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Research article

Endpoint Sobolev regularity of bilinear maximal commutators

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Abstract: In this paper our objective of investigation was the endpoint Sobolev regularity of the bilinear maximal commutator

$$\mathfrak{M}_{b,\alpha}(f,g)(x) = \sup_{r>0} \frac{1}{(2r)^{1-\alpha}} \int_{-r}^{r} |(b(x) - b(x+y))f(x+y)g(x-y)|dy,$$

where $\alpha \in [0, 1)$ and $b \in Lip(\mathbb{R})$ with $b' \in L^1(\mathbb{R})$. We showed that the map $\mathfrak{M}_{b,\alpha} : W^{1,1}(\mathbb{R}) \times W^{1,1}(\mathbb{R}) \to W^{1,q}(\mathbb{R})$ was bounded and continuous for $q \in (\frac{1}{1-\alpha}, \infty)$. The main result essentially answered a question motivated by Wang and Liu in 2022.

Keywords: bilinear maximal commutator; endpoint Sobolev space; boundedness; continuity
Mathematics Subject Classification: 42B25, 46E35

1. Introduction

Recently, the authors [26] established the boundedness of the following bilinear maximal commutator

$$\mathfrak{M}_{b,\alpha}(f,g)(x) = \sup_{r>0} \frac{1}{(2r)^{1-\alpha}} \int_{-r}^{r} |(b(x) - b(x+y))f(x+y)g(x-y)|dy,$$

on $W^{1,p_1}(\mathbb{R}) \times W^{1,p_2}(\mathbb{R})$ for $1 < p_1, p_2 < \infty$ where α, p_1, p_2, and b satisfy certain conditions. It is natural to wonder the differentiable behavior of $\mathfrak{M}_{b,\alpha}$ acting on a vector-valued function (f, g) with $f \in W^{1,1}(\mathbb{R})$ and $g \in W^{1,1}(\mathbb{R})$. This is the main motivation of this paper.

The study of regularity theory for maximal operators has become a focal point in numerous recent publications within the field of harmonic analysis. Kinnunen [12] first proved that the usual centered Hardy–Littlewood maximal operator M is bounded on the first order Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ for all $1 < p \le \infty$. This foundational result has been broadened to include various modifications of the maximal operator (see [7, 13, 14]). It is worth noting that the derivative of a maximal function does not inherently possess sublinearity. The continuity of $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ for $1 < p < \infty$ is indeed an intriguing issue. Luiro [20] affirmatively tackled this question, and it was later extended to the local version in [21] and the bilinear version in [7]. Owing to the absence of L^1-boundedness for M, the $W^{1,1}$-regularity for the maximal operator is a highly nontrivial issue. A pivotal question was raised by Hajłasz and Onninen in [11]:

Question 1. Is the map $f \mapsto |\nabla Mf|$ bounded from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$?

The above question was initially examined by Tanaka [25] who first considered the endpoint Sobolev regularity of the one-dimensional uncentered Hardy–Littlewood maximal operator $\widetilde{M}$ and showed that if $f \in W^{1,1}(\mathbb{R})$, then $\widetilde{M}f$ is weakly differentiable and

$$\|(\widetilde{M}f)'\|_{L^1(\mathbb{R})} \le 2\|f'\|_{L^1(\mathbb{R})}. \tag{1.1}$$

The constant $C = 2$ in (1.1) was improved by Aldaz and Pérez Lázaro [1] to the sharp constant $C = 1$. Later on, Kurka [15] established that if $f \in W^{1,1}(\mathbb{R})$, then inequality (1.1) holds for M (with constant $C = 240,004$). Based on the above bounds, Carneiro, Madrid, and Pierce [5] (resp., González-Riquelme [10]) proved that the map $f \mapsto (\widetilde{M}f)'$ (resp., $f \mapsto (Mf)'$) is continuous from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$, respectively. The aforementioned findings have been recently expanded to the fractional variants (see [3, 6, 24]). The higher dimensional $W^{1,1}$-regularity of the Hardy–Littlewood maximal operator and fractional maximal operator can be found in [2–4, 16, 22, 27].

On the other hand, the investigation on the regularity issues of maxmial commutators has similarly garnered considerable attention from numerous scholars (see [8, 9, 16–19]). Particularly, Chen and Liu [8, 9] studied the endpoint Sobolev regularity of the one dimensional maximal commutator and its fractional variant

$$M_{b,\alpha}f(x) = \sup_{r>0} \frac{1}{(2r)^{1-\alpha}} \int_{x-r}^{x+r} |(b(x) - b(y))f(y)|dy,$$

where $0 \le \alpha < 1$, $b \in L^1_{\text{loc}}(\mathbb{R})$, and $f \in L^1_{\text{loc}}(\mathbb{R})$. The main results of [8, 9] can be formulated as follows.
Theorem A. ([8, 9]) *Let $\alpha \in [0, 1)$, $q \in (1, \infty)$ and, $b \in W^{1,1}(\mathbb{R})$ with $b' \in L^\infty(\mathbb{R})$. Then, the map $f \mapsto (M_{b,\alpha}f)'$ is bounded and continuous from $W^{1,1}(\mathbb{R})$ to $L^q(\mathbb{R})$. Particularly, if $f \in W^{1,1}(\mathbb{R})$, then $M_{b,\alpha}f$ is differentiable almost everywhere in $\mathbb{R}$. Moreover,*

$$\|(M_{b,\alpha}f)'\|_{L^q(\mathbb{R})} \le C_b\|f\|_{W^{1,1}(\mathbb{R})}.$$

Later on, Liu and Ma [16] improved Theorem A by weakening the condition of b. Let us recall one definition. We denote by $Lip(\mathbb{R})$ the *homogeneous* Lipschitz space, i.e.,

$$Lip(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \text{ continuous} : \ \|f\|_{Lip(\mathbb{R})} < \infty\},$$

where

$$\|f\|_{Lip(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R},\\ h\in\mathbb{R}\setminus\{0\}}} \frac{|f(x+h) - f(x)|}{|h|} < \infty.$$

The *inhomogeneous* Lipschitz space $\mathrm{Lip}(\mathbb{R})$ is given by

$$\mathrm{Lip}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \text{ continuous} : \ \|f\|_{\mathrm{Lip}(\mathbb{R})} := \|f\|_{Lip(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})} < \infty\}.$$

The improvement of Theorem A can be enumerated as follows.

Theorem B. ([16]) *Let $q \in (1,\infty)$, $\alpha \in [0,1)$, $b \in Lip(\mathbb{R})$, and $b' \in L^1(\mathbb{R})$. Then, the map $f \mapsto (M_{b,\alpha}f)'$ is bounded and continuous from $W^{1,1}(\mathbb{R})$ to $L^q(\mathbb{R})$. Particularly, if $f \in W^{1,1}(\mathbb{R})$, then $\mathfrak{M}_{b,\alpha}f \in Lip(\mathbb{R})$. Moreover,*

$$\|(M_{b,\alpha}f)'\|_{L^q(\mathbb{R})} \leq C_b\|f\|_{W^{1,1}(\mathbb{R})}.$$

Remark 2. Let

$$\mathfrak{F}_1 := \{f \in W^{1,1}(\mathbb{R}) : f' \in L^\infty(\mathbb{R})\}, \quad \mathfrak{F}_2 := \{f \in Lip(\mathbb{R}) : f' \in L^1(\mathbb{R})\}.$$

It was noted in [16] that $\mathfrak{F}_1 \subsetneq \mathfrak{F}_2$, which is a proper inclusion. We also point out that if $b \in Lip(\mathbb{R})$, then the derivative b' exists almost everywhere. Moreover, we have that $b'(x) = \lim_{h\to 0}\frac{b(x+h)-b(x)}{h}$ and $|b'(x)| \leq \|b\|_{Lip(\mathbb{R})}$ for almost every $x \in \mathbb{R}$. It follows that $\|b'\|_{L^\infty(\mathbb{R})} \leq \|b\|_{Lip(\mathbb{R})}$. Particularly, if $b \in Lip(\mathbb{R})$ and $b' \in L^1(\mathbb{R})$, then $|b(x) - b(y)| \leq \|b'\|_{L^1(\mathbb{R})}$ for any $x, y \in \mathbb{R}$.

In this paper we focus on the endpoint Sobolev regularity of bilinear maximal commutator $\mathfrak{M}_{b,\alpha}$. This type of commutator was original introduced by Wang and Liu [26], who established the following result.

Theorem C. ([26]) *Let $1 < p_1, p_2, p_1p_2/(p_1+p_2) < \infty$, $0 \leq \alpha < 1/p_1 + 1/p_2$, $1/q = 1/p_1 + 1/p_2 - \alpha$, and $b \in \mathrm{Lip}(\mathbb{R})$. If $f \in W^{1,p_1}(\mathbb{R})$ and $g \in W^{1,p_2}(\mathbb{R})$, then we have*

$$\|\mathfrak{M}_{b,\alpha}(f,g)\|_{W^{1,q}(\mathbb{R})} \leq C_{\alpha,p_1,p_2}\|b\|_{\mathrm{Lip}(\mathbb{R})}\|f\|_{W^{1,p_1}(\mathbb{R})}\|g\|_{W^{1,p_2}(\mathbb{R})}.$$

Based on the above, it is interesting to ask the following question.

Question 3. Let $0 \leq \alpha < 1$. Is $\mathfrak{M}_{b,\alpha}$ bounded and continuous from $W^{1,1}(\mathbb{R}) \times W^{1,1}(\mathbb{R})$ to $W^{1,q}(\mathbb{R})$ for some $q \in (1,\infty)$ if $b \in Lip(\mathbb{R})$ with $b' \in L^1(\mathbb{R})$?

This question can be addressed by the following result.

