioneering research opened the door for subsequent investigations into a variety of other chaotic properties, such as supercyclicity [27], f-frequent hypercyclicity [28], multi-dimensional Li–Yorke chaos [19], $\mathcal{F}$-transitivity [20], $d\mathcal{F}$-transitivity [21], recurrent hypercyclicity [29], and distributional chaos [30]. Alongside these specific investigations, the general theory has been systematically developed, notably in the monograph by Kostić [31]. Reinforcing the fundamental principles in this setting, it was also recently confirmed that topological transitivity remains equivalent to hypercyclicity for these semigroups on separable spaces [32].

We recall several key dynamical properties for a C_0-semigroup $\{T_t\}_{t\in\Delta}$ on a Banach space X. The foundational property of complex dynamics is *topological transitivity*, which requires that for any nonempty open sets $U, V \subset X$, the set of meeting times $N(U, V) = \{t \in \Delta : T_t(U) \cap V \neq \emptyset\}$ is nonempty. For separable spaces, this is equivalent to *hypercyclicity*—the existence of a vector with a dense orbit. This can be strengthened to *frequent hypercyclicity*. Frequent hypercyclicity [28] requires an orbit to visit every nonempty open set with positive lower density. In [28], the *lower density* of a measurable set $A \subset \Delta$ is defined with respect to the Lebesgue measure μ as $\underline{d}(A) = \liminf_{r\to\infty} \mu(A \cap \Delta_{r+1})/r$, where $\Delta_{r+1} := \{\tau \in \Delta : |\tau| \leq r + 1\}$. A stronger condition than transitivity is *topological mixing*, which demands that each set $N(U, V)$ in the definition of topological transitivity is co-bounded in Δ. A central notion in the field is *Devaney chaos*, introduced by Conejero and Peris in [26], which couples hypercyclicity with the condition that the set of periodic points is dense in the space. In their work, a point $x \in X$ was defined as periodic for $\{T_t\}_{t\in\Delta}$ if its orbit returns to itself at some nonzero time $t \in \Delta$; that is, if $T_t x = x$.

However, regarding with the definitions of frequent hypercyclicity and Devaney chaos proposed in [28] and [26], many results on these two properties for a C_0-semigroup $\{T_t\}_{t\geq 0}$ no longer hold in the sectorial setting. In particular, characterizations of frequent hypercyclicity and Devaney chaos for the translation semigroup $\{T_t\}_{t\geq 0}$ fail to hold for the translation C_0-semigroup $\{T_t\}_{t\in\Delta}$. Recall that the translation semigroup $\{T_t\}_{t\in\Delta}$ acting on an L^p-space $L^p_\rho(\Delta, \mathbb{K})$, $1 \leq p < \infty$, is defined by $(T_t f)(\tau) = f(\tau + t)$. A function $f : \Delta \to \mathbb{K} \in L^p_\rho(\Delta, \mathbb{K})$ if it is measurable and its norm

$$\|f\| := \left(\int_\Delta |f(\tau)|^p \rho(\tau)\, d\tau \right)^{1/p},$$

is finite. This norm $\|\cdot\|$ is determined by an *admissible weight function* ρ, a positive measurable function that is constrained by the inequality $\rho(t) \leq Me^{w|t'|}\rho(t+t'), t, t' \in \Delta$ for some constants $M \geq 1$ and $w \in \mathbb{R}$. This family of operators is known to form a C_0-semigroup on X. A family of linear and continuous operators $\{T_t\}_{t\in\Delta}$ on a Banach space X is called a C_0-semigroup if it satisfies the semigroup law ($T_0 = I$, $T_{t+s} = T_tT_s$) and is strongly continuous (i.e., the map $t \mapsto T_tx$ is continuous for every $x \in X$). Throughout this paper, we always assume that $1 \leq p < \infty$ when considering L^p-spaces.

Let us list several results of frequent hypercyclicity and Devaney chaos for the translation semigroup $\{T_t\}_{t\geq 0}$, which no longer hold for the translation semigroup $\{T_t\}_{t\in\Delta}$.

(1) The translation semigroup $\{T_t\}_{t\geq 0}$ on $L^p_\rho(\mathbb{R}_{\geq 0}, \mathbb{K})$ is Devaney chaotic if and only if it has a nontrivial periodic point, and if and only if $\int_0^\infty \rho(s)\,ds < \infty$ [33]. However, these equivalences break down for the translation semigroup $\{T_t\}_{t\in\Delta}$ on $L^p_\rho(\Delta, \mathbb{K})$. In [26], Conejero and Peris provided an example of the translation semigroup $\{T_t\}_{t\in\Delta}$ that possesses nontrivial periodic points but is not Devaney chaotic, and they proved that Devaney chaos is equivalent to the existence of a ray $R \subseteq \Delta$ such that ρ is integrable on any stripe

$$F_{R,m} := \left\{ t \in \Delta : d(t, R) = \inf_{s\in R} |t-s| \leq m \right\},$$

where $m \in \mathbb{N}$.

(2) The translation semigroup $\{T_t\}_{t\geq 0}$ on $L^p_\rho(\mathbb{R}_{\geq 0}, \mathbb{K})$ is frequently hypercyclic if and only if $\int_0^\infty \rho(s)\,ds < \infty$ (see [34]), while Chaouchi et al. [28] showed that the translation semigroup $\{T_t\}_{t\in\Delta}$ on $L^p_\rho(\Delta, \mathbb{K})$ is frequently hypercyclic if a ray $R \subseteq \Delta$ exists such that ρ is integrable on any stripe $F_{R,m}$.

(3) The relationships among Devaney chaos, distributional chaos, and topological mixing are altered for translation semigroups. For the translation semigroup $\{T_t\}_{t\geq 0}$ on $X = L^p_\rho(\mathbb{R}_{\geq 0}, \mathbb{K})$, Devaney chaos is a stronger property that implies distributional chaos [35] and topological mixing [4]. In the sectorial case, there is a translation semigroup that is Devaney chaotic but is not distributionally chaotic [30], and a translation semigroup that is Devaney chaotic but is not topologically mixing (Example 2 in [26]).

To further investigate the frequent hypercyclicity and Devaney chaos of a C_0-semigroup $\{T_t\}_{t\in\Delta}$, two observations in these definitions are worth noting.

The first one is that the definition of lower density leads to paradoxical results. The definition provided in [28], $\underline{d}(A) = \liminf_{r\to\infty} \mu(A \cap \Delta_{r+1})/r$, assigns positive density to arbitrarily thin stripes along a ray. This is a counterintuitive outcome for sets that are clearly sparse within the two-dimensional geometry of the sector. Furthermore, it yields an infinite density for the entire space Δ, as the numerator $\mu(\Delta_{r+1}) = \alpha(r+1)^2$ is a higher-order infinity than the denominator r. A robust definition should not result in the density of a space within itself being infinite.

The second one is the definition of periodic points given in [26] for a C_0-semigroup $\{T_t\}_{t\in\Delta}$, which is inspired by the definition of periodic points for a single operator T and a C_0-semigroup $\{T_t\}_{t\geq 0}$. However, there are significant differences between Δ and the other two semigroups (i.e., $\mathbb{N}_0$ and $\mathbb{R}_{\geq 0}$). More precisely, the set of periods corresponding to a periodic point $\{nt_0 : n \in \mathbb{N}\}$ is syndetic in the other two semigroups but it is not syndetic in Δ. Let S be an additive topological abelian semigroup. A subset $A \subseteq S$ is said to be *syndetic* in S [36] if a compact subset $K \subseteq S$ exists such that

$$S = A - K = A + K,$$

where

$$A - K := \{t \in S : t + \tau \in A \text{ for some } \tau \in K\},$$

$$A + K := \{s + \tau : s \in A, \tau \in K\}.$$

In light of the aforementioned results and observations, it would be meaningful to propose some other definitions of frequent hypercyclicity and Devaney chaos for a C_0-semigroup $\{T_t\}_{t\in\Delta}$, which should be more natural and reasonable. For this purpose, we introduce the following two definitions.

Definition 1.1. *A C_0-semigroup $\{T_t\}_{t\in\Delta}$ on a separable Banach space X is said to be* frequently hypercyclic *if a vector $x \in X$ exists such that for every nonempty open set $U \subset X$, the set of return times $\{t \in \Delta : T_t x \in U\}$ has a positive lower density. In this case, x is called a frequently hypercyclic vector for the semigroup $\{T_t\}_{t\in\Delta}$. The lower density $\underline{D}(A)$ of a measurable set $A \subset \Delta$ is defined as:*

$$\underline{D}(A) = \liminf_{t\to\infty} \frac{\mu(A \cap \Delta_t)}{\mu(\Delta_t)}.$$

Definition 1.2. *Let $\{T_t\}_{t\in\Delta}$ be a C_0-semigroup on a Banach space X.*

- *We say that the semigroup is Devaney chaotic if it satisfies two conditions: It is topologically transitive, and its set of periodic points is dense in X.*
- *A vector $x \in X$ is defined as periodic for this semigroup if its set of return times, $\{t \in \Delta : T_t x = x\}$, forms a syndetic subset of Δ.*

It should be mentioned that the definition of periodic points requiring a syndetic set of periods is inspired by [36], and the definition of lower density was also proposed independently of this paper in [30]. Jiang et al. [30] focused on the distributional chaos of a C_0-semigroup $\{T_t\}_{t\in\Delta}$; they also proposed the *upper density* $\overline{D}(A)$ of a measurable set $A \subset \Delta$ as

$$\overline{D}(A) = \limsup_{t\to\infty} \frac{\mu(A \cap \Delta_t)}{\mu(\Delta_t)}.$$

In this paper, we show that the aforementioned characterizations of frequent hypercyclicity and Devaney chaos for the translation semigroup $\{T_t\}_{t\geq 0}$, which fail under the original definitions from [28] and [26], now hold true under our new definitions. In the remainder of this paper, the notions of 'frequent hypercyclicity', 'Devaney chaos', and a 'periodic point' will refer to Definitions 1.1 and 1.2. The 'original' definitions correspond to those presented in [28] and [26].

