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Research article

Resolution of an isolated case of Pfaff hypergeometric transformation and new application of integer sequences

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Abstract: A case of a Pfaff transformation is given by the following:

$$ {}_2F_1\left(\begin{matrix} l, m \\ 2m \end{matrix} ; x\right) = (1-x)^{-l}\, {}_2F_1\left(\begin{matrix} l, m \\ 2m \end{matrix} ; \frac{x}{x-1}\right). $$

In this paper, when m is a negative integer, we define the Gaussian hypergeometric series as follows:

$$ {}_2F_1^*\left(\begin{matrix} l, m \\ 2m \end{matrix} ; x\right) = \sum_{k=0}^{-m} \frac{(l)_k (m)_k}{k!(2m)_k} x^k, $$

which is well-defined, as it is a terminating hypergeometric series since the summation is only for $k = 0, .., -m$; additionally, the fact that $2m$ is a negative integer does not make any harm. With this definition, if we take $m = -1$ and $l = 1$, then the left-hand side is a terminating hypergeometric series equal to $1 + \frac{x}{2}$, while the right-hand side is also a terminating hypergeometric series, but has 1 as the pole of multiplicity 2 given by $-\frac{3x-2}{2(x-1)^2}$. More generally, with the definition above, we prove that this case of the Pfaff transformation does not hold for any positive integer l and for any negative integer m. Additionally, an analysis aims to solve this situation. In fact, we give a new expression $V^{(l,m)}(x)$ depending on l, m, and x such that

$$ (1-x)^{-l}\, {}_2F_1^*\left(\begin{matrix} l, m \\ 2m \end{matrix} ; \frac{x}{x-1}\right) = {}_2F_1^*\left(\begin{matrix} l, m \\ 2m \end{matrix} ; x\right) + V^{(l,m)}(x), $$

for any positive integer l and for any negative integer m. As a very interesting consequence we present a corollary from the boundary conditions, thereby providing the following:
(1) an expansion of x^{2n+1} as a sum of two terminating hypergeometric series (with symmetric values) with the coefficients given in the integer sequence $A046899$ (these coefficients can be found in Pascal's triangle as an inclined column);

(2) an expansion of $x^{2n+1}(x-2)$ as a sum of two terminating hypergeometric series (with symmetric values) with the coefficients given in the integer sequence $A033184$.

Keywords: hypergeometric series; terminating hypergeometric series; Pfaff transformation; Gauss's hypergeometric theorem; binomial sums; integer sequences; differential equation
Mathematics Subject Classification: 05A10, 05A19, 15A24, 33C05

1. Introduction

In 1797, Johann Friedrich Pfaff stated the following transformation:

$$ {}_2F_1\left(\begin{matrix} l,m \\ n \end{matrix};x\right)=(1-x)^{-l}\,{}_2F_1\left(\begin{matrix} l,n-m \\ n \end{matrix};\frac{x}{x-1}\right). \tag{1.1}$$

If we take $n=2m$, then (1.1) becomes

$$ {}_2F_1\left(\begin{matrix} l,m \\ 2m \end{matrix};x\right)=(1-x)^{-l}\,{}_2F_1\left(\begin{matrix} l,m \\ 2m \end{matrix};\frac{x}{x-1}\right).$$

In this paper, when m is a negative integer, we define the Gaussian hypergeometric series as follows:

$$ {}_2F_1^*\left(\begin{matrix} l,m \\ 2m \end{matrix};x\right)=\sum_{k=0}^{-m}\frac{(l)_k(m)_k}{k!(2m)_k}x^k,$$

which is well-defined, as it is a terminating hypergeometric series since the summation is only for $k=0,..,-m$; additionally, the fact that $2m$ is a negative integer does not make any harm. If we take $m=-1$ and $l=1$, then the left-hand side is a terminating hypergeometric series equal to $1+\frac{x}{2}$, while the right-hand side is also a terminating hypergeometric series, but has 1 as the pole of multiplicity 2 given by $-\frac{3x-2}{2(x-1)^2}$.

Remark 1. *When m is a negative integer, then the Gaussian hypergeometric series can be defined as follows:*

- $$ {}_2F_1\left(\begin{matrix} l,m \\ 2m \end{matrix};x\right)=\lim_{y\to m}\sum_{k=0}^{\infty}\frac{(l)_k(y)_k}{k!(2y)_k}x^k.$$

  Then the series is not a terminating hypergeometric series,
- $$ {}_2F_1^*\left(\begin{matrix} l,m \\ 2m \end{matrix};x\right)=\sum_{k=0}^{-m}\frac{(l)_k(m)_k}{k!(2m)_k}x^k,$$

  which is well-defined, as it is a terminating hypergeometric series since the summation is only for $k=0,..,-m$; additionally, and the fact that $2m$ is also a negative integer does not make any harm.

More generally, if we take l as a positive integer and m as a negative integer, then the left-hand side is a terminating hypergeometric series equal to ${}_2F_1^*\left(\begin{matrix} l,m \\ 2m \end{matrix};x\right) = \sum_{k=0}^{-m} \frac{(l)_k(m)_k x^k}{(2m)_k k!}$, which is a polynomial in x of degree m; whereas, the right-hand side $(1-x)^{-l}\,{}_2F_1^*\left(\begin{matrix} l,m \\ 2m \end{matrix};\frac{x}{x-1}\right) = \frac{1}{(1-x)^l}\sum_{k=0}^{-m}\frac{(l)_k(m)_k x^k}{(2m)_k(x-1)^k k!}$, is also a terminating hypergeometric series, but has 1 as the pole of multiplicity $l-m$ ($-m$ is a positive integer).

Thus, when $n = 2m$, (1.1) does not hold for a positive integer l and for a negative integer m. This paper seeks to solve this situation. We give a new expression $V^{(l,m)}(x)$ depending on l, m, and x such that

$$(1-x)^{-l}\,{}_2F_1^*\left(\begin{matrix} l,m \\ 2m \end{matrix};\frac{x}{x-1}\right) = {}_2F_1^*\left(\begin{matrix} l,m \\ 2m \end{matrix};x\right) + V^{(l,m)}(x),$$

for any positive integer l and for any negative integer m.

As a very interesting consequence, we give a corollary coming from the boundary conditions thereby providing
(1) an expansion of x^{2n+1} as a sum of two terminating hypergeometric series (with symmetric values) with the coefficients given in the integer sequence $A046899$

$$x^{2n+1} = (x-1)^{n+1}\sum_{k=0}^{n}\binom{n+k}{k}x^{n-k} + \sum_{k=0}^{n}\binom{n+k}{k}x^{n-k}(x-1)^k,$$

where the first terms are written as

$$x = 1(x-1) + 1,$$

$$x^3 = (1x+2)(x-1)^2 + 1x + 2(x-1),$$

$$x^5 = (1x^2+3x+6)(x-1)^3 + 1x^2 + 3x(x-1) + 6(x-1)^2,$$

$$x^7 = (1x^3+4x^2+10x+20)(x-1)^4 + 1x^3 + 4x^2(x-1) + 10x(x-1)^2 + 20(x-1)^3,$$

and the colored coefficients can be found in Pascal's triangle

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & \ddots & & \ddots & & 20 & & 15 & & 6 & & 1;
\end{array}$$

and (2) an expansion of $x^{2n+1}(x-2)$ as a sum of two terminating hypergeometric series (with symmetric values) with the coefficients given in the integer sequence $A033184$

$$x^{2n+1}(x-2) = (x-1)^{n+2}\sum_{k=0}^{n}\frac{k+1}{n+1}\binom{2n-k}{n-k}x^{n-k} - \sum_{k=0}^{n}\frac{k+1}{n+1}\binom{2n-k}{n-k}x^k(x-1)^{n-k},$$

where the first terms are written as

$$x(x-2) = 1(x-1)^2 - 1,$$

$$x^3(x-2) = (1x+1)(x-1)^3 - 1x - 1(x-1),$$

$$x^5(x-2) = (1x^2+2x+2)(x-1)^4 - 1x^2 - 2x(x-1) - 2(x-1)^2,$$

$$x^7(x-2) = (1x^3+3x^2+5x+5)(x-1)^4 - 1x^3 - 3x^2(x-1) - 5x(x-1)^2 - 5(x-1)^3,$$

and the colored sequence of numbers is exactly the integer sequence $A033184$.

