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Research article

Analytical valuation of vulnerable options under a stochastic volatility model with a stochastic long-term mean

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Abstract: We derive the explicit pricing formulas for vulnerable options under a stochastic volatility model with stochastic long-term mean. We extend the He and Chen model to incorporate counterparty default risk and derive explicit solutions for option prices using the characteristic function of the underlying asset's log-price. The option writer defaults when their asset value falls below a predetermined boundary, reducing the option payoff. Our numerical examples show that option prices are highly sensitive to default boundaries and exhibit asymmetric responses to volatility parameters.

Keywords: vulnerable option; stochastic volatility; characteristic function; stochastic long-term mean
Mathematics Subject Classification: 91G20

1. Introduction

The Black-Scholes model [1] has had a significant impact on financial economics by providing a fundamental framework for option pricing. Despite its theoretical and computational simplicity, the model is based on several assumptions that significantly contrast with real market conditions. These limitations include constant volatility, constant interest rates, no transaction costs, and the absence of arbitrage opportunities. Perhaps one of the most important but often ignored limitations of the Black-Scholes model was its assumption of a risk-free environment in which counterparty default is impossible. In real markets, financial derivatives are contracts between two parties, and either party may default before settlement, resulting in significant losses to the non-defaulting party. This vulnerability became painfully evident during the 2008 global financial crisis, which exposed the systemic consequences of neglecting counterparty default risk.

To overcome the limitations associated with neglecting default risk, the concept of "vulnerable options" was introduced. Vulnerable options are derivatives whose payoffs explicitly account for the possibility of default by the counterparty. Johnson and Stulz [9] pioneered the modeling of vulnerable options using structural approaches. Subsequently, many studies have developed their framework by incorporating various extended models such as correlated default risk, stochastic default barriers, and jump-diffusion processes to capture default dynamics more accurately [10, 12, 21]. Recent literature has emphasized the importance of incorporating stochastic volatility into vulnerable option pricing models. Stochastic volatility models, such as those developed by Heston [7], effectively capture the dynamic nature of market volatility and significantly improve the accuracy of option pricing. Numerous researchers have therefore combined stochastic volatility frameworks with default risk modeling to more effectively price vulnerable options.

Among these contributions, Yang et al. [19] first studied the application of stochastic volatility to vulnerable option valuation, employing a fast mean-reverting framework to derive asymptotic pricing expansions. This line of research has been developed by various additional investigations. Wang et al. [13] examined models characterized by short-term mean-reverting volatility, while Lee and Kim [11] employed a multiscale generalization of the Heston model to address defaultable options. Further extensions include the work of Wang [15], who introduced a stochastic volatility model that captures leverage effects and stochastic correlation, and Xie and Deng [18], who proposed a stochastic volatility with Markov regime-switching and utilized the fast Fourier transform (FFT) for efficient computation. Yun and Kim [20] used a two-factor stochastic volatility model for valuing the vulnerable option with hybrid default risk. In addition, several researchers recently investigated the valuation of vulnerable options with liquidity risk under the stochastic volatility model [5, 8].

In addition to stochastic volatility models, recent studies have highlighted the importance of rough volatility frameworks. Empirical evidence shows that volatility is not smooth but rather rough, a feature that has been extensively studied in the literature. Gatheral et al. [3] provided strong evidence that volatility is rough using high-frequency data. Xiao and Yu [16, 17] developed asymptotic theories for fractional Vasicek models, while Wang et al. [14] applied the fractional Ornstein–Uhlenbeck process to model and forecast realized volatility. Although our study focuses on stochastic volatility with a stochastic long-term mean, the integration of rough volatility into the pricing of vulnerable options remains an interesting direction for future research. Motivated by these results, this paper further develops the pricing of vulnerable options by adopting an improved stochastic volatility model proposed by He and Chen [4], which incorporates a stochastic long-term mean in volatility dynamics. Their model addressed a critical limitation in the Heston [7] framework by allowing the long-term mean of volatility to follow a stochastic process, rather than remaining constant. This modification enables more accurate modeling of the term structure of implied volatility and variance swap curves. That is, the stochastic long-term mean enhances the model's flexibility, better capturing the empirical features of the volatility surface and implied volatility term structure. Despite these advancements, the model of He and Chen [4] only deals with default-free European options, missing an important feature of counterparty default risk. In this paper, we address this essential absence. Specifically, we provide the explicit pricing formula of a vulnerable option under the stochastic volatility model with a stochastic long-term mean.

The rest of this paper is organized as follows. In Section 2, we introduce the mathematical model for vulnerable options and the stochastic volatility model of [4] to incorporate counterparty default risk. In Section 3, we derive a formula for the characteristic function of the underlying log-price and an explicit pricing formula for vulnerable option prices. In Section 4, we provide some numerical examples to show the movements of the option prices, including the impact of mean reversion using the formula in Section 3. Finally, we present the concluding remarks in Section 5.

2. Model

In this section, based on the work of He and Chen [4], we introduce a stochastic volatility model that incorporates a stochastic long-term mean for pricing of vulnerable options.

