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Research article

Characterizations of normaloid operators in Hilbert spaces via Birkhoff–James orthogonality

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Abstract: Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. An operator T is said to be *normaloid* if its numerical radius $w(T)$ equals its operator norm $\|T\|$. In this paper, we establish several characterizations of normaloid operators in Hilbert spaces. In particular, we investigate these operators through the framework of Birkhoff–James orthogonality and norm-parallelism. Mainly, we show that T is normaloid if, and only if, there exists $\xi_0 \in \mathbb{C}$ with $|\xi_0| = \|T\|$ such that

$$I \perp_{BJ} (T - \xi_0 I),$$

where $\perp_{BJ}$ denotes Birkhoff–James orthogonality. We also present further equivalent formulations and explore various structural consequences of these characterizations.

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1. Introduction

The study of specific classes of Hilbert space operators, such as normal, hyponormal, and normaloid operators, is central to operator theory due to their rich spectral and geometric properties. Fundamental work on normal operators in Hilbert spaces was established by Putnam [1], while Stampfli [2] developed the theory of hyponormal operators and their spectral density properties. More recent contributions include characterizations of normaloid operators by Chan and Chan [3], and studies of maximal numerical range properties by Spitkovsky [4]. Normal operators are particularly wellunderstood, but many important operators arising in applications do not satisfy normality, as discussed in Halmos's comprehensive treatment of Hilbert space problems [5] and recent developments in numerical radius theory [6]. This has motivated the investigation of broader operator classes that retain some desirable features of normal operators while allowing for more general behavior. Among these, normaloid operators are of particular interest. Normaloid operators, defined by the equality of their numerical radius and operator norm, play a key role in bridging normal and non-normal operators. This equality, closely related to the spectral radius, provides insights into operator structure and has several applications [7–9]. Motivated by these connections, we investigate normaloid operators through the concept of Birkhoff–James orthogonality [10], establishing a novel characterization that links their defining property to a single orthogonality condition, as highlighted in our main result.

Before stating our investigation, we need to recall some notations and terminology. Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, where the corresponding norm is given by $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in \mathcal{H}$. Denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$, equipped with the operator norm $\| \cdot \|$ defined by

$$\|T\| = \sup\{\|Tx\| : x \in \mathbb{S}_{\mathcal{H}}\},$$

for all $T \in \mathcal{B}(\mathcal{H})$, where

$$\mathbb{S}_{\mathcal{H}} = \{x \in \mathcal{H} : \|x\| = 1\}$$

denotes the unit sphere of $\mathcal{H}$. The spectral radius of $T \in \mathcal{B}(\mathcal{H})$ is given by

$$r(T) = \sup\{|\gamma| : \gamma \in \sigma(T)\},$$

where $\sigma(T)$ denotes the spectrum of T. Furthermore, the numerical range of an operator T is defined by

$$W(T) = \{\langle Tx, x \rangle : x \in \mathbb{S}_{\mathcal{H}}\},$$

and its numerical radius is given by

$$w(T) = \sup\{|\gamma| : \gamma \in W(T)\}.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is called normaloid if

$$\|T\| = w(T).$$

It is known that the inequalities

$$r(T) \leq w(T) \leq \|T\|$$

hold for every bounded linear operator T.

Thus, an operator T is normaloid if, and only if, $r(T) = \|T\|$. In fact, using Gelfand's formula for the spectral radius together with the characterization of normaloid operators (see Lemma 1.3 (2)), one concludes that every normaloid operator T satisfies

$$r(T) = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} = \|T\|.$$

Hence, equality between the spectral radius and the norm provides an alternative characterization of normaloid operators. For further information on the numerical radius, the algebraic numerical range, and basic properties of normaloid operators, we refer the reader to the foundational treatments by Bonsall and Duncan on numerical ranges of operators on normed spaces and their continuation on numerical ranges [6, 11, 12]. Additional comprehensive resources include Gustafson and Rao's monograph on numerical range theory and the field of values of linear operators, and Halmos's classical problem book covering Hilbert space operator theory [5, 13].

For the sequel, for any $T \in \mathcal{B}(\mathcal{H})$, the adjoint of T is denoted by T^*. An operator T is called positive, denoted by $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for every $x \in \mathcal{H}$. This induces the usual partial order on $\mathcal{B}(\mathcal{H})$: one writes $T \leq S$ precisely when $S - T \geq 0$. Further, recall that an operator T is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T - TT^* \geq 0$, and paranormal if $\|Tx\|^2 \leq \|T^2x\| \cdot \|x\|$ for all $x \in \mathcal{H}$. Every normal operator is hyponormal, and every hyponormal operator is paranormal; moreover, every paranormal operator is normaloid (see [2, 14]). The study of normaloid operators dates back to Wintner [15], who conjectured that $r(T) = \|T\|$ if, and only if, the numerical range $W(T)$ equals the convex hull of the spectrum of T. Although this was later shown to be false in general [5], the concept of normaloid operators remains fundamental in understanding the geometry and spectral behavior of non-normal operators.

We also need further notions for our investigation. The norm-attainment set of T is

$$\mathbb{M}_T = \{x \in \mathbb{S}_{\mathcal{H}} : \|Tx\| = \|T\|\},$$

that is, the collection of unit vectors at which T achieves its norm.