Our first result provides a sufficient condition for frequent hypercyclicity by extending the classical Frequent Hypercyclicity Criterion [37, 38] to the sectorial framework.

Theorem 1.3. *Let $\{T_t\}_{t\in\Delta}$ be a C_0-semigroup on a separable Banach space X. The semigroup is frequently hypercyclic if there exists a dense subset $X_0 \subset X$ and a family of mappings $S_t : X_0 \to X$ for $t \in \Delta$ that jointly satisfy the following two conditions.*

(i) Given $x \in X_0$ and $\epsilon > 0$, a real number $r_{\epsilon,x} > 0$ exists such that for any finite set $J \subset \mathbb{N}$ and any collection of pairs $\{(\lambda_n, \mu_n)\}_{n\in J} \subset \Delta \times \Delta$ satisfying the separation properties

$$|\lambda_n - \mu_n| \geq r_{\epsilon,x} \quad \textit{and} \quad \left|(\lambda_n - \mu_n) - (\lambda_m - \mu_m)\right| \geq 1 \quad \textit{for all distinct } n, m \in J,$$

the following inequality holds:

$$\left\| \sum_{n \in J} T_{\lambda_n} S_{\mu_n} x \right\| < \epsilon.$$

(ii)

$$\lim_{t \to \infty, t \in \Delta} T_t S_t x = x, \ x \in X_0.$$

As an application, we obtain a more applicable condition for the translation semigroup to be frequently hypercyclic.

Theorem 1.4. *The translation semigroup $\{T_t\}_{t\in\Delta}$ on $X = L^p_\rho(\Delta, \mathbb{K})$ is frequently hypercyclic if*

$$\int_\Delta \rho(s)\, ds < \infty.$$

Lastly, we give clean and effective characterizations of Devaney chaotic translation semigroups.

Theorem 1.5. *Let $\Delta = \Delta(\alpha)$ and $\{T_t\}_{t\in\Delta}$ be the translation semigroup on $X = L^p_\rho(\Delta, \mathbb{K})$. The following assertions are equivalent.*

(i) $\{T_t\}_{t\in\Delta}$ *is Devaney chaotic.*

(ii) The boundary semigroups, $\{T_{re^{\pm i\alpha}}\}_{r\geq 0}$, are themselves Devaney chaotic and possess a dense set of common periodic points.

(iii) $\{T_t\}_{t\in\Delta}$ *admits a nontrivial periodic point.*

(iv) The semigroups $\{T_{re^{\pm i\alpha}}\}_{r\geq 0}$ have a common nontrivial periodic point.

(v) Positive real numbers r_1, r_2 exist such that

$$\sum_{n,m\in\mathbb{N}_0}^{\infty} \rho(nr_1e^{i\alpha} + mr_2e^{-i\alpha}) < \infty.$$

(vi) $\int_\Delta \rho(t)\, dt < \infty.$

This paper is organized as follows. Section 2 contains the proofs of Theorems 1.3 and 1.4. We also provide an example to show that our definition of frequent hypercyclicity is a strict strengthening of the one given in [28]. Section 3 then presents the proof of Theorem 1.5. It is shown that Devaney chaos of the translation semigroup on $L^p_\rho(\Delta, \mathbb{K})$ implies topological mixing, frequent hypercyclicity, and distributional chaos. Explicit counterexamples are constructed to demonstrate that the topological mixing or distributional chaos of the translation semigroup cannot imply Devaney chaos.

2. Characterizations of frequent hypercyclicity

In this section, we investigate the conditions for a frequently hypercyclic C_0-semigroup $\{T_t\}_{t\in\Delta}$. The Frequent Hypercyclicity Criterion, established in [37, 38], is a cornerstone for proving frequent hypercyclicity for single operators. Following a similar idea, we extend the Frequent Hypercyclicity Criterion to the sectorial setting and study the frequent hypercyclicity of the translation semigroup.

2.1. Characterizations of frequently hypercyclic C_0-semigroups

Prior to the proof, we first recall the definition of the lower density for a set $A \subset \mathbb{N}_0$:

$$\underline{\text{dens}}(A) = \liminf_{N\to\infty} \frac{\text{card}\,\{0 \le n \le N \,:\, n \in A\}}{N+1}.$$

Proof of Theorem 1.3. Since X_0 is a dense subset of the separable space X, we can select a sequence (y_k) from X_0 that is also dense in X. Condition (i) then guarantees the existence of a sequence of positive integers $\{r_l\}_{l\in\mathbb{N}}$, such that for any $l \in \mathbb{N}$, $j \le l$ and any finite collection of pairs $\{(\lambda_n, \mu_n)\}_{n\in J} \subset \Delta \times \Delta$ satisfying the separation properties

$$|\lambda_n - \mu_n| \ge r_l \quad \text{and} \quad \left|(\lambda_n - \mu_n) - (\lambda_m - \mu_m)\right| \ge 1 \quad \text{for all distinct } n, m \in J,$$

the following inequality holds:

$$\left\| \sum_{n\in J} T_{\lambda_n} S_{\mu_n} y_j \right\| < \frac{1}{l2^l}. \tag{2.1}$$

In particular, setting $\lambda_n \equiv 0$, we obtain

$$\sup\left\{ \left\| \sum_{n\in J} S_{\mu_n} y_j \right\| : J \text{ finite }, \mu_n \in \Delta, |\mu_n| \ge r_l, \text{ and } |\mu_n - \mu_m| \ge 1,\ \forall n \ne m \right\} < \frac{1}{l2^l} \tag{2.2}$$

for any $j \le l$.

Condition (ii) allows us to assume that

$$\|T_t S_t y_l - y_l\| < \frac{1}{2^l}, \quad \forall |t| \ge r_l,\ \forall l \in \mathbb{N}. \tag{2.3}$$

Our next step requires the construction of a collection of well-separated integer sets. A direct application of [38, Lemma 2.5] and [17, Lemma 9.5] guarantees the existence of pairwise disjoint subsets $A(l, r_l) \subset \mathbb{N}, l \in \mathbb{N}$ with $\underline{\text{dens}}(A(l, r_l)) > 0$. These sets satisfy the crucial separation property that for any $n \in A(l, r_l)$ and $m \in A(k, r_k)$ with $n \ne m$, we have $n \ge r_l$ and $|n - m| \ge r_l + r_k$.

Now, fix an integer $l \ge 1$ and select any $n \in A(l, r_l)$. Assume that $\Delta = \Delta(\alpha)$. We focus on the vertical line segment $L_n := \{x + iy \in \Delta : x = n\}$. Since the set $A(l, r_l)$ is unbounded, we can always choose a sufficiently large n to ensure that the length of this segment, $R_n = 2n \tan\alpha$, is greater than any given value. Therefore, we may assume that $R_n > r_l$. Then we can select $\lfloor R_n/r_l \rfloor + 1$ evenly spaced points along L_n, given by

$$n + in\tan\alpha,\ n + i(n\tan\alpha - r_l),\ \ldots,\ n + i(n\tan\alpha - \lfloor R_n/r_l \rfloor r_l),$$

denoted as $B^n := \{b_1^n, b_2^n, \ldots, b^n_{\lfloor R_n/r_l \rfloor + 1}\}$.

As l varies over $\mathbb{N}$ and n varies over $A = \bigcup_{l\in\mathbb{N}} A(l, r_l)$, we construct the sequences $B^n, n \in A$. Concatenating these sequences in the order of n produces a new sequence

$$(b_k) = \bigcup_{n\in A} B^n.$$

Define $B(l, r_l) = \bigcup_{n \in A(l, r_l)} B^n$ for $l = 1, 2, \ldots$. It can be straightforwardly verified that the sets $B(l, r_l)$, $l \geq 1$, inherit the separation property of the sets $A(l, r_l)$. Specifically, for any distinct $b_k \in B(v, r_v)$ and $b_j \in B(\mu, r_\mu)$, we have

$$|b_k - b_j| \geq r_v.$$

We construct a frequently hypercyclic vector x for the semigroup $\{T_t\}_{t\in\Delta}$ by setting $z_k = y_l$ if $b_k \in B(l, r_l)$, and defining

$$x = \sum_{k\in\mathbb{N}} S_{b_k} z_k. \tag{2.4}$$

Then we divide the proof into two steps.

Step 1: Well-definedness of the vector x

We begin by showing that the series $x = \sum_{k\in\mathbb{N}} S_{b_k} z_k$ is well-defined in X.