Theorem 1. *Let $\alpha \in [0,1)$, $q \in (\frac{1}{1-\alpha},\infty)$ and $b \in Lip(\mathbb{R})$ with $b' \in L^1(\mathbb{R})$. Then $\mathfrak{M}_{b,\alpha}$ is bounded and continuous from $W^{1,1}(\mathbb{R}) \times W^{1,1}(\mathbb{R})$ to $L^q(\mathbb{R})$. Particularly, if $f \in W^{1,1}(\mathbb{R})$ and $g \in W^{1,1}(\mathbb{R})$, then $\mathfrak{M}_{b,\alpha}(f,g)$ is differentiable almost everywhere in $\mathbb{R}$ and*

$$\|(\mathfrak{M}_{b,\alpha}(f,g))'\|_{L^q(\mathbb{R})} \leq C_{\alpha,q}(\|b\|_{Lip(\mathbb{R})}^{1-1/q-\alpha}\|b'\|_{L^1(\mathbb{R})}^{1/q+\alpha}\|f'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})} + \|b\|_{Lip(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}\|f\|_{W^{1,1}(\mathbb{R})}).$$

Remark 4. It is worth noting that the conclusions of Theorem 1 also hold for the uncentered bilinear maximal commutator

$$\widetilde{\mathfrak{M}}_{b,\alpha}(f,g)(x) = \sup_{\substack{r,s\geq 0,\\ r+s>0}} \frac{1}{(r+s)^{1-\alpha}}\int_{-r}^{s}|(b(x)-b(x+y))f(x+y)g(x-y)|dy, \quad x \in \mathbb{R}.$$

More precisely, it can be proved that $\widetilde{\mathfrak{M}}_{b,\alpha}$ is bounded and continuous from $W^{1,1}(\mathbb{R}) \times W^{1,1}(\mathbb{R})$ to $L^q(\mathbb{R})$ if $\alpha \in [0,1)$, $q \in (\frac{1}{1-\alpha},\infty)$, and $b \in Lip(\mathbb{R})$ with $b' \in L^1(\mathbb{R})$.

This paper is organized as follows. In Section 2 we shall establish some preliminary lemmas, which contains some formulas and pointwise convergence of the derivatives of bilinear maximal commutators (see Lemmas 4 and 5). These are the main ingredients of proving Theorem 1. The proof of Theorem 1 will be given in Section 3. It should be pointed out that the methods used to prove the main theorem are motivated by [5, 8, 9].

Throughout the paper, the letter C, which may be accompanied by specific parameters, denotes positive constants not necessarily the same one at each occurrence, but are independent of the essential variables. For a set $A \subset \mathbb{R}$, the notation $|A| = 0$ means that A is a set of measure zero. For any function $f : \mathbb{R} \to \mathbb{R}$ and $h \in \mathbb{R}$, we define

$$d_h f(x) = \frac{f_{\tau(h)}(x) - f(x)}{h} \quad \text{and} \quad f_{\tau(h)}(x) = f(x+h).$$

For any arbitrary functions $F(x, y)$ defined on $\mathbb{R} \times \mathbb{R}$, we denote by $D_x F$ (resp., $D_y F$) as the partial derivative of F in x (resp., y).

2. Preliminaries

In this section we shall establish some lemmas, which are the main ingredients of proving Theorem 1. Let us begin with some properties of a $W^{1,1}(\mathbb{R})$ function.

Lemma 2. *Let $f \in W^{1,1}(\mathbb{R})$. Then,*

(i) $\sup_{x\in\mathbb{R}} |f(x)| \le \|f'\|_{L^1(\mathbb{R})}$.

(ii) $\|f\|_{Lip(\mathbb{R})} \le \|f'\|_{L^\infty(\mathbb{R})}$.

(iii) $d_h f \to f'$ *in* $L^1(\mathbb{R})$ *as* $h \to 0$.

(iv) *Let* $0 \le \alpha < 1$. *Then, the fractional maximal function*

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{(2r)^{1-\alpha}} \int_{x-r}^{x+r} |f(y)|dy,$$

satisfies the estimate $\sup_{x\in\mathbb{R}} M_\alpha f(x) \le \|f\|_{W^{1,1}(\mathbb{R})}$.

(v) *Let* $\{f_j\}_{j\ge1} \subset W^{1,1}(\mathbb{R})$ *be such that* $\|f_j - f\|_{W^{1,1}(\mathbb{R})} \to 0$ *as* $j \to \infty$. *Then,* $\||f_j| - |f|\|_{W^{1,1}(\mathbb{R})} \to 0$ *as* $j \to \infty$.

(vi) *Let* $\alpha \in [0, 1)$ *and* $b \in Lip(\mathbb{R})$ *with* $b' \in L^1(\mathbb{R})$. *Let* $\{f_j\}_{j\ge1} \subset W^{1,1}(\mathbb{R})$ *be such that* $\|f_j - f\|_{W^{1,1}(\mathbb{R})} \to 0$ *as* $j \to \infty$. *Let* $g \in W^{1,1}(\mathbb{R})$ *and* $\{g_j\}_{j\ge1} \subset W^{1,1}(\mathbb{R})$ *be such that* $g_j \to g$ *in* $W^{1,1}(\mathbb{R})$ *as* $j \to \infty$. *Then* $\mathfrak{M}_{b,\alpha}(f_j, g_j)$ *converges uniformly to* $\mathfrak{M}_{b,\alpha}(f, g)$ *on* $\mathbb{R}$.

Proof. Parts (i)–(iii) were shown in [8, Lemma 2.2]. Part (iv) follows from Remark 1.1 in [9, Remark 1.1]. Part (v) follows from [5]. Part (vi) follows from the following inequality:

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} |\mathfrak{M}_{b,\alpha}(f_j, g_j)(x) - \mathfrak{M}_{b,\alpha}(f, g)(x)| \\
&\le \sup_{x\in\mathbb{R}} (\mathfrak{M}_{b,\alpha}(f_j - f, g_j)(x) + \mathfrak{M}_{b,\alpha}(f, g_j - g)(x)) \\
&\le 2\|b'\|_{L^1(\mathbb{R})} \sup_{x\in\mathbb{R}} (\|g_j\|_{L^\infty(\mathbb{R})} M_\alpha(f_j - f)(x) + \|f\|_{L^\infty(\mathbb{R})} M_\alpha(g_j - g)(x)) \\
&\le 2\|b'\|_{L^1(\mathbb{R})} (\|g_j'\|_{L^1(\mathbb{R})} \|f_j - f\|_{W^{1,1}(\mathbb{R})} + \|f'\|_{L^1(\mathbb{R})} \|g_j - g\|_{W^{1,1}(\mathbb{R})}) \to 0 \ \text{as} \ j \to \infty.
\end{aligned}$$

□

Let $\alpha \in [0,1)$, $b \in Lip(\mathbb{R})$ and $b' \in L^1(\mathbb{R})$, $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. For any $x \in \mathbb{R}$, we define the function $A_{x,b,f,g} : [0,\infty) \to \mathbb{R}$ by

$$A_{x,b,f,g}(r) = \begin{cases} 0, & \text{if } r = 0; \\ \dfrac{1}{(2r)^{1-\alpha}} \displaystyle\int_{-r}^{r} |(b(x) - b(x+y))||f(x+y)g(x-y)|dy, & \text{if } r > 0. \end{cases}$$

Given a point $x \in \mathbb{R}$, we define the family of good radii for a pair (f,g) at x as

$$\mathcal{R}_\alpha(f,g)(x) = \{r \geq 0 : \mathfrak{M}_{b,\alpha}(f,g)(x) = A_{x,b,f,g}(r)\}.$$

Observe that for any $x \in \mathbb{R}$, the function $A_{x,b,f,g}$ is continuous on $[0,\infty)$. In addition, we get by Remark 2 that

$$A_{x,b,f,g}(r) \leq (2r)^{\alpha-1}\|b'\|_{L^1(\mathbb{R})}\|f\|_{L^1(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } r \to \infty. \tag{2.1}$$

It follows that for every $x \in \mathbb{R}$, the function $A_{x,b,f,g}$ has at least one maximum point in $[0,\infty)$. Consequently, the set $\mathcal{R}_\alpha(f,g)(x)$ is nonempty for every $x \in \mathbb{R}$.

Lemma 3. *Let $\alpha \in [0,1)$ and $b \in Lip(\mathbb{R})$ with $b' \in L^1(\mathbb{R})$. Assume that $f \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. Then,*

(i) *For any $x \in \mathbb{R}$ for which $\mathfrak{M}_{b,\alpha}(f,g)(x) > 0$, we have* $\inf \mathcal{R}_\alpha(f,g)(x) > 0$ *and* $\sup \mathcal{R}_\alpha(f,g)(x) < \infty$.

(ii) *Let $\{f_j\}_{j\geq1} \subset W^{1,1}(\mathbb{R})$ and $\{g_j\}_{j\geq1} \subset W^{1,1}(\mathbb{R})$. Assume that $f_j \to f$ in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $g_j \to g$ in $L^\infty(\mathbb{R})$ as $j \to \infty$. For any fixed $x \in \mathbb{R}$, let $r_j \in \mathcal{R}_\alpha(f_j,g_j)(x)$ for $j \geq 1$. If r is an accumulation point of $\{r_j\}_{j\geq1}$, then $r \in \mathcal{R}_\alpha(f,g)(x)$.*

Proof. At first we prove part (i). Let $x \in \mathbb{R}$ for which $\mathfrak{M}_{b,\alpha}(f,g)(x) > 0$. If $\inf \mathcal{R}_\alpha(f,g)(x) = 0$, then there exists $\{r_k\}_{k\geq1} \subset \mathcal{R}_\alpha(f,g)(x) \cap (0,\infty)$ such that $\lim_{k\to\infty} r_k = 0$. Hence, we have

$$A_{x,b,f,g}(r_k) \leq 2^\alpha r_k^{\alpha+1}\|b\|_{Lip(\mathbb{R})}\|f\|_{L^\infty(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } k \to \infty.$$

This implies $\mathfrak{M}_{b,\alpha}(f,g)(x) = 0$, which leads to a contradiction. So, $\inf \mathcal{R}_\alpha(f,g)(x) > 0$. The claim $\sup \mathcal{R}_\alpha(f,g)(x) < \infty$ follows by (2.1).