Stampfli [16] introduced the maximal numerical range

$$W_0(T) = \{\gamma \in \mathbb{C} : \exists\, \{x_n\} \subseteq \mathbb{S}_{\mathcal{H}}, \|Tx_n\| \to \|T\|, \langle Tx_n, x_n\rangle \to \gamma\},$$

and its maximal numerical radius

$$w_0(T) = \sup\{|\gamma| : \gamma \in W_0(T)\}.$$

Clearly, $W_0(T) \subseteq \overline{W(T)}$, where $\overline{W(T)}$ denotes the closure of the numerical range $W(T)$, and hence $w_0(T) \leq w(T)$.

The subclass of *normaloid operators* admits several elegant characterizations involving the maximal numerical range and spectral radius. A simple criterion, due to Chan and Chan [3], states that $T \in \mathcal{B}(\mathcal{H})$ is normaloid if, and only if,

$$w_0(T) = w(T). \tag{1.1}$$

More deeply, Spitkovsky [4] showed that T is normaloid precisely when the maximal numerical range $W_0(T)$ intersects the boundary of the numerical range $\partial W(T)$, i.e.,

$$W_0(T) \cap \partial W(T) \neq \varnothing. \tag{1.2}$$

Notably, (1.1) follows immediately from (1.2): Any $\gamma \in W_0(T) \cap \partial W(T)$ attains the supremum $|\gamma| = w(T)$, forcing $w_0(T) = w(T)$ by the inequality $w_0(T) \leq w(T)$.

We shall also use the following notions in $\mathcal{B}(\mathcal{H})$: for $X, Y \in \mathcal{B}(\mathcal{H})$,

$$X \perp_{BJ} Y \quad \text{if and only if} \quad \|X\| \leq \|X + \gamma Y\| \quad \text{for all } \gamma \in \mathbb{C},$$

and

$$X \parallel Y \quad \text{if and only if} \quad \|X + \gamma Y\| = \|X\| + \|Y\| \quad \text{for some } \gamma \in \mathbb{T},$$

where $\mathbb{T}$ denotes the unit circle in the complex plane, i.e., $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$.

Below, we present two lemmas that will be used throughout the manuscript. These lemmas provide characterizations of Birkhoff–James orthogonality and norm-parallelism within the context of $\mathcal{B}(\mathcal{H})$.

Lemma 1.1. *([17, Lemma 2.2] or [18, Remark 3.1]) Let $X, Y \in \mathcal{B}(\mathcal{H})$. Then, $X \perp_{BJ} Y$ if, and only if, there exists a sequence $\{z_n\} \subset \mathbb{S}_{\mathcal{H}}$ such that*

$$\|Xz_n\| \to \|X\| \quad \textit{and} \quad \langle Xz_n, Yx_n\rangle \to 0.$$

Lemma 1.2. *[9, Corollary 2.12] Let $X, Y \in \mathcal{B}(\mathcal{H})$. Then, $X \parallel Y$ if, and only if, there exists a sequence $\{z_n\} \subset \mathbb{S}_{\mathcal{H}}$ such that*

$$\lim_{n\to\infty} |\langle Xz_n, Yz_n\rangle| = \|X\|\,\|Y\|.$$

We present in the following lemma several well-known characterizations of normaloid operators in Hilbert spaces.

Lemma 1.3. *Let $T \in \mathcal{B}(\mathcal{H})$. The following statements are equivalent:*

1. *T is normaloid.*
2. *$\|T^n\| = \|T\|^n$ for any $n \in \mathbb{N}$.*
3. *T satisfies the Daugavet equation at a nonzero complex number γ, that is,*

$$\|T + \gamma I\| = \|T\| + |\gamma|.$$

4. *$T \parallel I$, i.e., there exists $\gamma \in \mathbb{T}$ such that*

$$\|T + \gamma I\| = \|T\| + 1.$$

5. *There exists a sequence $\{x_n\} \subseteq \mathcal{S}_{\mathcal{H}}$ such that*

$$\lim_{n\to\infty} \|Tx_n\| = \|T\| \quad \textit{and} \quad \lim_{n\to\infty} |\langle Tx_n, x_n\rangle| = \omega(T).$$

6. *$T^*T \leq \omega(T)^2 I$.*
7.

$$dw(T) = \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sqrt{|\langle Tx, x\rangle|^2 + \|Tx\|^4} = \sqrt{\omega^2(T) + \|T\|^4},$$

where $dw(T)$ is referred to as the Davis–Wielandt radius of T.

8. *For all nonnegative real scalars α, β with $\alpha\beta \neq 0$, the following weighted norm identity holds:*

$$\|T\|_{\alpha,\beta} := \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sqrt{\alpha|\langle Tx, x\rangle|^2 + \beta\|Tx\|^2} = \sqrt{\alpha\,\omega^2(T) + \beta\,\|T\|^2}.$$

Proof. The classical relationship between items (1) and (2) is established in [13, Theorem 6.2-1]. Item (3) was shown to be logically equivalent to (1) in [14, Proposition 5], while the connection between (1) and (5) was addressed in [3, Theorem 1]. Furthermore, item (4) reflects the notion of a normaloid operator, as noted in [10, Proposition 4.7], and its relevance to condition (3) was explored in [7, Theorem 2.3]. Finally, inequalities (6) and (7) were originally established in [19, Corollary 3.2], where additional connections with Davis–Wielandt radius inequalities are explored in greater detail. □

In this paper, we undertake a systematic study of normaloid operators using the geometric notions of Birkhoff–James orthogonality and norm-parallelism. These tools provide an effective framework for analyzing the structure of operators in Hilbert and Banach spaces, and have recently been employed to characterize various operator properties (see [9, 10, 19]).

We begin by reviewing several known characterizations of normaloid operators and then introduce new criteria based on Birkhoff–James orthogonality. Our approach bridges classical results with contemporary perspectives, yielding deeper insights into the nature of normaloid operators.