From the construction of (b_k), for any $l \in \mathbb{N}$, an integer N_l exists such that $|b_k| \geq r_l$ for all $k \geq N_l$. For any finite set $F \subset \mathbb{N}$, the corresponding partial sum can be expressed as

$$\sum_{k\in F} S_{b_k} z_k = \sum_{j=1}^{\infty} \sum_{\substack{b_k \in B(j, r_j) \\ k\in F}} S_{b_k} y_j.$$

We further split this into two parts

$$\sum_{k\in F} S_{b_k} z_k = \sum_{j=1}^{l} \sum_{\substack{b_k \in B(j, r_j) \\ k\in F}} S_{b_k} y_j + \sum_{j=l+1}^{\infty} \sum_{\substack{b_k \in B(j, r_j) \\ k\in F}} S_{b_k} y_j.$$

From the construction of (b_k), we know that $|b_k - b_{k'}| \geq r_j$ for any distinct pair $b_k, b_{k'} \in B(j, r_j)$. By inequality (2.2), for any $j \leq l$ and finite set F of integers greater than N_l, we have

$$\left\| \sum_{\substack{b_k \in B(j, r_j) \\ k\in F}} S_{b_k} y_j \right\| < \frac{1}{l2^l}.$$

Furthermore, since $|b_k| \geq r_j$ for all $b_k \in B(j, r_j)$, we also have, for any $j \geq 1$ and any finite set $F \subset \mathbb{N}$,

$$\left\| \sum_{\substack{b_k \in B(j, r_j) \\ k\in F}} S_{b_k} y_j \right\| < \frac{1}{j2^j}.$$

Combining these results, for any finite set F of integers greater than N_l, we obtain the total sum:

$$\begin{aligned} \left\| \sum_{k\in F} S_{b_k} z_k \right\| &\leq \sum_{j=1}^{l} \left\| \sum_{\substack{b_k \in B(j, r_j) \\ k\in F}} S_{b_k} y_j \right\| + \sum_{j=l+1}^{\infty} \left\| \sum_{\substack{b_k \in B(j, r_j) \\ k\in F}} S_{b_k} y_j \right\| \\ &\leq \sum_{j=1}^{l} \frac{1}{l2^l} + \sum_{j=l+1}^{\infty} \frac{1}{j2^j} < \frac{2}{2^l}. \end{aligned}$$

Since l is arbitrary, the series (2.4) converges unconditionally, proving that x is well-defined.

Step 2: x is frequently hypercyclic for the semigroup $\{T_t\}_{t\in\Delta}$

Next, we demonstrate that the vector x constructed above is frequently hypercyclic for the semigroup $\{T_t\}_{t\in\Delta}$. Let $l \geq 1$ be fixed. For any $t \in B(l, r_l)$, we can express the difference $T_t x - y_l$ as follows:

$$T_t x - y_l = \sum_{j=1}^{l-1} \sum_{b_k\in B(j,r_j)} T_t S_{b_k} y_j + \sum_{\substack{b_k\in B(l,r_l)\\ b_k\neq t}} T_t S_{b_k} y_l + \sum_{j=l+1}^{\infty} \sum_{b_k\in B(j,r_j)} T_t S_{b_k} y_j + T_t S_t y_l - y_l.$$

For all $j \neq l$, we have $|t - b_k| \geq \max\{r_l, r_j\}$ and $|b_k - b_{k'}| \geq r_j$ for any distinct $b_k, b_{k'} \in B(j, r_j)$. Using inequality (2.1), it follows that

$$\left\| \sum_{b_k\in B(j,r_j)} T_t S_{b_k} y_j \right\| \leq \frac{1}{l2^l}, \quad \forall j < l,$$

and

$$\left\| \sum_{b_k\in B(j,r_j)} T_t S_{b_k} y_j \right\| \leq \frac{1}{j2^j}, \quad \forall j \geq l+1.$$

For any $b_k \in B(l, r_l)$ with $b_k \neq t$, we have $|t - b_k| \geq r_l$. For any distinct $b_k, b_{k'} \in B(l, r_l)$, we have $|b_k - b_{k'}| \geq r_l$. Again, by (2.1)

$$\left\| \sum_{\substack{b_k\in B(l,r_l)\\ b_k\neq t}} T_t S_{b_k} y_l \right\| \leq \frac{1}{l2^l}.$$

Additionally, since $t \geq r_l$, it follows from condition (2.3) that

$$\|T_t S_t y_l - y_l\| \leq \frac{1}{2^l}.$$

Combining these results, we have

$$\|T_t x - y_l\| \leq \frac{1}{2^l} + \frac{1}{2^l} + \frac{1}{2^l} = \frac{3}{2^l}$$

for all $t \in B(l, r_l)$.

Let $U \subset X$ be a nonempty open subset. Since X has no isolated points and $(y_l)_{l\in\mathbb{N}}$ is dense in X, some $c > 0$ and infinitely many y_l with $l \in I \subset \mathbb{N}$ exist such that

$$V(y_l, c) \subset U, \quad \forall l \in I, \tag{2.5}$$

where $V(y_l, c)$ denotes the open ball centered at y_l with the radius c. By the local equicontinuity of $\{T_t\}_{t\in\Delta}$, a sufficiently large $l_0 \in I$ exists such that

$$\|T_s z\| \leq \frac{c}{2}, \quad \forall \|z\| \leq \frac{3}{2^{l_0}},\ |s| \leq 1.$$

Since $\{T_t\}_{t\in\Delta}$ is strongly continuous at y_{l_0}, some $0 < \delta_0 < 1$ exists such that

$$\left\|T_s y_{l_0} - y_{l_0}\right\| \le \frac{c}{2}, \quad \forall |s| \le \delta_0.$$

For any $t \in B(l_0, r_{l_0})$ and $s \in \Delta_{\delta_0}$, we have

$$\begin{aligned} \left\|T_{t+s}x - y_{l_0}\right\| &\le \left\|T_{t+s}x - T_s y_{l_0}\right\| + \left\|T_s y_{l_0} - y_{l_0}\right\| \\ &= \left\|T_s(T_t x - y_{l_0})\right\| + \left\|T_s y_{l_0} - y_{l_0}\right\| \\ &\le \frac{c}{2} + \frac{c}{2} = c. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6), it follows that

$$T_{t+s}x \in U, \quad \forall t \in B(l_0, r_{l_0}),\ \forall s \in \Delta_{\delta_0}.$$

It remains to show

$$\underline{D}\left(B(l_0, r_{l_0}) + \Delta_{\delta_0}\right) > 0.$$

In fact, we can prove

$$\underline{D}(B(l, r_l) + \Delta_\delta) > 0, \quad \forall l \in \mathbb{N}, 0 < \delta < 1.$$

Fix $l \in \mathbb{N}$ and $0 < \delta < 1$. Without loss of generality, we assume that the sets $b + \Delta_\delta, b \in B(l, r_l)$ are mutually disjoint. Let $d_l := \underline{\text{dens}}(A(l, r_l)) > 0$. For any $\epsilon > 0$, the intersection $A(l, r_l) \cap [0, N]$ contains at least $\lfloor (d_l - \epsilon)N \rfloor$ integers for a sufficiently large $N \in \mathbb{N}$. By the construction of $B(l, r_l)$, it consists of at least

$$\frac{\left(1 + \lfloor \frac{2\lfloor (d_l-\epsilon)N\rfloor \tan\alpha}{r_l} \rfloor\right) \lfloor (d_l - \epsilon)N \rfloor}{2}$$

points, leading to

$$\begin{aligned} \mu\left(\{t \in B(l, r_l) + \Delta_\delta : |t| \le N + 1\}\right) &\ge \frac{\left(1 + \lfloor \frac{2\lfloor (d_l-\epsilon)N\rfloor \tan\alpha}{r_l} \rfloor\right) \lfloor (d_l - \epsilon)N \rfloor}{2} \alpha\delta^2 \\ &\ge \frac{\left((d_l - \epsilon)N - 1\right)^2 \alpha\delta^2 \tan\alpha}{r_l}. \end{aligned}$$

Then

$$\begin{aligned} \liminf_{N\to\infty} \frac{\mu\left(\{t \in B(l, r_l) + \Delta_\delta : |t| \le N\}\right)}{\mu\left(\Delta_N\right)} &= \liminf_{N\to\infty} \frac{\mu\left(\{t \in B(l, r_l) + \Delta_\delta : |t| \le N + 1\}\right)}{\alpha N^2} \\ &\ge \frac{(d_l - \epsilon)\delta^2 \tan\alpha}{r_l}. \end{aligned} \tag{2.7}$$

Since $\epsilon > 0$ was arbitrary, it follows from (2.7) that

$$\liminf_{N\to\infty} \frac{\mu\left(\{t \in B(l, r_l) + \Delta_\delta : |t| \le N\}\right)}{\mu\left(\Delta_N\right)} \ge \frac{d_l\delta^2 \tan\alpha}{r_l}.$$

Therefore

$$\underline{D}(B(l, r_l) + \Delta_\delta) = \liminf_{N\to\infty} \frac{\mu\left(\{t \in B(l, r_l) + \Delta_\delta : |t| \le N\}\right)}{\mu\left(\Delta_N\right)} > 0. \tag{2.8}$$

Thus x is frequently hypercyclic for the semigroup $\{T_t\}_{t\in\Delta}$. □

Remark 2.1. *The conditions in Theorem 1.3 also imply that the semigroup is topologically mixing. To see this, we apply the $\mathcal{F}$-transitivity Criterion from [20]. It suffices to show that for any $x \in X_0$ and any $\varepsilon > 0$, a real number $r_{x,\varepsilon} > 0$ exists such that for all $|t| > r_{x,\varepsilon}$, $t \in \Delta$, we have*

$$||T_t x|| < \varepsilon, \quad ||S_t x|| < \varepsilon, \quad \textit{and} \quad ||T_t S_t x - x|| < \varepsilon.$$

These three conditions are a direct consequence of the hypotheses in Theorem 1.3 by choosing $J = \{1\}$ and setting (λ_1, μ_1) to be $(0, t)$ or $(t, 0)$.