Next, we prove part (ii). We may suppose, without loss of generality, that $r_j \to r$ as $j \to \infty$. Two cases will be examined:

Case 1 ($r = 0$). To prove $r \in \mathcal{R}_\alpha(f,g)(x)$, it suffices to show that $\mathfrak{M}_{b,\alpha}(f,g)(x) = 0$. If there exists $N_0 \in \mathbb{N}$ such that $r_j = 0$ for any $j \geq N_0$, then $\mathfrak{M}_{b,\alpha}(f_j,g_j)(x) = 0$ for all $j \geq N_0$. This together with Lemma 2(vi) implies $\mathfrak{M}_{b,\alpha}(f,g)(x) = 0$. If there exists a subsequence $\{j_k\}_{k\geq1} \subset \{j\}_{j\geq1}$ such that $r_{j_k} > 0$, then

$$A_{x,b,f_{j_k},g_{j_k}}(r_k) \leq 2^\alpha r_{j_k}^{\alpha+1}\|b\|_{Lip(\mathbb{R})}\|f_{j_k}\|_{L^\infty(\mathbb{R})}\|g_{j_k}\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } k \to \infty.$$

This together with Lemma 2(vi) implies that

$$\mathfrak{M}_{b,\alpha}(f,g)(x) = \lim_{k\to\infty} \mathfrak{M}_{b,\alpha}(f_{j_k},g_{j_k})(x) = \lim_{k\to\infty} A_{x,b,f_{j_k},g_{j_k}}(r_k) = 0.$$

Case 2 ($r > 0$). We may assume, without loss of generality, that all $r_j > 0$. By Lemma 2 and

Remark 2, one has

$$\begin{aligned}
&\Big|\int_{-r_j}^{r_j}|(b(x)-b(x+y))f_j(x+y)g_j(x-y)|dy\\
&\quad-\int_{-r}^{r}|(b(x)-b(x+y))f(x+y)g(x-y)|dy\Big|\\
&\le\int_{-r_j}^{r_j}|(b(x)-b(x+y))||f_j(x+y)g_j(x-y)|-|f(x+y)g(x-y)||dy\\
&\quad+\int_{\mathbb{R}}|(b(x)-b(x+y))||f(x+y)g(x-y)||\chi_{[-r_j,r_j]}(y)-\chi_{[-r,r]}(y)|dy\\
&\le\|b'\|_{L^1(\mathbb{R})}\int_{-r_j}^{r_j}|f_j(x+y)-f(x+y)||g_j|(x-y)dy\\
&\quad+\|b'\|_{L^1(\mathbb{R})}\int_{-r_j}^{r_j}|f|(x+y)|g_j(x-y)-g(x-y)|dy+2\|b'\|_{L^1(\mathbb{R})}\|f\|_{L^\infty(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})}|r_j-r|\\
&\le\|b'\|_{L^1(\mathbb{R})}(\|g_j\|_{L^\infty(\mathbb{R})}\|f_j-f\|_{L^1(\mathbb{R})}+\|f\|_{L^\infty(\mathbb{R})}\|g_j-g\|_{L^\infty(\mathbb{R})})\\
&\quad+2\|b'\|_{L^1(\mathbb{R})}\|f\|_{L^\infty(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})}|r_j-r|\to 0\ \text{ as }\ j\to\infty.
\end{aligned}$$

This together with Lemma 2(vi) yields that

$$\mathfrak{M}_{b,\alpha}(f,g)(x)=\lim_{j\to\infty}\mathfrak{M}_{b,\alpha}(f_j,g_j)(x)=\lim_{j\to\infty}A_{x,b,f_j,g_j}(r_j)=A_{x,b,f,g}(r),$$

which leads to $r\in\mathcal{R}_\alpha(f,g)(x)$. □

The following lemma presents the differentiability and derivative formulas of bilinear maximal commutator.

Lemma 4. *Let $\alpha\in[0,1)$, $b\in Lip(\mathbb{R})$ with $b'\in L^1(\mathbb{R})$, $f\in W^{1,1}(\mathbb{R})$, and $g\in W^{1,1}(\mathbb{R})$. Then, $\mathfrak{M}_{b,\alpha}(f,g)\in Lip(\mathbb{R})$. Let $E=\{x\in\mathbb{R}:\mathfrak{M}_{b,\alpha}(f,g)(x)>0\}$. Then, we have*

(a) *Let $x\in\mathbb{R}\setminus E$ for which $\mathfrak{M}_{b,\alpha}(f,g)$ is differentiable at x. Then*

$$(\mathfrak{M}_{b,\alpha}(f,g))'(x)=0. \tag{2.2}$$

(b) *For almost every $x\in E$ for which $\mathfrak{M}_{b,\alpha}(f,g)$ is differentiable at x, we have that if $r\in\mathcal{R}_\alpha(f,g)(x)$ and $r>0$, then*

$$\begin{aligned}(\mathfrak{M}_{b,\alpha}(f,g))'(x)=&\frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}|b(x)-b(x+y)|(|f|'(x+y)|g|(x-y)+|f|(x+y)|g|'(x-y))dy\\
&+\frac{1}{(2r)^{1-\alpha}}\int_{x-r}^{x+r}(D_x|b(x)-b(y)|+D_y|b(x)-b(y)|)|f(y)g(2x-y)|dy.\end{aligned}\tag{2.3}$$

Proof. Let x, $h\in\mathbb{R}$. Observe that

$$\begin{aligned}
&|\mathfrak{M}_{b,\alpha}(f,g)(x+h)-\mathfrak{M}_{b,\alpha}(f,g)(x)|\\
&\le\sup_{r>0}\frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}|(b(x+h)-b(x+h+y))f(x+h+y)g(x+h-y)\\
&\quad-(b(x)-b(x+y))f(x+y)g(x-y)|dy\\
&\le 2\|b\|_{Lip(\mathbb{R})}M_\alpha f(x+h)\|g\|_{L^\infty(\mathbb{R})}|h|\\
&\quad+\|g\|_{L^\infty(\mathbb{R})}\sup_{r>0}\frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}|b(x)-b(x+y)||f(x+y+h)-f(x+y)|dy\\
&\quad+\|f\|_{L^\infty(\mathbb{R})}\sup_{r>0}\frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}|b(x)-b(x+y)||g(x-y+h)-g(x-y)|dy.
\end{aligned}$$

By Remark 2, one gets

$$\begin{aligned}
&\sup_{r>0} \frac{1}{(2r)^{1-\alpha}} \int_{-r}^{r} |b(x) - b(x+y)||f(x+y+h) - f(x+y)|dy \\
&= \sup_{r>0} \frac{1}{(2r)^{1-\alpha}} \int_{-r}^{r} |b(x) - b(x+y)||d_h f(x+y)|dy|h| \\
&\leq \sup_{r>1/2} \frac{1}{(2r)^{1-\alpha}} \int_{-r}^{r} |b(x) - b(x+y)||d_h f(x+y)|dy|h| \\
&\quad + \sup_{0<r\leq 1/2} \frac{1}{(2r)^{1-\alpha}} \int_{-r}^{r} |b(x) - b(x+y)||d_h f(x+y)|dy|h| \\
&\leq (\|b'\|_{L^1(\mathbb{R})} + \|b\|_{Lip(\mathbb{R})})\|d_h f\|_{L^1(\mathbb{R})}|h|.
\end{aligned}$$

Similarly, we obtain

$$\sup_{r>0} \frac{1}{(2r)^{1-\alpha}} \int_{-r}^{r} |b(x) - b(x+y)||g(x-y+h) - g(x-y)|dy \leq (\|b'\|_{L^1(\mathbb{R})} + \|b\|_{Lip(\mathbb{R})})\|d_h g\|_{L^1(\mathbb{R})}|h|.$$

These estimates together with Lemma 2(iv) imply that

$$\begin{aligned}
&|\mathfrak{M}_{b,\alpha}(f,g)(x+h) - \mathfrak{M}_{b,\alpha}(f,g)(x)| \\
&\leq \|b\|_{Lip(\mathbb{R})}\|f\|_{W^{1,1}(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}|h| + (\|b'\|_{L^1(\mathbb{R})} + \|b\|_{Lip(\mathbb{R})}(\|g'\|_{L^1(\mathbb{R})}\|d_h f\|_{L^1(\mathbb{R})} + \|f'\|_{L^1(\mathbb{R})}\|d_h g\|_{L^1(\mathbb{R})})|h|.
\end{aligned}$$

By Lemma 2(iii), we observe that $\|d_h f - f'\|_{L^1(\mathbb{R})} \to 0$ and $\|d_h g - g'\|_{L^1(\mathbb{R})} \to 0$ as $h \to 0$. Consequently, there exists $\delta > 0$ such that for any $|h| < \delta$,

$$\|d_h f\|_{L^1(\mathbb{R})} \leq \|d_h f - f'\|_{L^1(\mathbb{R})} + \|f'\|_{L^1(\mathbb{R})} \leq 1 + \|f'\|_{L^1(\mathbb{R})},$$

$$\|d_h g\|_{L^1(\mathbb{R})} \leq \|d_h g - g'\|_{L^1(\mathbb{R})} + \|g'\|_{L^1(\mathbb{R})} \leq 1 + \|g'\|_{L^1(\mathbb{R})}.$$

Furthermore, for any $z \in \mathbb{R}$,

$$\mathfrak{M}_{b,\alpha}(f,g)(z) \leq \|b'\|_{L^1(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} M_\alpha f(x) \leq \|b'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}\|f\|_{W^{1,1}(\mathbb{R})}.$$

Hence, we have

$$\begin{aligned}
\|\mathfrak{M}_{b,\alpha}(f,g)\|_{Lip(\mathbb{R})} \leq\ & \|b\|_{Lip(\mathbb{R})}\|f\|_{W^{1,1}(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})} + 2\|b'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}\|f\|_{W^{1,1}(\mathbb{R})}\delta^{-1} \\
&+(\|b'\|_{L^1(\mathbb{R})} + \|b\|_{Lip}(\mathbb{R}))(\|g'\|_{L^1(\mathbb{R})}(1 + \|f'\|_{L^1(\mathbb{R})}) + \|f'\|_{L^1(\mathbb{R})}(1 + \|g'\|_{L^1(\mathbb{R})})).
\end{aligned}$$

This yields $\mathfrak{M}_{b,\alpha}(f,g) \in Lip(\mathbb{R})$.

Let $x \in \mathbb{R} \setminus E$ for which $\mathfrak{M}_{b,\alpha}(f,g)$ is differentiable at x. Observe that

$$(\mathfrak{M}_{b,\alpha}(f,g))'(x) = \lim_{h\to 0} \frac{\mathfrak{M}_{b,\alpha}(f,g)(x+h) - \mathfrak{M}_{b,\alpha}(f,g)(x)}{h} = \lim_{h\to 0} \frac{\mathfrak{M}_{b,\alpha}(f,g)(x+h)}{h}.$$

Then, we have

$$0 \leq \lim_{h\to 0^+} \frac{\mathfrak{M}_{b,\alpha}(f,g)(x+h)}{h} = (\mathfrak{M}_{b,\alpha}(f,g))'(x) = \lim_{h\to 0^-} \frac{\mathfrak{M}_{b,\alpha}(f,g)(x+h)}{h} \leq 0.$$

This yields (2.2).

