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Research article

Modified $\mathfrak{R}$-rational contractions and fixed point results with applications to boundary value problems

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Abstract: This manuscript introduces modified $\mathfrak{R}$-rational contractions for single self-maps, leveraging ω-distance in a relational-theoretic metric space. This novel approach establishes the existence and uniqueness of a fixed point for self-maps, specifically by applying the locally Θ-transitivity property. We support our theoretical advancements with compelling examples and demonstrate their practical significance by solving a fourth-order boundary value problem related to transverse oscillation in a homogeneous bar and a first-order periodic boundary value problem.

Keywords: fixed point technique; $\mathfrak{R}$-rational contraction mapping; transverse oscillation bar; boundary value problem; evaluation metrics
Mathematics Subject Classification: 47H09, 47H10

Abbreviations

FP⇒fixed point; MS⇒metric space; LSC⇒lower semicontinuous; TVO⇒transverse oscillations; BVP⇒boundary value problem

1. Introduction

The traditional view of scientific disciplines as isolated fields has evolved. Today, thanks to remarkable advancements in modern theories of basic science, these disciplines are deeply interconnected. Mathematics, for instance, has seen various levels of development across its branches in contemporary times. Fixed point (FP) theory stands out as a prime example, offering a fundamental concept with widespread applications. It remains a crucial theoretical tool in diverse fields such as topology, game theory, optimal control, artificial intelligence, logic programming, dynamical systems, functional analysis, differential equations, and economics. Specifically, FP techniques are employed to solve equilibrium problems in economics and game theory, and to find analytical and numerical solutions to nonlinear integral equations like Fredholm integral equations [1–3].

FP theory is fundamentally built upon the concept of contraction mappings, which are functions that effectively shrink distances between points in a metric space (MS) [4, 5]. The foundational example is Banach contraction, also known as strict contraction, where the distance between the images of two points is strictly less than a constant multiple (less than one) of the distance between the original points. This classical concept guarantees a unique FP and provides a constructive iterative method to find it. Building upon this, numerous generalizations have emerged to broaden the applicability of FP theorems. These include Kannan-type contractions, which relate the distance between images to distances involving the points and their images, and Ciric-type contractions, which offer more general conditions by involving distances between various combinations of points and their images. Further extensions encompass Ψ-contractions or ϕ-contractions, which use auxiliary functions to modulate the contraction condition, allowing for more flexible distance relationships; see [6–8] for more information.

The exploration of FPs for monotone mappings in MSs endowed with a partial order has significantly advanced FP theory, providing powerful tools for solving various problems in applied mathematics. This area of research gained substantial traction following the work of Ran and Reurings, who established a foundational theorem for contractions in partially ordered MSs, which are particularly useful in matrix equations [9]. Subsequent developments have extended these initial results to a broader class of monotone mappings and generalized contraction conditions, often under suitable compatibility or continuity assumptions related to the partial order and metric. These advancements have allowed for the investigation of FPs in diverse settings, including integral and differential equations, where the underlying operators exhibit monotonicity properties. Furthermore, the theory has been expanded to encompass more abstract structures like partially ordered partial MSs, further broadening its scope and applicability [10, 11].

Recent advancements have extended the classical Banach contraction principle to complete MSs equipped with a binary relation, offering a more generalized framework for studying FPs. This approach typically involves replacing the standard contractive condition with a weaker one that holds only for elements related by the given binary relation, while ensuring that the contraction mapping preserves this relation. These extensions provide powerful tools for solving various nonlinear problems in diverse fields such as differential equations, integral equations, and optimization. For instance, Alam and Imdad [12] introduced relation-theoretic metrical coincidence theorems, Samet and Turinici [13] obtained soem fixed point theorems on a metric space and arbitrary binary relations, Prasad and Khantwal [14] presented fixed point theorems in relational metric spaces and Antal et al. [15] proved some fixed points theorems under generalized contraction mappings via w-distance in relational metric spaces with applications.

This article aims to introduce modified $\mathfrak{R}$-rational contractions using suitable auxiliary functions, and to explore new FP results within the framework of relational metric spaces. We define and provide examples of these modified $\mathfrak{R}$-rational contractions, then prove some theorems establishing the existence and uniqueness of the FPs in relational metric spaces. To illustrate our findings, we include several examples, and conclude with an application demonstrating the existence of a solution for a boundary value problem (BVP) that governs transverse oscillation in a homogeneous bar and and a first-order periodic BVP.

2. Preliminary work

Before delving into the main results of this study, we will lay out some fundamental definitions, key examples, and crucial lemmas that will be instrumental in understanding our primary findings. Throughout this paper, the following notations will be used: Q represents a nonempty set, $\mathfrak{R}$ denotes a nonempty binary relation on Q, $\mathbb{N}$ signifies the set of positive integers, and $\mathbb{N}_0$ refers to the set of whole numbers.

Definition 2.1. *[12] Consider a nonempty set Q and a binary relation $\mathfrak{R}$ defined on it. For any $\xi, \zeta \in Q$:*

i) We say that ξ is an $\mathfrak{R}$-related to ζ if and only if $(\xi, \zeta) \in \mathfrak{R}$, provided that $\mathfrak{R} \subseteq Q \times Q$;
ii) If either $(\xi, \zeta) \in \mathfrak{R}$ or $(\zeta, \xi) \in \mathfrak{R}$, we say that ξ and ζ are R-comparable and we write $[\xi, \zeta] \in \mathfrak{R}$.

Definition 2.2. *[16] Assume that Q is a non-empty set and $\mathfrak{R}$ is a binary relation on Q. The dual, inverse, or transpose relation $\mathfrak{R}^{-1}$ is described for any $\xi, \zeta \in Q$ as follows:*

$$\mathfrak{R}^{-1} = \{(\xi, \zeta) \in Q \times Q : (\zeta, \xi) \in \mathfrak{R}\}.$$

Proposition 2.1. *[17] Consider $\mathfrak{R}^{sy}$ to be the minimal symmetric relation that includes $\mathfrak{R}$. For any binary relation $\mathfrak{R}$ on a non-empty set Q, we have*

$$(\xi, \zeta) \in \mathfrak{R}^{sy} \text{ if and only if } [\xi, \zeta] \in \mathfrak{R}.$$

Definition 2.3. *[17] Suppose that (Q, ϖ) is an MS and $\Theta : Q \to Q$ is a given mapping. It then follows that for any binary relation $\mathfrak{R}$ on Q, we have the following:*

1) $\mathfrak{R}$ is called a Θ-closed, if for any $\xi, \zeta \in Q$, and $(\xi, \zeta) \in \mathfrak{R}$ implies $(\Theta\xi, \Theta\zeta) \in \mathfrak{R}$.
2) For $j \in \mathbb{N}_0$, $\mathfrak{R}$ is Θ^j-closed, provided that $\mathfrak{R}$ is Θ-closed, where Θ^j represents the i-th iteration of Θ.
3) If $\left(\xi_j, \xi_{j+1}\right) \in \mathfrak{R}$, $j \in \mathbb{N}_0$, then a sequence $\{\xi_j\}$ is said to be $\mathfrak{R}$-preserving.
4) We say $\mathfrak{R}$ is a ϖ-self closed if an $\mathfrak{R}$-preserving sequence $\{\xi_j\}$ with $\lim_{j\to\infty} \xi_j = \xi$ on ϖ always implies the existence of a subsequence $\{\xi_{j_k}\}$ such that $[\xi_{j_k}, \xi] \in \mathfrak{R}$ for all $k \in \mathbb{N}_0$.

Definition 2.4. *[12] Assume that (Q, ϖ) is an MS and $\mathfrak{R}$ is a binary relation defined on it.*

i. (Q, ϖ) is said to be $\mathfrak{R}$-complete, if every $\mathfrak{R}$-preserving Cauchy sequence in Q converges to an element in Q.
ii. The mapping $\Theta : Q \to Q$ is called $\mathfrak{R}$-continuous at ξ if, whenever an $\mathfrak{R}$-preserving sequence $\{\xi_j\}$ converges to ξ, the sequence of images $\{\Theta\left(\xi_j\right)\}$ must converge to $\{\Theta(\xi)\}$ under the distance ϖ.
iii. The mapping Θ is simply called $\mathfrak{R}$-continuous if it exhibits $\mathfrak{R}$-continuity at every point in its domain Q.

Remark 2.1. *It is worth noting the following.*

($\heartsuit_1$) Every complete MS is also $\mathfrak{R}$-complete for any binary relation $\mathfrak{R}$, but the converse is not always true.
($\heartsuit_2$) Any continuous mapping is $\mathfrak{R}$-continuous, irrespective of the specific binary relation $\mathfrak{R}$.

Example 2.1. *[18] Assume that $Q = [0, 1)$ with the distance $\varpi(\xi, \zeta) = |\xi - \zeta|$. Describe a binary relation $\Re$ on Z as*

$$R = \left\{(\xi, \zeta) : \frac{1}{3} \geq \zeta \geq \xi \geq \frac{1}{4} \text{ or } \frac{1}{2} \leq \xi \leq \zeta \leq 1\right\}.$$

Clearly, (Q, ϖ) is an $\Re$-complete MS, but it fails to be a complete MS. Furthermore, define the mapping $\Theta : Q \to Q$ by

$$\Theta(\xi) = \begin{cases} \frac{1}{4}, & at\ \xi \in [0, \frac{1}{2}), \\ 1, & at\ \xi \in [\frac{1}{2}, 1]. \end{cases}$$

Then, Θ is $\Re$-continuous but it is not continuous.

Definition 2.5. *[12] For a mapping $\Theta : Q \to Q$ on an MS (Q, ϖ) with a binary relation $\Re$, we say that Θ is $\Re$-continuous at ξ if, whenever an $\Re$-preserving sequence $\{\xi_j\}$ converges to ξ, the sequence of images $\{\Theta\left(\xi_j\right)\}$ must converge to $\{\Theta(\xi)\}$ under the distance ϖ. Furthermore, the mapping Θ is simply called $\Re$-continuous if it exhibits $\Re$-continuity at every point in its domain Q.*

Definition 2.6. *For each $\xi, \zeta \in P$, the following hold.*

(i) If there is a path in $\Re$ from ξ to ζ, a subset $P \subseteq \Re$ is called $\Re$-connected [12].
(ii) If there exists $\rho \in Q$ such that $(\xi, \rho) \in \Re$ and $(\zeta, \rho) \in \Re$, a subset $P \subseteq \Re$ is called $\Re$-directed [13].

Definition 2.7. *[19] A path of length u in a binary relation $\Re$ on a nonempty set Q from ξ to ζ is a finite sequence $\{q_0, q_1, q_2, \cdots, q_u\} \subset Q$ such that*

(a) $q_0 = \xi$ and $q_u = \zeta$,
(b) for each $i\,(0 \leq i \leq u - 1)$, $(q_i, q_{i+1}) \in \Re$.

Remark 2.2. *Let $\Xi(\xi, \zeta, \Re)$ denote the collection of all paths from ξ to ζ within the relation $\Re$. When the set of paths from $\Theta(\xi)$ to $\Theta(\zeta)$ in $\Re$, denoted $\Xi(\Theta\xi, \Theta\zeta, \Re)$, is non-empty, this signifies precisely that $\Theta(Q)$ is $\Re$-connected.*

Definition 2.8. *[19] Consider a nonempty set Q, equipped with a binary relation $\Re$ and a self-map Θ. In this case*

1) If for all $\xi, \zeta, \rho \in Q$ such that

$$(\Theta\xi, \Theta\zeta) \in \Re, \ (\Theta\zeta, \Theta\rho) \in \Re \text{ implies } (\Theta\xi, \Theta\rho) \in \Re,$$

we say that $\Re$ is a Θ-transitive.
2) If for each $\Re$-preserving sequence, $\{\xi_j\} \subset Q$ (with the range $\Xi = \{\xi_j : j \in \mathbb{N}_0\}$) and the restriction $\Re\big|_\Xi$ is transitive, we say that $\Re$ is locally transitive.
3) If for each $\Re$-preserving sequence $\{\xi_j\} \subset \Theta(Q)$ (with range Ξ) and the restriction $\Re\big|_\Xi$ is transitive, we say that $\Re$ is locally Θ-transitive.