Remark 2.2. *It remains an open question whether the conditions in Theorem 1.3 imply Devaney chaos or distributional chaos. We note that the classical Frequent Hypercyclicity Criterion for a single operator implies these two forms of chaos [14, 38].*

In [20, 21], the $\mathcal{F}_{pld}$-transitivity of a C_0-semigroup $\{T_t\}_{t\in\Delta}$ was characterized, where $\mathcal{F}_{pld} := \{A \subset \Delta : \underline{D}(A) > 0\}$. A C_0-semigroup $\{T_t\}_{t\in\Delta}$ is *$\mathcal{F}_{pld}$-transitive* if, for any nonempty open sets $U, V \subset X$, the meeting time set $N(U, V) \in \mathcal{F}_{pld}$.

The following proposition establishes a direct relationship between frequent hypercyclicity and $\mathcal{F}_{pld}$-transitivity.

Proposition 2.3. *Let $\{T_t\}_{t\in\Delta}$ be a C_0-semigroup on a separable Banach space X. If $\{T_t\}_{t\in\Delta}$ is frequently hypercyclic, then it is $\mathcal{F}_{pld}$-transitive.*

Proof. Let $x \in X$ be a frequently hypercyclic vector of $\{T_t\}_{t\in\Delta}$. For any nonempty open subsets $U, V \subset X$, we only need to prove that

$$\underline{D}(N(U, V)) > 0.$$

This can be deduced from the observation that $N(x, V) - N(x, U) \subset N(U, V)$, where $N(x, U) = \{t \in \Delta : T_t x \in U\}$ and $N(x, V) = \{t \in \Delta : T_t x \in V\}$. □

We now establish the sectorial-time analog of a result from [10, Theorem 2.2], namely that *every frequently hypercyclic C_0-semigroup is weakly mixing*. For clarity, we recall that a C_0-semigroup $(T(t))_{t\in\Delta}$ on a Banach space X is *weakly mixing* if, for any nonempty open sets $U_1, U_2, V_1,$ and $V_2 \subset X$, the intersection of the corresponding meeting time sets, $N(U_1, V_1) \cap N(U_2, V_2)$, is nonempty.

Proposition 2.4. *Let $\{T_t\}_{t\in\Delta}$ be a C_0-semigroup on a separable Banach space X. If $\{T_t\}_{t\in\Delta}$ is frequently hypercyclic, then it is weakly mixing.*

Proof. The proof consists of three steps.

First, we establish that the difference set $D := A - A$ of a subset $A \subset \Delta$ of positive lower density is relatively dense in Δ. This means a positive constant $r_D > 0$ exists such that for any $t \in \Delta$, $(t+\Delta_{r_D}) \cap D \neq \emptyset$. This is a straightforward adaptation of the arguments for the discrete case (cf. [17, Theorem 9.7], [39, Theorem 2]). Hence we omit the detailed arguments.

Secondly, let U, V be arbitrary nonempty open subsets of X and let $W \subset X$ be an arbitrary 0-neighborhood. We will show that

$$N(U, W) \cap N(W, V) \neq \emptyset. \tag{2.9}$$

This result is the sectorial analog of [10, Theorem 2.2]. However, the adaptation of the proof to the sectorial setting is nontrivial, so we provide the details below.

Since a frequently hypercyclic semigroup is topologically transitive, some $t_0 \in N(U, W)$ exists. By the continuity of T_{t_0}, a nonempty open subset $U_0 \subset U$ exists such that $T_{t_0}(U_0) \subset W$. Let x be an arbitrary frequently hypercyclic vector for $\{T_t\}_{t\in\Delta}$. Then $A := N(x, U_0)$ has a positive lower density. For any $t, s \in A$ with $t - s \in \Delta$, we have $T_t x \in U_0$ and $T_s x \in U_0$, which implies

$$T_{t_0+t-s}(T_s x) = T_{t_0}(T_t x) \in W.$$

This shows that

$$t_0 + (A - A) \subset N(U_0, W) \subset N(U, W).$$

It follows from the result of the first step that $N(U, W)$ is relatively dense, which means a positive constant $r_0 > 0$ exists such that for any $t \in \Delta$, $(t + \Delta_{r_0}) \cap N(U, W) \neq \emptyset$.

By the local equicontinuity and linearity of $\{T_t\}_{t\in\Delta}$, a 0-neighborhood W_0 exists such that $T_s(W_0) \subset W$ for any $s \in \Delta_{2r_0}$. From Proposition 2.3, $\{T_t\}_{t\in\Delta}$ is $\mathcal{F}_{pld}$-transitive, which implies that $N(W_0, V)$ has a positive lower density. We note that any set with a positive lower density is unbounded and is not confined to any neighborhood of the boundary rays of Δ. Hence, we can find some $\tau_0 \in N(W_0, V)$ such that $\tau_0 - t \in \Delta$, for all $t \in \Delta_{2r_0}$. Let $y \in W_0$ be such that $T_{\tau_0} y \in V$. For any $t \in \Delta_{2r_0}$, we have

$$T_{\tau_0 - t}(T_t y) \in T_{\tau_0 - t}(W) \cap V.$$

This demonstrates that $\tau_0 - \Delta_{2r_0} \subset N(W, V)$. Setting $s_0 = \tau_0 - 2r_0$, we have $s_0 + \Delta_{r_0} \subset \tau_0 - \Delta_{2r_0} \subset N(W, V)$. From the previous discussion, we have shown that $s_0 + \Delta_{r_0} \cap N(U, W) \neq \emptyset$, which further implies that $N(U, W) \cap N(W, V) \neq \emptyset$.

Finally, let U_1, U_2, V_1, and V_2 be nonempty open subsets of X. By the standard topological arguments, we can choose nonempty open subsets $U_i' \subset U_i$, $V_i' \subset V_i$ for $i = 1, 2$ and a 0-neighborhood W such that $U_i' + W \subset U_i$ and $V_i' + W \subset V_i$. Applying [19, Lemma 2.9], we obtain the nonempty open sets $U \subset U_1'$, $V \subset V_1'$ and a 0-neighborhood $W' \subset W$ satisfying

$$N(U, W') \subset N(U_1', W) \cap N(U_2', W) \quad \text{and} \quad N(W', V) \subset N(W, V_1') \cap N(W, V_2').$$

Combining these inclusions with the result (2.9), we obtain the chain of inclusions

$$\emptyset \neq N(U, W') \cap N(W', V) \subset N(U_1' + W, W + V_1') \cap N(U_2' + W, W + V_2') \subset N(U_1, V_1) \cap N(U_2, V_2),$$

which completes the proof. □

2.2. Frequently hypercyclic translation semigroups on complex sectors

In this subsection, we discuss the frequent hypercyclicity of translation semigroups on complex sectors. Specifically, we prove Theorem 1.4, showing that the condition $\int_\Delta \rho(s)\, ds < \infty$ is sufficient for frequent hypercyclicity. Using this theorem, we demonstrate that the translation semigroups in Example 2.5 and Example 2.6 are frequently hypercyclic. Finally, we present a necessary condition for frequently hypercyclic translation semigroups, under which we provide an example of a translation semigroup, showing that our definition of frequently hypercyclic is strictly stronger than the one given by [28].

Proof of Theorem 1.4. We show that the translation semigroup $\{T_t\}_{t\in\Delta}$ with

$$\int_\Delta \rho(s)\,ds < \infty,$$

satisfies the condition in Theorem 1.3.

To do this, use X_0 to denote the set of continuous functions with compact support, which is dense in X. For any $t \in \Delta$ and any $f \in X_0$, define $S_t f$ as follows:

$$S_t f(s) := \begin{cases} f(s-t) & \text{if } s \in t+\Delta, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $T_t S_t f = f, \forall f \in X_0, t \in \Delta$. It suffices to prove that Condition (i) of Theorem 1.3 is satisfied.

Fix $f \in X_0$. Let $\text{supp} f$ denote the support of f. Let $t_f := \sup_{t\in \text{supp} f} |t|$ and $M_f := \max_{s\in\Delta} |f(s)|$. For any $\lambda, \mu, s \in \Delta$, we have

$$T_\lambda S_\mu f(s) = \begin{cases} f(s+\lambda-\mu) & \text{if } s+\lambda \in \mu + \text{supp} f, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \|T_\lambda S_\mu f\|^p &= \int_{\{s\in\Delta: s+\lambda-\mu \in \text{supp} f\}} |f(s+\lambda-\mu)|^p \rho(s) ds \\ &\le M_f^p \int_{\{s\in\Delta: s\in \text{supp} f + \mu - \lambda\}} \rho(s) ds. \end{aligned} \tag{2.10}$$

Note that $\text{supp} f \subset \Delta$ is compact. Then an integer $N \in \mathbb{N}$ exists such that

$$\text{card}\{n \in \mathbb{N} : s \in \text{supp} f + \gamma_n\} < N, \tag{2.11}$$

for any $s \in \Delta$ and any sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ satisfying $|\gamma_n - \gamma_m| \ge 1, n \neq m$.

For any $\epsilon > 0$, let $r_{\epsilon,f} > 0$ be such that

$$\left(\int_{\{s\in\Delta: |s| \ge |t| - t_f\}} \rho(s) ds\right)^{1/p} < \frac{\epsilon}{M_f N}, \tag{2.12}$$

whenever $|t| \ge r_{\epsilon,f}$.