We define S_t as the market value of the underlying asset and $V(t)$ as the market value of the asset of the option writer. The dynamics of S_t and $V(t)$ are modeled directly under the risk-neutral measure Q. Following the model of He and Chen [4], the dynamics of S_t and $V(t)$ are given by the following stochastic differential equations (SDEs):

$$\frac{dS(t)}{S(t)} = r\,dt + \sqrt{v_1(t)}\,W^1(t), \tag{2.1}$$

$$\frac{dV(t)}{V(t)} = r\,dt + \sqrt{v_2(t)}\,W^2(t), \tag{2.2}$$

$$dv_i(t) = k_i\left(\theta_i(t) - v_i(t)\right)\,dt + \sigma_i\sqrt{v_i(t)}\,dB^i(t), \tag{2.3}$$

where r is the risk-free rate, k_i $(i = 1, 2)$ denotes the mean-reversion speed, and σ_i $(i = 1, 2)$ are the constant volatilities of volatilities. In addition, $\theta_i(t)$ $(i = 1, 2)$ are the stochastic long-term means of the variance process and are given by

$$d\theta_i(t) = \lambda_i dt + \gamma_i dZ^i(t), \quad i = 1, 2, \tag{2.4}$$

where λ_i $(i = 1, 2)$ and γ_i $(i = 1, 2)$ are constants governing the drifts and volatilities of the long-term mean processes. Moreover, the correlation structure among the standard Brownian motions is given by

$$\begin{aligned} dW^1(t)\,dW^2(t) &= \rho\,dt, \\ dW^i(t)\,dB^i_t &= \rho_i\,dt, \quad i = 1, 2, \end{aligned} \tag{2.5}$$

with all other pairs being uncorrelated.

Note that if $x(t) = \ln(S(t))$, then

$$dx(t) = \left(r - \frac{1}{2}v_1(t)\right)dt + \sqrt{v_1(t)}\,dW^1(t), \tag{2.6}$$

and if $y(t) = \ln(V(t))$, then

$$dy(t) = \left(r - \frac{1}{2}v_2(t)\right)dt + \sqrt{v_2(t)}\,dW^2(t). \tag{2.7}$$

3. Valuation of vulnerable options

In this section, we derive the characteristic function for the log-prices $x(t)$ and $y(t)$. Using this function, we provide an explicit pricing formula for vulnerable options.

3.1. A characteristic function

We consider the conditional joint characteristic function $h(\phi_1, \phi_2, \tau)$, which is defined as:

$$h(\phi_1, \phi_2, \tau) = \mathbb{E}^Q \left[e^{-\int_0^\tau r\,ds + i\phi_1 x(t) + i\phi_2 y(t)} \mid x(t), y(t), v_1(t), v_2(t), \theta_1(t), \theta_2(t) \right], \tag{3.1}$$

where $\tau = T - t$, T is time to maturity, and $\mathbb{E}^Q[\cdot]$ denotes the expectation under the measure Q.

Theorem 1. *Suppose that the underlying asset prices follow the dynamics given in* (2.1) *and* (2.2). *Then the joint characteristic function of* $(x(t), y(t))$ *is given by*

$$\begin{aligned} h(\phi_1, \phi_2, \tau) = \exp\Big(& A(\phi_1, \phi_2; \tau) + \sum_{i=1}^{2} B_i(\phi_1, \phi_2; \tau) v_i(t) \\ & + \sum_{i=1}^{2} C_i(\phi_1, \phi_2; \tau)\theta_i(t) + i\phi_1 x(t) + i\phi_2 y(t) \Big), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} A(\phi_1, \phi_2; \tau) &= \int_0^\tau \Big\{ \sum_{i=1}^{2} \Big(\lambda_i C_i(\phi_1, \phi_2; s) + \frac{1}{2}\gamma_i^2 C_i(\phi_1, \phi_2; s)^2 \Big) \\ &\qquad + ir \sum_{i=1}^{2} \phi_i - \rho\phi_1\phi_2 \sqrt{\psi_1 s + \beta_1}\sqrt{\psi_2 s + \beta_2} - r \Big\} ds, \\ B_i(\phi_1, \phi_2; \tau) &= \frac{2}{\sigma_i^2} \left(\tilde{\alpha}_i - \delta_i \frac{\sinh(\delta_i \tau) + \tilde{\delta}_i \cosh(\delta_i \tau)}{\cosh(\delta_i \tau) + \tilde{\delta}_i \sinh(\delta_i \tau)} \right), \\ C_i(\phi_1, \phi_2; \tau) &= \frac{2k_i}{\sigma_i^2} \left(\tilde{\alpha}_i \tau - \ln\left(\cosh(\delta_i \tau) + \tilde{\delta}_i \sinh(\delta_i \tau) \right) \right), \end{aligned} \tag{3.3}$$

with

$$\tilde{\alpha}_i = \frac{1}{2}(k_i - i\rho_i\sigma_i\phi_i), \tag{3.4}$$

$$\delta_i = \sqrt{\tilde{\alpha}_i^2 + \frac{\sigma_i^2}{4}\phi_i(\phi_i + i)}, \tag{3.5}$$

$$\tilde{\delta}_i = \frac{\tilde{\alpha}_i}{\delta_i}, \quad \textit{for } i = 1, 2. \tag{3.6}$$

$$\beta_i = \theta_i(t) \quad \textit{and} \quad \psi_i = \lambda_i, \quad \textit{for } i = 1, 2. \tag{3.7}$$

Proof. By applying the Feynman–Kac formula, we derive the following PDE that governs the characteristic function $h(\phi_1, \phi_2; \tau)$

$$\begin{aligned} -\frac{\partial h}{\partial t} + rh = \frac{1}{2}v_1\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}v_2\frac{\partial^2 h}{\partial y^2} + \frac{1}{2}\sum_{i=1}^{2}\sigma_i^2 v_i \frac{\partial^2 h}{\partial v_i^2} + \frac{1}{2}\sum_{i=1}^{2}\gamma_i^2\frac{\partial^2 h}{\partial \theta_i^2} \\ + \rho_1\sigma_1 v_1 \frac{\partial^2 h}{\partial x \partial v_1} + \rho_2\sigma_2 v_2 \frac{\partial^2 h}{\partial y \partial v_2} + \rho\sqrt{v_1 v_2}\frac{\partial^2 h}{\partial x \partial y} \end{aligned}$$

$$
\begin{aligned}
&+\left(r-\frac{1}{2}v_1\right)\frac{\partial h}{\partial x}+\left(r-\frac{1}{2}v_2\right)\frac{\partial h}{\partial y}\\
&+\sum_{i=1}^{2}k_i\left(\theta_i-v_i\right)\frac{\partial h}{\partial v_i}+\sum_{i=1}^{2}\lambda_i\frac{\partial h}{\partial \theta_i},
\end{aligned}
\tag{3.8}
$$