Appearing frequently in physical problems, Gauss's hypergeometric function (Gauss 1812, Barnes 1908) is

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{1.2}$$

where $(a)_n$ is the Pochhammer symbol defined for any complex number $a \neq 0$ by

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1), & n \in \mathbb{N}, \\ 1, & n = 0, \end{cases} \tag{1.3}$$

which, in terms of the well-known Gamma function, $(a)_n$, is represented by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

The three parameters a, b, and $c$are typically rational numbers and examine (1.2) in terms of the complex variable z. This expression is well defined, provided that c is not a negative integer. Its radius of convergence is 1, assuming that a and b are not negative integers. If a or b is a negative integer, then the infinite series becomes a polynomial. Gauss's hypergeometric function is a solution of the following second-order linear ordinary differential equation (ODE):

$$z(z-1)g'' + \Big((a+b+1)z - c\Big)g' + abg = 0. \tag{1.4}$$

Researchers in the fields of classical orthogonal polynomials, special functions, and related disciplines have undoubtedly relied on foundational results, including linear and quadratic transformations of (1.2) such as the Gauss identity, the Chu-Vandermonde identity, and the quadratic transformation formulas for (1.2). These identities and transformation formulas serve as essential tools for both students and researchers. Their origins trace back to the pioneering work of Gauss [10] and Kummer [14]. Subsequently, Whipple [19–21] and Bailey [6–8], among others, expanded these results to include higher-order hypergeometric functions, which have since found broad applications.

However, none of the former researchers have considered isolated and/or non-defined cases, including, but not limited to, the following:

$$ {}_2F_1^*\left(\begin{matrix} a, -n \\ -2n \end{matrix}; x\right),$$

where n is a positive integer and

$$(1-x)^{-a}\, {}_2F_1^*\left(\begin{matrix} a, -n \\ -2n \end{matrix}; \frac{x}{x-1}\right),$$

where n and a are negative integers, which forms the central focus of this paper.

Hypergeometric function transformations, in which the argument is a free variable, are more manageable than general identities. One could reasonably hope to enumerate or otherwise characterize the class, say, of all two-term function transformations. In fact, the two-term transformations related to the Gauss hypergeometric function ${}_2F_1(x)$ to ${}_2F_1(R(x))$, where R is a rational map of the Riemann sphere to itself, are now fully classified. Besides the celebrated transformation of Euler,

$$ {}_2F_1\left(\begin{matrix} l, m \\ n \end{matrix}; x\right) = (1-x)^{n-l-m}\, {}_2F_1\left(\begin{matrix} n-l, n-m \\ n \end{matrix}; x\right), \tag{1.5} $$

in which $R(x) = x$, and (1.1), in which $R(x) = \dfrac{x}{x-1}$, there are transformations of a larger mapping degree ($deg(R) > 1$), which were classified by Goursat [11]; for example,

$$ {}_2F_1\left(\begin{matrix} l, m \\ 2m \end{matrix}; x\right) = (1-\frac{x}{2})^{-l}\, {}_2F_1\left(\begin{matrix} \frac{l}{2}, \frac{l+1}{2} \\ m+\frac{1}{2} \end{matrix}; \left(\frac{x}{2-x}\right)^2\right), \tag{1.6} $$

where $R(x) = \left(\dfrac{x}{2-x}\right)^2$, see [2, 4, 5].

Quadratic transformations are among the most popular, which were originally worked out by Gauss [9] and Kummer [14], and were concisely proven by Riemann [17]. Recently, Goursat's classification was completed by enumerating the transformations of ${}_2F_1(x)$ without a free parameter, most of which have a quite large degree. Several of the quadratic and cubic transformations of ${}_2F_1(x)$ have analogues on the level ${}_3F_2(x)$ which were discovered by Whipple [19, 20] and Bailey [6–8]. However, no clear analogues have previously been found at levels above ${}_2F_1(x)$ of the remaining ${}_2F_1(x)$ transformations, in particular, the degree-1 transformations of Euler and Pfaff.

Let us go back to (1.1), which was given in [1, p. 68], and [13, Formula 9.131] or [15, Formula 22]. This identity comes as a direct consequence of the Euler integral by replacing t by $1-s$:

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \tag{1.7} $$

where $Re(c) > Re(b) > 0$.

The first most striking and intriguing fact in (1.1) is that when we take $n = 2m$ in (1.1), we find the same values of the parameters l and m in both sides of (1.1):

$$ {}_2F_1^*\left(\begin{matrix} l, m \\ 2m \end{matrix}; z\right) = (1-z)^{-l}\, {}_2F_1^*\left(\begin{matrix} l, m \\ 2m \end{matrix}; \frac{z}{z-1}\right). \tag{1.8} $$

The second most striking and intriguing fact in the Pfaff transformation is that when we take $m = -n$, where n is any positive integer, the left hand side (LHS) of the identity (1.8) is a polynomial in z, whereas the right hand side (RHS) is a product of two rational fractions $\dfrac{1}{(1-z)^l}$ and a polynomial with the variable $\dfrac{z}{z-1}$ for any positive integer u.

This paper aims to solve this situation. In fact, we give a new expression $V^{(l,m)}(z)$ depending on l, m, and z such that

$$ (1-z)^{-l}\, {}_2F_1^*\left(\begin{matrix} l, m \\ 2m \end{matrix}; \frac{z}{z-1}\right) = {}_2F_1^*\left(\begin{matrix} l, m \\ 2m \end{matrix}; z\right) + V^{(l,m)}(z), $$

for any positive integer l and for any negative integer m.

Now, let us now recall some preliminaries on the convergence of (1.2). With the ratio test, it is not difficult to verify the following for the infinite series (1.2):

- it is convergent for all values of z, provided $|z| < 1$, and divergent when $|z| > 1$;
- it is convergent for $z = 1$, provided $Re(c - a - b) > 0$, and divergent when $Re(c - a - b) \leq 0$;
- it is absolutely convergent for $z = -1$, provided $Re(c - a - b) > 0$, convergent but not absolutely for $-1 < Re(c - a - b) \leq 0$, and divergent $Re(c - a - b) < -1$.