Now we show that for any finite set $J \subset \mathbb{N}$ and any collection of pairs $\{(\lambda_n, \mu_n)\}_{n\in J} \subset \Delta\times\Delta$ satisfying the separation properties

$$|\lambda_n - \mu_n| \ge r_{\epsilon,f} \quad \text{and} \quad \left|(\lambda_n - \mu_n) - (\lambda_m - \mu_m)\right| \ge 1 \quad \text{for all distinct } n, m \in J,$$

the following inequality holds:

$$\left\|\sum_{n\in J} T_{\lambda_n} S_{\mu_n} f\right\| < \epsilon.$$

From the estimation (2.10), it is clear that

$$\left\|\sum_{n\in J} T_{\lambda_n} S_{\mu_n} f\right\| \le \sum_{n\in J} \left\|T_{\lambda_n} S_{\mu_n} f\right\| \le \sum_{n\in J} M_f \left(\int_{\{s\in\Delta: s\in \text{supp} f + \mu_n - \lambda_n\}} \rho(s) ds\right)^{1/p}. \tag{2.13}$$

Since $|(\mu_n - \lambda_n) - (\mu_m - \lambda_m)| \geq 1, \forall n \neq m$, the condition (2.11) holds for the sequence $(\gamma_n) = (\mu_n - \lambda_n)$. Hence for any finite subset $J \subset \mathbb{N}$ and any complex number $\tau \in \Delta$

$$\text{card}\{n \in J : s \in \text{supp} f + \mu_n - \lambda_n\} < N.$$

Note also that $s \in \text{supp} f + \mu_n - \lambda_n$ implies $|s| \geq |\mu_n - \lambda_n| - t_f$. As $|\mu_n - \lambda_n| \geq r_{\epsilon,f}$, from (2.12) and (2.13), we obtain

$$\begin{aligned} \|\sum_{n\in J} T_{\lambda_n} S_{\mu_n} f\| &\leq \sum_{n\in J} M_f \left(\int_{\{s\in\Delta: s\in\text{supp} f + \mu_n - \lambda_n\}} \rho(s) ds \right)^{1/p} \\ &\leq M_f N \left(\int_{\{s\in\Delta: |s| \geq r_{\epsilon,f} - t_f\}} \rho(s) ds \right)^{1/p} < \epsilon, \end{aligned} \tag{2.14}$$

which completes the proof. □

From Theorem 1.4, we can easily obtain the following examples.

Example 2.5. *Let* $\Delta = \Delta(\pi/4)$, *and* $\rho(s) = e^{-|s|}, \forall s \in \Delta$. *The corresponding translation semigroup* $\{T_t\}_{t\in\Delta}$ *on* $X = L^p_\rho(\Delta, \mathbb{K})$ *is frequently hypercyclic.*

Example 2.6. *Let* $\Delta = \Delta(\pi/4)$ *and the weight function be given by*

$$\rho(s) = \frac{1}{(1 + |s|^2)^2}.$$

The translation semigroup $\{T_t\}_{t\in\Delta}$ *on* $X = L^p_\rho(\Delta, \mathbb{K})$ *is frequently hypercyclic.*

In Theorem 2.3, we show that a frequently hypercyclic C_0-semigroup $\{T_t\}_{t\in\Delta}$ is $\mathcal{F}_{pld}$-transitive. Theorem 6 in [20] characterized the $\mathcal{F}$-transitivity of the translation semigroup $\{T_t\}_{t\in\Delta}$. Combined with Theorem 2.3 and [20, Theorem 6], we can obtain the following necessary condition of frequently hypercyclic translation semigroups.

Proposition 2.7. *If the translation semigroup* $\{T_t\}_{t\in\Delta}$ *on* $L^p_\rho(\Delta, \mathbb{K})$ *is frequently hypercyclic, then* $\{t \in \Delta : \rho(t) \leq \epsilon\}$ *has a positive lower density for any* $\epsilon > 0$.

Proof. From Theorem 2.3, a frequently hypercyclic C_0-semigroup $\{T_t\}_{t\in\Delta}$ is $\mathcal{F}_{pld}$-transitive. From [20, Theorem 6], if the translation semigroup on $L^p_\rho(\Delta, \mathbb{K})$ is $\mathcal{F}_{pld}$-transitive, then the weight function satisfies the condition that for any $\epsilon > 0$, $\{t \in \Delta : \rho(t) \leq \epsilon\} \subset \mathcal{F}_{pld}$. □

From Proposition 2.7, we can show the following example is not frequently hypercyclic under our new definition, which was previously shown to be frequently hypercyclic in [28] under the definition there.

Example 2.8. *Let* $\Delta = \Delta(\pi/4)$ *and*

$$\rho(x + iy) = \begin{cases} 1 & \text{if } x + y \geq \sqrt{x - y}, \\ e^{x+y-\sqrt{x-y}} & \text{if } x + y < \sqrt{x - y}. \end{cases}$$

Then the translation semigroup $\{T_t\}_{t\in\Delta}$ *on* $X = L^p_\rho(\Delta, \mathbb{K})$ *is frequently hypercyclic under the definition in [28], but is not frequently hypercyclic under our new definition.*

Proof. First, the translation semigroup in this example was shown to be frequently hypercyclic under the original definition in [28] in the discussion in Example I (i) in [28].

Next, we prove that this semigroup is not frequently hypercyclic under our new definition. Indeed, for any fixed $\epsilon < 1$, the lower density of $\{t \in \Delta : \rho(t) \leq \epsilon\}$ is zero. For any $-1 < k < 1$, let

$$\Delta_{k,1} := \{t = x + yi \in \Delta : kx \leq y \leq x\}.$$

A key geometric observation is that for all points $t = x + iy$ in this region with a sufficiently large modulus (i.e., for $x > N$ for some $N > 0$), the inequality $x + y \geq \sqrt{x-y}$ holds.

Hence

$$\underline{D}(\{t \in \Delta : \rho(t) \leq \epsilon\} \leq \underline{D}(\Delta/\Delta_{k,1}) = \frac{k+1}{2}.$$

Letting $k \to -1$, we have $\underline{D}(\{t \in \Delta : \rho(t) \leq \epsilon\} = 0$. Therefore, from Proposition 2.7, $\{T_t\}_{t\in\Delta}$ is not frequently hypercyclic. □

3. Devaney chaotic translation semigroups on complex sectors

In this section, we investigate the Devaney chaotic translation C_0-semigroup on $L^p_\rho(\Delta, \mathbb{K})$. In Subsection 3.1, we present the proof of Theorem 1.5. In Subsection 3.2, we restore the expected hierarchy among the chaotic properties. We show that Devaney chaos implies topological mixing, frequent hypercyclicity, and distributional chaos under our revised framework in Proposition 3.2 and construct explicit counterexamples (Example 3.3 and Example 3.5) to demonstrate that the converse implications do not hold for topological mixing and distributional chaos.

3.1. Proof of Theorem 1.5

Prior to the proof, we present some basic facts about the set of periods of a periodic point and clarify some related notations. Let $\Delta = \Delta(\alpha)$. We use $\partial\Delta$ to denote the boundary of Δ. For any $-\alpha \leq \theta \leq \alpha$, we define $R_\theta := \left\{\rho e^{i\theta} : \rho \geq 0\right\}$, i.e., the ray in the complex plane originating from the origin and consisting of all complex numbers with the argument θ.

Basic fact 1. *Let $A \subseteq \Delta$. Then A is syndetic in Δ if and only if $\Delta = A - \Delta_r = A + \Delta_r$ for some $r > 0$.*

Proof. We only need to note that any Δ_r is compact in Δ and that any compact subset of Δ must be contained in some Δ_r. □

Basic fact 2. *Let $\Delta = \Delta(\alpha)$ and $A \subset \Delta$ be a syndetic set of Δ. Then $A \cap R_\alpha$ and $A \cap R_{-\alpha}$ are syndetic in R_α and $R_{-\alpha}$, respectively.*

Proof. Let $\Delta_r \subseteq \Delta$ be a compact subset such that

$$\Delta = A - \Delta_r = A + \Delta_r. \tag{3.1}$$

For any $t, t' \in \Delta$, we observe that as long as at least one of t or t' does not belong to R_α, $t + t' \notin R_\alpha$. From Equality (3.1), we have

$$R_\alpha = A \cap R_\alpha + \Delta_r \cap R_\alpha. \tag{3.2}$$

Note that $\Delta_r \cap R_\alpha$ is the set of complex numbers on the ray R_α whose modulus is no greater than r. This is a compact set, and we denote it by K_α, that is,

$$K_\alpha := \left\{\rho e^{i\alpha} : 0 \le \rho \le r\right\}.$$

Hence from Equality (3.2), we have $R_\alpha = A \cap R_\alpha + K_\alpha$. Now we only need to show that $R_\alpha = A \cap R_\alpha - K_\alpha$. Since $R_\alpha = A \cap R_\alpha + K_\alpha$, then for any $t \in R_\alpha$, there exists some $s \in A \cap R_\alpha$ and $s' \in K_\alpha$ such that $t + re^{i\alpha} = s + s'$. Note that $s'' = re^{i\alpha} - s' \in K_\alpha$. Hence, we have $t + s'' = s \in A \cap R_\alpha$; that is, $t \in A \cap R_\alpha - K_\alpha$. Thus, we have shown $A \cap R_\alpha$ is a syndetic set on R_α. Using the same method, we can also show that $A \cap R_{-\alpha}$ is a syndetic set on $R_{-\alpha}$.

□

Basic fact 3. *For a vector $x \in X$ and a C_0-semigroup $\{T_t\}_{t\in\Delta}$, the following statements are equivalent:*

(i) The vector x is periodic.

(ii) The vector x is periodic and the set of its periods contains a lattice-like subset of the form $\{nr_1e^{i\alpha} + mr_2e^{-i\alpha} : (n,m) \in \mathbb{N}_0 \times \mathbb{N}_0\}$, where $r_1, r_2 \in \mathbb{R}^+$.