with the terminal condition

$$
h(\phi_1,\phi_2;0)=\exp\left(i\phi_1 x(T)+i\phi_2 y(T)\right). \tag{3.9}
$$

Following the affine framework in [2], we assume that the characteristic function h takes the following form:

$$
\begin{aligned}
h(\phi_1,\phi_2;\tau)=\exp\Big(&A(\phi_1,\phi_2;\tau)+\sum_{i=1}^{2}B_i(\phi_1,\phi_2;\tau)v_i(t)+\sum_{i=1}^{2}C_i(\phi_1,\phi_2;\tau)\theta_i(t)\\
&+i\phi_1 x(t)+i\phi_2 y(t)\Big).
\end{aligned}
\tag{3.10}
$$

Substituting this affine form into the PDE gives:

$$
\begin{aligned}
&-\frac{\partial A}{\partial t}-\sum_{i=1}^{2}\frac{\partial B_i}{\partial t}v_i-\sum_{i=1}^{2}\frac{\partial C_i}{\partial t}\theta_i+r\\
&=-\frac{1}{2}\sum_{i=1}^{2}v_i\phi_i^2+\frac{1}{2}\sum_{i=1}^{2}\sigma_i^2 v_i B_i^2+\frac{1}{2}\sum_{i=1}^{2}\gamma_i^2 C_i^2+i\sum_{i=1}^{2}\sigma_i\rho_i\phi_i v_i B_i\\
&\quad-\rho\phi_1\phi_2\sqrt{v_1v_2}+i\sum_{i=1}^{2}\left(r-\frac{1}{2}v_i\right)\phi_i+\sum_{i=1}^{2}k_i(\theta_i-v_i)B_i+\sum_{i=1}^{2}\lambda_i C_i.
\end{aligned}
\tag{3.11}
$$

Since $\sqrt{v_1(t)v_2(t)}$ is nonlinear in $v_1(t)$ and $v_2(t)$, we adopt the approximation method proposed by [6], which exploits the mean-reverting nature of the volatility process. Accordingly, for $i = 1, 2$, we approximate

$$
\sqrt{v_i(t)}\approx\sqrt{\theta_i(t)}\approx\sqrt{\psi_i\tau+\beta_i},
$$

where $\beta_i = \theta_i(t)$ and $\psi_i = \lambda_i$. This approximation leads to a system of five ordinary differential equations with respect to τ derived from (3.11), associated with $A(\phi_1,\phi_2;\tau)$, $B_i(\phi_1,\phi_2;\tau)$, and $C_i(\phi_1,\phi_2;\tau)$ for $i = 1, 2$, which are given as follows:

$$
\frac{\partial A}{\partial \tau}=\sum_{i=1}^{2}\left(\lambda_i C_i+\frac{1}{2}\gamma_i^2C_i^2\right)+ir\sum_{i=1}^{2}\phi_i-\rho\phi_1\phi_2\sqrt{\psi_1\tau+\beta_1}\sqrt{\psi_2\tau+\beta_2}-r, \tag{3.12}
$$

$$
\frac{\partial B_i}{\partial \tau}=\frac{1}{2}\sigma_i^2B_i^2+\left(i\rho_i\sigma_i\phi_i-k_i\right)B_i-\frac{\phi_i}{2}(\phi_i+i),\qquad i=1,2, \tag{3.13}
$$

$$
\frac{\partial C_i}{\partial \tau}=k_iB_i,\qquad i=1,2. \tag{3.14}
$$

Note that the corresponding boundary conditions are specified by

$$
A(\phi_1,\phi_2;0)=B_1(\phi_1,\phi_2;0)=B_2(\phi_1,\phi_2;0)=C_1(\phi_1,\phi_2;0)=C_2(\phi_1,\phi_2;0)=0. \tag{3.15}
$$

The ODE for $B_i(\phi_1, \phi_2; \tau)$, $i = 1, 2$, is a standard Riccati equation with constant coefficients, and its closed-form solution is given by

$$B_i(\phi_1, \phi_2; \tau) = \frac{2}{\sigma_i^2}\left(\tilde{\alpha}_i - \delta_i \frac{\sinh(\delta_i \tau) + \tilde{\delta}_i \cosh(\delta_i \tau)}{\cosh(\delta_i \tau) + \tilde{\delta}_i \sinh(\delta_i \tau)}\right), \tag{3.16}$$

where, for $i = 1, 2$,

$$\tilde{\alpha}_i = \frac{1}{2}(k_i - i\rho_i \sigma_i \phi_i), \tag{3.17}$$

$$\delta_i = \sqrt{\tilde{\alpha}_i^2 + \frac{\sigma_i^2}{4}\phi_i(\phi_i + i)}, \tag{3.18}$$

$$\tilde{\delta}_i = \frac{\tilde{\alpha}_i}{\delta_i}. \tag{3.19}$$

The solution for $C_i(\phi_1, \phi_2; \tau)$ is obtained by solving the first-order linear ODE:

$$C_i(\phi_1, \phi_2; \tau) = \frac{2k_i}{\sigma_i^2}\left(\tilde{\alpha}_i \tau - \ln\left(\cosh(\delta_i \tau) + \tilde{\delta}_i \sinh(\delta_i \tau)\right)\right). \tag{3.20}$$

Finally, $A(\phi_1, \phi_2; \tau)$ is obtained by directly integrating its differential equation:

$$\begin{aligned} A(\phi_1, \phi_2; \tau) = \int_0^\tau \Bigg\{ & \sum_{i=1}^{2}\left(\lambda_i C_i(\phi_1, \phi_2; s) + \frac{1}{2}\gamma_i^2 C_i(\phi_1, \phi_2; s)^2\right) \\ & + ir\sum_{i=1}^{2}\phi_i - \rho\phi_1\phi_2\sqrt{\psi_1 s + \beta_1}\sqrt{\psi_2 s + \beta_2} - r\Bigg\} ds. \end{aligned}$$