We remark that almost all elementary functions of mathematics and mathematical physics are either special cases or limiting cases of Gauss's hypergeometric function. For more details on Gauss's hypergeometric function, we refer to the standard text of Rainville [18]. The reminder of this paper is organized as follows: in Section 2, we give the main result together with the boundary conditions. Section 3, will be devoted to a corollary and an application; in Section 4 we summarize new findings and future recommendations.

2. Main result

We have explained that the Pfaff transformation

$$ {}_2F_1^*\left(\begin{matrix} l, m \\ 2m \end{matrix} ; z \right) = (1-z)^{-l} \, {}_2F_1^*\left(\begin{matrix} l, m \\ 2m \end{matrix} ; \frac{z}{z-1} \right) $$

is NOT true for l and $-m$ positive integers. In the following theorem, we give the right transformation for this situation, which constitutes our main result.

Theorem 1. *For $m \in \{..., -4, -3, -2, -1\}$, $l \in \{1, 2, 3, ...\}$,*

$$ (1-z)^{-l} \, {}_2F_1^*\left(\begin{matrix} l, m \\ 2m \end{matrix} ; \frac{z}{z-1} \right) = {}_2F_1^*\left(\begin{matrix} l, m \\ 2m \end{matrix} ; z \right) + V^{(l,m)}(z), \tag{2.1} $$

where

$$ \begin{aligned} V^{(l,m)}(z) &= \frac{2\sqrt{\pi}(\frac{z}{2})^{l-2m-1}}{(z-1)^{l-m}\Gamma(-m+\frac{1}{2})} \left[\frac{1+(-1)^l}{2}(2-z)(\frac{l}{2})_{-m+1} \times {}_2F_1\left(\begin{matrix} 1-\frac{l}{2}, \frac{1-l}{2}+m \\ \frac{3}{2} \end{matrix} ; \left(\frac{2}{z}-1\right)^2 \right) \right. \\ &\quad \left. - \frac{1-(-1)^l}{2} \frac{z}{2} (\frac{l+1}{2})_{-m} \, {}_2F_1\left(\begin{matrix} \frac{1-l}{2}, m-\frac{l}{2} \\ \frac{1}{2} \end{matrix} ; \left(\frac{2}{z}-1\right)^2 \right) \right]. \end{aligned} \tag{2.2} $$

Remark 2.
- *The new result $V^{(l,m)}(z)$ presented in this theorem is different from the one given in [2, 3, 5]. In fact, the expression given here is the double (two times) of the expression given therein.*
- *All the series that will be considered in the sequel are terminating hypergeometric series, (i.e., a series with finitely many terms) and the variable z should not be 0 and 1.*
- *In the sequel, we take $l = u$, and $m = -n$.*
- *For numerical application, please refer to Appendix 1.*

Proof of Theorem 1. Let us replace m by $-n$, where n is a positive integer. Let us denote by the following:

$$Q_1(u,n,z) = {}_2F_1^*\left(\begin{matrix} u,-n \\ -2n \end{matrix};z\right),$$

$$Q_2(u,n,z) = (1-z)^{-u}\,{}_2F_1^*\left(\begin{matrix} u,-n \\ -2n \end{matrix};\frac{z}{z-1}\right),$$

and

$$\begin{aligned} Q_3(u,n,z) &= \frac{2\sqrt{\pi}(\frac{z}{2})^{u+2n-1}}{(z-1)^{u+n}\Gamma(n+\frac{1}{2})}\left[\frac{1+(-1)^u}{2}(2-z)(\frac{u}{2})_{n+1}\,{}_2F_1\left(\begin{matrix} 1-\frac{u}{2},\frac{1-u}{2}-n \\ \frac{3}{2} \end{matrix};\left(\frac{2}{z}-1\right)^2\right)\right. \\ &\quad \left. -\frac{1-(-1)^u}{2}\frac{z}{2}(\frac{u+1}{2})_n\,{}_2F_1\left(\begin{matrix} \frac{1-u}{2},-n-\frac{u}{2} \\ \frac{1}{2} \end{matrix};\left(\frac{2}{z}-1\right)^2\right)\right]. \end{aligned} \tag{2.3}$$

The following proof does not include the case where u is not a positive integer.

For any positive integer u greater than 2 ($u = 3, 4, 5, ...$), we consider the following relation:

$$(z-1)f(n,u) - \frac{n+u-1}{2(2n-1)}z^2 f(n-1,u) + f(n,u-2) = 0,\ n \geq 2. \tag{2.4}$$

To complete the proof of Theorem 1, we need the following steps:

Step 1. We prove that $Q_1(u,n,z)$, $Q_2(u,n,z)$, and $Q_3(u,n,z)$ fulfill this relation (2.4).

Step 2. Then, we prove that

⊙ $Q_1(u,1,z) - Q_2(u,1,z) - Q_3(u,1,z) = 0$, depending on whether u is odd or even,

⊙ $Q_1(2,n,z) - Q_2(2,n,z) - Q_3(2,n,z) = 0$, for $n \geq 1$,

⊙ $Q_1(1,n,z) - Q_2(1,n,z) - Q_3(1,n,z) = 0$, for $n \geq 1$. □

Step 1. Let us begin by proving that $Q_1(u,n,z)$ fulfills the relation (2.4). In fact, for $n \geq 2$, we have the following

$$\begin{aligned} & (z-1)Q_1(u,n,z) - \frac{n+u-1}{2(2n-1)}z^2 Q_1(u,n-1,z) + Q_1(u-2,n,z) \\ = \quad & (z-1)\,{}_2F_1^*\left(\begin{matrix} u,-n \\ -2n \end{matrix};z\right) - \frac{n+u-1}{4n-2}z^2\,{}_2F_1^*\left(\begin{matrix} u,-n+1 \\ -2n+2 \end{matrix};z\right) + {}_2F_1^*\left(\begin{matrix} u-2,-n \\ -2n \end{matrix};z\right) \\ = \quad & z\,{}_2F_1^*\left(\begin{matrix} u,-n \\ -2n \end{matrix};z\right) + {}_2F_1^*\left(\begin{matrix} u-2,-n \\ -2n \end{matrix};z\right) - {}_2F_1\left(\begin{matrix} u,-n \\ -2n \end{matrix};z\right) - \frac{n+u-1}{4n-2}z^2\,{}_2F_1^*\left(\begin{matrix} u,-n+1 \\ -2n+2 \end{matrix};z\right) \\ = \quad & \left(z + \frac{(u)_1(-n)_1}{(-2n)_1}\frac{z^2}{1!} + \frac{(u)_2(-n)_2}{(-2n)_2}\frac{z^3}{2!} + ... + \frac{(u)_{n-1}(-n)_{n-1}}{(-2n)_{n-1}}\frac{z^n}{(n-1)!} + \frac{(u)_n(-n)_n}{(-2n)_n}\frac{z^{n+1}}{n!}\right) \\ & + \left(\frac{((u-2)_1-(u)_1)(-n)_1}{(-2n)_1}\frac{z}{1!} + \frac{((u-2)_2-(u)_2)(-n)_2}{(-2n)_2}\frac{z^2}{2!} + ... + \frac{((u-2)_n-(u)_n)(-n)_n}{(-2n)_n}\frac{z^n}{n!}\right) \\ & - \frac{n+u-1}{4n-2}\left(z^2 + \frac{(u)_1(-n+1)_1}{(-2n+2)_1}\frac{z^3}{1!} + \frac{(u)_2(-n+1)_2}{(-2n+2)_2}\frac{z^4}{2!} + ... + \frac{(u)_{n-1}(-n+1)_{n-1}}{(-2n+2)_{n-1}}\frac{z^{n+1}}{(n-1)!}\right). \end{aligned}$$

(1) The terms of z^1 are 1 and $-\frac{((u-2)_1-(u)_1)(-n)_1}{(-2n)_1}\frac{1}{1!}$, with sum 0.