(iii) x is periodic for the semigroups $\{T_{re^{i\alpha}}\}_{r\ge0}$ and $\{T_{re^{-i\alpha}}\}_{r\ge0}$.

Proof. $(ii) \Rightarrow (i)$. Note that this type of set

$$\{nr_1e^{i\alpha} + mr_2e^{-i\alpha} : (n,m) \in \mathbb{N}_0 \times \mathbb{N}_0\},$$

where $r_1, r_2 \in \mathbb{R}^+$, is syndetic in Δ.

$(i) \Rightarrow (ii)$. Let $A = \{t \in \Delta : T_t x = x\}$. From Basic Fact 2, $A \cap R_\alpha$ and $A \cap R_{-\alpha}$ are syndetic in R_α and $R_{-\alpha}$, respectively. Take $r_1e^{i\alpha} \in A \cap R_\alpha$ and $r_2e^{-i\alpha} \in A \cap R_{-\alpha}$, with $r_1, r_2 > 0$. Since $T_{r_1e^{i\alpha}}x = x$ and $T_{r_2e^{-i\alpha}}x = x$, we have $T_{nr_1e^{i\alpha}+mr_2e^{-i\alpha}}x = x$ for any $n, m \in \mathbb{N}_0$.

$(ii) \Rightarrow (iii)$. The implication is trivial by noting that $T_{nr_1e^{i\alpha}}x = T_{nr_2e^{-i\alpha}}x = x$, for any $n \in \mathbb{N}$.

$(iii) \Rightarrow (ii)$. Let $T_{r_1e^{i\alpha}}x = x$ and $T_{r_2e^{-i\alpha}}x = x$ with $r_1, r_2 > 0$. We then have

$$T_{nr_1e^{i\alpha}+mr_2e^{-i\alpha}}x = x$$

for any $n, m \in \mathbb{N}_0$. □

Now we prove Theorem 1.5.

Proof of Theorem 1.5. The proof proceeds by establishing the equivalence of all six statements. Several implications are direct consequences of the definitions or previous results.

- The implications $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iv)$ are straightforward.
- The equivalence of a nontrivial periodic point for the full semigroup and a common one for its boundary rays, $(iii) \Leftrightarrow (iv)$, follows immediately from our characterization of periodicity in Fact 3.

Therefore, the core of the argument is to demonstrate the following cycle of implications: $(iii) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i)$. Finally, we will show that (vi) also implies (ii), which completes the proof.

$(iii) \Rightarrow (v)$. Let $f \in L^p_\rho(\Delta, \mathbb{K})$ be a nontrivial (i.e. satisfying $\|f\| \ne 0$) periodic point for the semigroup $\{T_t\}_{t\in\Delta}$. From Basic Fact 3, the set of periods of f contains a set $\{nr'_1e^{i\alpha} + mr'_2e^{-i\alpha} : (n,m) \in \mathbb{N}_0 \times \mathbb{N}_0\}$, where $r_1', r_2' \in \mathbb{R}^+$. Then we can find a sufficiently large $n_0 \in \mathbb{N}$, such that $\mu\left(supp(f) \cap \Delta_{\min\{n_0 r_1', n_0 r_2'\}}\right) > 0$.

Denote $r_0 := \min\{n_0 r_1', n_0 r_2'\}$, and $r_1 = 3n_0 r_1'/\cos\alpha, r_2 = 3n_0 r_2'/\cos\alpha$. Now we will show

$$\sum_{n,m\in\mathbb{N}_0}^{\infty} \rho(nr_1 e^{i\alpha} + mr_2 e^{-i\alpha)} < \infty.$$

The condition $\mu(\text{supp}(f) \cap \Delta_{r_0}) > 0$ implies that f is nonzero on a set of a positive measure. Consequently, we can find a positive integer l for which the sublevel set $A := \{t \in \Delta_{r_0} : |f(t)| \geq 1/l\}$ also has a positive measure.

Consider the restriction of f, denoted by $\tilde{f}$, to the set

$$\left\{nr_1 e^{i\alpha} + mr_2 e^{-i\alpha} + t : t \in \Delta_{r_0},\ (n,m) \in \mathbb{N}_0 \times \mathbb{N}_0\right\}.$$

Note that

$$\begin{aligned}|(nr_1 e^{i\alpha} + mr_2 e^{-i\alpha}) - (n' r_1 e^{i\alpha} + m' r_2 e^{-i\alpha})| &\geq \min\{r_1, r_2, r_1\cos\alpha + r_2\cos\alpha\}\\ &\geq \min\{3n_0 r_1', 3n_0 r_2', 3n_0 r_1' + 3n_0 r_2'\}\\ &\geq 3r_0,\end{aligned}$$

for any $(n,m) \neq (n',m')$.

Hence the sets $\{nr_1 e^{i\alpha} + mr_2 e^{-i\alpha} + \Delta_{r_0}\}, n, m \in \mathbb{N}_0$ are mutually disjoint for distinct pairs of indices (n,m). Furthermore, applying Lemma 4.2 from [18] yields a constant $\alpha > 0$ such that the weight function satisfies the inequality $\rho(t+\tau) \geq \alpha\rho(t)$ for all $t \in \Delta$ and $\tau \in \Delta_{r_0}$.

Then

$$\begin{aligned}||f||^p \geq ||\tilde{f}||^p &= \sum_{n,m\in\mathbb{N}_0}^{\infty} \int_{nr_1 e^{i\alpha} + mr_2 e^{-i\alpha} + \Delta_{r_0}} |f(\tau)|^p \rho(\tau)\, d\tau\\ &\geq \sum_{n,m\in\mathbb{N}_0}^{\infty} \alpha\rho(nr_1 e^{i\alpha} + mr_2 e^{-i\alpha}) \int_{nr_1 e^{i\alpha} + mr_2 e^{-i\alpha} + A} |f(\tau)|^p\, d\tau\\ &\geq \frac{\alpha\mu(A)}{l^p} \sum_{n,m\in\mathbb{N}_0}^{\infty} \rho(nr_1 e^{i\alpha} + mr_2 e^{-i\alpha}).\end{aligned}$$

Therefore

$$\sum_{n,m\in\mathbb{N}_0}^{\infty} \rho(nr_1 e^{i\alpha} + mr_2 e^{-i\alpha}) < \infty.$$

$(v) \Rightarrow (vi)$. To prove the implication $(v) \Rightarrow (vi)$, we leverage the syndeticity of the set $A = \{nr_1 e^{i\alpha} + mr_2 e^{-i\alpha} : (n,m) \in \mathbb{N}_0 \times \mathbb{N}_0\}$. By Fact 2, this property guarantees the existence of a radius $r > 0$ such that $\Delta = A - \Delta_r$. This allows us to relate the value of ρ at any point $t \in \Delta$ to its values on the lattice-like set A.

Specifically, from Lemma 4.2 [18], there is a constant $\beta > 0$ such that $\rho(t) \leq \beta\rho(t+\tau)$ for any $t \in \Delta$ and $\tau \in \Delta_r$. Since any $t \in \Delta$ can be written as $t = a - \tau$ for some $a \in A$ and $\tau \in \Delta_r$, we have $a = t + \tau$. This gives us the crucial upper bound

$$\rho(t) \leq \beta\rho(t+\tau) = \beta\rho(a).$$

We can now use this bound to estimate the integral of the weight function over the entire sector. By covering Δ with the sets $\{a - \Delta_r\}_{a\in A}$, we get:

$$\begin{aligned}\int_\Delta \rho(t)\,dt &\le \sum_{n,m\in\mathbb{N}_0} \int_{nr_1e^{i\alpha}+mr_2e^{-i\alpha}-\Delta_r} \rho(t)\,dt \\ &\le \sum_{n,m\in\mathbb{N}_0} \int_{nr_1e^{i\alpha}+mr_2e^{-i\alpha}-\Delta_r} \beta\rho(nr_1e^{i\alpha}+mr_2e^{-i\alpha})\,dt \\ &= \beta\cdot\mu(\Delta_r)\sum_{n,m\in\mathbb{N}_0}\rho(nr_1e^{i\alpha}+mr_2e^{-i\alpha}).\end{aligned}$$

By our initial premise (v), the sum on the right-hand side converges. Since β and $\mu(\Delta_r)$ are finite constants, the entire expression is finite, which completes the proof that $\int_\Delta \rho(t)\,dt < \infty$.

$(vi) \Rightarrow (i)$. First, we show that under Condition (vi), $\int_\Delta \rho(t)\,dt < \infty$, the semigroup $\{T_t\}_{t\in\Delta}$ is not only topologically mixing (a direct consequence of Theorem 4.10 in [18]) but also has a dense set of periodic points.

To prove the density of periodic points, our strategy is to show that for any continuous function $g \in L^p_\rho(\Delta, \mathbb{K})$ with compact support and any $\varepsilon > 0$, we can construct a periodic function f such that $\|f - g\| \le \varepsilon$.

Assume that $\mathrm{supp}(g) \subset \Delta_{r_0}$ for some $r_0 > 0$ and let $M := \sup_{t\in\Delta_{r_0}} |g(t)| > 0$. The integrability of ρ allows us to choose a radius $N > r_0$ large enough to make the integral over the tail sufficiently small

$$\int_{|t|\ge N} \rho(t)\,dt < \frac{\varepsilon^p}{M^p}.$$

Next, we choose R to be large enough such that $\min\{R, 2R\cos\alpha\} > 2N$. Then the sets $\{nRe^{i\alpha} + mRe^{-i\alpha} + \Delta_{r_0}\}, n, m \in \mathbb{N}_0$ are mutually disjoint for distinct pairs of indices (n, m), by noting that

$$\begin{aligned}|(nRe^{i\alpha} + mRe^{-i\alpha}) - (n'Re^{i\alpha} + m'Re^{-i\alpha})| &\ge \min\{R, 2R\cos\alpha\} \\ &> 2N > 2r_0,\end{aligned}$$

for any $(n, m) \ne (n', m')$.