In particular, we prove that an operator T is normaloid if and only if there exists $\xi_0 \in \mathbb{C}$ with $|\xi_0| = \|T\|$ such that

$$I \perp_{BJ} (T - \xi_0 I),$$

where $\perp_{BJ}$ denotes Birkhoff–James orthogonality. Several further equivalent conditions are established, along with their structural implications.

2. Main results

In this section, we provide new characterizations of the normaloid property for bounded linear operators on a complex Hilbert space. These characterizations are formulated in terms of Birkhoff–James orthogonality, the partial order on the set of bounded linear operators, and the norms of the real and imaginary parts of operators.

For greater clarity, we organize the presentation into two subsections, each focusing on one of these perspectives.

2.1. Characterizations via Birkhoff–James orthogonality

Prior to our first main result, we establish a lemma describing the numerical range and radius of bounded linear operators using Birkhoff–James orthogonality. This geometric approach supports characterizing normaloid operators by their norm-attaining behavior.

Lemma 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then,*

$$\overline{W(T)} = \{\gamma \in \mathbb{C} : I \perp_{BJ} T - \gamma I\},$$

and

$$\omega(T) = \sup\{ |\gamma| : \gamma \in \mathbb{C},\ I \perp_{BJ} (T - \gamma I)\}.$$

Proof. It is enough to show that $\gamma \in \overline{W(T)}$ if, and only if, $I \perp_{BJ} S$, where $S = T - \gamma I$.

If $\gamma \in \overline{W(T)}$, there is a sequence of unit vectors x_n with $\langle T x_n, x_n\rangle \to \gamma$. Equivalently

$$\langle S x_n, x_n\rangle = \langle T x_n, x_n\rangle - \gamma \longrightarrow 0,$$

and trivially $\|x_n\| = 1 \to \|I\|$. By Lemma 1.1, these two facts imply $I \perp_{BJ} S$.
Conversely, assume $I \perp_{BJ} S$. By definition, there exists a sequence of unit vectors x_n such that

$$\|I x_n\| \to \|I\| \quad \text{and} \quad \langle I x_n,\ S x_n\rangle \to 0.$$

Since $I x_n = x_n$, the second condition reads $\langle S x_n, x_n\rangle \to 0$, but then

$$\langle T x_n, x_n\rangle = \langle S x_n, x_n\rangle + \gamma \longrightarrow \gamma,$$

showing $\gamma \in \overline{W(T)}$. This completes the proof. □

Remark 2.1. *Observe that, by the previous proposition, if $\gamma \in \mathbb{C}$ is such that $I \perp_{BJ} T - \gamma I$, then it necessarily holds that*

$$|\gamma| \leq \omega(T) \leq \|T\|, \quad \textit{for any } T \in \mathcal{B}(\mathcal{H}).$$

This inequality leads to the following criterion, which fully characterizes normaloid operators in terms of Birkhoff–James orthogonality.

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then, the following statements are equivalent:*

(1) T is normaloid.
(2) There exists $\xi_0 \in \mathbb{C}$ such that $|\xi_0| = \|T\|$ and

$$I \perp_{BJ} (T - \xi_0 I).$$

Proof. Assume first that T is normaloid, so $\|T\| = w(T)$. By the Birkhoff–James characterization of the numerical radius,

$$w(T) = \sup\{|\xi| : I \perp_{BJ} (T - \xi I)\}.$$

Hence there exists a sequence $\{\xi_n\} \subseteq \mathbb{C}$ such that

$$I \perp_{BJ} (T - \xi_n I) \quad \text{and} \quad |\xi_n| \longrightarrow w(T).$$

Since the closed disk $\{\xi : |\xi| \leq w(T)\}$ is compact, by passing to a subsequence if necessary, we may assume $\xi_n \to \xi_0$ with $|\xi_0| = w(T)$. We now claim that

$$I \perp_{BJ} (T - \xi_0 I).$$

To justify the passage to the limit in the Birkhoff–James condition, recall that for each n we have $I \perp_{BJ} (T - \xi_n I)$, so

$$\|I\| \leq \|I + \gamma(T - \xi_n I)\| \quad \forall \gamma \in \mathbb{C}.$$

However, the operator-norm is continuous in its argument. Hence, for each fixed γ,

$$\|I + \gamma(T - \xi_0 I)\| = \lim_{n\to\infty} \|I + \gamma(T - \xi_n I)\| \geq \|I\|.$$

Since this holds for every γ, it follows that $I \perp_{BJ} (T - \xi_0 I)$. Combined with $|\xi_0| = w(T) = \|T\|$, this completes the argument.

Conversely, if there exists ξ_0 with $I \perp_{BJ} (T - \xi_0 I)$ and $|\xi_0| = \|T\|$, then by definition of $w(T)$

$$w(T) \geq |\xi_0| = \|T\|,$$

while always $w(T) \leq \|T\|$. Hence, $w(T) = \|T\|$, so T is normaloid. □

Remark 2.2. *It is worth noting that, although Lemma 1.3, item* (4)*, and Theorem 2.4 of Zamani–Moslehian [9] together yield an alternative proof of Theorem 2.1, we have chosen instead to present a more direct demonstration based on Lemma 2.1, since this approach makes explicit how extremal numerical values arise from Birkhoff–James orthogonality, and thus streamlines the logical flow. Since*

$$T \parallel I \quad \Leftrightarrow \quad \exists\, \gamma \in \mathbb{T} : I \perp_{BJ} \|T\| I + \overline{\gamma} T$$

$$\Leftrightarrow \quad \exists\, \gamma \in \mathbb{T} : I \perp_{BJ} \overline{\gamma}\left(T - \frac{\|T\|}{-\overline{\gamma}} I\right)$$

$$\Leftrightarrow \quad \exists\, \gamma \in \mathbb{T} : I \perp_{BJ} \left(T - \frac{\|T\|}{-\overline{\gamma}} I\right).$$

Then it suffices to choose $\xi_0 = -\frac{\|T\|}{\overline{\gamma}}$, *with* $\left|\xi_0\right| = \|T\|$.