(2) For $2 \leq k \leq n$, the terms of z^k are $\frac{(u)_k(-n)_k}{(-2n)_k}\frac{1}{k!}$, $-\frac{((u-2)_{k+1}-(u)_{k+1})(-n)_{k+1}}{(-2n)_{k+1}}\frac{1}{(k+1)!}$, and

$-\dfrac{n+u-1}{4n-2}\dfrac{(u)_{k-1}(-n+1)_{k-1}}{(-2n+2)_{k-1}}\dfrac{1}{(k-1)!}$, with sum 0.

(3) The terms of z^{n+1} are $-\dfrac{(n+u-1)(u)_{n-1}(-n+1)_{n-1}}{(4n-2)(-2n+2)_{n-1}(n-1)!}$ and $\dfrac{(u)_n(-n)_n}{n!(-2n)_n}$, with sum 0.

Now, we prove that $Q_2(u,n,z)$ fulfills the relation (2.4). In fact (2.4) with $Q_2(u,n,z)$ becomes the following:

$$\begin{aligned}&(z-1)Q_2(u,n,z)-\frac{n+u-1}{2(2n-1)}z^2Q_2(u,n-1,z)+Q_2(u-2,n,z)\\ =&(z-1)(1-z)^{-u}\,{}_2F_1\left(\begin{matrix}u,-n\\-2n\end{matrix};\frac{z}{z-1}\right)-\frac{n+u-1}{4n-2}z^2(1-z)^{-u}\,{}_2F_1\left(\begin{matrix}u,-n+1\\-2n+2\end{matrix};z\right)\\ &+(1-z)^{-u+2}\,{}_2F_1\left(\begin{matrix}u-2,-n\\-2n\end{matrix};\frac{z}{z-1}\right).\end{aligned}$$

Here, a change of variable $t=\dfrac{z}{z-1}$ and a simplification by $(1-t)^{u-2}$ give the following:

$$(t-1)\,{}_2F_1^*\left(\begin{matrix}u,-n\\-2n\end{matrix};t\right)-\frac{n+u-1}{4n-2}t^2\,{}_2F_1^*\left(\begin{matrix}u,-n+1\\-2n+2\end{matrix};t\right)+\,{}_2F_1^*\left(\begin{matrix}u-2,-n\\-2n\end{matrix};t\right),$$

which is exactly (2.4) with $Q_1(u,n,t)$.

The last step is to prove that $Q_3(u,n,z)$ fulfills the relation (2.4). We share $Q_3(u,n,z)$ into two quantities, depending on whether u is even or odd, as follows:

$$Q_{31}(u,n,z)=\frac{2\sqrt{\pi}(\frac{z}{2})^{u+2n-1}(2-z)(\frac{u}{2})_{n+1}}{(z-1)^{u+n}\Gamma(n+\frac{1}{2})}\,{}_2F_1\left(\begin{matrix}1-\frac{u}{2},\frac{1-u}{2}-n\\\frac{3}{2}\end{matrix};\left(\frac{2}{z}-1\right)^2\right),$$

and

$$Q_{32}(u,n,z)=-\frac{2\sqrt{\pi}(\frac{z}{2})^{u+2n-1}}{(z-1)^{u+n}\Gamma(n+\frac{1}{2})}\frac{z}{2}(\frac{u+1}{2})_n\,{}_2F_1\left(\begin{matrix}\frac{1-u}{2},-n-\frac{u}{2}\\\frac{1}{2}\end{matrix};\left(\frac{2}{z}-1\right)^2\right).$$

We prove the result for Q_{31} and follow the same steps for the proof of Q_{32}.

Let us prove that $Q_{31}(u,n,z)$ fulfills the relation (2.4). In fact, (2.4) with $Q_{31}(u,n,z)$ becomes the following:

$$\begin{aligned}&(z-1)Q_{31}(u,n,z)-\frac{n+u-1}{2(2n-1)}z^2Q_{31}(u,n-1,z)+Q_{31}(u-2,n,z)\\ =&(z-1)\frac{2\sqrt{\pi}(\frac{z}{2})^{u+2n-1}(2-z)(\frac{u}{2})_{n+1}}{(z-1)^{u+n}\Gamma(n+\frac{1}{2})}\,{}_2F_1\left(\begin{matrix}1-\frac{u}{2},\frac{1-u}{2}-n\\\frac{3}{2}\end{matrix};\left(\frac{2}{z}-1\right)^2\right)\\ &-\frac{n+u-1}{2(2n-1)}z^2\frac{2\sqrt{\pi}(\frac{z}{2})^{u+2n-3}(2-z)(\frac{u}{2})_n}{(z-1)^{u+n}\Gamma(n-\frac{1}{2})}\,{}_2F_1\left(\begin{matrix}1-\frac{u}{2},\frac{3-u}{2}-n\\\frac{3}{2}\end{matrix};\left(\frac{2}{z}-1\right)^2\right)\\ &+\frac{2\sqrt{\pi}(\frac{z}{2})^{u+2n-3}(2-z)(\frac{u}{2})_{n+1}}{\Gamma(n+\frac{1}{2})}\,{}_2F_1\left(\begin{matrix}2-\frac{u}{2},\frac{3-u}{2}-n\\\frac{3}{2}\end{matrix};\left(\frac{2}{z}-1\right)^2\right).\end{aligned}$$

If we take the common factor $\dfrac{4\sqrt{\pi}(\frac{u}{2})_n(\frac{z}{2})^{u+2n-3}(2-z)}{\Gamma(\frac{n}{2}+1)(z-1)^{n+u-1})}$ away, we obtain the following:

$$(z-1)Q_{31}(u,n,z)-\frac{n+u-1}{2(2n-1)}z^2Q_{31}(u,n-1,z)+Q_{31}(u-2,n,z)$$

$$= \frac{(2n+u)z^2}{2}\,{}_2F_1\left(\begin{matrix}1-\frac{u}{2},\frac{1-u}{2}-n\\ \frac{3}{2}\end{matrix};\left(\frac{2}{z}-1\right)^2\right)-(n+u-1)z^2\,{}_2F_1\left(\begin{matrix}1-\frac{u}{2},\frac{3-u}{2}-n\\ \frac{3}{2}\end{matrix};\left(\frac{2}{z}-1\right)^2\right)$$
$$+2(z-1)(u-2)\,{}_2F_1\left(\begin{matrix}2-\frac{u}{2},\frac{3-u}{2}-n\\ \frac{3}{2}\end{matrix};\left(\frac{2}{z}-1\right)^2\right).$$