Next we prove (2.3). Without loss of generality, we may assume that $f,\ g \geq 0$. Let F be the set of all $x \in \mathbb{R}$ for which $\mathfrak{M}_{b,\alpha}(f,g)$ is differentiable at x. By part (i), it has been observed that $|\mathbb{R} \setminus F| = 0$. Let $x \in E \cap F$. Write

$$(\mathfrak{M}_{b,\alpha}(f,g))'(x) = \lim_{h\to 0} \frac{\mathfrak{M}_{b,\alpha}(f,g)(x+h) - \mathfrak{M}_{b,\alpha}(f,g)(x)}{h}. \tag{2.4}$$

Let $r \in \mathcal{R}_\alpha(f,g)(x)$. Note that $r > 0$. The application of a change of variable yields that $\mathfrak{M}_{b,\alpha}(f,g)(x-h) = \mathfrak{M}_{b_{\tau(-h)},\alpha}(f_{\tau(-h)}, g_{\tau(-h)})(x)$. Subsequently, we obtain the following:

$$\begin{aligned}
&\mathfrak{M}_{b,\alpha}(f,g)(x) - \mathfrak{M}_{b,\alpha}(f,g)(x-h)\\
&= \mathfrak{M}_{b,\alpha}(f,g)(x) - \mathfrak{M}_{b_{\tau(-h)},\alpha}(f_{\tau(-h)}, g_{\tau(-h)})(x)\\
&\leq A_{x,b,f,g}(r) - A_{x,b_{\tau(-h)},f_{\tau(-h)},g_{\tau(-h)}}(r)\\
&\leq \frac{1}{(2r)^{1-\alpha}} \int_{-r}^{r} |b(x)-b(x+y)|(f(x+y)g(x-y) - f_{\tau(-h)}(x+y)g_{\tau(-h)}(x-y))dy\\
&\quad + \frac{1}{(2r)^{1-\alpha}} \int_{-r}^{r} (|b(x)-b(x+y)| - |b(x-h)-b(x+y-h)|) f_{\tau(-h)}(x+y)g_{\tau(-h)}(x-y)dy
\end{aligned} \tag{2.5}$$

for all $h > 0$. Note that

$$\frac{f(x+y)g(x-y) - f_{\tau(-h)}(x+y)g_{\tau(-h)}(x-y)}{h} = f_{-h}(x+y)g(x-y) + f_{\tau(-h)}(x+y)g_{-h}(x-y).$$

Consequently, for any $h > 0$,

$$\begin{aligned}
&\frac{1}{h}\int_{-r}^{r} |b(x)-b(x+y)|(f(x+y)g(x-y) - f_{\tau(-h)}(x+y)g_{\tau(-h)}(x-y))dy\\
&= \int_{-r}^{r} |b(x)-b(x+y)|(d_{-h}f(x+y)g(x-y) + f(x+y)d_{-h}g(x-y))dy\\
&\quad + \int_{-r}^{r} |b(x)-b(x+y)|(f_{\tau(-h)}(x+y) - f(x+y))d_{-h}g(x-y)dy.
\end{aligned}$$

By Remark 2 and Lemma 2,

$$\begin{aligned}
&\Big| \int_{-r}^{r} |b(x)-b(x+y)|(f_{-h}(x+y)g(x-y) + f(x+y)d_{-h}g(x-y))dy\\
&\quad - \int_{-r}^{r} |b(x)-b(x+y)|(f'(x+y)g(x-y) + f(x+y)g'(x-y))dy \Big|\\
&\leq \|b'\|_{L^1(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} \int_{-r}^{r} |d_{-h}f(x+y) - f'(x+y)|dy\\
&\quad + \|b'\|_{L^1(\mathbb{R})}\|f\|_{L^\infty(\mathbb{R})} \int_{-r}^{r} |d_{-h}g(x-y) - g'(x-y)|dy\\
&\leq \|b'\|_{L^1(\mathbb{R})}(\|g'\|_{L^1(\mathbb{R})}\|d_{-h}f - f'\|_{L^1(\mathbb{R})} + \|f'\|_{L^1(\mathbb{R})}\|d_{-h}g - g'\|_{L^1(\mathbb{R})})\\
&\to 0 \ \text{ as } h \to 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\lim_{h\to 0}\int_{-r}^{r} |b(x)-b(x+y)|(d_{-h}f(x+y)g(x-y) + f(x+y)d_{-h}g(x-y))dy\\
&= \int_{-r}^{r} |b(x)-b(x+y)|(f'(x+y)g(x-y) + f(x+y)g'(x-y))dy.
\end{aligned} \tag{2.6}$$

We also note that

$$
\begin{aligned}
&\int_{-r}^{r} |b(x)-b(x+y)||(f_{\tau(-h)}(x+y)-f(x+y))d_{-h}f(x-y)|dy \\
&\leq \int_{-r}^{r} |b(x)-b(x+y)||(f_{\tau(-h)}(x+y)-f(x+y))||d_{-h}g(x-y)-g'(x-y)|dy \\
&\quad + \int_{-r}^{r} |b(x)-b(x+y)||(f_{\tau(-h)}(x+y)-f(x+y))||g'(x-y)|dy \\
&\leq 2\|b\|_{Lip(\mathbb{R})}\|f'\|_{L^1(\mathbb{R})} r \|d_{-h}g-g'\|_{L^1(\mathbb{R})} + \|b\|_{Lip(\mathbb{R})} r \int_{-r}^{r} |(f_{\tau(-h)}(x+y)-f(x+y))||g'(x-y)|dy.
\end{aligned}
$$

Since $|(f_{\tau(-h)}(x+y)-f(x+y))||g'(x-y)| \leq 2\|f'\|_{L^1(\mathbb{R})}|g'(x-y)|$, $g' \in L^1(\mathbb{R})$, and $\lim_{h\to 0}(f_{\tau(-h)}(x+y) - f(x+y)) = 0$, then we derive from the dominated convergence theorem that

$$
\lim_{h\to 0} \int_{-r}^{r} |(f_{\tau(-h)}(x+y)-f(x+y))||g'(x-y)|dy = 0.
$$

Hence, we obtain

$$
\lim_{h\to 0} \int_{-r}^{r} |b(x)-b(x+y)|(f_{\tau(-h)}(x+y)-f(x+y))d_{-h}g(x-y)dy = 0. \tag{2.7}
$$

In view of (2.6) and (2.7), the following conclusion can be drawn.

$$
\begin{aligned}
&\lim_{h\to 0} \frac{1}{h} \int_{-r}^{r} |b(x)-b(x+y)|(f(x+y)g(x-y)-f_{\tau(-h)}(x+y)g_{\tau(-h)}(x-y))dy \\
&= \int_{-r}^{r} |b(x)-b(x+y)|(f'(x+y)g(x-y)+f(x+y)g'(x-y))dy.
\end{aligned} \tag{2.8}
$$

Subsequently, we proceed to prove that

$$
\begin{aligned}
&\lim_{h\to 0} \frac{1}{h} \int_{-r}^{r} (|b(x)-b(x+y)|-|b(x-h)-b(x+y-h)|)f_{\tau(-h)}(x+y)g_{\tau(-h)}(x-y)dy \\
&= \int_{x-r}^{x+r} (D_x|b(x)-b(y)|+D_y|b(x)-b(y)|)f(y)g(2x-y)dy.
\end{aligned} \tag{2.9}
$$

For convenience, we set $F_b(x,y) = |b(x)-b(y)|$ and

$$
(F_{x,b})_h(y) = \frac{1}{h}(F_b(x,y+h)-F_b(x,y)), \quad (F_{y,b})_h(x) = \frac{1}{h}(F_b(x+h,y)-F_b(x,y)).
$$

Observe that

$$
\begin{aligned}
\frac{F_b(x,y)-F_b(x-h,y-h)}{h} &= \frac{F_b(x,y)-F_b(x,y-h)}{h} + \frac{F_b(x,y-h)-F_b(x-h,y-h)}{h} \\
&= (F_{x,b})_{-h}(y) + (F_{y-h,b})_{-h}(x).
\end{aligned}
$$

By a change of variable, we have

$$\begin{aligned}
&\frac{1}{h}\int_{-r}^{r}(|b(x)-b(x+y)|-|b(x-h)-b(x+y-h)|)f_{\tau(-h)}(x+y)g_{\tau(-h)}(x-y)dy\\
&=\int_{x-r}^{x+r}\frac{F_b(x,y)-F_b(x-h,y-h)}{h}f_{\tau(-h)}(y)g_{\tau(-h)}(2x-y)dy\\
&=\int_{x-r}^{x+r}((F_{x,b})_{-h}(y)+(F_{y-h,b})_{-h}(x))f_{\tau(-h)}(y)g_{\tau(-h)}(2x-y)dy \qquad (2.10)\\
&=\int_{x-r}^{x+r}((F_{x,b})_{-h}(y)+(F_{y-h,b})_{-h}(x))(f_{\tau(-h)}(y)g_{\tau(-h)}(2x-y)-f(y)g(2x-y))dy\\
&\quad+\int_{x-r}^{x+r}((F_{x,b})_{-h}(y)+(F_{y-h,b})_{-h}(x))f(y)g(2x-y)dy.
\end{aligned}$$

Since $b\in Lip(\mathbb{R})$, then $\|F_b(x,\cdot)\|_{Lip(\mathbb{R})}\le\|b\|_{Lip(\mathbb{R})}$ and $\|F_b(\cdot,y)\|_{Lip(\mathbb{R})}\le\|b\|_{Lip(\mathbb{R})}$ for any $x\in\mathbb{R}$ and $y\in\mathbb{R}$. By Lemma 2, we see that $|f_{\tau(-h)}(y)g_{\tau(-h)}(2x-y)-f(y)g(2x-y)|\le 2\|f'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}$ and $f_{\tau(-h)}(y)g_{\tau(-h)}(2x-y)-f(y)g(2x-y)\to 0$ as $h\to 0$. According to the dominated convergence theorem, it can be deduced that

$$\begin{aligned}
&\Big|\int_{x-r}^{x+r}((F_{x,b})_{-h}(y)+(F_{y-h,b})_{-h}(x))(f_{\tau(-h)}(y)g_{\tau(-h)}(2x-y)-f(y)g(2x-y))dy\Big|\\
&\le 2\|b\|_{Lip(\mathbb{R})}\int_{x-r}^{x+r}|f_{\tau(-h)}(y)g_{\tau(-h)}(2x-y)-f(y)g(2x-y)|dy\to 0 \text{ as } h\to 0.
\end{aligned} \qquad (2.11)$$