Definition 2.9. *[20] A function $\Theta : Q \to \mathbb{R} \cup \{-\infty, \infty\}$ is defined as $\Re$-lower semicontinuous (LSC) at ξ, where $\Re$ is a binary relation on the MS (Q, ϖ), if, for any $\Re$-preserving sequence, $\{\xi_j\}$ converges to ξ, it must be true that*

$$\liminf_{j \to \infty} \Theta\left(\xi_j\right) \geq \Theta(\xi).$$

Definition 2.10. *[20] A map $q : Q \times Q \to [0, \infty)$ is said to be an ω-distance on Q where $\Re$ is a binary relation on the MS (Q, ϖ), if it possesses the following characteristics for all $\xi, \zeta, \rho \in Q$:*

i) $q(\xi, .) : Q \to [0, \infty)$ *is $\Re$-LSC;*
ii) $q(\xi, \zeta) \leq q(\xi, \rho) + q(\rho, \zeta)$;
iii) for each $\varepsilon > 0$, there is a $\vartheta > 0$ such that

$$q(\rho, \xi) \leq \vartheta,\ q(\rho, \zeta) \leq \vartheta \text{ implies } q(\xi, \zeta) \leq \varepsilon.$$

Lemma 2.1. *[20] Assume that (Q, ϖ) is an MS and q is a ω-distance defined on it. Let $\{\xi_j\}$ be a sequence in Q such that*

$$\lim_{j \to \infty} q\left(\xi_j, \xi_{j+1}\right) = 0.$$

Should $\{\xi_j\}$ fail to be a Cauchy sequence in Q, it implies the existence of an $\varepsilon > 0$ and two subsequences, $\{\xi_{j_k}\}$ and $\{\xi_{i_k}\}$ of $\{\xi_j\}$ satisfying $k \leq j_k \leq i_k$ such that

$$q(\xi_{j_k}, \xi_{i_k}) \geq \varepsilon,\ q(\xi_{j_k}, \xi_{i_k-1}) < \varepsilon,$$

and the following inequalities are true:

$$\lim_{j \to \infty} q\left(\xi_{j_k}, \xi_{i_k}\right) = \varepsilon,\ \lim_{j \to \infty} q\left(\xi_{j_k}, \xi_{i_k-1}\right) = \varepsilon,\ \lim_{j \to \infty} q\left(\xi_{j_k+1}, \xi_{i_k+1}\right) = \varepsilon,\ \lim_{j \to \infty} q\left(\xi_{j_k+1}, \xi_{i_k-1}\right) = \varepsilon.$$

The concepts of local Θ-transitivity, ω-distance, and $\Re$-preserving sequences are crucial generalizations that extend the applicability of classical FP theorems, such as the Banach contraction principle, to a wider range of spaces and mappings. Local Θ-transitivity weakens the strong requirement of global transitivity in a binary relation, which is a common assumption in relation-theoretic FP results. By only requiring transitivity on specific parts of the space, it allows for the study of FPs in more complex and less structured settings. The introduction of an ω-distance provides a more flexible alternative to a traditional metric. Unlike a metric, an ω-distance does not necessarily satisfy the triangular inequality, making it possible to work with spaces that lack this property while still retaining crucial convergence concepts. Finally, an $\Re$-preserving sequence ensures that the iterative process used to find an FP remains within the constraints of the defined binary relation $\Re$. This is a vital condition for guaranteeing that a contraction inequality, which is often defined only for related points, can be applied to the sequence generated by the mapping, thereby allowing for proof of the existence and the uniqueness of FPs in these more generalized spaces. Together, these tools enable mathematicians to prove FP theorems in a richer variety of abstract spaces, with direct applications to solving complex equations in fields like differential equations and integral equations.

3. Modified $\Re$-rational contractions

We begin this part with defining the function Ψ as follows:

$$\begin{aligned} \Psi = \{&\psi : \psi : [0, \infty) \to [0, \infty) \text{ is continuous and monotone increasing function} \\ &\text{fulfills } \psi(\gamma) = 0 \Leftrightarrow \gamma = 0\}. \end{aligned}$$

Definition 3.1. *Consider an MS (Q,ϖ) equipped with an arbitrary binary relation $\mathfrak{R}$. Let p be an ω-distance on Q. A map $\Theta : Q \to Q$ is defined as $\omega_{\psi}^{L} - \mathfrak{R}$-contraction if for all $\xi,\zeta \in Q$ with $(\xi,\zeta) \in \mathfrak{R}$, the following holds:*

$$\psi\left(q\left(\Theta\xi,\Theta\zeta\right)\right) \leq \psi\left(U_{\Theta}\left(\xi,\zeta\right)\right) - LU_{\Theta}\left(\xi,\zeta\right), \tag{3.1}$$

where $\psi \in \Psi$, $L \in (0,1)$, and

$$U_{\Theta}\left(\xi,\zeta\right) = \max\left\{q\left(\xi,\zeta\right), \frac{q\left(\xi,\Theta\xi\right)\left[1+q\left(\zeta,\Theta\xi\right)\right]}{1+q\left(\xi,\zeta\right)}, \frac{q\left(\zeta,\Theta\zeta\right)\left[1+q\left(\xi,\Theta\xi\right)\right]}{1+q\left(\xi,\zeta\right)}, \frac{q\left(\xi,\Theta\zeta\right)+q\left(\zeta,\Theta\xi\right)}{2}\right\}.$$

Example 3.1. *Let (Q,ϖ) be an MS where $Q = [1,\infty) \cup \{0\}$ and ϖ is the usual metric. Define a binary relation $\mathfrak{R}$ on Q and a map $\Theta : Q \to Q$ such that for all $\xi,\zeta \in Q$,*

$$\mathfrak{R} = \{(\xi,\zeta) : \xi \leq \zeta\}, \text{ and } \Theta\left(\xi\right) = \frac{\xi}{3}.$$

Moreover, define the function $\psi : [0,\infty) \to [0,\infty)$ by $\psi(\gamma) = \gamma$ for all $\gamma \in [0,\infty)$ and an ω-distance $q : Q \times Q \to Q$ by $q(\xi,\zeta) = \zeta$. Hence, for all $\xi,\zeta \in Q$ with $(\xi,\zeta) \in \mathfrak{R}$, one has

$$\begin{aligned}
\psi\left(q\left(\Theta\xi,\Theta\zeta\right)\right) &= \psi\left(\Theta\left(\zeta\right)\right) = \Theta\left(\zeta\right) \\
&= \frac{\zeta}{3} \leq \frac{\zeta}{2} = \zeta - \frac{\zeta}{2} \\
&= q\left(\xi,\zeta\right) - Lq\left(\xi,\zeta\right) \\
&= \psi\left(U_{\Theta}\left(\xi,\zeta\right)\right) - LU_{\Theta}\left(\xi,\zeta\right).
\end{aligned}$$

Thus, Θ is an $\omega_{\psi}^{L} - \mathfrak{R}$-contraction with $L = \frac{1}{2}$.

This next definition extends and generalizes the C-condition (as considered by [21]) by incorporating auxiliary functions, ω-distance, and a binary relation.

Definition 3.2. *Consider an MS (Q,ϖ) equipped with an arbitrary binary relation $\mathfrak{R}$. Let p be an ω-distance on Q. We say that a map $\Theta : Q \to Q$ is $\omega C_{\phi}^{L} - \mathfrak{R}$-contraction if for all $\xi,\zeta \in Q$ with $(\xi,\zeta) \in \mathfrak{R}$,*

$$\frac{1}{2}q\left(\xi,\Theta\xi\right) \leq q\left(\xi,\zeta\right) \Longrightarrow L\left(q\left(\Theta\xi,\Theta\zeta\right)\right) \leq \phi\left(U_{\Theta}\left(\xi,\zeta\right)\right), \tag{3.2}$$

where $L \in (0,1)$ and a function $\phi : [0,\infty) \to [0,\infty)$ is continuous fulfilling $\phi(\gamma) < L$ for all $\gamma > 0$

The next example supports Definition 3.2.

Example 3.2. *Assume that $Q = [0,\infty)$ is an MS endowed with the usual metric ω. Define a binary relation $\mathfrak{R}$ on Q and a map $\Theta : Q \to Q$ such that for all $\xi,\zeta \in Q$,*

$$\mathfrak{R} = \{(\xi,\zeta) : \xi \leq \zeta\}, \text{ and } \Theta\left(\xi\right) = \sqrt{\xi}.$$

Clearly, $\mathfrak{R}$ is Θ-closed and and Θ is $\mathfrak{R}$-continuous. Furthermore, we describe the function ϕ : $[0,\infty) \to [0,\infty)$ as $\phi(\gamma) = \gamma$ for all $\gamma > 0$ and an ω-distance $q : Q \times Q \to Q$ by $q(\xi,\zeta) = \zeta$. Thus, for all $\xi,\zeta \in Q$ with $(\xi,\zeta) \in \mathfrak{R}$, we can write

$$\frac{1}{2}q\left(\xi,\Theta\xi\right) = \frac{1}{2}\Theta\left(\xi\right) = \frac{1}{2}\sqrt{\xi} \leq \sqrt{\xi} \leq \sqrt{\zeta} \leq \zeta = q\left(\xi,\zeta\right),$$

implies

$$L\left(q\left(\Theta\xi,\Theta\zeta\right)\right) = L\Theta\left(\zeta\right) = L\sqrt{\xi} \leq \sqrt{\xi} \leq \xi \leq \zeta = q\left(\xi,\zeta\right) = \phi\left(U_{\Theta}\left(\xi,\zeta\right)\right).$$

Thus, Θ is an $\omega C_{\phi}^{L} - \mathfrak{R}$-contraction with $L \in (0,1)$.

4. Fixed point results

This section delves into the existence and uniqueness of FPs for modified $\Re$-contraction mappings. We will establish several new theorems that provide conditions for these FPs to exist and be unique.

Theorem 4.1. *Assume that (Q, ϖ) is an $\Re$-complete MS, where $\Re$ is a binary relation described on Q. Suppose that p is an ω-distance on Q, $\Theta : Q \to Q$ is a mapping, and the conditions are satisfied:*

i) $Q(\Theta, \Re) \neq \emptyset$, *where* $Q(\Theta, \Re) = \{\xi \in Q : (\xi, \Theta\xi) \in \Re\}$,
ii) $\Re$ *is* Θ*-closed and locally* Θ*-transitive,*
iii) either Θ *is* $\Re$*-continuous or* $\Re$ *is* ϖ*-self closed,*
iv) Θ *is an* $\omega_\psi^L - \Re$*-contraction.*

Then Θ *has an FP.*

Proof. We split the proof into the following steps.
Step 1. We prove that $\lim_{j\to\infty} q\left(\xi_j, \xi_{j+1}\right) = 0$. Since $Q(\Theta, \Re) \neq \emptyset$, assume that $\xi_0 \in Q(\Theta, \Re)$ then $(\xi_0, \Theta\xi_0) \in \Re$. By iteratively applying this process, we generate a sequence of Picard iterations $\{\xi_j\}$ with an initial point ξ_0 satisfying

$$\xi_j = \Theta^j(\xi_0), \text{ for all } j \in \mathbb{N}_0.$$

Because $(\xi_0, \Theta\xi_0) \in \Re$ and $\Re$ is Θ-closed, one has

$$\left(\Theta\xi_0, \Theta^2\xi_0\right), \left(\Theta^2\xi_0, \Theta^3\xi_0\right), \cdots, \left(\Theta^j\xi_0, \Theta^{j+1}\xi_0\right), \cdots \in \Re,$$

and $\left(\xi_j, \xi_{j+1}\right) \in \Re$, for all $j \in \mathbb{N}_0$, which implies that the sequence $\{\xi_j\}$ is an $\Re$-preserving.