The change of variable $z=\frac{2}{\sqrt{t+1}}$, (as we are considering only real cases and terminating hypergeometric series we assume $t\geq 0$), and some simplifications lead to the following:

$$\frac{(2n+u)}{u-2}\,{}_2F_1\left(\begin{matrix}1-\frac{u}{2},\frac{1-u}{2}-n\\ \frac{3}{2}\end{matrix};t\right)-\frac{2(n+u-1)}{u-2}\,{}_2F_1\left(\begin{matrix}1-\frac{u}{2},\frac{3-u}{2}-n\\ \frac{3}{2}\end{matrix};t\right)-(t-1)\,{}_2F_1\left(\begin{matrix}2-\frac{u}{2},\frac{3-u}{2}-n\\ \frac{3}{2}\end{matrix};t\right)$$
$$=\frac{(2n+u)}{u-2}\Bigg(1+\frac{(1-\frac{u}{2})(\frac{1-u}{2}-n)}{\frac{3}{2}}\frac{t^1}{1!}+\frac{(1-\frac{u}{2})(2-\frac{u}{2})(\frac{1-u}{2}-n)(\frac{3-u}{2}-n)}{\frac{3*5}{2*2}}\frac{t^2}{2!}+$$
$$\ldots+\frac{(1-\frac{u}{2})_{\frac{u}{2}-2}(\frac{1-u}{2}-n)_{\frac{u}{2}-2}}{(\frac{3}{2})_{\frac{u}{2}-2}}\frac{t^{\frac{u}{2}-2}}{(\frac{u}{2}-2)!}+\frac{(1-\frac{u}{2})_{\frac{u}{2}-1}(\frac{1-u}{2}-n)_{\frac{u}{2}-1}}{(\frac{3}{2})_{\frac{u}{2}-1}}\frac{t^{\frac{u}{2}-1}}{(\frac{u}{2}-1)!}\Bigg)$$
$$-\frac{(2n+2u+2)}{u-2}\Bigg(1+\frac{(1-\frac{u}{2})(\frac{3-u}{2}-n)}{\frac{3}{2}}\frac{t^1}{1!}+\frac{(1-\frac{u}{2})(2-\frac{u}{2})(\frac{3-u}{2}-n)(\frac{5-u}{2}-n)}{\frac{3*5}{2*2}}\frac{t^2}{2!}+$$
$$\ldots+\frac{(1-\frac{u}{2})_{\frac{u}{2}-2}(\frac{3-u}{2}-n)_{\frac{u}{2}-2}}{(\frac{3}{2})_{\frac{u}{2}-2}}\frac{t^{\frac{u}{2}-2}}{(\frac{u}{2}-2)!}+\frac{(1-\frac{u}{2})_{\frac{u}{2}-1}(\frac{3-u}{2}-n)_{\frac{u}{2}-1}}{(\frac{3}{2})_{\frac{u}{2}-1}}\frac{t^{\frac{u}{2}-1}}{(\frac{u}{2}-1)!}\Bigg)$$
$$+\Bigg(1+\frac{(2-\frac{u}{2})(\frac{3-u}{2}-n)}{\frac{3}{2}}\frac{t^1}{1!}+\frac{(2-\frac{u}{2})(3-\frac{u}{2})(\frac{3-u}{2}-n)(\frac{5-u}{2}-n)}{\frac{3*5}{2*2}}\frac{t^2}{2!}+$$
$$\ldots+\frac{(2-\frac{u}{2})_{\frac{u}{2}-3}(\frac{3-u}{2}-n)_{\frac{u}{2}-3}}{(\frac{3}{2})_{\frac{u}{2}-3}}\frac{t^{\frac{u}{2}-3}}{(\frac{u}{2}-3)!}+\frac{(1-\frac{u}{2})_{\frac{u}{2}-2}(\frac{3-u}{2}-n)_{\frac{u}{2}-2}}{(\frac{3}{2})_{\frac{u}{2}-2}}\frac{t^{\frac{u}{2}-2}}{(\frac{u}{2}-2)!}\Bigg)$$
$$-\Bigg(t+\frac{(2-\frac{u}{2})(\frac{3-u}{2}-n)}{\frac{3}{2}}\frac{t^2}{1!}+\frac{(2-\frac{u}{2})(3-\frac{u}{2})(\frac{3-u}{2}-n)(\frac{5-u}{2}-n)}{\frac{3*5}{2*2}}\frac{t^3}{2!}+$$
$$\ldots+\frac{(2-\frac{u}{2})_{\frac{u}{2}-3}(\frac{3-u}{2}-n)_{\frac{u}{2}-3}}{(\frac{3}{2})_{\frac{u}{2}-3}}\frac{t^{\frac{u}{2}-2}}{(\frac{u}{2}-3)!}+\frac{(1-\frac{u}{2})_{\frac{u}{2}-2}(\frac{3-u}{2}-n)_{\frac{u}{2}-2}}{(\frac{3}{2})_{\frac{u}{2}-2}}\frac{t^{\frac{u}{2}-1}}{(\frac{u}{2}-2)!}\Bigg),$$

where u is an odd positive integer. In the aforementioned expression, we have the following:

- the terms with t^0 are $\frac{(u+2n)}{(u-2)}$, $-\frac{(2u-2+2n)}{(u-2)}$, and 1, with sum 0;
- the sum of terms with t^k, and $1\leq k\leq\frac{u}{2}-2$ is given by

$$\frac{(2n+u)}{u-2}\frac{(1-\frac{u}{2})_k(\frac{1-u}{2}-n)_k}{(\frac{3}{2})_k k!}-\frac{(2n+2u+2)}{u-2}\frac{(1-\frac{u}{2})_k(\frac{3-u}{2}-n)_k}{(\frac{3}{2})_k k!}$$
$$+\frac{(2-\frac{u}{2})_k(\frac{3-u}{2}-n)_k}{(\frac{3}{2})_k k!}-\frac{(2-\frac{u}{2})_{k-1}(\frac{3-u}{2}-n)_{k-1}}{(\frac{3}{2})_{k-1}(k-1)!}=0;$$

- the sum of terms with $t^{\frac{u}{2}-1}$ is given by

$$\frac{(2n+u)}{u-2}\frac{(1-\frac{u}{2})_{\frac{u}{2}-1}(\frac{1-u}{2}-n)_{\frac{u}{2}-1}}{(\frac{3}{2})_{\frac{u}{2}-1}(\frac{u}{2}-1)!}-\frac{(2n+2u+2)}{u-2}\frac{(1-\frac{u}{2})_{\frac{u}{2}-1}(\frac{3-u}{2}-n)_{\frac{u}{2}-1}}{(\frac{3}{2})_{\frac{u}{2}-1}(\frac{u}{2}-1)!}-\frac{(2-\frac{u}{2})_{\frac{u}{2}-2}(\frac{3-u}{2}-n)_{\frac{u}{2}-2}}{(\frac{3}{2})_{\frac{u}{2}-2}(\frac{u}{2}-2)!}=0.$$

Remark 3. *We follow the same steps to prove that $Q_{32}(u, n, z)$ fulfills the relation* (2.4) *when u is an odd positive integer.*

Step 2. We need to prove the following:

⊙ $Q_1(u, 1, z) - Q_2(u, 1, z) - Q_3(u, 1, z) = 0$, for u is odd or even;
⊙ $Q_1(2, n, z) - Q_2(2, n, z) - Q_3(2, n, z) = 0$ for $n \geq 1$;
⊙ $Q_1(1, n, z) - Q_2(1, n, z) - Q_3(1, n, z) = 0$ for $n \geq 1$.

Let us prove the first one, i.e.,

⊙ $Q_1(1, n, z) - Q_2(1, n, z) - Q_3(1, n, z) = 0$ for $n \geq 1$.