By the fact that $F_b(x,\cdot)\in Lip(\mathbb{R})$ and Remark 2, it follows that for almost every $y\in\mathbb{R}$,

$$(F_{x,b})_{-h}(y)\to D_y|b(x)-b(y)| \text{ as } h\to 0^+$$

and

$$|(F_{x,b})_{-h}(\cdot)||f(\cdot)g(2x-\cdot)|\le\|b\|_{Lip(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}|f(\cdot)|\in L^1(\mathbb{R}).$$

These facts together with the dominated convergence theorem imply

$$\lim_{h\to 0^+}\int_{x-r}^{x+r}(F_{x,b})_{-h}(y)f(y)g(2x-y)dy=\int_{x-r}^{x+r}D_y|b(x)-b(y)|f(y)g(2x-y)dy. \qquad (2.12)$$

On the other hand, by a change of variable, we can write

$$\int_{x-r}^{x+r}(F_{y-h,b})_{-h}(x)f(y)g(2x-y)dy=\int_{x-r-h}^{x+r-h}(F_{y,b})_{-h}(x)f_{\tau(h)}(y)g_{\tau(-h)}(2x-y)dy. \qquad (2.13)$$

Observe that

$$\begin{aligned}
&\Big|\int_{x-r-h}^{x+r-h}(F_{y,b})_{-h}(x)f_{\tau(h)}(y)g_{\tau(-h)}(2x-y)dy-\int_{x-r}^{x+r}(F_{y,b})_{-h}(x)f_{\tau(h)}(y)g_{\tau(-h)}(2x-y)dy\Big|\\
&\le 2\|b\|_{Lip(\mathbb{R})}\|f'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}|h|\to 0 \text{ as } h\to 0.
\end{aligned} \qquad (2.14)$$

Note that $F_b(\cdot,y)\in Lip(\mathbb{R})$ for all $y\in\mathbb{R}$. Then, by Remark 2, we have that for almost every $x\in\mathbb{R}$,

$$\lim_{h\to 0^+}(F_{y,b})_{-h}(x)=D_x|b(x)-b(y)|,\quad \forall y\in\mathbb{R}.$$

By employing arguments analogous to those utilized in the derivation (2.12), we have that for almost every $x \in \mathbb{R}$,

$$\lim_{h\to 0^+}\int_{x-r}^{x+r}(F_{y,b})_{-h}(x)f(y)g(2x-y)dy=\int_{x-r}^{x+r}D_x|b(x)-b(y)|f(y)g(2x-y)dy. \tag{2.15}$$

An argument similar to (2.11) leads to

$$\Big|\int_{x-r}^{x+r}(F_{y,b})_{-h}(x)(f_{\tau(h)}(y)g_{\tau(-h)}(2x-y)-f(y)g(2x-y))dy\to 0 \text{ as } h\to 0.$$

This, in conjunction with (2.13)–(2.15), suggests that

$$\lim_{h\to 0^+}\int_{x-r}^{x+r}(F_{y-h,b})_{-h}(x)f(y)g(2x-y)dy=\int_{x-r}^{x+r}D_x|b(x)-b(y)|f(y)g(2x-y)dy. \tag{2.16}$$

Then, (2.9) follows from (2.10)–(2.12) and (2.16).

It follows from (2.4), (2.5), (2.8), and (2.9) that

$$\begin{aligned}(\mathfrak{M}_{b,\alpha}(f,g))'(x)&=\lim_{h\to 0^+}\frac{\mathfrak{M}_{b,\alpha}(f,g)(x)-\mathfrak{M}_{b,\alpha}(f,g)(x-h)}{h}\\&\le\frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}|b(x)-b(x+y)|(f'(x+y)g(x-y)+f(x+y)g'(x-y))dy\\&\quad+\frac{1}{(2r)^{1-\alpha}}\int_{x-r}^{x+r}(D_x|b(x)-b(y)|+D_y|b(x)-b(y)|)|f(y)g(2x-y)|dy.\end{aligned} \tag{2.17}$$

On the other hand, we have

$$\begin{aligned}&\mathfrak{M}_{b,\alpha}(f,g)(x+h)-\mathfrak{M}_{b,\alpha}(f,g)(x)\\&=\mathfrak{M}_{b_{\tau(h)},\alpha}(f_{\tau(h)},g_{\tau(h)})(x)-\mathfrak{M}_{b,\alpha}(f,g)(x)\\&\ge\frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}|b_{\tau(h)}(x)-b_{\tau(h)}(x+y)|f_{\tau(h)}(x+y)g_{\tau(h)}(x-y)dy\\&\quad-\frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}|b(x)-b(x+y)|f(x+y)g(x-y)dy\\&\ge\frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}|b(x)-b(x+y)|(f_{\tau(h)}(x+y)g_{\tau(h)}(x-y)-f(x+y)g(x-y))dy\\&\quad+\frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}(|b(x+h)-b(x+y+h)|-|b(x)-b(x+y)|)f_{\tau(h)}(x+y)g_{\tau(h)}(x-y)dy\end{aligned}$$

for all $h>0$. By (2.4) and the arguments similar to those used in getting (2.8) and (2.9),

$$\begin{aligned}(\mathfrak{M}_{b,\alpha}(f,g))'(x)&=\lim_{h\to 0^+}\frac{\mathfrak{M}_{b,\alpha}(f,g)(x+h)-\mathfrak{M}_{b,\alpha}(f,g)(x)}{h}\\&\ge\frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}|b(x)-b(x+y)|(f'(x+y)g(x-y)+f(x+y)g'(x-y))dy\\&\quad+\frac{1}{(2r)^{1-\alpha}}\int_{x-r}^{x+r}(D_x|b(x)-b(y)|+D_y|b(x)-b(y)|)f(y)g(2x-y)dy.\end{aligned} \tag{2.18}$$

Combining (2.18) with (2.17) leads to (2.3). This completes the proof. □

We end this section by establishing some pointwise convergence of the derivative of bilinear maximal functions.

Lemma 5. *Let $\alpha \in [0,1)$ and $b \in Lip(\mathbb{R})$ with $b' \in L^1(\mathbb{R})$. Let $f,\ g \in W^{1,1}(\mathbb{R})$, $\{f_j\}_{j\geq 1} \subset W^{1,1}(\mathbb{R})$, and $\{g_j\}_{j\geq 1} \subset W^{1,1}(\mathbb{R})$. Assume that $f_j \to f$ and $g_j \to g$ in $W^{1,1}(\mathbb{R})$ as $j \to \infty$. Then, for almost every $x \in \mathbb{R}$,*

$$(\mathfrak{M}_{b,\alpha}(f_j, g_j))'(x) \to (\mathfrak{M}_{b,\alpha}(f, g))'(x) \ \text{ as } \ j \to \infty. \tag{2.19}$$

Proof. Without loss of generality, we may assume all $f_j,\ g_j,\ f,\ g \geq 0$ because of Lemma 2(iv). Set $D_0 = \{x \in \mathbb{R} : \mathfrak{M}_{b,\alpha}(f,g) > 0\}$. The proof of (2.19) can be divided into two steps:

Step 1. Proof of (2.19) for almost every $x \in D_0$.

We shall adapt the method as in the proof of [8, Lemma 2.10] to prove (2.19) for almost every $x \in D_0$. Given $k \in \mathbb{Z}$, it suffices to show that (2.19) holds for almost every $x \in D_{0,k} := \{x \in \mathbb{R} : 2^k < \mathfrak{M}_{b,\alpha}(f,g)(x) \leq 2^{k+1}\}$. By Lemma 2(vi), we see that $\mathfrak{M}_{b,\alpha}(f_j, g_j)$ converges uniformly to $\mathfrak{M}_{b,\alpha}(f,g)$ on $\mathbb{R}$. Without loss of generality, we may assume $\mathfrak{M}_{b,\alpha}(f_j, g_j)(x) > 0$ for all $x \in D_{0,k}$. Let us fix $k \in \mathbb{Z}$. Let A_0 (resp., A_j) be the set for which the function $\mathfrak{M}_{b,\alpha}(f,g)$ (resp., $\mathfrak{M}_{b,\alpha}(f_j, g_j)$) is differentiable on A_0 (resp., A_j) for $j \geq 1$. Set $A = \bigcap_{j=0}^{\infty} A_j$. Invoking Lemma 4, we have that $|\mathbb{R}\backslash A_j| = 0$ for all $j \geq 0$. So, $|\mathbb{R}\backslash A| = 0$. Let G be the set for which b is differentiable on G. Let $H = \{x \in \mathbb{R} : |D_x|b(x)-b(y)|| \leq \|b\|_{Lip(\mathbb{R})}, \forall y \in \mathbb{R}\}$. It was pointed out in the proof of [8, Lemma 2.10] that $|\mathbb{R} \setminus G| = 0$, $|\mathbb{R} \setminus H| = 0$ and $|\mathbb{R} \setminus (A \cap G \cap H)| = 0$. Let B_0 (resp., B_j) be the set of all $x \in A \cap D_{0,k}$ for which (2.3) holds for (f,g) (resp., (f_j, g_j)) at x. Invoking Lemma 4, we see that $|(A \cap D_{0,k}) \setminus B_j| = 0$. Let $B = \bigcap_{j=0}^{\infty} B_j$. Clearly, $|(A \cap D_{0,k}) \setminus B| = 0$. Based on the above analyses, it is sufficient to demonstrate that (2.19) holds for $x \in B \cap G \cap H$.