Since Θ is an $\omega_\psi^L - \Re$-contraction, we get

$$\psi\left(q\left(\xi_j, \xi_{j+1}\right)\right) = \psi\left(q\left(\Theta\xi_{j-1}, \Theta\xi_j\right)\right) \leq \psi\left(U_\Theta\left(\xi_{j-1}, \xi_j\right)\right) - LU_\Theta\left(\xi_{j-1}, \xi_j\right), \tag{4.1}$$

where

$$\begin{aligned}
&U_\Theta\left(\xi_{j-1}, \xi_j\right)\\
=\ &\max\left\{q\left(\xi_{j-1},\xi_j\right), \frac{q\left(\xi_{j-1},\Theta\xi_{j-1}\right)\left[1+q\left(\xi_j,\Theta\xi_{j-1}\right)\right]}{1+q\left(\xi_{j-1}\xi_j\right)}, \frac{q\left(\xi_j,\Theta\xi_j\right)\left[1+q\left(\xi_{j-1},\Theta\xi_{j-1}\right)\right]}{1+q\left(\xi_{j-1}\xi_j\right)}, \frac{q\left(\xi_{j-1},\xi_{j+1}\right)+q\left(\xi_j,\xi_j\right)}{2}\right\}\\
=\ &\max\left\{q\left(\xi_{j-1},\xi_j\right), \frac{q\left(\xi_{j-1},\xi_j\right)\left[1+q\left(\xi_j,\xi_j\right)\right]}{1+q\left(\xi_{j-1}\xi_j\right)}, \frac{q\left(\xi_j,\xi_{j+1}\right)\left[1+q\left(\xi_{j-1},\xi_j\right)\right]}{1+q\left(\xi_{j-1}\xi_j\right)}, \frac{q\left(\xi_{j-1},\xi_{j+1}\right)+q\left(\xi_j,\xi_j\right)}{2}\right\}\\
=\ &\max\left\{q\left(\xi_{j-1},\xi_j\right), \frac{q\left(\xi_{j-1},\xi_j\right)}{1+q\left(\xi_{j-1}\xi_j\right)}, q\left(\xi_j,\xi_{j+1}\right), \frac{q\left(\xi_{j-1},\xi_{j+1}\right)}{2}\right\}\\
\leq\ &\max\left\{q\left(\xi_{j-1},\xi_j\right), q\left(\xi_{j-1},\xi_j\right), q\left(\xi_j,\xi_{j+1}\right), \frac{q\left(\xi_{j-1},\xi_j\right)+q\left(\xi_j,\xi_{j+1}\right)}{2}\right\}\\
=\ &\max\left\{q\left(\xi_{j-1},\xi_j\right), q\left(\xi_j,\xi_{j+1}\right)\right\}.
\end{aligned}$$

Now, if $U_\Theta\left(\xi_{j-1},\xi_j\right)=q\left(\xi_j,\xi_{j+1}\right)$, we have from (4.1) that

$$\psi\left(q\left(\xi_j,\xi_{j+1}\right)\right)\leq\psi\left(q\left(\xi_j,\xi_{j+1}\right)\right)-Lq\left(\xi_j,\xi_{j+1}\right)\leq\psi\left(q\left(\xi_j,\xi_{j+1}\right)\right).$$

By the properties of ψ, one has

$$q\left(\xi_j,\xi_{j+1}\right)\leq q\left(\xi_j,\xi_{j+1}\right),$$

which contradicts our hypothesis, and hence $U_\Theta\left(\xi_{j-1},\xi_j\right)=q\left(\xi_{j-1},\xi_j\right)$. Again, by (4.1), we get

$$\psi\left(q\left(\xi_j,\xi_{j+1}\right)\right)\leq\psi\left(q\left(\xi_{j-1},\xi_j\right)\right)-Lq\left(\xi_{j-1},\xi_j\right)\leq\psi\left(q\left(\xi_{j-1},\xi_j\right)\right).\tag{4.2}$$

Using the properties of ψ, we have

$$q\left(\xi_j,\xi_{j+1}\right)\leq q\left(\xi_{j-1},\xi_j\right).\tag{4.3}$$

Analogously, one can obtain

$$q\left(\xi_{j-1},\xi_j\right)\leq q\left(\xi_{j-2},\xi_{j-1}\right).\tag{4.4}$$

Applying both inequalities (4.3) and (4.4), we derive

$$q\left(\xi_j,\xi_{j+1}\right)\leq q\left(\xi_{j-1},\xi_j\right)\leq q\left(\xi_{j-2},\xi_{j-1}\right).$$

By virtue of being monotone nonincreasing and bounded, the sequence $\{q\left(\xi_j,\xi_{j+1}\right)\}$ converges to some $\tau\geq 0$, and thus

$$\lim_{j\to\infty}q\left(\xi_j,\xi_{j+1}\right)=\tau\geq 0.$$

As $j\to\infty$, the inequality (4.2) combined with the continuity of ψ, yields

$$\psi(\tau)\leq\psi(\tau).$$

This is a contradiction, from which it follows that $\tau=0$. Consequently, we conclude that

$$\lim_{j\to\infty}q\left(\xi_j,\xi_{j+1}\right)=0.\tag{4.5}$$

Step 2. We claim that $\{\xi_j\}$ is a Cauchy sequence. To the contrary, let us assume that the sequence $\{\xi_j\}$ is not Cauchy. Then, by Lemma 2.1, there exist $\varepsilon>0$ and subsequences $\{\xi_{j_k}\}$ and $\{\xi_{i_k}\}$ of $\{\xi_j\}$ satisfying $k\leq j_k\leq i_k$ such that

$$q(\xi_{j_k},\xi_{i_k})\geq\varepsilon,$$

and

$$\lim_{k\to\infty}q(\xi_{j_k},\xi_{i_k})=\varepsilon,\ \ \lim_{k\to\infty}q(\xi_{j_k+1},\xi_{i_k+1})=\varepsilon.\tag{4.6}$$

Because the binary relation $\mathfrak{R}$ is locally Θ-transitive, $(\xi_{j_k},\xi_{i_k})\in\mathfrak{R}$. Since Θ is an $\omega_\psi^L-\mathfrak{R}$-contraction, we have

$$\psi\left(q(\xi_{j_k+1},\xi_{i_k+1})\right)=\psi\left(q(\Theta\xi_{j_k},\Theta\xi_{i_k})\right)\leq\psi\left(U_\Theta\left(\xi_{j_k},\xi_{i_k}\right)\right)-LU_\Theta\left(\xi_{j_k},\xi_{i_k}\right),\tag{4.7}$$

where

$$\begin{aligned} U_\Theta\left(\xi_{j_k},\xi_{i_k}\right) &= \max\left\{\begin{array}{c} q\left(\xi_{j_k},\xi_{i_k}\right), \dfrac{q\left(\xi_{j_k},\Theta\xi_{j_k}\right)\left[1+q\left(\xi_{i_k},\Theta\xi_{i_k}\right)\right]}{1+q\left(\xi_{j_k},\xi_{i_k}\right)}, \dfrac{q\left(\xi_{i_k},\Theta\xi_{i_k}\right)\left[1+q\left(\xi_{j_k},\Theta\xi_{j_k}\right)\right]}{1+q\left(\xi_{j_k},\xi_{i_k}\right)}, \\ \dfrac{q\left(\xi_{j_k},\Theta\xi_{i_k}\right)+q\left(\xi_{i_k},\Theta\xi_{j_k}\right)}{2} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} q\left(\xi_{j_k},\xi_{i_k}\right), \dfrac{q\left(\xi_{j_k},\xi_{j_{k+1}}\right)\left[1+q\left(\xi_{i_k},\xi_{i_k+1}\right)\right]}{1+q\left(\xi_{j_k},\xi_{i_k}\right)}, \dfrac{q\left(\xi_{i_k},\xi_{i_k+1}\right)\left[1+q\left(\xi_{j_k},\xi_{j_k+1}\right)\right]}{1+q\left(\xi_{j_k},\xi_{i_k}\right)}, \\ \dfrac{q\left(\xi_{j_k},\xi_{i_k+1}\right)+q\left(\xi_{i_k},\xi_{j_k+1}\right)}{2} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} q\left(\xi_{j_k},\xi_{i_k}\right), \dfrac{q\left(\xi_{j_k},\xi_{j_{k+1}}\right)\left[1+q\left(\xi_{i_k},\xi_{i_k+1}\right)\right]}{1+q\left(\xi_{j_k},\xi_{i_k}\right)}, \dfrac{q\left(\xi_{i_k},\xi_{i_k+1}\right)\left[1+q\left(\xi_{j_k},\xi_{j_k+1}\right)\right]}{1+q\left(\xi_{j_k},\xi_{i_k}\right)}, \\ \dfrac{q\left(\xi_{j_k},\xi_{j_k+1}\right)+q\left(\xi_{j_k+1},\xi_{i_k+1}\right)+q\left(\xi_{i_k},\xi_{i_k+1}\right)+q\left(\xi_{i_k+1},\xi_{j_k+1}\right)}{2} \end{array}\right\}. \end{aligned}$$

Taking $k \to \infty$ in the inequality above and using (4.6), one can write

$$\lim_{k\to\infty} U_\Theta\left(\xi_{j_k},\xi_{i_k}\right) \leq \max\left\{\varepsilon, \frac{0\,[1+0]}{1+\varepsilon}, \frac{0\,[1+0]}{1+\varepsilon}, \frac{0+\varepsilon+0+\varepsilon}{2}\right\} = \max\{\varepsilon,0,0,\varepsilon\} = \varepsilon. \tag{4.8}$$

Letting $k \to \infty$ in (4.7) and utilizing (4.6) and (4.8), we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - L\varepsilon,$$

which is possible only if $L\varepsilon = 0$. The case of $L = 0$ leads to $\psi(\varepsilon) \leq \psi(\varepsilon)$, which is a contradiction. Hence, $\{\xi_j\}$ is a Cauchy sequence. In the case of $\varepsilon = 0$, Lemma 1.19 in [20] implies that $\{\xi_j\}$ is an $\mathfrak{R}$-preserving Cauchy sequence in Q.

Step 3. We show that ξ is an FP of Θ, that is, $\xi = \Theta\xi$. By Step 2, we have $\{\xi_j\}$ is an $\mathfrak{R}$-preserving Cauchy sequence in Q. The $\mathfrak{R}$-completeness of Q, leads to there exists $\xi \in Q$ such that

$$\lim_{j\to\infty} \xi_j = \xi. \tag{4.9}$$

There are two distinct cases to consider: $\mathfrak{R}$ is ϖ-self-closed, or Θ is $\mathfrak{R}$-continuous.

Case i. Suppose that $\mathfrak{R}$ is ϖ-self-closed, then there exists $\{\xi_{j_k}\}$ of $\{\xi_j\}$ such that $[\xi_{j_k},\xi] \in \mathfrak{R}$. Because Θ is an $\omega_\psi^L - \mathfrak{R}$-contraction, for all $k \in \mathbb{N}_0$, one has

$$\psi\left(q(\xi_{j_k+1},\Theta\xi)\right) = \psi\left(q(\Theta\xi_{j_k},\Theta\xi)\right) \leq \psi\left(U_\Theta\left(\xi_{j_k},\xi\right)\right) - LU_\Theta\left(\xi_{j_k},\xi\right), \tag{4.10}$$

where

$$\begin{aligned} &U_\Theta\left(\xi_{j_k},\xi\right) \\ =\ &\max\left\{ q\left(\xi_{j_k},\xi\right), \frac{q\left(\xi_{j_k},\Theta\xi_{j_k}\right)\left[1+q\left(\xi,\Theta\xi_{j_k}\right)\right]}{1+q\left(\xi_{j_k},\xi\right)}, \frac{q(\xi,\Theta\xi)\left[1+q\left(\xi_{j_k},\Theta\xi_{j_k}\right)\right]}{1+q\left(\xi_{j_k},\xi\right)}, \frac{q\left(\xi_{j_k},\Theta\xi\right)+q\left(\xi,\Theta\xi_{j_k}\right)}{2} \right\} \end{aligned}$$

$$
\begin{aligned}
&= \max\left\{ q\left(\xi_{j_k},\xi\right), \frac{q\left(\xi_{j_k},\xi_{j_k+1}\right)\left[1+q\left(\xi,\xi_{j_k+1}\right)\right]}{1+q\left(\xi_{j_k},\xi\right)}, \frac{q\left(\xi,\Theta\xi\right)\left[1+q\left(\xi_{j_k},\xi_{j_k+1}\right)\right]}{1+q\left(\xi_{j_k},\xi\right)}, \frac{q\left(\xi_{j_k},\Theta\xi\right)+q\left(\xi,\xi_{j_k+1}\right)}{2} \right\} \\
&\leq \max\left\{ q\left(\xi_{j_k},\xi\right), q\left(\xi,\Theta\xi\right), \frac{q\left(\xi_{j_k},\xi\right)+q\left(\xi,\Theta\xi\right)+q\left(\xi,\xi_{j_k}\right)+q\left(\xi_{j_k},\xi_{j_k+1}\right)}{2} \right\} \\
&= \max\left\{ q\left(\xi_{j_k},\xi\right), q\left(\xi,\Theta\xi\right) \right\}.
\end{aligned}
$$

This again presents two possibilities: Either $U_\Theta\left(\xi_{j_k},\xi\right) = q\left(\xi_{j_k},\xi\right)$ or $U_\Theta\left(\xi_{j_k},\xi\right) = q\left(\xi,\Theta\xi\right)$. Suppose that $U_\Theta\left(\xi_{j_k},\xi\right) = q\left(\xi,\Theta\xi\right)$. Then, by (4.10), we can write

$$
\psi\left(q(\xi_{j_k+1},\Theta\xi)\right) \leq \psi\left(q\left(\xi,\Theta\xi\right)\right) - Lq\left(\xi,\Theta\xi\right) \leq \psi\left(q\left(\xi,\Theta\xi\right)\right).
$$

Letting $k \to \infty$ and using (4.9), one has

$$
\psi\left(q(\xi,\Theta\xi)\right) \leq \psi\left(q\left(\xi,\Theta\xi\right)\right),
$$

which violates our assumption. Thus, we assume that $U_\Theta\left(\xi_{j_k},\xi\right) = q\left(\xi_{j_k},\xi\right)$, so by (4.10), we have

$$
\psi\left(q(\xi_{j_k+1},\Theta\xi)\right) = \psi\left(q(\Theta\xi_{j_k},\Theta\xi)\right) \leq \psi\left(q\left(\xi_{j_k},\xi\right)\right) - Lq\left(\xi_{j_k},\xi\right) \leq \psi\left(q\left(\xi_{j_k},\xi\right)\right).
$$

Due to the property of ψ, we find that

$$
q(\Theta\xi_{j_k},\Theta\xi) \leq q\left(\xi_{j_k},\xi\right), \text{ for all } k \in \mathbb{N}_0. \tag{4.11}
$$

We needs to prove that (4.11) holds for all $k \in \mathbb{N} = \mathbb{N}_0 \cup \mathbb{N}^+$. Clearly, if $k \in \mathbb{N}_0$, (4.11) is true. Therefore, we consider the case if $k \in \mathbb{N}^+$. This presents two more possibilities: Either $q\left(\xi_{j_k},\xi\right) = 0$ or $q\left(\xi_{j_k},\xi\right) > 0$.