This completes the proof. □

3.2. Vulnerable option pricing

In this subsection, we obtain the pricing formula of a vulnerable European option under the structural model, using the model proposed in the previous section. We now consider a vulnerable option with maturity T. Under the structural model, the default event occurs if the option issuer's asset $V(t)$ falls below the default boundary D. Then the option price at time 0 under measure Q is given by

$$C \quad = \quad e^{-\int_0^T r ds} \mathrm{E}^Q\left[(S(T) - K)^+ \left(\mathbf{1}_{\{V(T)>D\}} + \frac{(1-\alpha)V(T)}{D}\mathbf{1}_{\{V(T)<D\}}\right)\right], \tag{3.21}$$

where K is the strike price, α is the deadweight cost, and $\mathbf{1}_{\{\cdot\}}$ is the indicate function. Using the characteristic function in Theorem 1, we can obtain the analytic pricing formula of the vulnerable option under the proposed model shown in the following Theorem.

Theorem 2. *The pricing formula for a vulnerable European option at time* 0 *under the proposed model is presented as*

$$C = e^{-rT}\left[J_1 - J_2 + J_3 - J_4\right], \tag{3.22}$$

where

$$
\begin{aligned}
J_1 &= e^{rT} \times h(-i, 0, T) \times \Phi_1(x(T), y(T)),\\
J_2 &= K \times \Phi_2(x(T), y(T)),\\
J_3 &= e^{rT} \times \frac{1-\alpha}{D} \times h(-i, -i, T) \times \Phi_3(x(T), -y(T)),\\
J_4 &= e^{rT} \times K \times \frac{1-\alpha}{D} \times h(0, -i, T) \times \Phi_4(x(T), -y(T)),
\end{aligned}
$$

with

$$
\begin{aligned}
\Phi_1(x(T), y(T)) &= 1 - F_1(x(T); \ln K) - F_1(y(T); \ln D) + F_1(x(T), y(T); \ln K, \ln D),\\
\Phi_2(x(T), y(T)) &= 1 - F_2(x(T); \ln K) - F_2(y(T); \ln D) + F_2(x(T), y(T); \ln K, \ln D),\\
\Phi_3(x(T), -y(T)) &= 1 - F_3(x(T); \ln K) - F_3(-y(T); -\ln D) + F_3(x(T), -y(T); \ln K, -\ln D),\\
\Phi_4(x(T), -y(T)) &= 1 - F_4(x(T); \ln K) - F_4(-y(T); -\ln D) + F_4(x(T), -y(T); \ln K, -\ln D),
\end{aligned}
$$

and for $j = 1, 2, 3, 4,$

$$
\begin{aligned}
F_j(x(T), y(T); x, y) &= -\frac{1}{4} + \frac{1}{2}F_j(x(T); x) + \frac{1}{2}F_j(y(T); y)\\
&-\frac{1}{2\pi^2}\int_0^\infty \int_0^\infty \left(Re\left[\frac{e^{-i\phi_1 x - i\phi_2 y} f_j(i\phi_1, i\phi_2)}{\phi_1\phi_2}\right] - Re\left[\frac{e^{-i\phi_1 x + i\phi_2 y} f_j(i\phi_1, -i\phi_2)}{\phi_1\phi_2}\right]\right) d\phi_1 d\phi_2,\\
F_j(x(T); x) &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi_1 x} f_j(i\phi_1, 0)}{i\phi_1}\right] d\phi_1,\\
F_j(y(T); y) &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi_2 y} f_j(0, i\phi_2)}{i\phi_2}\right] d\phi_2,
\end{aligned}
$$

where $F_j(x(T), y(T); x, y)$ *is the distribution function of* $x(T)$ *and* $y(T)$, $F_j(x(T); x)$ *and* $F_j(y(T); y)$ *are the marginal distributions of of* $x(T)$ *and* $y(T)$, *respectively, and*

$$
\begin{aligned}
f_1(i\phi_1, i\phi_2) &:= \frac{h(\phi_1 - i, \phi_2, T)}{h(-i, 0, T)}, \quad f_2(i\phi_1, i\phi_2) := e^{rT} \times h(\phi_1, \phi_2, T),\\
f_3(i\phi_1, i\phi_2) &:= \frac{h(\phi_1 - i, -\phi_2 - i, T)}{h(-i, -i, T)}, \quad f_4(i\phi_1, i\phi_2) := \frac{h(\phi_1, -\phi_2 - i, T)}{h(0, -i, T)}.
\end{aligned}
$$

Proof. We first rewrite the price of the option in the following form

$$
\begin{aligned}
C &= e^{-rT}\mathrm{E}^Q\left[(S(T) - K)^+\left(\mathbf{1}_{\{V(T)>D\}} + \frac{(1-\alpha)V(T)}{D}\mathbf{1}_{\{V(T)<D\}}\right)\right]\\
&= e^{-rT}\mathrm{E}^Q\left[(e^{x(T)} - K)^+\mathbf{1}_{\{y(T)>\ln D\}}\right] + \frac{(1-\alpha)}{D}e^{-rT}\mathrm{E}^Q\left[e^{y(T)}(e^{x(T)} - K)^+\mathbf{1}_{\{y(T)<\ln D\}}\right]\\
&= e^{-rT}\mathrm{E}^Q\left[(e^{x(T)} - K)\mathbf{1}_{\{x(T)>\ln K, y(T)>\ln D\}}\right] + \frac{(1-\alpha)}{D}e^{-rT}\mathrm{E}^Q\left[e^{y(T)}(e^{x(T)} - K)\mathbf{1}_{\{x(T)>\ln K, y(T)<\ln D\}}\right]\\
&= e^{-rT}\left[J_1 - J_2 + J_3 - J_4\right],
\end{aligned} \tag{3.23}
$$