Let $x \in B \cap G \cap H$. By Lemma 3, there exist $\delta_1 = \inf \mathcal{R}_\alpha(f,g)(x) > 0$ and $\delta_2 = \sup \mathcal{R}_\alpha(f,g)(x) > 0$ such that $\delta_1 < r < \delta_2$ when $r \in \mathcal{R}_\alpha(f,g)(x)$. Invoking Lemma 4, there exists $\{r_j\}_{j\geq 1} \subset \mathcal{R}_\alpha(f_j, g_j)(x) \setminus \{0\}$ such that

$$\begin{aligned}
&(\mathfrak{M}_{b,\alpha}(f_j, g_j))'(x) \\
&= \frac{1}{(2r_j)^{1-\alpha}}\Big(\int_{-r_j}^{r_j} |b(x) - b(x+y)|(f_j'(x+y)g_j(x-y) + f_j(x+y)g_j'(x-y))dy \\
&\quad + \int_{x-r_j}^{x+r_j} (D_x|b(x) - b(y)| + D_y|b(x) - b(y)|)f_j(y)g_j(2x-y)dy\Big).
\end{aligned} \tag{2.20}$$

According to our assumption, there exists $C > 0$ such that $\|f_j\|_{W^{1,1}(\mathbb{R})} \leq C$ and $\|g_j\|_{W^{1,1}(\mathbb{R})} < C$ for all $j \geq 1$. By Lemmas 2(vi) and 3 and the arguments similar to those used to derive [8, Lemma 2.10], there exists $N \in \mathbb{N}$ such that $r_j \in [\delta_1/2, 2\delta_2]$ for any $j \geq N$. Note that $|b(x) - b(\cdot)| \in Lip(\mathbb{R})$. By Remark 2, we see that $|D_y|b(x) - b(y)|| \leq \|b\|_{Lip(\mathbb{R})}$. By Remark 2, Lemma 2, and (2.20), one gets

$$|(\mathfrak{M}_{b,\alpha}(f_j, g_j))'(x)| \leq 2(\|b'\|_{L^1(\mathbb{R})} + \|b\|_{Lip(\mathbb{R})})\|f'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})} + 2\|b\|_{Lip(\mathbb{R})}\|g_j'\|_{L^1(\mathbb{R})}\|f_j\|_{W^{1,1}(\mathbb{R})}.$$

This yields that the sequence $\{(\mathfrak{M}_{b,\alpha}(f_j, g_j))'(x)\}_{j\geq 1}$ is a bounded set.

Given a convergent subsequence $\{(\mathfrak{M}_{b,\alpha}(f_{j_\iota}, g_{j_\iota}))'(x)\}_{\iota\geq 1}$ of $\{(\mathfrak{M}_{b,\alpha}(f_j, g_j))'(x)\}_{j\geq 1}$, note that $\{r_{j_\iota}\}_{\iota\geq 1}$ is a bounded sequence. There exist $r > 0$ and a subsequence $\{r_{j_{\iota_\ell}}\}_{\ell\geq 1} \subset \{r_{j_\iota}\}_{\iota\geq 1}$ such that $\lim_{\ell\to\infty} r_{j_{\iota_\ell}} = r$. By Lemma 3(ii), we see that $r \in \mathcal{R}_\alpha(f,g)(x)$. Applying Lemma 4, one has

$$\begin{aligned}
(\mathfrak{M}_{b,\alpha}(f,g))'(x) = &\frac{1}{(2r)^{1-\alpha}} \int_{-r}^{r} |b(x) - b(x+y)|(f'(x+y)g(x-y) + f(x+y)g'(x-y))dy \\
&+ \frac{1}{(2r)^{1-\alpha}} \int_{x-r}^{x+r} (D_x|b(x) - b(y)| + D_y|b(x) - b(y)|)f(y)g(2x-y)dy.
\end{aligned} \tag{2.21}$$

By the arguments similar to those used to derive the proof of Lemma 3, we have

$$\int_{-r_{j_{\iota_\ell}}}^{r_{j_{\iota_\ell}}} |b(x)-b(x+y)|(f'_{j_{\iota_\ell}}(x+y)g_{j_{\iota_\ell}}(x-y)+f_{j_{\iota_\ell}}(x+y)g'_{j_{\iota_\ell}}(x-y))dy$$
$$\to \int_{-r}^{r} |b(x)-b(x+y)|(f'(x+y)g(x-y)+f(x+y)g'(x-y))dy \ \text{ as } \ell\to\infty,$$

$$\int_{x-r_{j_{\iota_\ell}}}^{x+r_{j_{\iota_\ell}}} (D_x|b(x)-b(y)|+D_y|b(x)-b(y)|)f_{j_{\iota_\ell}}(y)g_{j_{\iota_\ell}}(2x-y)dy$$
$$\to \int_{x-r}^{x+r} (D_x|b(x)-b(y)|+D_y|b(x)-b(y)|)f(y)g(2x-y)dy \ \text{ as } \ell\to\infty.$$

These together with (2.20) and (2.21) imply that

$$(\mathfrak{M}_{b,\alpha}(f_{j_{\iota_\ell}}, g_{j_{\iota_\ell}}))'(x) \to (\mathfrak{M}_{b,\alpha}(f,g))'(x) \ \text{ as } \ \ell\to\infty.$$

So, $(\mathfrak{M}_{b,\alpha}(f_{j_\iota}, g_{j_\iota}))'(x) \to (\mathfrak{M}_{b,\alpha}(f,g))'(x)$ as $\iota\to\infty$. Consequently, $(\mathfrak{M}_{b,\alpha}(f,g))'(x)$ is the unique accumulation point of $\{(\mathfrak{M}_{b,\alpha}(f_{j_\iota}, g_{j_\iota}))'(x)\}_{j\geq 1}$. This proves Step 1.

Step 2. Proof of (2.19) for almost every $x\in\mathbb{R}\setminus D_0$.

Let $D_j := \{x\in\mathbb{R} : \mathfrak{M}_{b,\alpha}(f,g)(x) > 0\}$. By Lemma 4, we see that $(\mathfrak{M}_{b,\alpha}(f,g))'(x) = 0$ for all $x\in A\cap(\mathbb{R}\setminus D_0)$. Moreover, $(\mathfrak{M}_{b,\alpha}(f_j,g_j))'(x) = 0$ for all $x\in A\cap(\mathbb{R}\setminus D_j)$. Thus, it suffices to show that for almost every $x\in\mathbb{R}\setminus D_0$,

$$(\mathfrak{M}_{b,\alpha}(f_j,g_j))'(x)\chi_{D_j}(x)\to 0 \ \text{ as } \ j\to\infty. \tag{2.22}$$

By Lemma 4, there exists a measurable set $E_j\subset D_j$ such that $|D_j\setminus E_j| = 0$, and for any $x\in E_j$, there exists $r_j\in\mathcal{R}_\alpha(f_j,g_j)(x)\setminus\{0\}$ such that

$$\begin{aligned}&(\mathfrak{M}_{b,\alpha}(f_j,g_j))'(x)\\ &= \frac{1}{(2r_j)^{1-\alpha}}\Big(\int_{-r_j}^{r_j}|b(x)-b(x+y)|(f'_j(x+y)g_j(x-y)+f_j(x+y)g'_j(x-y))dy\\ &\quad+\int_{x-r_j}^{x+r_j}(D_x|b(x)-b(y)|+D_y|b(x)-b(y)|)f_j(y)g_j(2x-y)dy\Big).\end{aligned} \tag{2.23}$$

Let $J := \{x\in\mathbb{R} : |b(x)-b(y)|$ be differentiable at x, $\forall y\in\mathbb{R}\}$. Since $|b(\cdot)-b(y)|\in Lip(\mathbb{R})$, then we have that for almost $x\in\mathbb{R}$, the function $|b(\cdot)-b(y)|$ is differentiable at x for all $y\in\mathbb{R}$. Hence, $|\mathbb{R}\setminus J| = 0$. Therefore, it is enough to show that for all $x\in A\cap G\cap H\cap J\cap(\mathbb{R}\setminus D_0)$,

$$(\mathfrak{M}_{b,\alpha}(f_j,g_j))'(x)\chi_{E_j}(x)\to 0 \ \text{ as } \ j\to\infty. \tag{2.24}$$

In view of (2.23), for (2.24) it suffices to prove that for all $x\in A\cap G\cap H\cap J\cap(\mathbb{R}\setminus D_0)$,

$$\frac{1}{(2r_j)^{1-\alpha}}\int_{-r_j}^{r_j}|b(x)-b(x+y)|(f'_j(x+y)g_j(x-y)+f_j(x+y)g'_j(x-y))dy\to 0 \ \text{ as } j\to\infty; \tag{2.25}$$

$$\frac{1}{(2r_j)^{1-\alpha}}\int_{x-r_j}^{x+r_j}(D_x|b(x)-b(y)|+D_y|b(x)-b(y)|)f_j(y)g_j(2x-y)dy\to 0 \ \text{ as } j\to\infty. \tag{2.26}$$

We first prove (2.25). Let us fix $j \geq 1$ and $x_0 \in A \cap G \cap H \cap J \cap (\mathbb{R} \setminus D_0) \cap E_j$. Since $\mathfrak{M}_{b,\alpha}(f,g)(x_0) = 0$, then $|b(x_0) - b(x_0+y)| f(x_0+y) g(x_0-y) = 0$ for almost every $y \in \mathbb{R}$. Let $B := \{y \in \mathbb{R}; |(b(x_0) - b(x_0 + y))| f(x_0+y)g(x_0-y) = 0\}$. It is readily apparent that $|\mathbb{R} \setminus B| = 0$. Let

$$B_1 := \{y \in B : |b(x_0) - b(x_0+y)| > 0\}, \quad B_2 := \{y \in B : f(x_0+y)g(x_0-y) > 0\}.$$

Clearly, $|B_1 \cap B_2| = 0$. Then, we have

$$\begin{aligned}
&\Big|\frac{1}{(2r_j)^{1-\alpha}} \int_{-r_j}^{r_j} |b(x_0) - b(x_0+y)|(f_j'(x_0+y)g_j(x_0-y) + f_j(x_0+y)g_j'(x_0-y))dy\Big| \\
&\leq \frac{1}{(2r_j)^{1-\alpha}} \\
&\quad \times \int_{[-r_j,r_j]\cap B_1 \cap (B\setminus B_2)} |b(x_0) - b(x_0+y)||f_j'(x_0+y)g_j(x_0-y) + f_j(x_0+y)g_j'(x_0-y)|dy \\
&\leq (\|b\|_{Lip(\mathbb{R})} + \|b'\|_{L^1(\mathbb{R})}) \int_{B\setminus B_2} |f_j'(x_0+y)g_j(x_0-y) + f_j(x_0+y)g_j'(x_0-y)|dy.
\end{aligned} \tag{2.27}$$

Let

$$B_{2,1} = \{y \in B \setminus B_2 : f(x_0+y) > 0\}, \quad B_{2,2} = \{y \in B \setminus B_2 : g(x_0-y) > 0\}.$$