Hence, we can construct the periodic function f by extending g periodically across the lattice $A = \{nRe^{i\alpha} + mRe^{-i\alpha} : (n, m) \in \mathbb{N}_0 \times \mathbb{N}_0\}$. Formally,

$$f(s) = \begin{cases} g(s) & \text{if } s \in \Delta_{r_0}, \\ g(\tau) & \text{if } s = nRe^{i\alpha} + mRe^{-i\alpha} + \tau \text{ for } \tau \in \Delta_{r_0} \text{ and } (n, m) \ne (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

By this construction, f is periodic with a syndetic set of periods A. It remains to verify that f approximates g. The error norm $\|f - g\|$ is determined by the integral over the regions where f and g differ

$$\|f - g\|^p = \sum_{(n,m)\in\mathbb{N}_0\times\mathbb{N}_0\setminus\{(0,0)\}} \int_{nRe^{i\alpha}+mRe^{-i\alpha}+\Delta_{r_0}} |f(s)|^p\rho(s)\,ds$$

$$
\begin{aligned}
&= \sum_{(n,m)\in\mathbb{N}_0\times\mathbb{N}_0\setminus\{(0,0)\}} \int_{nRe^{i\alpha}+mRe^{-i\alpha}+\Delta_{r_0}} |g(s-(nRe^{i\alpha}+mRe^{-i\alpha}))|^p \rho(s)\,ds \\
&\le M^p \sum_{(n,m)\in\mathbb{N}_0\times\mathbb{N}_0\setminus\{(0,0)\}} \int_{nRe^{i\alpha}+mRe^{-i\alpha}+\Delta_{r_0}} \rho(s)\,ds \\
&\le M^p \int_{|t|\ge N} \rho(t)\,dt < \varepsilon^p.
\end{aligned}
$$

The inequality $\|f-g\|<\varepsilon$ shows any continuous function with compact support can be approximated arbitrarily closely by a periodic function. Because the set of compactly supported continuous functions is itself dense in $L^p_\rho(\Delta,\mathbb{K})$, it follows that the set of periodic points is also dense in this space.

(vi) $\Rightarrow$ (ii). The assumption that the weight function ρ is integrable over Δ has two key consequences. First, by Theorem 4.10 in [18], it guarantees that the full semigroup $\{T_t\}_{t\in\Delta}$ is topologically mixing, which, in turn, implies that the boundary semigroups $\{T_{re^{\pm i\alpha}}\}_{r\ge 0}$ are also mixing.

Second, as we have already shown (vi) implies (i) (Devaney chaos for the full semigroup), the set of its periodic points is dense. By Fact 3, this is equivalent to the density of common periodic points for the boundary semigroups. A mixing semigroup with a dense set of periodic points is Devaney chaotic by definition. Therefore, both boundary semigroups are Devaney chaotic, which proves Assertion (ii). □

Remark 3.1. *From this theorem, one can easily verify that Examples 2.5 and 2.6 are Devaney chaotic.*

3.2. Relationships between Devaney chaos and other chaotic notions for translation semigroups

We will show that Devaney chaos implies topological mixing, frequent hypercyclicity, and distributional chaos for the translation semigroup $\{T_t\}_{t\in\Delta}$ on $L^p_\rho(\Delta,\mathbb{K})$, but the converse implications are not true for topological mixing and distributional chaos.

Proposition 3.2. *For the translation semigroup $\{T_t\}_{t\in\Delta}$ on $X=L^p_\rho(\Delta,\mathbb{K})$, Devaney chaos is a sufficient condition for topological mixing, frequent hypercyclicity, and distributional chaos.*

Proof. The proof relies on combining Theorem 1.5 (specifically, the equivalence (i) $\Leftrightarrow$ (vi)) with established characterizations of the other chaotic properties. From Theorem 1.5, we know that if $\{T_t\}_{t\in\Delta}$ is Devaney chaotic, then its weight function is integrable, i.e., $\int_\Delta \rho(t)\,dt<\infty$.

This integrability condition is known to be a sufficient condition for the other three properties.

(a) It implies topological mixing, as shown in [18, Theorem 4.10].

(b) It implies frequent hypercyclicity, as shown in Theorem 1.4.

(c) It implies distributional chaos, as shown in [30, Theorem 4.5].

Therefore, Devaney chaos implies all three properties, completing the proof. □

Having established this hierarchy, the remainder of the subsection is devoted to testing the converse implications. We will show that the hierarchy is strict by constructing key counterexamples: Example 3.3 provides a semigroup that is topologically mixing but not Devaney chaotic, while Example 3.5 presents one that is distributionally chaotic but fails to be Devaney chaotic.

Example 3.3. *Let* $\Delta = \Delta(\pi/4)$ *and*

$$\rho(\tau) = \begin{cases} 1 & \text{if } \tau \in \Delta, |\tau| \leq 1, \\ \frac{1}{|\tau|} & \text{otherwise}. \end{cases}$$

Then the translation semigroup $\{T_t\}_{t\in\Delta}$ *on* $X = L^p_\rho(\Delta, \mathbb{K})$ *is topologically mixing but not Devaney chaotic.*

Proof. The semigroup is topologically mixing, as this property is equivalent to the condition

$$\lim_{t\in\Delta, |t|\to\infty} \rho(t) = 0,$$

(by Theorem 4.10 in [18]), which is clearly satisfied.

However, a straightforward calculation

$$\int_\Delta \rho(s)\, ds > \int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \int_1^{+\infty} \frac{1}{r} r dr d\theta = \infty$$

shows that the weight function is not integrable over the sector. According to Theorem 1.5, the failure of this integrability condition means that the semigroup cannot be Devaney chaotic. □

Similarly, distributional chaos does not imply Devaney chaos, as we will demonstrate in Example 3.5. For the translation semigroups under consideration, distributional chaos is equivalent to the existence of a *distributionally unbounded orbit* [30]. The orbit of a vector x is *distributionally unbounded* if $\|T_t x\|$ tends to infinity along a set of times $A \subset \Delta$ that has an upper density of 1.

Inspired by [14, Proposition 7], we establish the following practical equivalent characterization. Since its proof closely follows the argument in [14], we omit the details here.

Proposition 3.4. *Let* $\{T_t\}_{t\in\Delta}$ *be a* C_0*-semigroup on a Banach space* X*. The following assertions are equivalent:*

(i) $\{T_t\}_{t\in\Delta}$ *admits a distributionally unbounded orbit.*

(ii) There exist $\delta > 0$*, a sequence* $(y_k) \subset X$ *converging to zero, and an increasing sequence* $(r_k) \subset \mathbb{R}^+$ *such that*
$$\lim_{k\to\infty} \frac{\mu\left(\{t \in \Delta_{r_k} : \|T_t y_k\| > \delta\}\right)}{\mu\left(\Delta_{r_k}\right)} = 1.$$

Example 3.5. *Let* $\Delta = \Delta(\pi/6)$*, and let* $(b_k)_{k\in\mathbb{N}} \subset \mathbb{R}^+$ *be a sequence satisfying* $b_{k+1} > k \cdot b_k$ *for all* $k \in \mathbb{N}$*, with* $b_0 := 0$*. We define the weight function* $\rho : \Delta \to \mathbb{R}^+$ *piecewise as*

$$\rho(t) = e^{(b_k - |t|)p} \quad \text{for } b_{k-1} \leq |t| < b_k,\ k \in \mathbb{N}.$$

Then the translation semigroup $\{T_t\}_{t\in\Delta}$ *on* $X = L^p_\rho(\Delta, \mathbb{K})$ *is distributionally chaotic but not hypercyclic and non-Devaney chaotic.*

Proof. At first, we will show that the semigroup is distributionally chaotic. Our strategy is to show that the semigroup admits a distributionally unbounded orbit by applying Proposition 3.4(ii). We will construct a sequence of functions (y_k) converging to zero and a sequence of radii (r_k) such that the orbit of y_k becomes large on a significant portion of the disk Δ_{r_k}.