where

$$\begin{aligned} J_1 &= \mathrm{E}^Q\left[e^{x(T)}\mathbf{1}_{\{x(T)>\ln K, y(T)>\ln D\}}\right], \\ J_2 &= K\cdot\mathrm{E}^Q\left[\mathbf{1}_{\{x(T)>\ln K, y(T)>\ln D\}}\right], \\ J_3 &= \frac{(1-\alpha)}{D}\cdot\mathrm{E}^Q\left[e^{x(T)+y(T)}\mathbf{1}_{\{x(T)>\ln K, y(T)<\ln D\}}\right], \\ J_4 &= \frac{(1-\alpha)}{D}K\cdot\mathrm{E}^Q\left[e^{y(T)}\mathbf{1}_{\{x(T)>\ln K, y(T)<\ln D\}}\right]. \end{aligned}$$

We complete the proof by calculating J_1, J_2, J_3, and J_4. To derive J_1, we introduce a new measure Q_1 defined by

$$\frac{dQ_1}{dQ} = \frac{e^{x(T)}}{\mathrm{E}^Q\left[e^{x(T)}\right]}.$$

Then, under the measure Q_1, the joint characteristic function of $x(T)$ and $y(T)$ is given by

$$\begin{aligned} f_1(i\phi_1, i\phi_2) = \mathrm{E}^{Q_1}\left[e^{i\phi_1 x(T)+i\phi_2 y(T)}\right] &= \mathrm{E}^Q\left[\frac{e^{x(T)}}{\mathrm{E}^Q\left[e^{x(T)}\right]}e^{i\phi_1 x(T)+i\phi_2 y(T)}\right] \\ &= \frac{h(\phi_1 - i, \phi_2, T)}{h(-i, 0, T)}, \end{aligned} \tag{3.24}$$

where the function h is defined in Theorem 1. Using the function $f_1(i\phi_1, i\phi_2)$, we can derive J_1 as follows.

$$\begin{aligned} J_1 &= \mathrm{E}^Q\left[e^{x(T)}\mathbf{1}_{\{x(T)>\ln K, y(T)>\ln D\}}\right] \\ &= \mathrm{E}^Q\left[e^{x(T)}\right]\mathrm{E}^Q\left[\frac{dQ_1}{dQ}\mathbf{1}_{\{x(T)>\ln K, y(T)>\ln D\}}\right] \\ &= \mathrm{E}^Q\left[e^{x(T)}\right]\mathrm{E}^{Q_1}\left[\mathbf{1}_{\{x(T)>\ln K, y(T)>\ln D\}}\right] \\ &= e^{rT}\times h(-i, 0, T)\times\Phi_1(x(T), y(T)). \end{aligned} \tag{3.25}$$

We calculate J_2 similarly as

$$\begin{aligned} J_2 &= K\mathrm{E}^Q\left[\mathbf{1}_{\{x(T)>\ln K, y(T)>\ln D\}}\right] \\ &= K\cdot\Phi_2(x(T), y(T)), \end{aligned} \tag{3.26}$$

where $f_2(i\phi_1, i\phi_2)$ is defined by

$$f_2(i\phi_1, i\phi_2) = \mathrm{E}^Q\left[e^{i\phi_1 x(T)+i\phi_2 y(T)}\right] = e^{rT}h(\phi_1, \phi_2, T).$$

To derive J_3, we introduce a new measure Q_3 defined by

$$\frac{dQ_3}{dQ} = \frac{e^{x(T)+y(T)}}{\mathrm{E}^Q\left[e^{x(T)+y(T)}\right]}.$$

Then, under the measure Q_3, the joint characteristic function of $x(T)$ and $y(T)$ is given by

$$f_3(i\phi_1, i\phi_2) = \mathrm{E}^{Q_3}\left[e^{i\phi_1 x(T) - i\phi_2 y(T)}\right] = \mathrm{E}^{Q}\left[\frac{e^{x(T)+y(T)}}{\mathrm{E}^{Q}\left[e^{x(T)+y(T)}\right]} e^{i\phi_1 x(T) - i\phi_2 y(T)}\right]$$
$$= \frac{h(\phi_1 - i, -\phi_2 - i, T)}{h(-i, -i, T)}. \tag{3.27}$$

Using the function $f_3(i\phi_1, i\phi_2)$, we can derive J_3 as follows.

$$\begin{aligned} J_3 &= \frac{1-\alpha}{D}\mathrm{E}^{Q}\left[e^{x(T)+y(T)}\mathbf{1}_{\{x(T)>\ln K, y(T)<\ln D\}}\right] \\ &= \frac{1-\alpha}{D}\mathrm{E}^{Q}\left[e^{x(T)+y(T)}\right]\mathrm{E}^{Q}\left[\frac{dQ_3}{dQ}\mathbf{1}_{\{x(T)>\ln K, y(T)<\ln D\}}\right] \\ &= \frac{1-\alpha}{D}\mathrm{E}^{Q}\left[e^{x(T)+y(T)}\right]\mathrm{E}^{Q_3}\left[\mathbf{1}_{\{x(T)>\ln K, -y(T)>-\ln D\}}\right] \\ &= e^{rT} \times \frac{1-\alpha}{D} \times h(-i, -i, T) \times \Phi_3(x(T), -y(T)). \end{aligned} \tag{3.28}$$

Similarly, we can obtain the form of J_4 as

$$\begin{aligned} J_4 &= \frac{(1-\alpha)}{D}K \cdot \mathrm{E}^{Q}\left[e^{y(T)}\mathbf{1}_{\{x(T)>\ln K, y(T)<\ln D\}}\right], \\ &= \frac{1-\alpha}{D}\mathrm{E}^{Q}\left[e^{y(T)}\right]\mathrm{E}^{Q}\left[\frac{dQ_4}{dQ}\mathbf{1}_{\{x(T)>\ln K, y(T)<\ln D\}}\right] \\ &= \frac{1-\alpha}{D}\mathrm{E}^{Q}\left[e^{y(T)}\right]\mathrm{E}^{Q_4}\left[\mathbf{1}_{\{x(T)>\ln K, -y(T)>-\ln D\}}\right] \\ &= e^{rT} \times \frac{1-\alpha}{D} \times h(0, -i, T) \times \Phi_4(x(T), -y(T)), \end{aligned} \tag{3.29}$$

where

$$\frac{dQ_4}{dQ} = \frac{e^{y(T)}}{\mathrm{E}^{Q}\left[e^{y(T)}\right]}.$$

This completes the derivation of the pricing formula presented in Theorem 2.