Lemma 1. *For $u = 1$ and for any $n \geq 1$, we have the following interesting result:*

$$(1-z)^{-1}\ {}_2F_1^*\left(\begin{array}{c}1,-n\\-2n\end{array};\frac{z}{z-1}\right)-\ {}_2F_1^*\left(\begin{array}{c}1,-n\\-2n\end{array};z\right)=\frac{(-n)_n}{(-2n)_n}\frac{z^{2n+1}}{(z-1)^{n+1}}. \tag{2.5}$$

Proof of Lemma 1. To prove

$$(1-z)^{-1}\ {}_2F_1^*\left(\begin{array}{c}1,-n\\-2n\end{array};\frac{z}{z-1}\right)-\ {}_2F_1^*\left(\begin{array}{c}1,-n\\-2n\end{array};z\right)=\frac{(-n)_n}{(-2n)_n}\frac{z^{2n+1}}{(z-1)^{n+1}},$$

it is equivalent to prove

$$z^{2n+1}=-\frac{(-2n)_n}{(-n)_n}\left((z-1)^n\ {}_2F_1^*\left(\begin{array}{c}1,-n\\-2n\end{array};\frac{z}{z-1}\right)+(z-1)^{n+1}\ {}_2F_1^*\left(\begin{array}{c}1,-n\\-2n\end{array};z\right)\right).$$

Expanding both ${}_2F_1^*$ above we should prove that (see [12] for the inverse summation)

$$,z^{2n+1}=(z-1)^{n+1}\sum_{k=0}^{n}\left(\begin{array}{c}n+k\\k\end{array}\right)z^{n-k}+\sum_{k=0}^{n}\left(\begin{array}{c}n+k\\k\end{array}\right)z^{n-k}(z-1)^k,\ n\geq 0, \tag{2.6}$$

which can be done by induction.

For $n = 0$, we have the following:

$$z^1=(z-1)+1.$$

We suppose the property fulfilled for n and we prove that it is true for $n + 1$.

Let us start from z^2z^{2n+1}, which can be written as $\Big(z(z-1)+z\Big)z^{2n+1}$, where

$$\begin{aligned}
z^{2n+3}=&\Big(z(z-1)+z\Big)\left((z-1)^{n+1}\sum_{k=0}^{n}\left(\begin{array}{c}n+k\\k\end{array}\right)z^{n-k}+\sum_{k=0}^{n}\left(\begin{array}{c}n+k\\k\end{array}\right)z^{n-k}(z-1)^k\right)\\
=&(z-1)^{n+2}\sum_{k=0}^{n}\left(\begin{array}{c}n+k\\k\end{array}\right)z^{n+1-k}+(z-1)^{n+1}\sum_{k=0}^{n}\left(\begin{array}{c}n+k\\k\end{array}\right)z^{n+1-k}+\sum_{k=0}^{n}\left(\begin{array}{c}n+k\\k\end{array}\right)z^{n+1-k}(z-1)^{k+1}\\
&+\sum_{k=0}^{n}\left(\begin{array}{c}n+k\\k\end{array}\right)z^{n+1-k}(z-1)^k\\
=&(z-1)^{n+2}z^{n+1}+(z-1)^{n+2}\sum_{k=1}^{n}\left(\begin{array}{c}n+k\\k\end{array}\right)z^{n+1-k}+z^{n+1}+\sum_{k=1}^{n}\left(\begin{array}{c}n+k\\k\end{array}\right)z^{n+1-k}(z-1)^k\\
&+(z-1)^{n+1}\sum_{k=0}^{n}\left(\begin{array}{c}n+k\\k\end{array}\right)z^{n+1-k}+\sum_{k=0}^{n}\left(\begin{array}{c}n+k\\k\end{array}\right)z^{n+1-k}(z-1)^{k+1}.
\end{aligned}$$

Using

$$\binom{n+k}{k-1}+\binom{n+k}{k}=\binom{n+k+1}{k},\ 1\leq k\leq n,$$

we obtain

$$\begin{aligned}z^{2n+3}=&(z-1)^{n+2}z^{n+1}+(z-1)^{n+2}\sum_{k=1}^{n}\left(\binom{n+k+1}{k}-\binom{n+k}{k-1}\right)z^{n+1-k}+z^{n+1}\\&+\sum_{k=1}^{n}\left(\binom{n+k+1}{k}-\binom{n+k}{k-1}\right)z^{n+1-k}(z-1)^{k}+(z-1)^{n+1}\sum_{k=0}^{n}\binom{n+k}{k}z^{n+1-k}\\&+\sum_{k=0}^{n}\binom{n+k}{k}z^{n+1-k}(z-1)^{k+1}\\=&(z-1)^{n+2}\sum_{k=0}^{n}\binom{n+k+1}{k}z^{n+1-k}-(z-1)^{n+2}\sum_{k=1}^{n}\binom{n+k}{k-1}z^{n+1-k}\\&+\sum_{k=0}^{n}\binom{n+k+1}{k}z^{n+1-k}(z-1)^{k}-\sum_{k=1}^{n}\binom{n+k}{k-1}z^{n+1-k}(z-1)^{k}\\&+(z-1)^{n+1}\sum_{k=0}^{n}\binom{n+k}{k}z^{n+1-k}+\sum_{k=0}^{n}\binom{n+k}{k}z^{n+1-k}(z-1)^{k+1}.\end{aligned}$$

When we add the $(n+1)^{th}$ term in the first and third summations, we obtain the following:

$$\begin{aligned}z^{2n+3}=&(z-1)^{n+2}\sum_{k=0}^{n+1}\binom{n+k+1}{k}z^{n+1-k}-(z-1)^{n+2}\binom{2n+2}{n+1}\\&-(z-1)^{n+2}\sum_{k=1}^{n}\binom{n+k}{k-1}z^{n+1-k}+\sum_{k=0}^{n+1}\binom{n+k+1}{k}z^{n+1-k}(z-1)^{k}-\binom{2n+2}{n+1}(z-1)^{n+1}\\&-\sum_{k=1}^{n}\binom{n+k}{k-1}z^{n+1-k}(z-1)^{k}+(z-1)^{n+1}\sum_{k=0}^{n}\binom{n+k}{k}z^{n+1-k}+\sum_{k=0}^{n}\binom{n+k}{k}z^{n+1-k}(z-1)^{k+1}.\end{aligned}$$

The last step is to prove that

$$\begin{aligned}&-(z-1)^{n+2}\binom{2n+2}{n+1}-(z-1)^{n+2}\sum_{k=1}^{n}\binom{n+k}{k-1}z^{n+1-k}-\binom{2n+2}{n+1}(z-1)^{n+1}\\&-\sum_{k=1}^{n}\binom{n+k}{k-1}z^{n+1-k}(z-1)^{k}+(z-1)^{n+1}\sum_{k=0}^{n}\binom{n+k}{k}z^{n+1-k}+\sum_{k=0}^{n}\binom{n+k}{k}z^{n+1-k}(z-1)^{k+1}=0.\end{aligned}$$