In what follows, we present a simplified characterization of normaloid operators, refining condition (5) of Lemma 1.3. As we shall demonstrate, this concise formulation directly entails the original condition.

Corollary 2.1. *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then,* T *is normaloid if, and only if, there exists a sequence of unit vectors* $\{x_n\} \subseteq \mathcal{H}$ *such that* $\lim_{n\to\infty} |\langle T x_n, x_n\rangle| = \|T\|$.

Proof. Assume first that T is normaloid. By Theorem 2.1, there exists $\xi_0 \in \mathbb{C}$ with

$$I \perp_{BJ} (T - \xi_0 I) \quad \text{and} \quad |\xi_0| = \|T\|.$$

By the definition of Birkhoff–James orthogonality in the complex setting, $I \perp_{BJ} (T - \xi_0 I)$ guarantees the existence of a sequence of unit vectors x_n satisfying

$$\langle (T - \xi_0 I)x_n, x_n\rangle \longrightarrow 0.$$

Hence

$$\langle T x_n, x_n\rangle = \langle (T - \xi_0 I)x_n, x_n\rangle + \xi_0 \longrightarrow \xi_0,$$

and taking moduli gives

$$\left|\langle T x_n, x_n\rangle\right| \longrightarrow |\xi_0| = \|T\|.$$

Conversely, suppose there is a sequence of unit vectors x_n with

$$|\langle T x_n, x_n\rangle| \to \|T\|.$$

By passing to a subsequence if necessary, we may suppose

$$\lim_{n\to\infty} \langle T x_n, x_n\rangle = \gamma.$$

Then, setting $\xi_0 = \gamma \in \mathbb{C}$, we obtain

$$\langle (T - \xi_0 I)x_n, x_n\rangle = \langle T x_n, x_n\rangle - \gamma \longrightarrow 0,$$

so by Lemma 1.1, it follows that $I \perp_{BJ} (T - \xi_0 I)$, since $|\xi_0| = |\gamma| = \|T\|$. Then, by Theorem 2.1, we conclude that T is normaloid.

□

It is almost immediate to observe that, if there exists a sequence of unit vectors $\{x_n\} \subseteq \mathcal{H}$ such that

$$\lim_{n\to\infty} \left|\langle T x_n,\ x_n\rangle\right| = \|T\|,$$

then this sequence satisfies the conditions of item (5) in Lemma 1.3. Indeed,

$$\|T\| = \lim_{n\to\infty}\left|\langle Tx_n,\ x_n\rangle\right| \ \leq\ w(T) = \sup_{\|x\|=1}\left|\langle Tx,\ x\rangle\right| \ \leq\ \|T\|,$$

and, therefore, $\lim_{n\to\infty}\left|\langle Tx_n,\ x_n\rangle\right| \ =\ w(T)$. Moreover, by the Cauchy–Schwarz inequality, we deduce

$$\|T\| = \lim_{n\to\infty}\left|\langle Tx_n,\ x_n\rangle\right| \ \leq\ \lim_{n\to\infty}\|Tx_n\| \ \leq\ \|T\|,$$

and, hence, $\lim_{n\to\infty}\|Tx_n\| \ =\ \|T\|$.
Observe that if $T \in \mathcal{B}(\mathcal{H})$ and $\gamma \in \overline{W(T)}$, then by the triangle inequality,

$$\|T + \gamma I\| \ \leq\ \|T\| + |\gamma| \ \leq\ 2\|T\|.$$

Hence,

$$\sup_{\gamma\in\overline{W(T)}}\|T + \gamma I\| \ \leq\ 2\|T\|.$$

We now present a new characterization of normaloid operators as precisely to those operators for which the foregoing inequality becomes an equality. More precisely:

Theorem 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then, T is normaloid if, and only if,*

$$\sup\left\{\|T + \gamma I\| : \gamma \in \overline{W(T)}\right\} = 2\|T\|.$$

Proof. Suppose the equality

$$\sup\{\|T + \gamma I\| : \gamma \in \overline{W(T)}\} = 2\|T\|$$

holds. Then, for each $n \in \mathbb{N}$, there exists $\gamma_n \in \overline{W(T)}$ such that

$$0 \ \leq\ 2\|T\| - \frac{\|T\|}{n} \ \leq\ \|T + \gamma_n I\|.$$

Hence,

$$\begin{aligned}\|T\| - \frac{\|T\|}{n} &\leq \|T + \gamma_n I\| - \|T\| \\ &= \|(T + \gamma_n I) - T\| = |\gamma_n| \ \leq\ \|T\|,\end{aligned}$$

where the last inequality follows from $\overline{W(T)} \subseteq \{z \in \mathbb{C} : |z| \leq \|T\|\}$. Letting $n \to \infty$, we conclude that

$$\lim_{n\to\infty}|\gamma_n| = \|T\|.$$

Since $\gamma_n \in \overline{W(T)}$ for all $n \in \mathbb{N}$, we obtain

$$\|T\| = \lim_{n\to\infty}|\gamma_n| \leq \omega(T) \leq \|T\|,$$

and it follows immediately that $\omega(T) = \|T\|$.