Assume that $q\left(\xi_{j_k},\xi\right) = 0$. Since $\mathfrak{R}$ is Θ-closed, we get $q(\Theta\xi_{j_k},\Theta\xi) = 0$. Therefore, the condition (4.11) is satisfied.

If $q\left(\xi_{j_k},\xi\right) > 0$, the monotonicity of ψ, as established in (4.11), leads to

$$
q(\Theta\xi_{j_k},\Theta\xi) < q(\xi_{j_k},\xi), \text{ for all } k \in \mathbb{N}^+.
$$

Consequently, the condition (4.11) holds for all $k \in \mathbb{N}^+$, which extends to all $k \in \mathbb{N}$. taking $k \to \infty$ in (4.11) and using (4.9), we have $\Theta\xi \xrightarrow{q} \Theta\xi$. The uniqueness of the limit guarantees that $\xi = \Theta\xi$.

Case ii. If the mapping Θ is $\mathfrak{R}$-continuous, because of the $\mathfrak{R}$-preserving property with $\xi_j \xrightarrow{q} \xi$, then the $\mathfrak{R}$-continuity of Θ leads to $\xi_{j+1} = \Theta\xi_j \xrightarrow{q} \Theta\xi$. Consequently, by the uniqueness of the limit, we derive $\xi = \Theta\xi$.

This completes the proof. □

The following examples support Theorem 4.1.

Example 4.1. *Consider the MS $Q = [1,\infty) \cup \{0\}$ with the usual metric ϖ. Let $\mathfrak{R}$ be a binary relation and $\Theta : Q \to Q$ be a mapping, defined for all $\xi, \zeta \in Q$ as follows:*

$$\mathfrak{R} = \{(\xi,\zeta) : \xi < \zeta\}, \text{ and } \Theta(\xi) = \frac{\xi}{4}.$$

Define the function $\psi : [0,\infty) \to [0,\infty)$ by $\psi(\gamma) = \gamma$ for all $\gamma \in [0,\infty)$ and an ω-distance $q : Q \times Q \to Q$ by

$$q(\xi,\zeta) = \zeta.$$

Hence, for all $\xi, \zeta \in Q$ with $(\xi,\zeta) \in \mathfrak{R}$, we get

$$\psi(q(\Theta\xi,\Theta\zeta)) = q(\Theta\xi,\Theta\zeta) = \Theta\zeta = \frac{\zeta}{4} \leq \zeta - \frac{\zeta}{2} = \psi(U_\Theta(\xi,\zeta)) - LU_\Theta(\xi,\zeta).$$

Thus, Θ is an $\omega^L_\psi - \mathfrak{R}$-contraction with $L = \frac{1}{2}$. Therefore, all requirements of Theorem 4.1 are fulfilled and $\xi = 0$ is a unique FP of Θ in Q.

Example 4.2. *Assume that Q, $\mathfrak{R}$, ψ and q are defined as in Example 4.1. Define the mapping $\Theta : Q \to Q$ by*

$$\Theta(\xi) = \frac{\xi}{1+\sqrt{2}\xi}, \text{ for all } \xi \in Q.$$

It is evident that $\mathfrak{R}$ is Θ-closed and Θ is $\mathfrak{R}$-continuous. Moreover

$$\psi(q(\Theta\xi,\Theta\zeta)) = q(\Theta\xi,\Theta\zeta) = \Theta\zeta = \frac{\zeta}{1+\sqrt{2}\zeta} \leq \zeta - \frac{\zeta}{3} = \psi(U_\Theta(\xi,\zeta)) - LU_\Theta(\xi,\zeta).$$

Hence, Θ is an $\omega^L_\psi - \mathfrak{R}$-contraction with $L = \frac{1}{3}$. Thus, all requirements of Theorem 4.1 are fulfilled and $\xi = 0$ is a unique FP of Θ in Q.

Theorem 4.2. *Assume that (Q,ϖ) is an $\mathfrak{R}$-complete MS, where $\mathfrak{R}$ is a binary relation described on Q. Suppose that p is an ω-distance on Q and $\Theta : Q \to Q$ is a mapping and that the following assertions are true:*

i) $Q(\Theta,\mathfrak{R}) \neq \emptyset$, where $Q(\Theta,\mathfrak{R}) = \{\xi \in Q : (\xi,\Theta\xi) \in \mathfrak{R}\}$,
ii) $\mathfrak{R}$ is Θ-closed and locally Θ-transitive,
iii) either Θ is $\mathfrak{R}$-continuous or $\mathfrak{R}$ is ϖ-self closed,
iv) Θ is an $\omega C^L_\phi - \mathfrak{R}$-contraction.

Then Θ has an FP.

Proof. In a similar way to the proof of Theorem 4.1, a sequence $\{\xi_j\}$ can be derived such that $\xi_j = \Theta^j(\xi_0)$, for all $j \in \mathbb{N}_0$ and $\{\xi_j\}$ is a $\mathfrak{R}$-preserving sequence.

Since Θ is an $\omega C^L_\phi - \mathfrak{R}$-contraction, we have

$$\frac{1}{2}q\left(\xi_j,\Theta\xi_j\right) = \frac{1}{2}q\left(\xi_j,\xi_{j+1}\right) \leq q\left(\xi_j,\xi_{j+1}\right),$$

which implies

$$Lq\left(\xi_{j+1},\xi_{j+2}\right) = Lq\left(\Theta\xi_j,\Theta\xi_{j+1}\right) \leq \phi\left(U_\Theta\left(\xi_j,\xi_{j+1}\right)\right), \tag{4.12}$$

where

$$\begin{aligned}
& U_\Theta\left(\xi_j,\xi_{j+1}\right) \\
= & \max\left\{q\left(\xi_j,\xi_{j+1}\right),\frac{q\left(\xi_j,\Theta\xi_j\right)\left[1+q\left(\xi_{j+1},\Theta\xi_j\right)\right]}{1+q\left(\xi_j,\xi_{j+1}\right)},\frac{q\left(\xi_{j+1},\Theta\xi_{j+1}\right)\left[1+q\left(\xi_j,\Theta\xi_j\right)\right]}{1+q\left(\xi_j,\xi_{j+1}\right)},\frac{q\left(\xi_j,\Theta\xi_{j+1}\right)+q\left(\xi_{j+1},\Theta\xi_j\right)}{2}\right\} \\
= & \max\left\{q\left(\xi_j,\xi_{j+1}\right),\frac{q\left(\xi_j,\xi_{j+1}\right)\left[1+q\left(\xi_{j+1},\xi_{j+1}\right)\right]}{1+q\left(\xi_j,\xi_{j+1}\right)},\frac{q\left(\xi_{j+1},\xi_{j+2}\right)\left[1+q\left(\xi_j,\xi_{j+1}\right)\right]}{1+q\left(\xi_j,\xi_{j+1}\right)},\frac{q\left(\xi_j,\xi_{j+2}\right)+q\left(\xi_{j+1},\xi_{j+1}\right)}{2}\right\} \\
\leq & \max\left\{q\left(\xi_j,\xi_{j+1}\right),\frac{q\left(\xi_j,\xi_{j+1}\right)}{1+q\left(\xi_j,\xi_{j+1}\right)},q\left(\xi_{j+1},\xi_{j+2}\right),\frac{q\left(\xi_j,\xi_{j+1}\right)+q\left(\xi_{j+1},\xi_{j+2}\right)}{2}\right\} \\
\leq & \max\left\{q\left(\xi_j,\xi_{j+1}\right),q\left(\xi_{j+1},\xi_{j+2}\right)\right\}.
\end{aligned}$$

Here, either $U_\Theta\left(\xi_j,\xi_{j+1}\right) = q\left(\xi_j,\xi_{j+1}\right)$ or $U_\Theta\left(\xi_j,\xi_{j+1}\right) = q\left(\xi_{j+1},\xi_{j+2}\right)$. In the case of $U_\Theta\left(\xi_j,\xi_{j+1}\right) = q\left(\xi_{j+1},\xi_{j+2}\right)$, we have, from (4.12)

$$Lq\left(\xi_{j+1},\xi_{j+2}\right) \leq \phi\left(q\left(\xi_{j+1},\xi_{j+2}\right)\right),$$

which implies that $L \leq \phi$. This contradicts our assumption that $\phi(\gamma) < L$, and hence $U_\Theta\left(\xi_j,\xi_{j+1}\right) = q\left(\xi_j,\xi_{j+1}\right)$. Again, by (4.12), we have

$$Lq\left(\xi_{j+1},\xi_{j+2}\right) \leq \phi\left(q\left(\xi_j,\xi_{j+1}\right)\right).$$

Using the properties of ϕ and by the condition $\phi(\gamma) < L$ for all $\gamma > 0$, we get

$$q\left(\xi_{j+1},\xi_{j+2}\right) \leq q\left(\xi_j,\xi_{j+1}\right).$$

Similarly, one can write

$$q\left(\xi_j,\xi_{j+1}\right) \leq q\left(\xi_{j-1},\xi_j\right).$$

Combining these two inequalities, we obtain

$$q\left(\xi_{j+1},\xi_{j+2}\right) \leq q\left(\xi_j,\xi_{j+1}\right) \leq q\left(\xi_{j-1},\xi_j\right).$$