Using

$$\begin{aligned}&-(z-1)^{n+2}\sum_{k=2}^{n}\binom{n+k}{k-1}z^{n+1-k}+(z-1)^{n+1}\sum_{k=1}^{n-1}\binom{n+k}{k}z^{n+1-k}\\&=-(z-1)^{n+1}\sum_{k=2}^{n}\binom{n+k}{k-1}z^{n+2-k}+(z-1)^{n+1}\sum_{k=2}^{n}\binom{n+k}{k-1}z^{n+1-k}+(z-1)^{n+1}\sum_{k=1}^{n-1}\binom{n+k}{k}z^{n+1-k}\end{aligned}$$

$$= -(z-1)^{n+1}\sum_{k=1}^{n-1}\binom{n+k+1}{k}z^{n+1-k} + (z-1)^{n+1}\sum_{k=2}^{n}\binom{n+k}{k-1}z^{n+1-k} + (z-1)^{n+1}\sum_{k=1}^{n-1}\binom{n+k}{k}z^{n+1-k}$$

$$= (z-1)^{n+1}\sum_{k=2}^{n-1}\left(-\binom{n+k+1}{k}+\binom{n+k}{k-1}+\binom{n+k}{k}\right)z^{n+1-k} - (z-1)^{n+1}\left(z^n - z\binom{2n}{n-1}\right)$$

$$= -(z-1)^{n+1}z^n + z(z-1)^{n+1}\binom{2n}{n-1},$$

and

$$\sum_{k=0}^{n}\binom{n+k}{k}z^{n+1-k}(z-1)^{k+1} - \sum_{k=1}^{n}\binom{n+k}{k-1}z^{n+1-k}(z-1)^{k} = z(z-1)^{n+1}\binom{2n+1}{n+1},$$

we get the desired result (for more information, see the Appendix 2). □

Another interesting case is when $u = 2$.

Lemma 2. *For $u = 2$ and for any $n \geq 1$, we have the following result:*

$$(1-z)^{-2}\,{}_2F_1^*\left(\begin{matrix}2,-n\\-2n\end{matrix};\frac{z}{z-1}\right) - {}_2F_1^*\left(\begin{matrix}2,-n\\-2n\end{matrix};z\right) = \frac{(-n-1)_n}{(-2n)_n}\frac{(z-2)z^{2n+1}}{(z-1)^{n+2}},$$

which is equivalent to

$$z^{2n+1}(z-2) = (z-1)^{n+2}\sum_{k=0}^{n}\frac{k+1}{n+1}\binom{2n-k}{n-k}z^{n-k} - \sum_{k=0}^{n}\frac{k+1}{n+1}\binom{2n-k}{n-k}z^{k}(z-1)^{n-k}.$$

Now, let us prove the following:

⊙ $Q_1(u,1,z) - Q_2(u,1,z) - Q_3(u,1,z) = 0$, for any odd number u (i.e., $u = 1,3,5,...$); and

⊙ $Q_1(u,1,z) - Q_2(u,1,z) - Q_3(u,1,z) = 0$, for any even number u (i.e., $u = 1,3,5,...$).

Lemma 3. *For $n = 1$, and for any odd number $u \geq 1$, we have the following interesting result:*

$$-2z\left(\frac{z}{2(z-1)}\right)^{u+1}(u+1)\,{}_2F_1\left(\begin{matrix}\frac{1-u}{2},-1-\frac{u}{2}\\\frac{1}{2}\end{matrix};\left(\frac{2}{z}-1\right)^2\right) = -\frac{1}{2}\frac{(u+2)z-2}{(z-1)^{u+1}} - \frac{uz+2}{2}.$$

For $n = 1$, and for any even number $u \geq 2$, we have the following interesting result:

$$(2-z)\left(\frac{z}{2(z-1)}\right)^{u+1}u(u+2)\,{}_2F_1\left(\begin{matrix}\frac{-1-u}{2},1-\frac{u}{2}\\\frac{1}{2}\end{matrix};\left(\frac{2}{z}-1\right)^2\right) = -\frac{1}{2}\frac{(u+2)z-2}{(z-1)^{u+1}} - \frac{uz+2}{2}.$$

If we put $\frac{2}{z}-1 = t$ and , $t \neq -1$, where $p \in \mathbb{N}$, then the two equations above become

$${}_2F_1\left(\begin{matrix}-p,-p-\frac{3}{2}\\\frac{1}{2}\end{matrix};t^2\right) = \frac{(2p+2-t)(t+1)^{2p+2} + (2p+2+t)(t-1)^{2p+2}}{4p+4}, \tag{2.7}$$

and

$$t\,{}_2F_1\left(\begin{matrix}-p,-p-\frac{3}{2}\\\frac{3}{2}\end{matrix};t^2\right) = \frac{(2p+3+t)(t-1)^{2p+3} + (2p+3-t)(t+1)^{2p+3}}{8(p+1)(p+2)} \tag{2.8}$$

when $u = 2p+1$ and $u = 2p+3$ respectively.

Proof of Lemma 3. The proof of this Lemma can be derived from the following known hypergeometric identities:

- some properties of Jacobi polynomials with negative parameters depending on the degree of the polynomials,

$$JacobiP(n,\alpha,\beta,x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-x}{2}\right); \tag{2.9}$$

- the three term recurrence relation fulfilled by Jacobi polynomials

$$\begin{aligned}&2(n+\alpha+1)JacobiP(n,\alpha,\beta,x) - 2(n+1)JacobiP(n+1,\alpha,\beta,x)\\ &= (2n+\alpha+\beta+2)(1-x)JacobiP(n,\alpha+1,\beta,x);\,\textit{and}\end{aligned}$$

- the identities [16, 15.4.7 and 15.4.9],

$$\begin{aligned}{}_2F_1\left(\begin{matrix}a, a+\frac{1}{2}\\ \frac{1}{2}\end{matrix};t^2\right) &= \frac{(1+t)^{-2a}+(1-t)^{-2a}}{2},\\ {}_2F_1\left(\begin{matrix}a, a+\frac{1}{2}\\ \frac{3}{2}\end{matrix};t^2\right) &= \frac{(1+t)^{1-2a}-(1-t)^{1-2a}}{2(1-2a)t}.\end{aligned}$$

□

Summary 1.

- We have proven that $Q_1(u,n,z)$, $Q_2(u,n,z)$, and $Q_3(u,n,z)$ fulfill the relation (2.4):

$$(z-1)f(n,u) - \frac{n+u-1}{2(2n-1)}z^2 f(n-1,u) + f(n,u-2) = 0,\ n \geq 2.$$

- From the two corollaries (i.e., the boundary conditions), if we have $f(n,1),\ n \geq 2$, then we get $f(n,3),\ n \geq 2$, with the only value of $f(1,3)$.
- If we have $f(n,3),\ n \geq 2$, then we get $f(n,5),\ n \geq 2$, with the only value of $f(1,5)$ etc...
- If we have $f(n,2),\ n \geq 2$, then we get $f(n,4),\ n \geq 2$, with the only value of $f(1,4)$.
- If we have $f(n,4),\ n \geq 2$, then we get $f(n,6),\ n \geq 2$, with the only value of $f(1,6)$ etc...
  This procedure generates all the terms $f(n,u),\ n,u \geq 1$.