Conversely, suppose T is normaloid. By the argument in the proof of Corollary 2.1, there exist $\xi_0 \in \mathbb{C}$ with $|\xi_0| = \|T\|$ and a sequence of unit vectors $\{x_n\} \subseteq \mathcal{H}$ such that

$$\langle T x_n, x_n \rangle \longrightarrow \xi_0 = e^{i\theta_0}\|T\| \quad \text{for some } \theta_0 \in [0, 2\pi),$$

and, hence, $\xi_0 \in \overline{W(T)}$. Therefore, by Lemma 1.2,

$$\left\| \frac{T}{\|T\|} + e^{i\theta_0} I \right\| = \left\| \frac{T}{\|T\|} \right\| + |e^{i\theta_0}| = 2,$$

or, equivalently,

$$\left\| T + e^{i\theta_0}\|T\|\, I \right\| = 2\|T\|.$$

Thus,

$$\sup\{\|T + \gamma I\| : \gamma \in \overline{W(T)}\} = 2\|T\|,$$

as claimed. □

2.2. Characterizations via real and imaginary decomposition

For the sequel, let us recall that every bounded linear operator T on a complex Hilbert space admits the Cartesian decomposition:

$$T = \Re(T) + i\,\Im(T),$$

where

$$\Re(T) = \frac{T + T^*}{2} \quad \text{and} \quad \Im(T) = \frac{T - T^*}{2i}.$$

From this decomposition, and from the well-known fact that $\Re(T)$ and $\Im(T)$ are self-adjoint operators, we obtain the basic estimates:

$$\|\Re(T)\| \le \omega(T), \qquad \|\Im(T)\| \le \omega(T),$$

where $\omega(T)$ denotes the numerical radius of the operator T.

To prove our next main result, we recall the following useful lemma due to Yamazaki [8]:

Lemma 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then, we have*

$$\omega(T) = \max_{\gamma \in \mathbb{T}} \|\Re(\gamma T)\| = \max_{\mu \in \mathbb{T}} \|\Im(\mu T)\|. \tag{2.1}$$

Using this result, we can now derive a new characterization of normaloid operators.

Theorem 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then, the following conditions are equivalent:*

1. *T is normaloid.*
2. *There exists $\gamma_0 \in \mathbb{T}$ such that $\|\Re(\gamma_0 T)\| = \|T\|$.*
3. *There exists $\mu_0 \in \mathbb{T}$ such that $\|\Im(\mu_0 T)\| = \|T\|$.*

Proof. First, suppose that (1) holds, i.e., $\|T\| = \omega(T)$. Then, by Yamazaki's formula (2.1), there exists $\gamma_0 \in \mathbb{T}$ such that

$$\omega(T) = \|\Re(\gamma_0 T)\|.$$

Hence,

$$\|T\| = \omega(T) = \|\Re(\gamma_0 T)\|,$$

so (2) is satisfied.

Conversely, assume (2), namely, that for some $\gamma_0 \in \mathbb{T}$ with $\|T\| = \|\Re(\gamma_0 T)\|$. Then, again by (2.1),

$$\omega(T) = \max_{\gamma \in \mathbb{T}} \|\Re(\gamma T)\| \geq \|\Re(\gamma_0 T)\| = \|T\|.$$

Since always $\omega(T) \leq \|T\|$, it follows that $\omega(T) = \|T\|$, i.e., T is normaloid.

Assume now that condition (1) holds, i.e., $\|T\| = \omega(T)$. By Yamazaki's formula (2.1), there exists $\mu_0 \in \mathbb{T}$ such that

$$\omega(T) = \|\mathrm{Im}(\mu_0 T)\|.$$

Thus,

$$\|T\| = \omega(T) = \|\mathrm{Im}(\mu_0 T)\|,$$

so condition (3) holds.

Conversely, assume condition (3), i.e., there exists $\mu_0 \in \mathbb{T}$ such that $\|\mathrm{Im}(\mu_0 T)\| = \|T\|$. By Yamazaki's formula (2.1),

$$\omega(T) = \max_{\mu \in \mathbb{T}} \|\mathrm{Im}(\mu T)\| \geq \|\mathrm{Im}(\mu_0 T)\| = \|T\|.$$

Since $\omega(T) \leq \|T\|$ always holds, it follows that $\omega(T) = \|T\|$, so T is normaloid. □

To prove another characterization of normaloid operators, we provide alternative formulas for the numerical radius that relates directly to norm-parallelism between operators.

Lemma 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then,*

$$\omega(T) = \max_{\gamma \in \mathbb{T}} \left\| \frac{T + \gamma T^*}{2} \right\| = \max_{\mu \in \mathbb{T}} \left\| \frac{T - \mu T^*}{2i} \right\|. \tag{2.2}$$

Proof. Let $\gamma = e^{i\theta} \in \mathbb{T}$. Then

$$\Re(\gamma T) = \frac{\gamma T + \overline{\gamma}\, T^*}{2} = \frac{e^{i\theta} T + e^{-i\theta} T^*}{2} = e^{i\theta}\, \frac{T + e^{-2i\theta} T^*}{2}.$$

Since multiplication by the unimodular scalar $e^{i\theta}$ does not change the operator norm, we obtain

$$\left\|\Re(\gamma T)\right\| = \left\| \frac{T + e^{-2i\theta} T^*}{2} \right\|.$$

Maximizing over all $\gamma = e^{i\theta}$ and invoking Yamazaki's formula (2.1) yields

$$\omega(T) = \max_{\gamma \in \mathbb{T}} \|\Re(\gamma T)\| = \max_{\mu \in \mathbb{T}} \left\| \frac{T + \mu\, T^*}{2} \right\|.$$

This completes the proof of the first equality. The second equality in the statement follows by an entirely analogous argument applied to the imaginary part.