By virtue of being monotone nonincreasing and bounded, the sequence $\{q\left(\xi_j,\xi_{j+1}\right)\}$ converges to some $\tau \geq 0$, and thus

$$\lim_{j\to\infty} q\left(\xi_j,\xi_{j+1}\right) = \tau \geq 0.$$

Letting $j \to \infty$, the inequality (4.12) implies that

$$L\tau \leq \phi(\tau),$$

which is a contradiction, from which it follows that $\tau = 0$. Consequently, we conclude that

$$\lim_{j\to\infty} q\left(\xi_j,\xi_{j+1}\right) = 0.$$

Next, we prove that the sequence $\{\xi_j\}$ is Cauchy. By contradiction, we assume that $\{\xi_j\}$ is not Cauchy. Then, there exist $\varepsilon > 0$ and subsequences $\{\xi_{j_k}\}$ and $\{\xi_{i_k}\}$ of $\{\xi_j\}$ satisfying $k \leq j_k \leq i_k$ such that

$$q(\xi_{j_k}, \xi_{i_k}) \geq \varepsilon.$$

Furthermore, the sequence of $\{q\left(\xi_j, \xi_{j+1}\right)\}$ means for each $\varepsilon > 0$, there exists $\mathbb{N}_0 \in \mathbb{N}$ such that $q\left(\xi_j, \xi_{j+1}\right) < \varepsilon$ for all $j \in \mathbb{N}_0$. Assume that $\mathbb{N}_1 = \max\left\{i_j, \mathbb{N}_0\right\}$, for $\mathbb{N}_1 \leq j_k < i_k$, we get

$$q(\xi_{j_k}, \xi_{j_k+1}) < \varepsilon \leq q(\xi_{j_k}, \xi_{i_k}).$$

Since $\mathfrak{R}$ is locally Θ-transitive, we have $(\xi_{j_k}, \xi_{i_k}) \in \mathfrak{R}$. Because Θ is an $\omega C_\phi^L - \mathfrak{R}$-contraction, we get

$$\frac{1}{2} q(\xi_{j_k}, \xi_{j_k+1}) \leq q(\xi_{j_k}, \xi_{i_k+1}),$$

which implies

$$Lq(\xi_{j_k+1}, \xi_{i_k+1}) = Lq(\Theta\xi_{j_k}, \Theta\xi_{i_k}) \leq \phi\left(U_\Theta\left(\xi_{j_k}, \xi_{i_k}\right)\right), \tag{4.13}$$

where

$$\begin{aligned} U_\Theta\left(\xi_{j_k}, \xi_{i_k}\right) &= \max\left\{\begin{array}{c} q\left(\xi_{j_k}, \xi_{i_k}\right), \dfrac{q\left(\xi_{j_k}, \Theta\xi_{j_k}\right)\left[1 + q\left(\xi_{i_k}, \Theta\xi_{i_k}\right)\right]}{1 + q\left(\xi_{j_k}, \xi_{i_k}\right)}, \dfrac{q\left(\xi_{i_k}, \Theta\xi_{i_k}\right)\left[1 + q\left(\xi_{j_k}, \Theta\xi_{j_k}\right)\right]}{1 + q\left(\xi_{j_k}, \xi_{i_k}\right)}, \\ \dfrac{q\left(\xi_{j_k}, \Theta\xi_{i_k}\right) + q\left(\xi_{i_k}, \Theta\xi_{j_k}\right)}{2} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} q\left(\xi_{j_k}, \xi_{i_k}\right), \dfrac{q\left(\xi_{j_k}, \xi_{j_{k+1}}\right)\left[1 + q\left(\xi_{i_k}, \xi_{i_k+1}\right)\right]}{1 + q\left(\xi_{j_k}, \xi_{i_k}\right)}, \dfrac{q\left(\xi_{i_k}, \xi_{i_k+1}\right)\left[1 + q\left(\xi_{j_k}, \xi_{j_k+1}\right)\right]}{1 + q\left(\xi_{j_k}, \xi_{i_k}\right)}, \\ \dfrac{q\left(\xi_{j_k}, \xi_{i_k+1}\right) + q\left(\xi_{i_k}, \xi_{j_k+1}\right)}{2} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} q\left(\xi_{j_k}, \xi_{i_k}\right), \frac{q\left(\xi_{j_k}, \xi_{j_{k+1}}\right)\left[1 + q\left(\xi_{i_k}, \xi_{i_k+1}\right)\right]}{1 + q\left(\xi_{j_k}, \xi_{i_k}\right)}, \dfrac{q\left(\xi_{i_k}, \xi_{i_k+1}\right)\left[1 + q\left(\xi_{j_k}, \xi_{j_k+1}\right)\right]}{1 + q\left(\xi_{j_k}, \xi_{i_k}\right)}, \\ \dfrac{q\left(\xi_{j_k}, \xi_{j_k+1}\right) + q\left(\xi_{j_k+1}, \xi_{i_k+1}\right) + q\left(\xi_{i_k}, \xi_{i_k+1}\right) + q\left(\xi_{i_k+1}, \xi_{j_k+1}\right)}{2} \end{array}\right\}. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using Lemma 2.1, we get

$$\lim_{k \to \infty} U_\Theta\left(\xi_{j_k}, \xi_{i_k}\right) \leq \max\left\{\varepsilon, \frac{0\,[1 + 0]}{1 + \varepsilon}, \frac{0\,[1 + 0]}{1 + \varepsilon}, \frac{0 + \varepsilon + 0 + \varepsilon}{2}\right\} = \max\{\varepsilon, 0, 0, \varepsilon\} = \varepsilon.$$

Letting $k \to \infty$ in (4.13) and utilizing Lemma 2.1, we conclude that

$$L\varepsilon \leq \phi\left(\varepsilon\right),$$

which is a contradiction. Hence, $\{\xi_j\}$ is an $\mathfrak{R}$-preserving Cauchy in Q.

The $\mathfrak{R}$-completeness of Q, leads to there exists $\xi^* \in Q$ such that

$$\lim_{j \to \infty} \xi_j = \xi^*. \tag{4.14}$$

Finally, to prove ξ^* is an FP of Θ, we have the following two scenarios: Either $\mathfrak{R}$ is ϖ-self-closed, or Θ is $\mathfrak{R}$-continuous.

Scenario 1. Suppose that $\mathfrak{R}$ is ϖ-self-closed. $\{\xi_{j_k}\}$ of $\{\xi_j\}$ exists such that $[\xi_{j_k}, \xi^*] \in \mathfrak{R}$. Since Θ is an $\omega C_\phi^L - \mathfrak{R}$-contraction, for all $k \in \mathbb{N}$, we have

$$\frac{1}{2}q(\xi_{j_k}, \Theta\xi_{j_k}) = \frac{1}{2}q(\xi_{j_k}, \xi_{j_k+1}) \leq q(\xi_{j_k}, \xi^*),$$

implies

$$Lq(\xi_{j_k+1}, \Theta\xi^*) = Lq(\Theta\xi_{j_k}, \Theta\xi^*) \leq \phi\left(U_\Theta\left(\xi_{j_k}, \xi^*\right)\right), \tag{4.15}$$

where

$$\begin{aligned} & U_\Theta\left(\xi_{j_k}, \xi^*\right) \\ = \;& \max\left\{q\left(\xi_{j_k}, \xi^*\right), \frac{q\left(\xi_{j_k}, \Theta\xi_{j_k}\right)\left[1+q\left(\xi^*, \Theta\xi_{j_k}\right)\right]}{1+q\left(\xi_{j_k}, \xi^*\right)}, \frac{q\left(\xi, \Theta\xi\right)\left[1+q\left(\xi_{j_k}, \Theta\xi_{j_k}\right)\right]}{1+q\left(\xi_{j_k}, \xi^*\right)}, \frac{q\left(\xi_{j_k}, \Theta\xi^*\right)+q\left(\xi^*, \Theta\xi_{j_k}\right)}{2}\right\} \\ = \;& \max\left\{q\left(\xi_{j_k}, \xi\right), \frac{q\left(\xi_{j_k}, \xi_{j_k+1}\right)\left[1+q\left(\xi^*, \xi_{j_k+1}\right)\right]}{1+q\left(\xi_{j_k}, \xi^*\right)}, \frac{q\left(\xi, \Theta\xi\right)\left[1+q\left(\xi_{j_k}, \xi_{j_k+1}\right)\right]}{1+q\left(\xi_{j_k}, \xi^*\right)}, \frac{q\left(\xi_{j_k}, \Theta\xi^*\right)+q\left(\xi^*, \xi_{j_k+1}\right)}{2}\right\} \\ \leq \;& \max\left\{q\left(\xi_{j_k}, \xi^*\right), q\left(\xi^*, \Theta\xi^*\right), \frac{q\left(\xi_{j_k}, \xi^*\right)+q\left(\xi^*, \Theta\xi^*\right)+q\left(\xi^*, \xi_{j_k}\right)+q\left(\xi_{j_k}, \xi_{j_k+1}\right)}{2}\right\} \\ = \;& \max\left\{q\left(\xi_{j_k}, \xi^*\right), q\left(\xi^*, \Theta\xi^*\right)\right\}. \end{aligned}$$

There are two possibilities: Either $U_\Theta\left(\xi_{j_k}, \xi^*\right) = q\left(\xi_{j_k}, \xi^*\right)$ or $U_\Theta\left(\xi_{j_k}, \xi^*\right) = q\left(\xi^*, \Theta\xi^*\right)$.

Assume that $U_\Theta\left(\xi_{j_k}, \xi^*\right) = q\left(\xi^*, \Theta\xi^*\right)$. Then, by (4.15), we have

$$Lq(\xi_{j_k+1}, \Theta\xi^*) \leq \phi\left(q\left(\xi^*, \Theta\xi^*\right)\right),$$

As $k \to \infty$ and using (4.14), we get

$$L\left(q(\xi^*, \Theta\xi^*)\right) \leq \phi\left(q\left(\xi^*, \Theta\xi^*\right)\right),$$

This violates our assumption. Therefore, we assume that $U_\Theta\left(\xi_{j_k}, \xi^*\right) = q\left(\xi_{j_k}, \xi^*\right)$, so by (4.15), we obtain

$$Lq(\xi_{j_k+1}, \Theta\xi^*) \leq \phi\left(q\left(\xi_{j_k}, \xi^*\right)\right).$$

Due to the property of ψ, we obtain

$$q(\Theta\xi_{j_k}, \Theta\xi) \leq q\left(\xi_{j_k}, \xi\right), \text{ for all } k \in \mathbb{N}_0. \tag{4.16}$$

We now prove that (4.16) is true for all $k \in \mathbb{N} = \mathbb{N}_0 \cup \mathbb{N}^+$. Clearly, if $k \in \mathbb{N}_0$, (4.16) holds. Therefore, we assume the case that $k \in \mathbb{N}^+$. This presents two more possibilities: Either $q\left(\xi_{j_k}, \xi^*\right) = 0$ or $q\left(\xi_{j_k}, \xi^*\right) > 0$.

Let $q\left(\xi_{j_k}, \xi^*\right) = 0$. Since $\mathfrak{R}$ is Θ-closed, we get $q(\Theta\xi_{j_k}, \Theta\xi^*) = 0$. Therefore, (4.16) is fulfilled.

If $q\left(\xi_{j_k}, \xi^*\right) > 0$, the properties of ϕ and (4.15) lead to

$$Lq(\Theta\xi_{j_k}, \Theta\xi^*) \leq \phi\left(q\left(\xi_{j_k}, \xi^*\right)\right) < Lq\left(\xi_{j_k}, \xi^*\right), \text{ for all } k \in \mathbb{N}^+,$$

which implies that

$$Lq(\Theta\xi_{j_k}, \Theta\xi^*) < Lq\left(\xi_{j_k}, \xi^*\right).$$

Hence,

$$q(\Theta\xi_{j_k}, \Theta\xi^*) < q\left(\xi_{j_k}, \xi^*\right) \text{ for all } k \in \mathbb{N}^+.$$

Consequently, the inequality (4.16) holds for all $k \in \mathbb{N}^+$, which extends to all $k \in \mathbb{N}$. Letting $k \to \infty$ in (4.16) and using (4.14), we have $\xi^* = \xi_{j_k+1} \xrightarrow{q} \Theta\xi$. The uniqueness of the limit guarantees that $\xi^* = \Theta\xi^*$.

Scenario 2. If the mapping Θ is $\mathfrak{R}$-continuous, because of the $\mathfrak{R}$-preserving property with $\xi_j \xrightarrow{q} \xi^*$, then the $\mathfrak{R}$-continuity of Θ leads to $\xi_{j+1} = \Theta\xi_j \xrightarrow{q} \Theta\xi^*$. Consequently, by the uniqueness of the limit, we conclude that $\xi^* = \Theta\xi^*$. This finishes the proof. □

The example below supports Theorem 4.2.

Example 4.3. *On the MS $Q = [0, \infty)$ with the usual metric ϖ, we define a binary relation $\mathfrak{R}$, where $(\xi, \zeta) \in \mathfrak{R}$ if and only if $\xi < \zeta$. Furthermore, a self-mapping Θ on Q is defined as*

$$\Theta(\xi) = \frac{\xi}{2} \textit{ for all } \xi \in Q.$$

Then $\mathfrak{R}$ is Θ-closed and Θ is $\mathfrak{R}$-continuous. Additionally, we describe the function $\phi : [0, \infty) \to [0, \infty)$ as

$$\phi(\gamma) = \gamma \textit{ for all } \gamma > 0,$$

and the ω-distance functions $q : Q \times Q \to Q$ as

$$q(\xi, \zeta) = \zeta.$$

Hence, for all $\xi, \zeta \in Q$ with $(\xi, \zeta) \in \mathfrak{R}$, we get

$$\frac{1}{2}q(\xi, \Theta\xi) = \frac{1}{2}\Theta(\xi) = \frac{\xi}{4} \leq \zeta = q(\xi, \zeta),$$

which implies

$$L(q(\Theta\xi, \Theta\zeta)) = L\Theta(\zeta) = L\frac{\zeta}{2} \leq \zeta = q(\xi, \zeta) = \phi(U_\Theta(\xi, \zeta)).$$

Thus, Θ is an $\omega C^L_\phi - \mathfrak{R}$-contraction with $L \in (0, 1)$, $\gamma > 0$. Therefore, all requirements of Theorem 4.2 are satisfied and $\xi = 0$ is a unique FP of Θ in Q.