Clearly, $B_{2,1} \cap B_{2,2} = \emptyset$. Moreover, $f'(x_0+y) = 0$ for almost every $y \in (B \setminus B_2) \setminus B_{2,1}$ and $g'(x_0-y) = 0$ for almost every $y \in (B \setminus B_2) \setminus B_{2,2}$ since $f,\ g \in W^{1,1}(\mathbb{R})$. These facts together with Lemma 2 imply that

$$\begin{aligned}
&\int_{B\setminus B_2} |(f_j'(x_0+y)g_j(x_0-y) + f_j(x_0+y)g_j'(x_0-y))|dy \\
&\leq \int_{(B\setminus B_2)\setminus B_{2,2}} |f_j'(x_0+y)g_j(x_0-y) + f_j(x_0+y)g_j'(x_0-y)|dy \\
&\quad + \int_{B_{2,2}\cap((B\setminus B_2)\setminus B_{2,1})} |f_j'(x_0+y)g_j(x_0-y) + f_j(x_0+y)g_j'(x_0-y)|dy \\
&= \int_{(B\setminus B_2)\setminus B_{2,2}} |f_j'(x_0+y)(g_j(x_0-y) - g(x_0-y)) + f_j(x_0+y)(g_j'(x_0-y) - g'(x_0-y))|dy \\
&\quad + \int_{B_{2,2}\cap((B\setminus B_2)\setminus B_{2,1})} |(f_j'(x_0+y) - f'(x_0+y))g_j(x_0-y) \\
&\quad +(f_j(x_0+y) - f(x_0+y))g_j'(x_0-y)|dy \\
&\leq \|(g_j - g)'\|_{L^1(\mathbb{R})}\|f_j'\|_{L^1(\mathbb{R})} + \|f_j'\|_{L^1(\mathbb{R})}\|g_j' - g'\|_{L^1(\mathbb{R})} \\
&\quad +\|g_j'\|_{L^1(\mathbb{R})}\|f_j' - f'\|_{L^1(\mathbb{R})} + \|(f_j - f)'\|_{L^1(\mathbb{R})}\|g_j'\|_{L^1(\mathbb{R})} \\
&\leq 2\|g_j' - g'\|_{L^1(\mathbb{R})}(\|f_j' - f'\|_{L^1(\mathbb{R})} + \|f'\|_{L^1(\mathbb{R})}) + 2(\|g_j' - g'\|_{L^1(\mathbb{R})} + \|g'\|_{L^1(\mathbb{R})})\|f_j' - f'\|_{L^1(\mathbb{R})} \\
&\to 0 \ \text{ as } \ j \to \infty.
\end{aligned} \tag{2.28}$$

Combining (2.28) with (2.27) implies (2.25).

Now we prove (2.26). The argument is analogous to (2.25). Since $|(b(x_0) - b(x_0+y))f(x_0+y)g(x_0-y)| = 0$ for almost every $y \in \mathbb{R}$, then $|(b(x_0) - b(y))f(y)g(2x_0-y)| = 0$ for almost every $y \in \mathbb{R}$. Let $I = \{y \in \mathbb{R} : |b(x_0) - b(y)|f(y)g(2x_0-y) = 0\}$. It is clear that $|\mathbb{R} \setminus I| = 0$. For convenience, we let $F(x,y) = |b(x) - b(y)|$ and denote

$$F_x(x,y) = D_x|b(x) - b(y)|, \quad F_y(x,y) = D_y|b(x) - b(y)|.$$

Let

$$I_1 = \{y \in I; |b(x_0) - b(y)| > 0\}, \quad I_2 = \{y \in I; f(y)g(2x_0-y) > 0\}.$$

We have $I_1 \cap I_2 = \emptyset$. Then, we have

$$\begin{aligned}
&\Big|\frac{1}{(2r_j)^{1-\alpha}}\int_{x_0-r_j}^{x_0+r_j}(F_x(x_0,y)+F_y(x_0,y))f_j(y)g_j(2x_0-y)dy\Big| \\
&\le \frac{1}{(2r_j)^{1-\alpha}}\Big(\Big|\int_{[x_0-r_j,x_0+r_j]\cap I_1\cap(I\setminus I_2)}(F_x(x_0,y)+F_y(x_0,y))f_j(y)g_j(2x_0-y)dy\Big| \\
&\quad+\Big|\int_{[x_0-r_j,x_0+r_j]\cap(I\setminus I_1)}(F_x(x_0,y)+F_y(x_0,y))f_j(y)g_j(2x_0-y)dy\Big|\Big).
\end{aligned} \tag{2.29}$$

Let

$$I_{2,1}=\{y\in I\setminus I_2 : f(y)>0\},\quad I_{2,2}=\{y\in I\setminus I_2 : g(2x_0-y)>0\}.$$

Clearly, $I_{2,1}\cap I_{2,2}=\emptyset$. We also note that $|F_x(x_0,y)|\le \|b\|_{Lip(\mathbb{R})}$ for any $y\in\mathbb{R}$ and $|F_y(x_0,y)|\le\|b\|_{Lip(\mathbb{R})}$ for almost every $y\in\mathbb{R}$. It follows that

$$\begin{aligned}
&\frac{1}{(2r_j)^{1-\alpha}}\int_{[x_0-r_j,x_0+r_j]\cap I_1\cap(I\setminus I_2)}|(F_x(x_0,y)+F_y(x_0,y))|f_j(y)g_j(2x_0-y)dy \\
&\le 2\|b\|_{Lip(\mathbb{R})}\frac{1}{(2r_j)^{1-\alpha}}\int_{[x_0-r_j,x_0+r_j]\cap(I\setminus I_2)}f_j(y)g_j(2x_0-y)dy.
\end{aligned}$$

By Lemma 2, one obtains

$$\begin{aligned}
&\frac{1}{(2r_j)^{1-\alpha}}\int_{[x_0-r_j,x_0+r_j]\cap I_1\cap(I\setminus I_2)}f_j(y)g_j(2x_0-y)dy \\
&\le \frac{1}{(2r_j)^{1-\alpha}}\int_{[x_0-r_j,x_0+r_j]\cap(I\setminus I_2)}f_j(y)g_j(2x_0-y)dy \\
&\le \frac{1}{(2r_j)^{1-\alpha}}\int_{[x_0-r_j,x_0+r_j]\cap(I\setminus I_2)\setminus I_{2,2}}f_j(y)g_j(2x_0-y)dy \\
&\quad+\frac{1}{(2r_j)^{1-\alpha}}\int_{[x_0-r_j,x_0+r_j]\cap I_{2,2}\cap((I\setminus I_2)\setminus I_{2,1})}f_j(y)g_j(2x_0-y)dy \\
&= \frac{1}{(2r_j)^{1-\alpha}}\int_{[x_0-r_j,x_0+r_j]\cap(I\setminus I_2)\setminus I_{2,2}}|f_j(y)(g_j(2x_0-y)-g(2x_0-y))|dy \\
&\quad+\frac{1}{(2r_j)^{1-\alpha}}\int_{[x_0-r_j,x_0+r_j]\cap I_{2,2}\cap((I\setminus I_2)\setminus I_{2,1})}|(f_j(y)-f(y))g_j(2x_0-y)|dy \\
&\le \|f_j\|_{L^\infty(\mathbb{R})}M_\alpha(g_j-g)(x_0)+\|g_j\|_{L^\infty(\mathbb{R})}M_\alpha(f_j-f)(x_0) \\
&\le \|f_j'\|_{L^1(\mathbb{R})}\|g_j-g\|_{W^{1,1}(\mathbb{R})}+\|g_j'\|_{L^1(\mathbb{R})}\|f_j-f\|_{W^{1,1}(\mathbb{R})}\to 0\ \text{ as }\ j\to\infty.
\end{aligned}$$

Hence, we conclude that

$$\frac{1}{(2r_j)^{1-\alpha}}\int_{[x_0-r_j,x_0+r_j]\cap I_1\cap(I\setminus I_2)}(F_x(x_0,y)+F_y(x_0,y))f_j(y)g_j(2x_0-y)dy\to 0\ \text{ as }\ j\to\infty. \tag{2.30}$$

On the other hand, we see that $F(x_0,y)\equiv 0$ for $y\in I\setminus I_1$. It is inferred that $F_y(x_0,y)=0$ for almost every $y\in I\setminus I_1$. Consequently,

$$\frac{1}{(2r_j)^{1-\alpha}}\int_{[x_0-r_j,x_0+r_j]\cap(I\setminus I_1)}F_y(x_0,y)f_j(y)g_j(2x_0-y)dy=0. \tag{2.31}$$

By (2.29)–(2.31), for (2.26), it is sufficient to demonstrate that

$$\int_{[x_0-r_j,x_0+r_j]\cap(I\setminus I_1)} F_x(x_0,y)f_j(y)g_j(2x_0-y)dy \to 0 \text{ as } j\to\infty. \tag{2.32}$$

Since $b(y)\equiv b(x_0)$ for $y\in I\setminus I_1$, then we have

$$F(x,y)\equiv F(x,x_0)=|b(x)-b(x_0)|,\quad \forall y\in I\setminus I_1.$$

Since $x_0\in J$, then for any $y\in I\setminus I_1$, we have that $F(\cdot,y)$ is differentiable at x_0. Fix $y\in I\setminus I_1$, and we note

$$0\geq \lim_{h\to 0^+}\frac{|b(x_0-h)-b(x_0)|}{-h}=F_x(x_0,y)=\lim_{h\to 0^+}\frac{|b(x_0+h)-b(x_0)|}{h}\geq 0.$$

Hence, we have $F_x(x_0,y)=0$ for any $y\in I\setminus I_1$. This yields (2.32). Then, Lemma 5 is proved. □

3. Proof of Theorem 1

We now present the proof of Theorem 1. We first prove the boundedness part in Theorem 1. Without loss of generality, we may assume that all $f,g\geq 0$. Let $\frac{1}{1-\alpha}<q<\infty$ and $0<\alpha<1$. Let $p=1/(1/q+\alpha)$. Clearly, $1<p<q<\infty$ and $1/q=1/p-\alpha$. Note that

$$\|f\|_{L^p(\mathbb{R})}\leq \|f\|_{L^\infty(\mathbb{R})}^{1-1/p}\|f\|_{L^1(\mathbb{R})}^{1/p}\leq \|f'\|_{L^1(\mathbb{R})}^{1-1/p}\|f\|_{L^1(\mathbb{R})}^{1/p}\leq \|f\|_{W^{1,1}(\mathbb{R})}.$$

Applying Remark 2 and Lemma 2.1, it is clear that

$$\mathfrak{M}_{b,\alpha}(f,g)(x)\leq \|b'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}M_\alpha f(x),\quad x\in\mathbb{R}.$$