□

We are now ready to state the following characterizations concerning normaloid operators.

Theorem 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then, the following conditions are equivalent:*

1. *T is normaloid, i.e., $\|T\| = \omega(T)$.*
2. *There exists $\gamma_0 \in \mathbb{T}$ such that*
$$\left\|\frac{T + \gamma_0 T^*}{2}\right\| = \|T\|.$$
3. *There exists $\mu_0 \in \mathbb{T}$ such that*
$$\left\|\frac{T - \mu_0 T^*}{2i}\right\| = \|T\|.$$
4. *$T \parallel T^*$.*

Proof. We will use the well-known identities

$$\omega(T) = \max_{\gamma\in\mathbb{T}} \left\|\frac{T + \gamma T^*}{2}\right\| = \max_{\mu\in\mathbb{T}} \left\|\frac{T - \mu T^*}{2i}\right\|. \tag{2.3}$$

Assume first that condition (1) holds. Then, $\|T\| = \omega(T)$, and by the second formula in (2.3) there exists $\mu_0 \in \mathbb{T}$ such that

$$\omega(T) = \left\|\frac{T - \mu_0 T^*}{2i}\right\|.$$

Hence, $\|T\| = \left\|\frac{T-\mu_0 T^*}{2i}\right\|$, which shows that condition (3) is satisfied.

Now suppose that (3) holds. Then, for some $\mu_0 \in \mathbb{T}$, we have

$$\|T\| = \left\|\frac{T - \mu_0 T^*}{2i}\right\|.$$

Since

$$\left\|\frac{T - \mu_0 T^*}{2i}\right\| \le \max_{\mu\in\mathbb{T}} \left\|\frac{T - \mu T^*}{2i}\right\| = \omega(T),$$

and always $\omega(T) \le \|T\|$, we conclude that $\omega(T) = \|T\|$, so T is normaloid. This proves that (1) and (3) are equivalent.

Assume again that (1) holds. Using the first formula in (2.3), there exists $\gamma_0 \in \mathbb{T}$ such that

$$\|T\| = \left\|\frac{T + \gamma_0 T^*}{2}\right\|,$$

so (2) is satisfied.

Next, suppose that (2) holds. Multiplying the equation

$$\left\|\frac{T + \gamma_0 T^*}{2}\right\| = \|T\|$$

by 2, we get

$$\|T + \gamma_0 T^*\| = 2\|T\|.$$

By the triangle inequality and the facts that $\|T^*\| = \|T\|$ and $|\gamma_0| = 1$, we have

$$\|T + \gamma_0 T^*\| \le \|T\| + |\gamma_0|\,\|T^*\| = 2\|T\|,$$

so the equality holds. Therefore, T is norm–parallel to T^*, which means that (4) holds.

Finally, assume that (4) is satisfied. Then, there exists $\lambda \in \mathbb{T}$ such that

$$\|T + \lambda T^*\| = 2\|T\|.$$

Dividing by 2 yields

$$\left\| \frac{T + \lambda T^*}{2} \right\| = \|T\|,$$

which is precisely (2) with $\gamma_0 = \lambda$. From (2) we also obtain

$$\omega(T) \geq \left\| \frac{T + \gamma_0 T^*}{2} \right\| = \|T\|,$$

and since $\omega(T) \leq \|T\|$ always holds, we conclude that $\omega(T) = \|T\|$, i.e., (1) is valid. This completes the proof. □

Based on the characterization of normaloid operators recently obtained via the notion of norm parallelism, and drawing from the manuscript by Barra and Bouzmagour—where the authors investigate when the norm of the sum of two bounded operators on a Hilbert space equals the sum of their norms—we obtain the following statement.

Corollary 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then, the following statements are equivalent:*

1. *T is normaloid.*
2. *$\omega(T^2) = \|T^2\| = \|T\|^2$.*
3. *$r(T^2) = \|T^2\| = \|T\|^2$.*

Proof. The equivalence between the three conditions is a direct consequence of the characterization of normaloid operators via norm parallelism, namely, $T \parallel T^*$, as established in Theorem 2.4, item (4), together with [20, Corollary 2.2]. □

Let us observe that item (2) in Corollary 2.2 can be reformulated using sequences and the definition of the numerical radius as follows: T is normaloid if, and only if, there exists a sequence $\{x_n\} \subseteq \mathbb{S}_{\mathcal{H}}$ such that

$$\lim_{n\to\infty} |\langle T x_n, T^* x_n\rangle| = \lim_{n\to\infty} |\langle T^2 x_n, x_n\rangle| = \|T\|^2.$$

Moreover, from the proof of [20, Theorem 2.1], we obtain the following necessary condition satisfied by every normaloid operator. We omit the proof.

Corollary 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. If T is normaloid, then there exists a sequence $\{x_n\} \subseteq \mathbb{S}_{\mathcal{H}}$ such that*

$$\lim_{n\to\infty} \|T x_n\| = \lim_{n\to\infty} \|T^* x_n\| = \|T\|.$$

In what follows, we demonstrate that the condition above is not sufficient to guarantee that an operator is normaloid.

Example 2.1. *Consider the nilpotent matrix*

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then $r(T) = 0$*. Now, we compute*

$$T^* = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \textit{and} \quad T^*T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, eigenvalues of T^*T *are* $\{0, 1, 1\}$*, hence* $\|T\| = 1$.

Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \textit{with} \quad \|x\| = 1.$$

Then,

$$Tx = \begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix}, \quad \|Tx\|^2 = |x_2|^2 + |x_3|^2,$$

and

$$T^*x = \begin{bmatrix} 0 \\ x_1 \\ x_2 \end{bmatrix}, \quad \|T^*x\|^2 = |x_1|^2 + |x_2|^2.$$

The vector

$$x = \begin{bmatrix} 0 \\ e^{i\theta} \\ 0 \end{bmatrix}, \quad \theta \in \mathbb{R},$$

is a unit vector such that

$$\|Tx\| = \|T^*x\| = 1 = \|T\|.$$

Thus, there exists a unit vector where T *and its adjoint* T^* *simultaneously attain their norm, even though* T *is nilpotent and therefore not normaloid.*

In the next corollary, we use Corollary 2.3 to characterize compact normaloid operators.

Theorem 2.5. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a compact operator. Then,* T *is normaloid if and only if*

$$\mathbb{M}_T \cap \mathbb{M}_{T^*} \neq \emptyset.$$

Proof. First note that if T is compact then so is its adjoint T^*.

Assume that $T \in \mathcal{B}(\mathcal{H})$ is a compact normaloid operator. Then, as consequence of Corollary 2.3, that there exists a sequence of unit vectors $\{x_n\} \subseteq \mathcal{H}$ such that

$$\|Tx_n\| \to \|T\|, \qquad \|T^*x_n\| \to \|T\|.$$

Since $\{x_n\}$ is a bounded sequence in the unit sphere of $\mathcal{H}$, by the Banach–Alaoglu theorem, it has a subsequence $\{x_{n_k}\}$ that converges weakly to some $x \in \mathcal{H}$. Because T and T^* are compact operators, they map weakly convergent sequences to strongly convergent sequences. That is,

$$Tx_{n_k} \to Tx \quad \text{and} \quad T^*x_{n_k} \to T^*x \quad \text{in norm.}$$

Hence, by continuity of the norm, we obtain

$$\|Tx\| = \lim_{k\to\infty} \|Tx_{n_k}\| = \|T\|, \qquad \|T^*x\| = \lim_{k\to\infty} \|T^*x_{n_k}\| = \|T\|.$$

To deduce that $\|x\| = 1$, note that for any bounded operator T, we have $\|Tx\| \leq \|T\| \cdot \|x\|$. Therefore,

$$\|T\| = \|Tx\| \leq \|T\|\|x\|,$$

which implies $\|x\| \geq 1$. On the other hand, since $x_{n_k} \rightharpoonup x$ weakly and $\|x_{n_k}\| = 1$, by weak lower semicontinuity of the norm, we have

$$\|x\| \leq \liminf_{k\to\infty} \|x_{n_k}\| = 1.$$

Combining both inequalities gives $\|x\| = 1$. Therefore, the vector x lies in the unit sphere of $\mathcal{H}$ and satisfies $\|Tx\| = \|T^*x\| = \|T\|$, i.e., $\mathbb{M}_T \cap \mathbb{M}_{T^*} \neq \emptyset$.

Conversely, if there exists $x \in \mathbb{S}_{\mathcal{H}}$, with $\|Tx\| = \|T\|$ and $\|T^*x\| = \|T\|$, then by Corollary 2.3, we conclude that T is normaloid. □

To conclude the paper, we highlight a connection between normaloid operators and the Daugavet-type equation involving the numerical radius. The following remark clarifies how this equation can be reformulated and analyzed in terms of known results in the literature. We then explore a natural question arising from our earlier characterizations: whether satisfying an ω-Daugavet-type equation implies that an operator is normaloid. As the subsequent example shows, the answer is negative, thus demonstrating the limitations of this property as a full characterization. These final observations offer a broader perspective on the structure of normaloid operators and point to potential directions for future study.

Remark 2.3. *We observe that T satisfies the Daugavet equation for some nonzero scalar $\gamma \in \mathbb{C}$ if, and only if, the operator $\frac{1}{\gamma}T$ satisfies the classical Daugavet equation, i.e.,*

$$\left\|\frac{1}{\gamma}T + I\right\| = \left\|\frac{1}{\gamma}T\right\| + 1.$$

This reduction allows one to apply general results from the classical work of Abramovich et al. on the Daugavet equation (see [21]). We refer the reader to that article for further characterizations of normaloid operators in terms of the approximate point spectrum.

Although items (3) *and* (4) *can be easily deduced from one another, we have included both in the previous result. This is because* (4) *was obtained independently of Gevorgyan's proof of* (3), *and in fact, it follows from a more general theorem valid in the setting of Banach spaces (see [10, Theorem 4.6]).*

Additionally, [14] presents related results in this direction. Finally, the equivalence between items (1) *and* (5) *in the previous theorem is also reproven in [7].*

A natural question, motivated by Lemma 1.3, is whether the requirement that a bounded linear operator T satisfies a version of the Daugavet equation involving the numerical radius ω, for some $\gamma \in \mathbb{T}$, also characterizes normaloid operators. More precisely, we shall prove that the ω-Daugavet equation holds for every $T \in \mathcal{B}(\mathcal{H})$ without imposing any further hypotheses. Indeed, given $T \in \mathcal{B}(\mathcal{H})$, consider a sequence $\{x_n\} \subseteq \mathcal{S}_{\mathcal{H}}$ such that $\lim_{n\to\infty} |\langle Tx_n, x_n\rangle| = \omega(T)$.