Remark 4.1. *It should be noted that Theorems 4.1 and 4.2 still hold if the condition of the local Θ-transitivity of $\mathfrak{R}$ is substituted with any of the following alternatives:*

- *$\mathfrak{R}$ is locally transitive,*
- *$\mathfrak{R}$ is transitive,*

- $\Re$ *is* Θ*-transitive.*

Now, by assuming the $\Re^{sy}$-connectedness of the range space $\Theta(Q)$, we extend our earlier theorems to prove the uniqueness of an FP for self-maps.

Theorem 4.3. *Under the hypotheses of Theorem 4.1, Θ possesses a unique FP, provided that $\Theta(Q)$ is $\Re^{sy}$-connected.*

Proof. In a relational MS (Q, ϖ), assume ξ and ζ are FPs of Θ in accordance with Theorem 4.1. Then, for all $j \in \mathbb{N}$,

$$\begin{cases} \Theta^j\xi = \xi \text{ and } \Theta^j\zeta = \zeta, \\ \Theta^{j-1}\xi = \xi \text{ and } \Theta^{j-1}\zeta = \zeta. \end{cases} \tag{4.17}$$

Our discussion will now focus on two separate options.

Option i. Suppose that $(\xi, \zeta) \in \Re^{sy}$. Since $\Re$ is Θ-closed, we have either $\left(\Theta^j\xi, \Theta^j\zeta\right) \in \Re$ or $\left(\Theta^j\zeta, \Theta^j\xi\right) \in \Re$ for all $j = 0, 1, 2, \cdots$.

It follows from (3.1) and (4.17) that

$$\psi\left(q\left(\xi,\zeta\right)\right) = \psi\left(q\left(\Theta^j\xi, \Theta^j\zeta\right)\right) \leq \psi\left(U_\Theta\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right)\right) - LU_\Theta\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right), \tag{4.18}$$

where

$$\begin{aligned} & U_\Theta\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right) \\ = & \max\left\{ \begin{array}{c} q\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right), \dfrac{q\left(\Theta^{j-1}\xi, \Theta\left(\Theta^{j-1}\xi\right)\right)\left[1 + q\left(\Theta^{j-1}\zeta, \Theta\left(\Theta^{j-1}\xi\right)\right)\right]}{1 + q\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right)}, \\ \dfrac{q\left(\Theta^{j-1}\zeta, \Theta\left(\Theta^{j-1}\zeta\right)\right)\left[1 + q\left(\Theta^{j-1}\xi, \Theta\left(\Theta^{j-1}\xi\right)\right)\right]}{1 + q\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right)}, \dfrac{q\left(\Theta^{j-1}\xi, \Theta\left(\Theta^{j-1}\zeta\right)\right) + q\left(\Theta^{j-1}\zeta, \Theta\left(\Theta^{j-1}\xi\right)\right)}{2} \end{array} \right\} \\ = & \max\left\{ \begin{array}{c} q\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right), \dfrac{q\left(\Theta^{j-1}\xi, \Theta^j\xi\right)\left[1 + q\left(\Theta^{j-1}\zeta, \Theta^j\xi\right)\right]}{1 + q\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right)}, \\ \dfrac{q\left(\Theta^{j-1}\zeta, \Theta^j\zeta\right)\left[1 + q\left(\Theta^{j-1}\xi, \Theta^j\xi\right)\right]}{1 + q\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right)}, \dfrac{q\left(\Theta^{j-1}\xi, \Theta^j\zeta\right) + q\left(\Theta^{j-1}\zeta, \Theta^j\xi\right)}{2} \end{array} \right\} \\ = & \max\left\{ q\left(\xi,\zeta\right), \frac{q(\xi,\xi)\left[1 + q(\zeta,\xi)\right]}{1 + q(\xi,\zeta)}, \frac{q\left(\zeta,\zeta\right)\left[1 + q\left(\xi,\xi\right)\right]}{1 + q\left(\xi,\zeta\right)}, \frac{q\left(\xi,\zeta\right) + q\left(\zeta,\xi\right)}{2} \right\} \\ = & q\left(\xi,\zeta\right). \end{aligned}$$

Thus, by (4.18), we get

$$\psi\left(q\left(\xi,\zeta\right)\right) \leq \psi\left(q\left(\xi,\zeta\right)\right) - Lq\left(\xi,\zeta\right) \leq \psi\left(q\left(\xi,\zeta\right)\right).$$

This valid only if $q\left(\xi,\zeta\right) = 0$, that is, $\xi = \zeta$.

Option ii. Suppose that $(\xi, \zeta) \notin \Re^{sy}$. Because $\Theta\left(Q\right)$ is $\Re^{sy}$-connected, for each $\xi, \zeta \in \Theta\left(Q\right)$ there exists a path (say $\{\rho_0, \rho_1, \cdots, \rho_k\}$ of some finite length k in $\Re^{sy}$) from ξ to ζ in order that

$$\rho_0 = \xi,\ \rho_k = \zeta \text{ and } [\rho_i, \rho_{i+1}] \in \Re,\ \text{for each } i\ \ (0 \leq i \leq (k-1)).$$

Since $\mathfrak{R}$ is Θ-closed for each i $(0 \le i \le k-1)$ and

$$[\Theta^j \rho_i, \Theta^j \rho_{i+1}] \in \mathfrak{R}, \text{ for all } j \in \mathbb{N}_0,$$

then, there exists a path of length $k > 1$ in $\mathfrak{R}^{sy}$. Denote $\gamma_j^i = q\left(\Theta^j \rho_i, \Theta^j \rho_{i+1}\right) \in \mathfrak{R}^{sy}$ for i $(0 \le i \le (k-1))$ and $j \in \mathbb{N}_0$.

It follows from (3.1) that

$$\psi\left(\gamma_j^i\right) = \psi\left(q\left(\Theta^j \rho_i, \Theta^j \rho_{i+1}\right)\right) \le \psi\left(U_\Theta\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right)\right) - LU_\Theta\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right), \tag{4.19}$$

where

$$\begin{aligned}
& U_\Theta\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right) \\
= & \max\left\{\begin{array}{c} q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right), \dfrac{q\left(\Theta^{j-1}\rho_i, \Theta\left(\Theta^{j-1}\rho_i\right)\right)\left[1 + q\left(\Theta^{j-1}\rho_{i+1}, \Theta\left(\Theta^{j-1}\rho_i\right)\right)\right]}{1 + q\left(\Theta^{j-1}\xi, \Theta^{j-1}\rho_{i+1}\right)}, \\ \dfrac{q\left(\Theta^{j-1}\rho_{i+1}, \Theta\left(\Theta^{j-1}\rho_{i+1}\right)\right)\left[1 + q\left(\Theta^{j-1}\rho_i, \Theta\left(\Theta^{j-1}\rho_i\right)\right)\right]}{1 + q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right)}, \\ \dfrac{q\left(\Theta^{j-1}\rho_i, \Theta\left(\Theta^{j-1}\rho_{i+1}\right)\right) + q\left(\Theta^{j-1}\rho_{i+1}, \Theta\left(\Theta^{j-1}\rho_i\right)\right)}{2} \end{array}\right\} \\
= & \max\left\{\begin{array}{c} q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right), \dfrac{q\left(\Theta^{j-1}\rho_i, \Theta^j\rho_i\right)\left[1 + q\left(\Theta^{j-1}\rho_{i+1}, \Theta^j\rho_i\right)\right]}{1 + q\left(\Theta^{j-1}\xi, \Theta^{j-1}\rho_{i+1}\right)}, \\ \dfrac{q\left(\Theta^{j-1}\rho_{i+1}, \Theta^j\rho_{i+1}\right)\left[1 + q\left(\Theta^{j-1}\rho_i, \Theta^j\rho_i\right)\right]}{1 + q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right)}, \dfrac{q\left(\Theta^{j-1}\rho_i, \Theta^j\rho_{i+1}\right) + q\left(\Theta^{j-1}\rho_{i+1}, \Theta^j\rho_i\right)}{2} \end{array}\right\} \\
\le & \max\left\{\begin{array}{c} q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right), \\ \dfrac{q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right) + q\left(\Theta^{j-1}\rho_{i+1}, \Theta^j\rho_{i+1}\right) + q\left(\Theta^{j-1}\rho_{i+1}, \Theta^j\rho_{i+1}\right) + q\left(\Theta^j\rho_{i+1}, \Theta^j\rho_i\right)}{2} \end{array}\right\} \\
= & \max\left\{q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right), q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right)\right\}.
\end{aligned}$$

Assume that $U_\Theta\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right) = q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right)$. Then, by (4.19), we get

$$\psi\left(\gamma_j^i\right) = \psi\left(q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right)\right) \le \psi\left(q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right)\right) - Lq\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right) \le \psi\left(q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right)\right) = \psi\left(\gamma_j^i\right),$$

which implies a contradiction. Then we consider $U_\Theta\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right) = q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right)$. Again, by (4.19), we have

$$\begin{aligned}
\psi\left(\gamma_j^i\right) &= \psi\left(q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right)\right) \le \psi\left(q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right)\right) - Lq\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right) \\
&\le \psi\left(q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right)\right) = \psi\left(\gamma_j^{i-1}\right).
\end{aligned} \tag{4.20}$$

As a result, the sequence $\left\{\psi\left(\gamma_j^i\right)\right\} = \left\{\psi\left(q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right)\right)\right\}$ is non-negative decreasing. Due to the monotonicity of ψ, the sequence $\{\gamma_j^i\}$ is also decreasing. Therefore, there exists $\gamma > 0$ such that $\lim_{j\to\infty} \gamma_j^i = \gamma$.

Taking $j \to \infty$ (4.20), and utilizing the monotonicity of ψ, we find that

$$\psi(\gamma) \leq \psi(\gamma) - L\gamma \leq \psi(\gamma).$$

This valid only if $\gamma = 0$ for each i $(0 \leq i \leq k-1)$. Therefore, the application of the triangular inequality to the foregoing conclusion leads to

$$q(\xi,\zeta) = q\left(\Theta^j \xi_0, \Theta^j \zeta_k\right) \leq \gamma_j^0 + \gamma_j^1 + \cdots + \gamma_j^{k-1} \to 0 \text{ as } j \to \infty,$$

which implies that $\xi = \zeta$. This completes the proof. □

Theorem 4.4. *From the assertions of Theorem 4.2, Θ has a unique FP, provided that $\Theta(Q)$ is $\mathfrak{R}^{sy}$-connected.*

Proof. In a relational MS (Q, ϖ), assume ξ and ζ are FPs of Θ in accordance with Theorem 4.2. Then, for all $j \in \mathbb{N}$, (4.17) is true. Now, we study the following situations:

Situation 1. Assume that $(\xi,\zeta) \in \mathfrak{R}^{sy}$. Since $\mathfrak{R}$ is Θ-closed, we have either $\left(\Theta^j\xi, \Theta^j\zeta\right) \in \mathfrak{R}$ or $\left(\Theta^j\zeta, \Theta^j\xi\right) \in \mathfrak{R}$ for all $j = 0, 1, 2, \cdots$.

It follows from (3.2) and (4.17) that

$$\frac{1}{2} q(\xi, \Theta\xi) \leq q(\xi,\zeta),$$

which implies

$$L(q(\xi,\zeta)) = L\left(q\left(\Theta^j\xi, \Theta^j\zeta\right)\right) \leq \phi\left(U_\Theta\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right)\right), \tag{4.21}$$

where (by using the same method as in Theorem 4.3)

$$U_\Theta\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right) = \max\left\{q\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right), q\left(\Theta^j\xi, \Theta^j\zeta\right)\right\}.$$

In the case of $U_\Theta\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right) = q\left(\Theta^j\xi, \Theta^j\zeta\right)$, we have, from (4.21)

$$L(q(\xi,\zeta)) = L\left(q\left(\Theta^j\xi, \Theta^j\zeta\right)\right) \leq \phi\left(q\left(\Theta^j\xi, \Theta^j\zeta\right)\right) = \phi(q(\xi,\zeta)). \tag{4.22}$$

Moreover, in the case of $U_\Theta\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right) = q\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right)$, from (4.21), we get

$$L(q(\xi,\zeta)) = L\left(q\left(\Theta^j\xi, \Theta^j\zeta\right)\right) \leq \phi\left(q\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right)\right) = \phi(q(\xi,\zeta)). \tag{4.23}$$

Clearly (4.22) and (4.23) are valid only if $q(\xi,\zeta) = 0$, that is, $\xi = \zeta$.