For each $k \in \mathbb{N}$, define a function $f_k \in L^p_\rho(\Delta, \mathbb{K})$ supported on the annulus $\{t \in \Delta : b_{k-1} \leq |t| < b_k\}$

$$f_k(t) = \begin{cases} e^{|t|}, & b_{k-1} \leq |t| < b_k, \\ 0, & \text{otherwise.} \end{cases}$$

The norm of f_k can be calculated directly. On the support of f_k, the integrand $|f_k(t)|^p \rho(t)$ is constant: $|f_k(t)|^p \rho(t) = (e^{|t|})^p \cdot e^{(b_k - |t|)p} = e^{p|t|} e^{pb_k - p|t|} = e^{pb_k}$. Since the area of the annulus is $\mu(\{t \in \Delta : b_{k-1} \leq |t| < b_k\}) = \frac{\pi}{6}(b_k^2 - b_{k-1}^2)$, the norm is given by

$$\|f_k\|^p = \int_\Delta |f_k(t)|^p \rho(t)\, dt = e^{pb_k} \cdot \frac{\pi}{6}(b_k^2 - b_{k-1}^2).$$

Moreover, for any $s = |s| \cos\theta_s + i|s| \sin\theta_s \in \Delta$, $t = |t| \cos\theta_t + i|t| \sin\theta_t \in \Delta$, it can be straightforwardly verified that

$$|s + t| = \sqrt{|s|^2 + |t|^2 + 2|s||t| \cos(\theta_t - \theta_s)} \geq \sqrt{|s|^2 + |t|^2 + 2|s||t| \cos\frac{\pi}{3}} \geq \frac{1}{2}|s| + |t|.$$

Then

$$\begin{aligned} \|T_s f_k\|^p &\geq \int_{b_{k-1} \leq |t| < b_k - |s|} |f_k(s + t)|^p \rho(t)\, dt \\ &= \int_{b_{k-1} \leq |t| < b_k - |s|} e^{|s+t|p} e^{(b_k - |t|)p}\, dt \\ &\geq \int_{b_{k-1} \leq |t| < b_k - |s|} e^{(\frac{|s|}{2} + |t|)p} e^{(b_k - |t|)p}\, dt \\ &= e^{\frac{p}{2}|s|} e^{b_k p} \frac{\pi}{6}((b_k - |s|)^2 - b_{k-1}^2), \end{aligned}$$

for any $s \in D_k := \{s \in \Delta : b_{k-1} \leq |s| \leq b_k - b_{k-1} - 1\}$.

Note that

$$\inf_{s \in D_k} \frac{e^{\frac{p}{2}|s|}((b_k - |s|)^2 - b_{k-1}^2)}{b_k^2 - b_{k-1}^2} \to \infty \text{ as } k \to \infty.$$

Hence

$$\inf_{s \in D_k} \frac{\|T_s f_k\|}{\|f_k\|} \to \infty \text{ as } k \to \infty.$$

Take a sequence (n_k) of positive integers such that

$$\frac{\|T_s f_{n_k}\|}{\|f_{n_k}\|} \geq k,$$

for any $k \in \mathbb{N}$, $s \in D_k$.

Let $y_k = \frac{f_{n_k}}{k\|f_{n_k}\|}$ and $r_k = b_{n_k} - b_{n_{k-1}} - 1$. Then

$$\|y_k\| < \frac{1}{k} \text{ and } \|T_s y_k\| \geq 1$$

for any $s \in \{t \in \Delta : b_{n_{k-1}} \leq |t| \leq r_k\}$.

Note also that

$$\frac{\mu\left(\{s \in \Delta : b_{n_{k-1}} \leq |s| \leq r_k\}\right)}{\mu(\Delta_{r_k})} \to 1 \text{ as } k \to \infty.$$

Now we have shown that $\{T_t\}_{t\in\Delta}$ satisfies Condition (*ii*) in Proposition 3.4 for (y_k), (r_k) and $\delta = 1$. Hence, it admits a distributionally unbounded orbit and is distributionally chaotic, according to [30, Theorem 4.4].

To show that the semigroup is not hypercyclic, we examine its weight function. By definition, $\rho(t) \geq 1$ for all $t \in \Delta$. However, a necessary condition for the hypercyclicity of a translation semigroup, as established in [18, Theorem 4.8] and [20, Theorem 6], is that the weight function must decay to zero along some path escaping to infinity. Since our weight function ρ is uniformly bounded below by 1, it fails to meet this essential decay requirement. Consequently, the semigroup is not hypercyclic and therefore cannot be Devaney chaotic. □

This subsection has clarified the hierarchy of the chaotic properties for translation semigroups on $L^p_\rho(\Delta, \mathbb{K})$. Proposition 3.2 confirms that Devaney chaos is the strongest notion in this hierarchy, implying topological mixing, distributional chaos, and frequent hypercyclicity. We have also demonstrated that this hierarchy is strict by providing counterexamples for two of the converse implications in Example 3.3 and Example 3.5.

However, it is still unknown whether frequent hypercyclicity implies Devaney chaos for translation semigroups on complex sectors.

Question 3.6. *Does any frequently hypercyclic translation semigroup* $\{T_t\}_{t\in\Delta}$ *on* $L^p_\rho(\Delta, \mathbb{K})$ *satisfy the condition* $\int_\Delta \rho(t)dt < \infty$*, i.e., is it Devaney chaotic?*

4. Conclusions

In this paper, we have revisited the definitions of frequent hypercyclicity and Devaney chaos for C_0-semigroups indexed by complex sectors. Our primary goal was to propose more natural and robust definitions that resolve the inconsistencies observed in the sectorial setting and align better with the classical theory for semigroups on $\mathbb{R}_{\geq 0}$. Our main results are summarized below.

Results for general C_0-semigroups

For a general C_0-semigroup $\{T_t\}_{t\in\Delta}$ on a separable Banach space, we extended the classical Frequent Hypercyclicity Criterion to the sectorial setting (Theorem 1.3). The conditions in Theorem 1.3 also imply topological mixing. Furthermore, we showed that frequent hypercyclicity implies weak mixing. These relationships are placed in the context of known results [19, 32] in Figure 1.

For the translation semigroup $\{T_t\}_{t\in\Delta}$ on $L^p_\rho(\Delta,\mathbb{K})$, our new framework yields precise characterizations. We showed that the integrability of the weight function, $\int_\Delta \rho(s)ds < \infty$, is a sufficient condition for frequent hypercyclicity. More strikingly, this same condition is equivalent to Devaney chaos, which we also proved is equivalent to the existence of a single nontrivial periodic point. These findings clarify the hierarchy of chaotic properties specifically for translation semigroups on sectors. We showed that Devaney chaos is a strictly stronger property that implies topological mixing, frequent hypercyclicity, and distributional chaos, while the converse for mixing and distributional chaos are false. The resulting hierarchy of these dynamic properties is visualized in Figure 2.

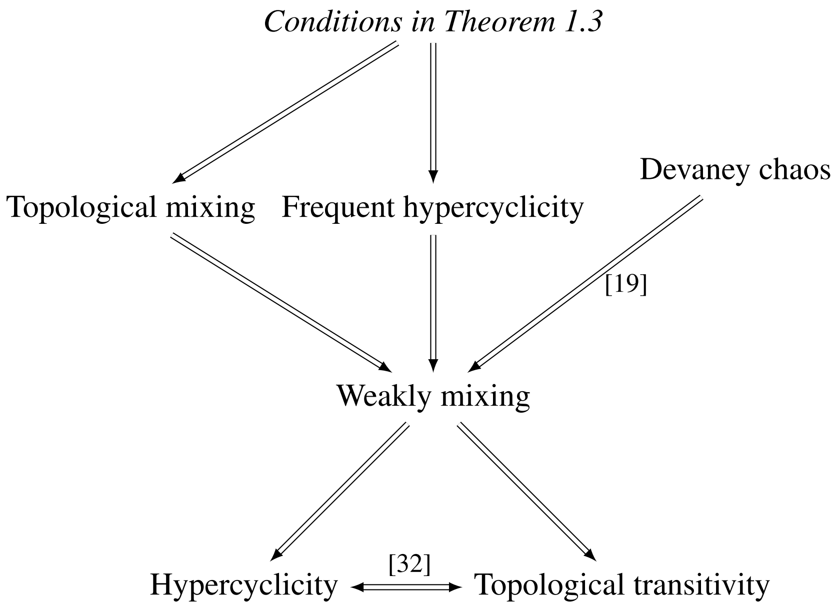

Figure 1. Relationships among various dynamic properties of a C_0-semigroup $\{T_t\}_{t\in\Delta}$ on a separable Banach space.

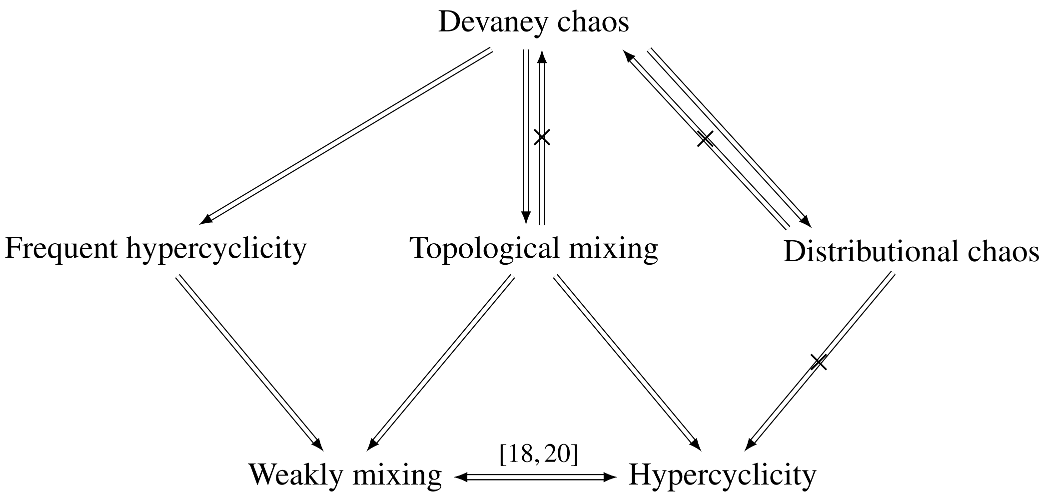

Figure 2. Relationships among various chaos and hypercyclicity properties for the translation semigroup on $L^p_\rho(\Delta, \mathbb{K})$.

Author contributions

Shengnan He: Conceptualization, methodology, investigation, formal analysis, validation, writing–original draft, supervision, writing–review and editing; Zongbin Yin: Conceptualization, formal analysis, validation, writing–original draft, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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