For each $n \in \mathbb{N}$, let

$$\theta_n := \arg\langle Tx_n, x_n\rangle,$$

and

$$z_n : z(\theta_n, x_n) = e^{i\theta_n} \left|\langle Tx_n, x_n\rangle\right|.$$

Note that z_n depends on both x_n and θ_n, where θ_n itself is determined by x_n.

As this sequence is bounded, the complex Bolzano–Weierstrass theorem guarantees the existence of a convergent subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ with limit $z_0 = e^{i\theta_0}|z_0| \in \mathbb{C}$. It follows that

$$\lim_{k\to\infty} z_{n_k} = \lim_{k\to\infty} e^{i\theta_{n_k}} |\langle Tx_{n_k}, x_{n_k}\rangle| = e^{i\theta_0}\omega(T),$$

for some $\theta_0 \in \mathbb{R}$.

Consequently,

$$\begin{aligned} \omega(T + e^{i\theta_0} I) &= \sup\{|\langle Tx, x\rangle + e^{i\theta_0}| : x \in \mathcal{S}_{\mathcal{H}}\} \\ &\geq \lim_{k\to\infty} |\langle Tx_{n_k}, x_{n_k}\rangle + e^{i\theta_0}| = \left|\lim_{k\to\infty}\langle Tx_{n_k}, x_{n_k}\rangle + e^{i\theta_0}\right| \\ &= \left|\lim_{k\to\infty} z_{n_k} + e^{i\theta_0}\right| = |e^{i\theta_0}\omega(T) + e^{i\theta_0}| = \omega(T) + 1. \end{aligned}$$

On the other hand, it is always true that $\omega(T + e^{i\theta_0} I) \leq \omega(T) + 1$. Therefore, we conclude that

$$\omega(T + e^{i\theta_0} I) = \omega(T) + 1.$$

Example 2.2. *In this example, we construct an operator that satisfies the Daugavet-type equation for ω, but does not satisfy $\omega(T) = \|T\|$, thus providing a counterexample to the converse of this property.*

Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})$. *For* $x = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, *a unit vector in* $\mathbb{C}^2$, *we have:*

$$Tx = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \sin\theta \\ 0 \end{pmatrix}.$$

Thus:

$$\langle Tx, x\rangle = \sin\theta\cos\theta = \frac{1}{2}\sin(2\theta).$$

The maximum value of $|\langle Tx, x\rangle|$ *occurs when* $\theta = \frac{\pi}{4}$ *or* $\theta = \frac{3\pi}{2}$ *and is:*

$$\omega(T) = \frac{1}{2}.$$

Now consider the operator $T+\gamma I$ for $\gamma \in \mathbb{T}$. Then,

$$(T+\gamma I)x = \begin{pmatrix} \gamma & 1 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \gamma\cos\theta + \sin\theta \\ \gamma\sin\theta \end{pmatrix}.$$

Thus:

$$\langle (T+\gamma I)x, x\rangle = \gamma\cos^2\theta + \sin\theta\cos\theta + \gamma\sin^2\theta = \gamma(\cos^2\theta + \sin^2\theta) + \sin\theta\cos\theta.$$

Since $\cos^2\theta + \sin^2\theta = 1$*, this simplifies to:*

$$\langle (T+\gamma I)x, x\rangle = \gamma + \frac{1}{2}\sin(2\theta).$$

Thus, the maximum value of $|\langle (T+\gamma I)x, x\rangle|$ *is*

$$\omega(T+\gamma I) = |\gamma| + \frac{1}{2} = \frac{3}{2}.$$

Therefore, T satisfies the ω-Daugavet-type equation

$$\omega(T+\gamma I) = \omega(T) + 1.$$

Finally, we show that T is not normaloid since $\omega(T) = \frac{1}{2} \neq \|T\| = 1$.

3. Conclusions

In this work, we have established several new characterizations of normaloid operators in complex Hilbert spaces, framed within the geometric concepts of Birkhoff–James orthogonality and norm–parallelism. Our approach unifies and extends existing results, providing concise and elegant equivalences that connect the numerical radius, the operator norm, and structural properties of the real and imaginary parts of operators.

In particular, we proved that the normaloid property is equivalent to the existence of a complex scalar ξ_0 with $|\xi_0| = \|T\|$ such that $I \perp_{\mathrm{BJ}} (T - \xi_0 I)$, offering a direct link between extremal numerical range elements and Birkhoff–James orthogonality. We also derived alternative formulations based on norm–parallelism, the Davis–Wielandt radius, and the norms of the Cartesian components of T. These results shed new light on how normaloid operators can be identified through norm-attaining sequences and geometric parallelism relations.

Furthermore, our analysis clarified the limitations of certain Daugavet-type identities involving the numerical radius as a complete characterization of normaloid operators, providing explicit counterexamples. For compact operators, we obtained a neat criterion in terms of the intersection of norm-attainment sets of T and T^*.

Overall, the results presented here contribute to a deeper understanding of the interplay between numerical range geometry, operator norms, and orthogonality notions, and open potential directions for further investigation in the setting of Banach space operators and Hilbert C^*-modules.

Author contributions

Feryal Aladsani: Conceptualization, visualization, funding, resources, writing–review & editing, formal analysis, project administration, validation, investigation; Asmahan Alajyan: Conceptualization, visualization, funding, writing–review & editing, formal analysis, project administration, validation, investigation; Cristian Conde: Conceptualization, visualization, funding, writing–review & editing, formal analysis, project administration, validation, investigation; Kais Feki: Conceptualization, visualization, funding, writing–review & editing, formal analysis, project administration, validation, investigation. All authors declare that they have contributed equally to this paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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