Situation 2. Suppose that $(\xi,\zeta) \notin \mathfrak{R}^{sy}$. Because $\Theta(Q)$ is $\mathfrak{R}^{sy}$-connected, in line with Theorem 4.3, we derive

$$[\Theta^j\rho_i, \Theta^j\rho_{i+1}] \in \mathfrak{R}, \text{ for all } j \in \mathbb{N}_0.$$

There is then a path of length $k > 1$ in $\mathfrak{R}^{sy}$. Let $\gamma_j^i = q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right) \in \mathfrak{R}^{sy}$ for i $(0 \leq i \leq (k-1))$ and $j \in \mathbb{N}_0$.

It follows from Theorem 4.2 that, for any fixed i,

$$L\left(\gamma_j^i\right) = L\left(q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right)\right) \leq \phi\left(U_\Theta\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right)\right), \tag{4.24}$$

where

$$U_\Theta\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right) = \max\left\{q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right), q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right)\right\}.$$

In the case of $U_\Theta\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right) = q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right)$, we have from (4.24)

$$L\left(\gamma_j^i\right) = L\left(q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right)\right) \leq \phi\left(q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right)\right) \leq \phi\left(\gamma_j^i\right),$$

which is a contradiction. Therefore, we select $U_\Theta\left(\Theta^{j-1}\xi, \Theta^{j-1}\zeta\right) = q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right)$. Again, by (4.24), one has

$$L\left(\gamma_j^i\right) = L\left(q\left(\Theta^j\rho_i, \Theta^j\rho_{i+1}\right)\right) \leq \phi\left(q\left(\Theta^{j-1}\rho_i, \Theta^{j-1}\rho_{i+1}\right)\right) \leq \phi\left(\gamma_{j-1}^i\right).$$

Leveraging the properties of ϕ, we obtain

$$\gamma_j^i \leq \gamma_{j-1}^i.$$

Consequently, the sequence $\{\gamma_j^i\}$ is decreasing, which implies the existence of a $\gamma > 0$ such that

$$L\gamma \leq \phi(\gamma).$$

This leads to $L\gamma = 0$; consequently, $\gamma = 0$ for each i $(0 \leq i \leq (k-1))$. Ultimately, by the conclusion above and the triangle inequality, we have

$$q(\xi, \zeta) = q\left(\Theta^j\xi_0, \Theta^j\zeta_k\right) \leq \gamma_j^0 + \gamma_j^1 + \cdots + \gamma_j^{k-1} \to 0 \text{ as } j \to \infty,$$

which implies that $\xi = \zeta$. Thus, Θ has a unique FP. □

Remark 4.2.
- *Examples 4.1–4.3 can be applied to support Theorems 4.3 and 4.4.*
- *If take $U_\Theta(\xi, \zeta) = V_\Theta(\xi, \zeta)$ and $L = \theta$, where*

$$V_\Theta(\xi, \zeta) = \max\left\{q(\xi, \zeta), q(\xi, \Theta\xi), q(\zeta, \Theta\zeta), \frac{q(\xi, \Theta\zeta) + q(\zeta, \Theta\xi)}{2}\right\},$$

and $\phi : [0, \infty) \to [0, \infty)$ is an LSC function with $\phi(\gamma) > 0$ for all $\gamma \in (0, \infty)$ and $\phi(0) = 0$. We have the following previous results:

  i. *Theorem 4.1 yields a generalized version of Dutta and Chaudhary's findings [22] through the ϖ-distance when $\Re = \preceq$.*
  ii. *Theorem 4.3, with $\Re = \preceq$, yields an extended and generalized version of the findings by Gupta et al. [23].*
  iii. *A generalized version of Ben-El-Mechaiekh's result [24] is achieved by specifying $\Re$ as the transitive relation in Theorems 4.1 and 4.3, utilizing a ϖ-distance.*

5. Application to transverse oscillations of a homogeneous bar

Let a homogeneous bar be fixed at one extremity and free at the other. Assuming its longitudinal axis coincides with the segment $(0, 1)$ of the x-axis, and that deflections occur parallel to the z-axis at a point γ, the transverse oscillations (TVOs) of the bar are consequently described by the following ordinary differential equation: (to unify the symbols, we consider $x = \xi$ and $z = \rho$):

$$\begin{cases} \xi^{(4)}(\gamma) = \lambda^4 B(\gamma, \xi(\gamma)); \ \gamma \in [0,1], \\ \xi(0) = \xi'(0) = \xi''(1) = \xi'''(1) = 0, \end{cases} \tag{5.1}$$

where $\xi^{(4)}(\gamma) = \frac{d^4\xi}{d\gamma^4}$, $\lambda > 0$ and $B : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $\Omega = C([0, 1], \mathbb{R})$ be the space of real continuous functions on $[0, 1]$, endowed with the metric $\varpi : \Omega \times \Omega \to [0, \infty)$ defined by

$$\varpi(\xi, \zeta) = \sup_{\gamma \in [0,1]} |\xi(\gamma) - \zeta(\gamma)|, \text{ for all } \xi, \zeta \in \Omega.$$

Define a binary relation $\mathfrak{R}$ on Ω by

$$\mathfrak{R} = \{(\xi, \zeta) \in \Omega \times \Omega : \xi(\gamma) \le \zeta(\gamma), \ \gamma \in [0, 1]\}.$$

Then (Ω, ϖ) forms a complete MS, and it is also $\mathfrak{R}$-complete. Assume that $q : \Omega \times \Omega \to [0, \infty)$ such that

$$q(\xi, \zeta) = \|\zeta\|_\infty = \sup_{\gamma \in [0,1]} |\zeta(\gamma)|, \text{ for all } \xi, \zeta \in \Omega.$$

Then q is an ω-distance on Ω.

We are now prepared to state the theorem regarding the existence of a solution for the problem (5.1).

Theorem 5.1. *Let the problem (5.1) represent the governing equation for the TVOs of a homogeneous bar where $\lambda > 0$ is a constant and $B : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Furthermore, assume that there exists $\varphi > 0$ such that*

$$\varphi(\gamma) = \frac{24}{\gamma^2 \lambda^4 (\lambda^2 - 4\lambda + 6)}, \tag{5.2}$$

and $\xi \le \zeta$ for all $\xi, \zeta \in \Omega$ fulfilling

$$\sup_{\gamma \in [0,1]} B(\gamma, \zeta(\gamma)) \le \sup_{\gamma \in [0,1]} \varphi \frac{\zeta(\gamma)}{1 + \zeta(\gamma)}.$$

Then the problem (5.1) has a unique non-negative solution.

Proof. Problem (5.1) can be reformulated as the following integral equation:

$$\xi(\gamma) = \lambda^4 \int_0^1 \mho(\gamma, \rho) B(\rho, \xi(\rho)) d\rho, \tag{5.3}$$

where $\mho(\gamma, \rho)$ represent a Green's function defined as

$$\mho(\gamma, \rho) = \begin{cases} \frac{3\rho^2\gamma - \rho^3}{6}, & 0 \le \rho \le \gamma \le 1, \\ \frac{3\rho\gamma^2 - \rho^3}{6}, & 0 \le \gamma \le \rho \le 1. \end{cases}$$

It can be readily calculated that

$$\int_0^1 \backepsilon(\gamma,\rho)\,d\rho = \frac{\gamma^4 - 4\gamma^3 + 6\gamma^2}{24}.$$

Describe the mapping $\Theta : \Omega \to \Omega$ as

$$\Theta\xi(\gamma) = \lambda^4 \int_0^1 \backepsilon(\gamma,\rho)\, B(\gamma,\xi(\gamma))\,d\rho, \text{ for all } \xi \in \Omega.$$

The condition that $\xi \in C([0,1])$ is an FP of Θ implies that $\xi \in C([0,1])$ is a solution to the problem (5.1).

We now assume that $\zeta \in (\Omega, \Re)$ is a solution of (5.3), i.e.,

$$\zeta(\gamma) \le \lambda^4 \int_0^1 \backepsilon(\gamma,\rho)\, B(\gamma,\zeta(\gamma))\,d\rho,$$

which leads to $(\zeta, \Theta\zeta) \in \Re$ and $\zeta \in \Omega(\Theta, \Re) \neq \emptyset$. Consider that $(\xi,\zeta) \in \Re$. Then for all $\gamma \in ([0,1],\mathbb{R})$, one has

$$\begin{aligned}
\xi(\gamma) &\le \zeta(\gamma) \\
&\Longrightarrow B(\gamma,\xi(\gamma)) \le B(\gamma,\zeta(\gamma)) \text{ (since } B \text{ is continuous)} \\
&\Longrightarrow \backepsilon(\gamma,\rho)\, B(\gamma,\xi(\gamma)) \le \backepsilon(\gamma,\rho)\, B(\gamma,\zeta(\gamma)) \\
&\Longrightarrow \lambda^4 \int_0^1 \backepsilon(\gamma,\rho)\, B(\gamma,\xi(\gamma)) \le \lambda^4 \int_0^1 \backepsilon(\gamma,\rho)\, B(\gamma,\zeta(\gamma)) \\
&\Longrightarrow (\Theta\xi)(\gamma) \le (\Theta\zeta)(\gamma).
\end{aligned}$$

It follows that $(\Theta\xi, \Theta\zeta) \in \Re$, i.e., $\Re$ is Θ-closed.

In addition, for all $\xi, \zeta \in C(([0,1],\mathbb{R}),\mathbb{R})$ with $\xi \le \zeta$,

$$\frac{1}{2} q(\Theta\xi, \xi) \le q(\Theta\xi, \xi) \le q(\xi, \zeta),$$

which implies

$$\begin{aligned}
q(\Theta\xi(\gamma), \Theta\zeta(\gamma)) &= \sup_{\gamma\in[0,1]} |\Theta\zeta(\gamma)| \\
&= \sup_{\gamma\in[0,1]} \left| \lambda^4 \int_0^1 \backepsilon(\gamma,\rho)\, B(\gamma,\zeta(\gamma))\,d\rho \right| \\
&\le \lambda^4 \sup_{\gamma\in[0,1]} \int_0^1 |\backepsilon(\gamma,\rho)\, B(\gamma,\zeta(\gamma))|\,d\rho \\
&\le \lambda^4 \sup_{\gamma\in[0,1]} \int_0^1 \backepsilon(\gamma,\rho)\, \varphi \frac{\zeta(\gamma)}{1+\zeta(\gamma)}\,d\rho \\
&\le \lambda^4 \varphi \frac{q(\xi(\gamma), \zeta(\gamma))}{1 + q(\xi(\gamma), \zeta(\gamma))} \sup_{\gamma\in[0,1]} \int_0^1 \backepsilon(\gamma,\rho)\,d\rho
\end{aligned}$$

$$= \lambda^4\varphi\frac{q(\xi(\gamma),\zeta(\gamma))}{1+q(\xi(\gamma),\zeta(\gamma))}\left(\frac{\gamma^4-4\gamma^3+6\gamma^2}{24}\right). \tag{5.4}$$

Substituting (5.2) into (5.4), we get

$$q(\Theta\xi(\gamma),\Theta\zeta(\gamma)) \leq \frac{q(\xi(\gamma),\zeta(\gamma))}{1+q(\xi(\gamma),\zeta(\gamma))}. \tag{5.5}$$

Suppose that $L \in (0,1)$ and $\phi(\gamma) = \frac{\gamma}{1+\gamma}$ such that $\phi(\gamma) < L$ for all $\gamma > 0$. Obviously, $\phi : [0,\infty) \to [0,\infty)$ is continuous. Hence, by (5.5), we have

$$\begin{aligned} & Lq(\Theta\xi(\gamma),\Theta\zeta(\gamma)) \\ \leq\ & q(\Theta\xi(\gamma),\Theta\zeta(\gamma)) \leq \phi(q(\xi(\gamma),\zeta(\gamma))) \\ \leq\ & \phi\max\left\{ q(\xi,\zeta), \frac{q(\xi,\Theta\xi)[1+q(\zeta,\Theta\xi)]}{1+q(\xi,\zeta)}, \frac{q(\zeta,\Theta\zeta)[1+q(\xi,\Theta\xi)]}{1+q(\xi,\zeta)}, \frac{q(\xi,\Theta\zeta)+q(\zeta,\Theta\xi)}{2} \right\}. \end{aligned}$$

Therefore, the requirements of both Theorems 4.2 and 4.4 are fulfilled. This guarantees the existence of a unique FP of Θ, from which it follows that $\xi \in C([0,1])$ is a solution to the problem (5.1). □

6. Application to the first-order periodic BVP

This section presents an application of our primary findings to a first-order periodic boundary value problem (BVP), specifically regarding the existence of a unique solution characterized by a binary relation. Let us consider the following problem:

$$\begin{cases} \xi'(\gamma) = \Im(\gamma,\xi(\gamma)), \ \gamma \in [0,U], \\ \xi(0) = \xi(U), \end{cases} \tag{6.1}$$

where $\Im : [0,U]\times\mathbb{R} \to \mathbb{R}$ is continuous function.