This together with the bounds for M_α yields that

$$\begin{aligned}\|\mathfrak{M}_{b,\alpha}(f,g)\|_{L^q(\mathbb{R})}&\leq \|b'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}\|M_\alpha f\|_{L^q(\mathbb{R})}\\&\leq C_{\alpha,q}\|b'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}\|f\|_{L^p(\mathbb{R})}\leq C_{\alpha,q}\|b'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}\|f\|_{W^{1,1}(\mathbb{R})}.\end{aligned}\tag{3.1}$$

Hence, to prove the boundedness, it is adequate to establish that

$$\|(\mathfrak{M}_{b,\alpha}(f,g))'\|_{L^q(\mathbb{R})}\leq C_{\alpha,q}(\|b\|_{Lip(\mathbb{R})}^{1-1/q-\alpha}\|b'\|_{L^1(\mathbb{R})}^{1/q+\alpha}\|f'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}+\|b\|_{Lip(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}\|f\|_{W^{1,1}(\mathbb{R})}).\tag{3.2}$$

Let E be the set of all points $x\in\mathbb{R}$ for which $\mathfrak{M}_{b,\alpha}(f,g)$ is differentiable at x. In view of Lemma 4, we have $|\mathbb{R}\setminus E|=0$. Let $F=\{x\in\mathbb{R}:|D_x|b(x)-b(y)||=|b'(x)|,\ \forall y\in\mathbb{R}\}$. It was shown in [9] that $|\mathbb{R}\setminus F|=0$. Let $G=\{x\in\mathbb{R}:\mathfrak{M}_{b,\alpha}(f,g)(x)>0\}$. By Lemma 4 we see that for almost every $x\in G^c$,

$$(\mathfrak{M}_{b,\alpha}(f,g))'(x)=0.\tag{3.3}$$

Moreover, for almost every $x\in G$, there exists $r\in\mathcal{R}_\alpha(f,g)(x)\setminus\{0\}$ such that

$$\begin{aligned}(\mathfrak{M}_{b,\alpha}(f,g))'(x)=&\frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}|b(x)-b(x+z)|(f'(x+z)g(x-z)+f(x+z)g'(x-z))dz\\&+\frac{1}{(2r)^{1-\alpha}}\int_{x-r}^{x+r}(D_x|b(x)-b(z)|+D_z|b(x)-b(z)|)f(z)g(2x-z)dz.\end{aligned}\tag{3.4}$$

Note that $b \in Lip(\mathbb{R})$. By the fundamental theorem of calculus and Lemma 2(i), one gets

$$\begin{aligned}
&\Big|\frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}|b(x)-b(x+z)|f'(x+z)g(x-z)dz\Big| \\
&\leq \frac{1}{(2r)^{1-\alpha}}\int_{-r}^{r}\Big|\int_{x}^{x+z}b'(t)dt\Big||f'(x+z)g(x-z)|dz \\
&\leq \|g'\|_{L^1(\mathbb{R})}\int_{-r}^{r}\frac{1}{(2r)^{1-\alpha}}\int_{x-r}^{x+r}|b'(t)|dt|f'(x+z)|dz \\
&\leq \|f'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}M_\alpha|b'|(x).
\end{aligned}$$

Hence, we obtain that for almost every $x \in \mathbb{R}$,

$$|(\mathfrak{M}_{b,\alpha}(f,g))'(x)| \leq 2\|g'\|_{L^1(\mathbb{R})}(\|f'\|_{L^1(\mathbb{R})}M_\alpha|b'|(x) + \|b\|_{Lip(\mathbb{R})}M_\alpha f(x)). \tag{3.5}$$

By (3.5) and the L^q bounds for M_α, one gets

$$\begin{aligned}
\|(\mathfrak{M}_{b,\alpha}(f,g))'\|_{L^q(\mathbb{R})} &\leq 2\|f'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}\|M_\alpha|b'|\|_{L^q(\mathbb{R})} + 2\|b\|_{Lip(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}\|M_\alpha f\|_{L^q(\mathbb{R})} \\
&\leq C_{\alpha,q}\|f'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}\|b'\|_{L^p(\mathbb{R})} + 2\|b\|_{Lip(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}\|f\|_{L^p(\mathbb{R})} \\
&\leq C_{\alpha,q}(\|b\|_{Lip(\mathbb{R})}^{1-1/q-\alpha}\|b'\|_{L^1(\mathbb{R})}^{1/q+\alpha}\|f'\|_{L^1(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})} + \|b\|_{Lip(\mathbb{R})}\|g'\|_{L^1(\mathbb{R})}\|f\|_{W^{1,1}(\mathbb{R})}).
\end{aligned}$$

This proves (3.2).

Next, we prove the continuity part in Theorem 1. Let $\alpha \in [0,1)$, $\frac{1}{1-\alpha} < q < \infty$, $f \in W^{1,1}(\mathbb{R})$, and $g \in W^{1,1}(\mathbb{R})$. Let $\{f_j\}_{j\geq 1} \subset W^{1,1}(\mathbb{R})$ be such that $f_j \to f$ in $W^{1,1}(\mathbb{R})$ and $\{g_j\}_{j\geq 1} \subset W^{1,1}(\mathbb{R})$ be such that $g_j \to g$ in $W^{1,1}(\mathbb{R})$ as $j \to \infty$. By the sublinearity of $\mathfrak{M}_{b,\alpha}$, one obtains

$$|\mathfrak{M}_{b,\alpha}(f_j,g_j)(x) - \mathfrak{M}_{b,\alpha}(f,g)(x)| \leq \mathfrak{M}_{b,\alpha}(f_j-f,g_j-g)(x) + \mathfrak{M}_{b,\alpha}(f_j-f,g)(x) + \mathfrak{M}_{b,\alpha}(f,g_j-g)(x).$$

Consequently, in conjunction with Section (3.1) implies that

$$\begin{aligned}
&\|\mathfrak{M}_{b,\alpha}(f_j,g_j) - \mathfrak{M}_{b,\alpha}(f,g)\|_{L^q(\mathbb{R})} \\
&\leq C_{\alpha,q}(\|f_j-f\|_{W^{1,1}(\mathbb{R})}(\|g_j-g\|_{W^{1,1}(\mathbb{R})} + \|g\|_{W^{1,1}(\mathbb{R})}) + \|f\|_{W^{1,1}(\mathbb{R})}\|g_j-g\|_{W^{1,1}(\mathbb{R})}) \to 0 \text{ as } j\to\infty.
\end{aligned}$$

Thus, to establish continuity, it is sufficient to demonstrate that

$$\|(\mathfrak{M}_{b,\alpha}(f_j,g_j))' - (\mathfrak{M}_{b,\alpha}(f,g))'\|_{L^q(\mathbb{R})} \to 0 \text{ as } j\to\infty. \tag{3.6}$$

Presume that (3.6) is incorrect. We may assume, without loss of generality, that

$$\|(\mathfrak{M}_{b,\alpha}(f_j,g_j))' - (\mathfrak{M}_{b,\alpha}(f,g))'\|_{L^q(\mathbb{R})} > c, \quad \forall j \geq 1 \tag{3.7}$$

for some $c > 0$. By our assumption, there exists $C > 0$ such that

$$\|f_j\|_{W^{1,1}(\mathbb{R})} + \|g_j\|_{W^{1,1}(\mathbb{R})} \leq C, \quad \forall j \geq 1. \tag{3.8}$$

Through the proof of the boundedness segment and (3.8), we have that for any $j \geq 1$ and almost every $x \in \mathbb{R}$,

$$\begin{aligned}
|(\mathfrak{M}_{b,\alpha}(f_j,g_j))'(x)| &\leq 2\|g_j'\|_{L^1(\mathbb{R})}(\|f_j'\|_{L^1(\mathbb{R})}M_\alpha|b'|(x) + \|b\|_{Lip(\mathbb{R})}M_\alpha f_j(x)) \\
&\leq 2C^2M_\alpha|b'|(x) + 2C\|b\|_{Lip(\mathbb{R})}M_\alpha f(x) + 2C\|b\|_{Lip(\mathbb{R})}M_\alpha(f_j-f)(x).
\end{aligned} \tag{3.9}$$

Let $p = 1/(1/q + \alpha)$. Clearly, $1/q = 1/p - \alpha$ and $1 < p < q < \infty$. Note that

$$\|M_\alpha(f_j - f)\|_{L^q(\mathbb{R})} \leq C_{q,\alpha}\|f_j - f\|_{L^p(\mathbb{R})} \leq C_{q,\alpha}\|f_j - f\|_{W^{1,1}(\mathbb{R})} \to 0 \text{ as } j \to \infty.$$

This yields that there exists $\{j_k\}_{k\geq 1}$, satisfying the condition that

$$\Big\|\sum_{k=1}^{\infty} M_\alpha(f_{j_k} - f)\Big\|_{L^q(\mathbb{R})} \leq \sum_{k=1}^{\infty} \|M_\alpha(f_{j_k} - f)\|_{L^q(\mathbb{R})} \leq 1.$$

By (3.9), we see that

$$\begin{aligned}
&|(\mathfrak{M}_{b,\alpha}(f_{j_k}, g_{j_k}))'(x) - (\mathfrak{M}_{b,\alpha}(f, g))'(x)| \\
&\leq 2C^2 M_\alpha|b'|(x) + 2C\|b\|_{Lip(\mathbb{R})}\Big(M_\alpha f(x) + \sum_{k=1}^{\infty} M_\alpha(f_{j_k} - f)(x)\Big) + |(\mathfrak{M}_{b,\alpha}(f, g))'(x)| =: \Phi(x).
\end{aligned}$$

Note that $\Phi \in L^q(\mathbb{R})$. By Lemma 5 and the dominated convergence theorem, we have (3.6). This completes the proof of Theorem 1. □

4. Conclusions

In this paper we study the endpoint Sobolev regularity of the bilinear maximal commutator and its fractional variant $\mathfrak{M}_{b,\alpha}$ with $\alpha \in [0, 1)$ and the symbol function b. We prove that the above commutator $\mathfrak{M}_{b,\alpha}$ is bounded and continuous from $W^{1,1}(\mathbb{R}) \times W^{1,1}(\mathbb{R})$ to $W^{1,q}(\mathbb{R})$ if $q \in (\frac{1}{1-\alpha}, \infty)$ and $b \in Lip(\mathbb{R})$ with $b' \in L^1(\mathbb{R})$. Our main result essentially answered a question motivated by Wang and Liu in 2022.

Author contributions

F. Liu: Writing-review and editing, Conceptualization; X. Zhu: Writing-original draft, Methodology. Both authors have been working together in the mathematical development of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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