□

4. Numerical examples

In this section, we examine the behavior of vulnerable option prices under the proposed stochastic volatility model with stochastic long-term mean. The numerical results demonstrate how various model parameters affect option values and highlight the importance of incorporating both stochastic volatility and stochastic long-term mean in vulnerable option pricing.

Based on the study of He and Chen [4] using the data of the real market, we choose the base case parameters used in our numerical experiments in Table 1. We set both the underlying asset $S(0)$ and the option writer's asset $V(0)$ to 100, with a strike price $K = 100$ and default boundary $D = 80$. The risk-free rate is $r = 0.01$, maturity $T = 0.5$ years, and deadweight cost $\alpha = 0.2$. The volatility parameters are initialized with $v_1(0) = v_2(0) = 0.1$ and long-term means $\theta_1(0) = \theta_2(0) = 0.2$. The mean-reversion speeds are set to $\kappa_1 = \kappa_2 = 5$, indicating relatively fast mean reversion. The correlation parameters $\rho_1 = \rho_2 = 0.1$ and $\rho = -0.05$ capture the interdependencies between the Brownian motions, while the drift and volatility parameters of the long-term mean processes are set to $\lambda_1 = \lambda_2 = 0.1$ and $\gamma_1 = \gamma_2 = 0.01$, respectively.

Table 1. Parameter values of options in the base case.

Parameter	Value	Parameter	Value
$S(0), V(0)$	100	r	0.01
α	0.2	T	0.5
K	100	D	80
ρ_1, ρ_2	0.1	ρ	-0.05
$v_1(0), v_2(0)$	0.1	λ_1, λ_2	0.1
$\theta_1(0), \theta_2(0)$	0.2	σ_1, σ_2	0.1
κ_1, κ_2	5	γ_1, γ_2	0.01

Unless otherwise stated, all numerical results are generated using the analytical pricing formula from Theorem 2. The double integrals within the formula are evaluated using "double quadrature methods", which allowed for efficient and stable computation of the option values. This methodology confirms that our model can be implemented with reasonable computational effort. We carry out Monte Carlo simulation with time step $\Delta t = 1/252$ and 50,000 sample paths using the Euler-Maruyama scheme to verify our analytical pricing formula. The results are presented in Table 2.

Table 2. Monte carlo simulation for option pricing.

Parameter	Value	Analytical	Monte Carlo	Std. Error	Relative Error
T	0.5	10.83	11.05	0.09	0.020
T	1	15.43	15.7	0.14	0.017
T	1.5	18.58	19	0.19	0.022
D	70	11.4	11.55	0.09	0.013
D	80	10.83	11.02	0.09	0.017
D	90	10.12	10.45	0.08	0.032
$V(0)$	70	8.32	8.47	0.07	0.018
$V(0)$	80	9.37	9.58	0.08	0.022
$V(0)$	90	10.21	10.54	0.08	0.032

Figure 1 illustrates the relationship between option prices and time to maturity for different default boundaries ($D = 70, 80, 90$), including no default risk. The results show that option prices increase with time to maturity for all default boundaries, exhibiting a concave shape where the rate of price increase diminishes as maturity extends. Lower default boundaries result in higher option prices, as the probability of default decreases when the option writer's asset has more room to fluctuate before reaching the default threshold. The differences in option prices across various default boundaries become more pronounced for longer maturities, suggesting that the impact of credit risk becomes more significant as the time horizon extends.

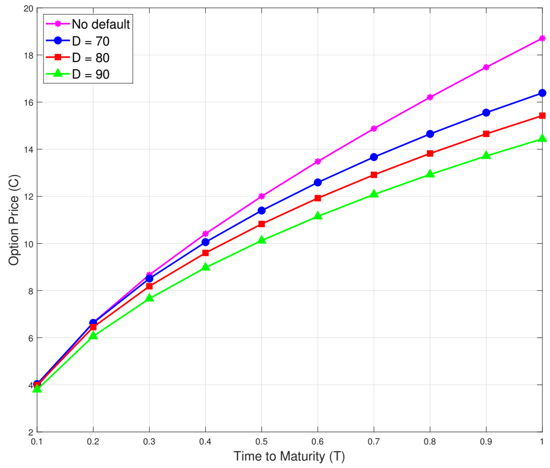


Figure 1. Option prices against the time to maturity with different D.

Figure 2 demonstrates how option prices vary with the initial value of the option writer's asset $V(0)$ for different default boundaries, including no default risk.. The option prices increase consistently with $V(0)$, as higher initial asset values reduce the probability of default. The sensitivity to $V(0)$ is particularly pronounced when $V(0)$ is close to the default boundary, where small changes in the initial asset value can significantly affect the default probability. For $V(0)$ values well above the default boundary, the option prices converge toward the risk-free value, indicating that credit risk becomes negligible when the option writer is financially strong. Lower default boundaries show steeper price increases, particularly in the region near the boundary, reflecting the nonlinear nature of default risk.