Remark 4. *Gauss stated the following result on hypergeometric series:*

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix};1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ c-a-b>0.$$

When $t=\pm1$, the result of Lemma 3 coincides with Gauss's result. In fact, for $t=\pm1$, we have

$${}_2F_1\left(\begin{matrix}-p, -p-\frac{3}{2}\\ \frac{1}{2}\end{matrix};1\right) = 2^{2p}\frac{(2p+1)}{p+1}, \tag{2.10}$$

and

$$\pm\ {}_2F_1\left(\begin{matrix}-p, -p-\frac{3}{2}\\ \frac{3}{2}\end{matrix};1\right) = \pm\frac{2^{2p+1}}{(p+2)}. \tag{2.11}$$

3. Corollary and application

The first corollary comes from the change of variable $z \to \dfrac{z}{z-1}$

Corollary 1. *We have the following equality:*

$$\begin{aligned} Q_2(u,n,z) &= (1-z)^{-u} Q_1(u,n,\frac{z}{z-1}), \\ Q_1(u,n,z) &= (1-z)^{-u} Q_2(u,n,\frac{z}{z-1}), \\ Q_3(u,n,z) &= -(1-z)^{-u} Q_3(u,n,\frac{z}{z-1}). \end{aligned}$$

Proof. In fact, we have the same expressions for the parameters in Q_1 and Q_2 but with different variables, i.e.,

$$\begin{aligned} Q_1(u,n,z) &= {}_2F_1^*\left(\begin{matrix} u,-n \\ -2n \end{matrix}\ ;z\right), \\ Q_2(u,n,z) &= (1-z)^{-u}\ {}_2F_1^*\left(\begin{matrix} u,-n \\ -2n \end{matrix}\ ;\frac{z}{z-1}\right), \end{aligned}$$

which become

$$\begin{aligned} Q_1(u,n,\frac{z}{z-1}) &= {}_2F_1^*\left(\begin{matrix} u,-n \\ -2n \end{matrix}\ ;\frac{z}{z-1}\right), \\ Q_2(u,n,\frac{z}{z-1}) &= (1-z)^{u}\ {}_2F_1^*\left(\begin{matrix} u,-n \\ -2n \end{matrix}\ ;z\right). \end{aligned}$$

Finally,

$$Q_3(u,n,\frac{z}{z-1}) = -(1-z)^{u} Q_3(u,n,z)$$

can be easily checked. If we denote by $T(z) = \dfrac{z}{z-1}$, then T is a Mobius transformation such that $T \circ T = id$. □

4. Conclusions

The Pfaff transformation

$${}_2F_1^*\left(\begin{matrix} a,b \\ c \end{matrix}\ ;x\right) = (1-x)^{-a}\ {}_2F_1^*\left(\begin{matrix} a,c-b \\ c \end{matrix}\ ;\frac{x}{x-1}\right)$$

is not true for $c = 2b$ and the negative integer b. In this paper, a new expression $V^{(a,b)}(x)$, depending on a, b, and x was given, in this paper so that

$${}_2F_1^*\left(\begin{matrix} a,b \\ 2b \end{matrix}\ ;x\right) = (1-x)^{-a}\ {}_2F_1^*\left(\begin{matrix} a,b \\ 2b \end{matrix}\ ;\frac{x}{x-1}\right) + V^{(a,b)}(z)$$

becomes true for every positive integer a and for every negative integer b.

We are sure that this new result can be widely applied, especially in number theory. Moreover, we can state the following two open problems:

- With the expression $V^{(a,b)}(x)$, depending on a, b, and x, we believe that

$$ {}_2F_1^*\left(\begin{matrix} \frac{1}{2p+1}, -m \\ -2m \end{matrix} ; x \right) = +V^{(\frac{1}{2p+1}, -m)}(z), $$

for p, m positives integers; and

- If we take a positive integer and b, c negative integers such that $c \leq b < 0$ in the general Pfaff transformation can be written as

$$ {}_2F_1^*\left(\begin{matrix} a, b \\ c \end{matrix} ; x \right) = (1-x)^{-a}\, {}_2F_1^*\left(\begin{matrix} a, c-b \\ c \end{matrix} ; \frac{x}{x-1} \right), $$

then the above transformation does not hold. For example, if we take $a = 1$, $b = -1$, and $c = -3$, the left-hand side is a polynomial of degree 2, whereas the right-hand side is a rational fraction with a pole of multiplicity 2. Is it possible to explicitly give the expression $V^{(a,b,c)}(x)$ such that

$$ {}_2F_1^*\left(\begin{matrix} a, b \\ c \end{matrix} ; x \right) = (1-x)^{-a}\, {}_2F_1^*\left(\begin{matrix} a, c-b \\ c \end{matrix} ; \frac{x}{x-1} \right) + V^{(a,b,c)}(x) $$

holds for any a positive integer and b, c negative integers such that $c \leq b < 0$.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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Appendix 1.

In this appendix, we write the expressions of $Q1$, $Q2$, and $Q3$ (this latter is split into two expressions depending on the parity of u) given in the proof of Theorem 1 and the reader can be assured by the correctness of our result.

> restart;
> $Q1 := (u, n, z)- > hypergeom([u, -n], [-2*n], z);$
> $Q2 := (u, n, z)- > (1-z)^{(-u)} * hypergeom([u, -n], [-2*n], z/(z-1));$
> $Q31 := (u, n, z)- > 2*((1/2)*z)^{(u+2*n-1)} * (z-1)^{(-n-u)} * (2-z) * \sqrt{\pi}$
$*pochhammer((1/2)*u, n+1) * hypergeom([1-(1/2)*u, 1/2-n-(1/2)*u],$
$[3/2], (2-z)^2/z^2)/GAMMA(n+1/2);$
> $Q32 := (u, n, z)- > -2*((1/2)*z)^{(u+2*n)} * (z-1)^{(-n-u)} * \sqrt{\pi}$
$*pochhammer((1/2)*u+1/2, n) * hypergeom([-n-(1/2)*u, 1/2-(1/2)*u],$
$[1/2], (2-z)^2/z^2)/GAMMA(n+1/2);$
> $factor(simplify(Q2(1, 1, z) - Q1(1, 1, z) - Q32(1, 1, z)));$
> $factor(simplify(Q2(2, 1, z) - Q1(2, 1, z) - Q31(2, 1, z))).$

Appendix 2.

In this appendix we provide the expressions given in page 12 and the reader can be assured by the correctness of our result.

> restart;
> EXPRESSION1 := n→ $-(z-1)^{(n+2)}*binomial(2*n+2, n+1)-(z-1)^{(n+2)}*(sum(binomial(n+k, k-1)*z^{(n+1-k)}, k = 1..n))-(z-1)^{(n+1)}*binomial(2*n+2, n+1)-(sum(binomial(n+k, k-1)*(z-1)^k*z^{(n+1-k)}, k = 1..n)) + (z-1)^{(n+1)} * (sum(binomial(n+k, k) * z^{(n+1-k)}, k = 0..n)) + sum(binomial(n+k, k) * (z-1)^{(k+1)} * z^{(n+1-k)}, k = 0..n);$
> factor(simplify(EXPRESSION1(3))); (This compilation will give 0)
> restart;
> EXPRESSION2 := n→ $-(z-1)^{(n+1)} * z * binomial(2*n+1, n+1) - (sum(binomial(n+k, k-1) * (z-1)^k * z^{(n+1-k)}, k = 1..n)) + sum(binomial(n+k, k) * (z-1)^{(k+1)} * z^{(n+1-k)}, k = 0..n);$
> factor(simplify(EXPRESSION2(5))); (This compilation will give 0).