Let $\widetilde{\Omega} = C([0,U],\mathbb{R})$ be the space of real continuous functions on $[0,U]$, endowed with the metric $\varpi : \widetilde{\Omega}\times\widetilde{\Omega} \to [0,\infty)$ defined by

$$\varpi(\xi,\zeta) = \sup_{\gamma\in[0,U]} |\xi(\gamma)-\zeta(\gamma)|, \text{ for all } \xi,\zeta \in \widetilde{\Omega}.$$

Define a binary relation $\mathfrak{R}$ on $\widetilde{\Omega}$ as

$$\mathfrak{R} = \left\{(\xi,\zeta) \in \widetilde{\Omega}\times\widetilde{\Omega} : \xi(\gamma) \leq \zeta(\gamma), \ \gamma \in [0,1]\right\}.$$

Then, $(\widetilde{\Omega},\varpi)$ forms a complete MS, and it is also $\mathfrak{R}$-complete. Assume that $q : \widetilde{\Omega}\times\widetilde{\Omega} \to [0,\infty)$ such that

$$q(\xi,\zeta) = \|\zeta\|_\infty = \sup_{\gamma\in[0,U]} |\zeta(\gamma)|, \text{ for all } \xi,\zeta \in \widetilde{\Omega}.$$

Then, q is an ω-distance on $\widetilde{\Omega}$.

Definition 6.1. *[25] Let $\varrho \in \widetilde{\Omega}$ be a given function. Then* *i) ϱ is called a lower solution of (6.1) if*

$$\begin{cases} \xi'(\gamma) \leq \Im(\gamma, \xi(\gamma)); \ \gamma \in [0, U], \\ \xi(0) \leq \xi(U), \end{cases}$$

ii) ϱ is called an upper solution of (6.1) if

$$\begin{cases} \xi'(\gamma) \geq \Im(\gamma, \xi(\gamma)); \ \gamma \in [0, U], \\ \xi(0) \geq \xi(U). \end{cases}$$

Our main theorem in this section is as follows.

Theorem 6.1. *The periodic BVP (6.1) has a unique solution, provided that $\xi \leq \zeta$,*

$$\Im\left(\zeta(r), \zeta^{\hbar}(r)\right) \leq \frac{\sqrt{\zeta(\gamma)}}{\hbar} \text{ and } \int_0^U \widetilde{\eth}(\gamma, r)\, dr \leq \hbar, \tag{6.2}$$

where $\widetilde{\eth}(\gamma, r)$ is defined below, $\hbar \in \mathbb{N}_0$, and $U > 0$.

Proof. The equation derived from the BVP can be written as

$$\begin{cases} \xi'(\gamma) = \Im\left(\xi(\gamma), \xi^{\hbar}(\gamma)\right); \ \gamma \in [0, U], \ \hbar \in \mathbb{N}_0, \\ \xi(0) = \xi(U). \end{cases} \tag{6.3}$$

Problem (6.3) can be expressed as the following integral equation:

$$\xi(\gamma) = \int_0^U \widetilde{\eth}(\gamma, r)\, \Im\left(\xi(r), \xi^{\hbar}(r)\right) dr,$$

where $\widetilde{\eth}(\gamma, r)$ is given by

$$\widetilde{\eth}(\gamma, r) = \begin{cases} \frac{e^{2(U+r-\gamma)}}{e^{-\gamma}-1}, & 0 \leq r \leq \gamma \leq 1, \\ \frac{e^{2(r-\gamma)}}{e^{-\gamma}-1}, & 0 \leq \gamma \leq r \leq 1. \end{cases}$$

Define the mapping $\Theta : \widetilde{\Omega} \to \widetilde{\Omega}$ as

$$\Theta\xi(\gamma) = \int_0^U \widetilde{\eth}(\gamma, r)\, \Im\left(\xi(r), \xi^{\hbar}(r)\right) dr.$$

Clearly, the unique FP of Θ is equivalent to the solution of the BVP (6.1).

Now, we select an $\mathfrak{R}$-preserving sequence $\{\xi_{\hbar}\}$ such that $\xi_{\hbar} \xrightarrow{\varpi} \xi^*$, for all $\hbar \in \mathbb{N}_0$. Then,

$$\xi_0(\gamma) \leq \xi_1(\gamma) \leq \cdots \leq \xi_{\hbar}(\gamma) \leq \xi_{\hbar+1}(\gamma) \leq \cdots,$$

and is convergent to $\xi(\gamma)$, which leads to $\xi_{\hbar}(\gamma) \leq \xi^*(\gamma)$ for all $\gamma \in [0, U]$ and $\hbar \in \mathbb{N}_0$. Thus, $[\xi_{\hbar}, \xi^*] \in \mathfrak{R}$ for all $\hbar \in \mathbb{N}_0$. Hence, $\mathfrak{R}$ is $\mathfrak{R}$-continuous.

Assume that $\xi \in \widetilde{\Omega}$ is a lower solution of (6.1). Then, we have

$$\xi'(\gamma) \leq \Im\left(\xi(\gamma), \xi^{\hbar}(\gamma)\right) \text{ for all } \gamma \in [0, U].$$

It follows after multiplying the two sides in $e^{2\gamma}$ that

$$\xi(\gamma)e^{2\gamma} \leq \xi(0) + \int_0^{\gamma} \left[\Im\left(\xi(r), \xi^{\hbar}(r)\right)\right] e^{2r} dr. \tag{6.4}$$

Since $\xi(0) \leq \xi(U)$, we have

$$\xi(0)e^{2\gamma} \leq \xi(U)e^{2\gamma} \leq \xi(0) + \int_0^{U} \left[\Im\left(\xi(r), \xi^{\hbar}(r)\right)\right] e^{2r} dr.$$

Thus,

$$\xi(0) \leq \int_0^{U} \frac{e^{2r}}{e^{2\gamma}-1} \left[\Im\left(\xi(r), \xi^{\hbar}(r)\right)\right] dr. \tag{6.5}$$

From (6.4) and (6.5), we have

$$\begin{aligned}
\xi(\gamma)e^{2\gamma} &\leq \int_0^{U} \frac{e^{2r}}{e^{2\gamma}-1} \left[\Im\left(\xi(r), \xi^{\hbar}(r)\right)\right] dr + \int_0^{\gamma} \left[\Im\left(\xi(r), \xi^{\hbar}(r)\right)\right] e^{2r} dr \\
&\leq \int_0^{U} \frac{e^{2(1+r)}}{e^{-\gamma}-1} \left[\Im\left(\xi(r), \xi^{\hbar}(r)\right)\right] dr + \int_0^{U} \frac{e^{2r}}{e^{-\gamma}-1} \left[\Im\left(\xi(r), \xi^{\hbar}(r)\right)\right] dr,
\end{aligned}$$

which implies that

$$\begin{aligned}
\xi(\gamma) &\leq \int_0^{U} \frac{e^{2(1+r-\gamma)}}{e^{\gamma}-1} \left[\Im\left(\xi(r), \xi^{\hbar}(r)\right)\right] dr + \int_0^{U} \frac{e^{2(r--\gamma)}}{e^{\gamma}-1} \left[\Im\left(\xi(r), \xi^{\hbar}(r)\right)\right] dr \\
&= \int_0^{U} \widetilde{\eth}(\gamma, r) \left[\Im\left(\xi(r), \xi^{\hbar}(r)\right)\right] dr \\
&= \Theta\xi(\gamma).
\end{aligned}$$

Hence, $(\xi(\gamma), \Theta\xi(\gamma)) \in \mathfrak{R}$ for all $\gamma \in [0, U]$, which proves that $\widetilde{\Omega}(\Theta, \mathfrak{R}) \neq \emptyset$.

Next, for any $(\xi, \zeta) \in \mathfrak{R}$, that is, $\xi(\gamma) \leq \zeta(\gamma)$ and

$$\Im\left(\xi(\gamma), \xi^{\hbar}(\gamma)\right) \leq \Im\left(\zeta(\gamma), \zeta^{\hbar}(\gamma)\right) \text{ for all } \gamma \in [0, U] \text{ and } \hbar \in \mathbb{N}_0,$$

and $\widetilde{\eth}(\gamma, r) > 0$ for $(\gamma, r) \in [0, U] \times [0, U]$. Also, we have

$$\begin{aligned}
(\Theta\xi)(\gamma) &= \int_0^{U} \widetilde{\eth}(\gamma, r) \left[\Im\left(\xi(r), \xi^{\hbar}(r)\right)\right] dr \\
&\leq \int_0^{U} \widetilde{\eth}(\gamma, r) \left[\Im\left(\zeta(r), \zeta^{\hbar}(r)\right)\right] dr \\
&= (\Theta\zeta)(\gamma), \text{ for all } \gamma \in [0, 1] \text{ and } \hbar \in \mathbb{N}_0.
\end{aligned}$$

Thus, $(\Theta\xi, \Theta\zeta) \in \mathfrak{R}$; that is, $\mathfrak{R}$ is Θ-closed.

Finally, for all $(\xi, \zeta) \in \mathfrak{R}$, one has

$$\frac{1}{2} q(\Theta\xi, \xi) \leq q(\Theta\xi, \xi) \leq q(\xi, \zeta),$$

which implies that

$$\begin{aligned} q(\Theta\xi(\gamma), \Theta\zeta(\gamma)) &= \sup_{\gamma\in[0,U]} |\Theta\zeta(\gamma)| \\ &= \sup_{\gamma\in[0,U]} \int_0^U \widetilde{\eth}(\gamma, r)\left[\Im\left(\zeta(r), \zeta^{\hbar}(r)\right)\right] dr \\ &\leq \sup_{\gamma\in[0,U]} \int_0^U \widetilde{\eth}(\gamma, r)\, \frac{\sqrt{\zeta(\gamma)}}{\hbar} dr \\ &\leq \hbar \frac{\sqrt{\zeta(\gamma)}}{\hbar} \\ &= \sqrt{\zeta(\gamma)} \\ &= \sqrt{q(\xi(\gamma), \zeta(\gamma))}. \end{aligned}$$

Assume that $L \in (0, 1)$ and $\phi(\gamma) = \sqrt{\gamma}$ such that $\phi(\gamma) < L$ for all $\gamma > 0$. Obviously, $\phi : [0, \infty) \to [0, \infty)$ is continuous and

$$\begin{aligned} & Lq(\Theta\xi(\gamma), \Theta\zeta(\gamma)) \\ \leq\ & q(\Theta\xi(\gamma), \Theta\zeta(\gamma)) \leq \phi(q(\xi(\gamma), \zeta(\gamma))) \\ \leq\ & \phi \max\left\{ q(\xi, \zeta), \frac{q(\xi, \Theta\xi)[1 + q(\zeta, \Theta\xi)]}{1 + q(\xi, \zeta)}, \frac{q(\zeta, \Theta\zeta)[1 + q(\xi, \Theta\xi)]}{1 + q(\xi, \zeta)}, \frac{q(\xi, \Theta\zeta) + q(\zeta, \Theta\xi)}{2} \right\}. \end{aligned}$$

The fulfillment of the requirements of both Theorems 4.2 and 4.4 ensure the existence of a unique FP of Θ. From this, it follows that $\xi \in C([0, U])$ constitutes a unique solution to the BVP (6.1). □

7. Conclusions

This manuscript introduced modified $\mathfrak{R}$-rational contractions for single self-maps, leveraging ω-distance within a relational-theoretic metric space. This novel approach established the existence and uniqueness of a fixed point for such self-maps, specifically through the application of the locally Θ-transitivity property. We provided compelling examples to support our theoretical advancements and demonstrated their practical significance. For instance, we successfully solved a fourth-order BVP that concerned the transverse oscillation in a homogeneous bar. Furthermore, we addressed and resolved a first-order periodic boundary value problem, showcasing the broad applicability of our findings.

Authors contributions

Hasanen A. Hammad and Manal Elzain Mohamed Abdalla: Conceptualization, methodology, validation, writing-original draft, writing-review & editing. All authors contributed equally and significantly to writing this article.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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