The impact of the stochastic long-term mean parameters is examined in Figure 3, which shows the effects of the drift parameters λ_1 and λ_2. Option prices increase linearly with λ_1, ranging from approximately 8 to 12.5 as λ_1 varies from -0.5 to 0.5. This positive relationship reflects the fact that a higher drift in the underlying asset's long-term volatility mean tends to increase the option value through enhanced upside potential. In contrast, option prices decrease with λ_2, falling from about 11.4 to 10.5 over the same range. This negative relationship occurs because a higher drift in the option writer's long-term volatility mean increases the uncertainty about the writer's asset value, thereby raising default risk and reducing the option value. The opposite effects of λ_1 and λ_2 highlight the asymmetric roles of volatility in vulnerable option pricing.

Figure 4 analyzes the sensitivity to initial volatility values $v_1(0)$ and $v_2(0)$. Option prices increase substantially with $v_1(0)$, rising from about 10 to 13 as $v_1(0)$ increases from 0.05 to 0.3. This strong positive relationship reflects the fundamental option pricing principle that higher volatility of the underlying asset increases option value due to the asymmetric payoff structure. The effect of $v_2(0)$ on option prices is much more modest, with prices increasing only from 10.45 to 10.95 over the same range. This weaker sensitivity suggests that when the option writer's asset is sufficiently above the default boundary, variations in its volatility have limited impact on the option value. The contrasting sensitivities to $v_1(0)$ and $v_2(0)$ underscore the different mechanisms through which volatility affects vulnerable option prices.

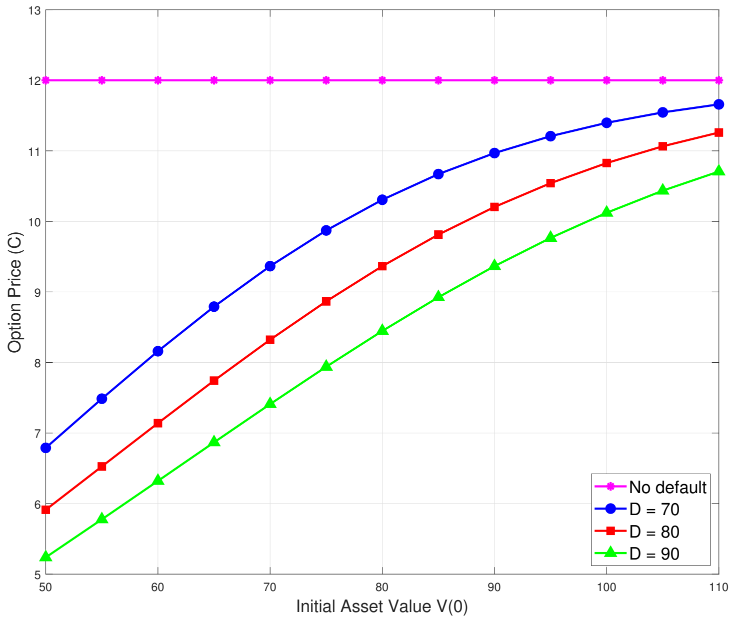
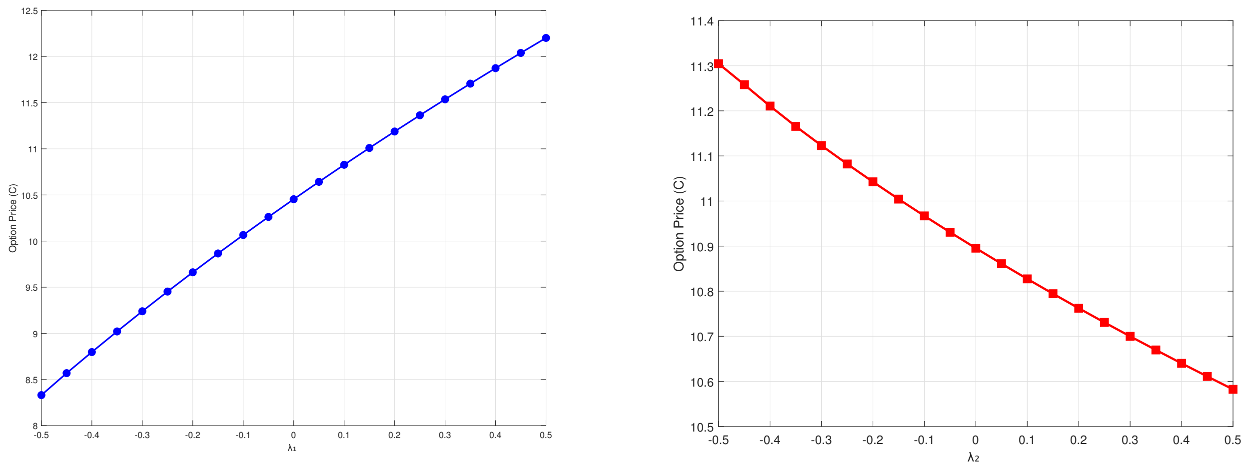


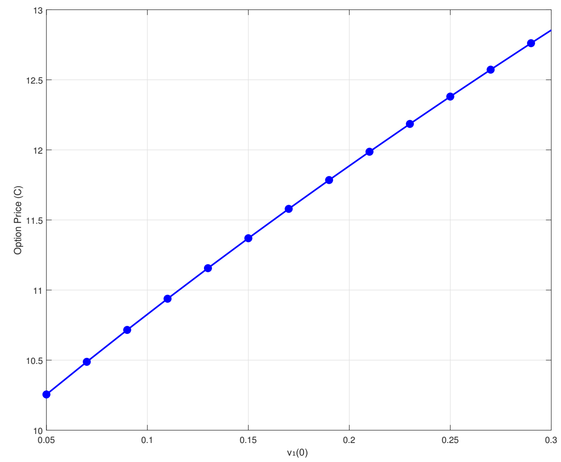
Figure 2. Option prices against the initial values $V(0)$ with different D.



(a) Price of the option varying with λ_1.

(b) Price of the option varying with λ_2.

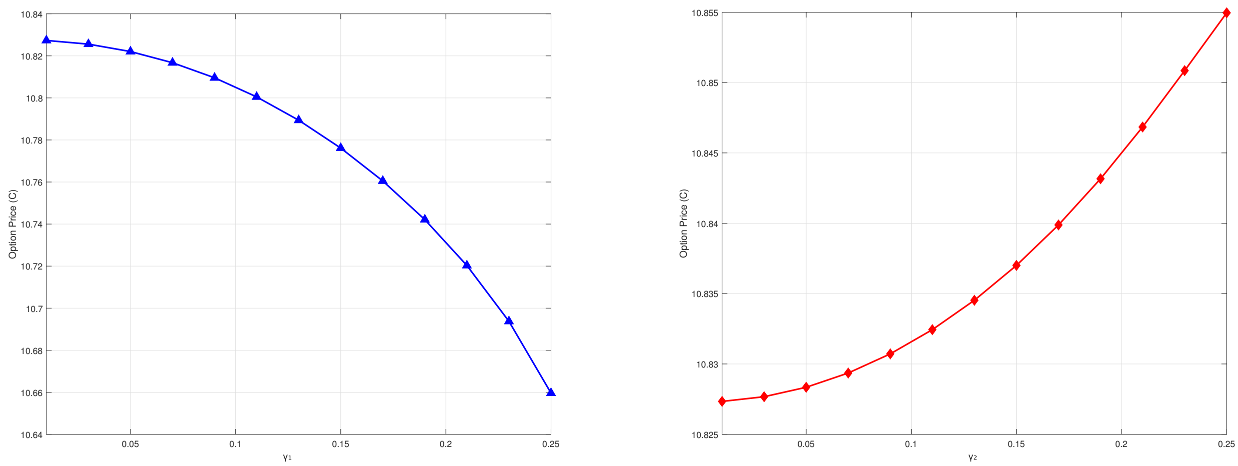
Figure 3. Option prices against the drifts of the long-term mean processes.



(a) Price of the option varying with $v_1(0)$. (b) Price of the option varying with $v_2(0)$.

Figure 4. Option prices against the initial values of the volatility processes.

Figure 5 investigates the effects of the volatility parameters γ_1 and γ_2 of the long-term mean processes. The results show a moderate positive relationship between γ_1 and option prices, with values increasing from 10.64 to 10.84 as γ_1 rises from 0.05 to 0.25. Similarly, γ_2 has a slight positive effect, with prices rising from 10.825 to 10.855. The relatively small sensitivities to these parameters suggest that the volatility of the long-term mean has a second-order effect compared to the level parameters. This finding indicates that while the stochastic nature of the long-term mean is important for model completeness and empirical fit, its volatility has a more subtle impact on option prices compared to other model parameters.



(a) Price of the option varying with γ_1. (b) Price of the option varying with γ_2.

Figure 5. Option prices against the volatilities of the long-term mean processes.

These numerical results provide important insights for practitioners in vulnerable option pricing and risk management. The strong sensitivity to default boundaries emphasizes the critical importance of accurately assessing and monitoring the credit quality of option writers. The asymmetric effects of various volatility parameters highlight the complex interactions between market risk and credit risk in vulnerable options. The results also demonstrate that the proposed model with stochastic long-term mean can capture rich dynamics that simpler models might miss, particularly in scenarios where volatility clustering and time-varying volatility means are relevant. Furthermore, the analysis reveals that parameters affecting the underlying asset generally have stronger impacts on option prices than corresponding parameters for the option writer's asset, except when the writer is close to financial distress.

5. Conclusions

In this paper, we have developed a comprehensive framework for pricing vulnerable options under a stochastic volatility model with stochastic long-term mean. By extending the model of He and Chen [4] to incorporate counterparty default risk, we provide a more realistic and flexible approach to valuing options in the presence of both market risk and credit risk. Our main contributions and findings can be summarized as follows.

We derived closed-form solutions for the characteristic function and explicit pricing formulas for vulnerable European options under the proposed model. The incorporation of stochastic long-term mean in the volatility dynamics allows our model to better capture empirical features of financial markets, including the term structure of implied volatility and time-varying volatility clustering. The analytical tractability of our solutions makes the model practical for real-world applications, avoiding the computational burden often associated with numerical methods.

Our numerical analysis reveals several important insights into the behavior of vulnerable option prices. First, the default boundary has a crucial role in determining option values, with lower boundaries leading to significantly higher prices due to reduced default risk. This finding emphasizes the importance of careful credit assessment and monitoring in vulnerable option pricing. Second, we observe asymmetric effects of volatility parameters, where parameters related to the underlying asset generally have stronger impacts than those related to the option writer's asset. This asymmetry becomes less pronounced when the writer's asset approaches the default boundary, highlighting the nonlinear nature of credit risk.

The incorporation of stochastic long-term mean parameters introduces additional flexibility in capturing market dynamics. We find that the drift parameters λ_1 and λ_2 have opposite effects on option prices, reflecting their different roles in the pricing. While λ_1 increases option values through enhanced volatility of the underlying asset, λ_2 decreases values by increasing uncertainty about the writer's creditworthiness. In addition, the volatility parameters of the long-term mean processes (γ_1 and γ_2) have more modest effects.

This research significantly advances the analytical framework for valuing vulnerable options, particularly in environments characterized by stochastic volatility and a stochastic long-term mean. The explicit pricing formula offers a computationally efficient method for financial practitioners to assess counterparty credit risk embedded in derivative contracts. Future research could explore extensions to include other complexities such as jump-diffusion processes, stochastic interest rates, or liquidity risk.

Authors contributions

So-Yoon Cho: Writing – original draft, Software, Methodology, Conceptualization. Geonwoo Kim: Writing – review & editing, Supervision, Funding acquisition, Conceptualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

Authors declare that they have no conflicts of interest.

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