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Research article

Multi-dimensional rational recurrence models: Local analysis with nonlinear effects

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Abstract: In this paper, we investigate the dynamic behavior of a class of p-dimensional rational difference equation systems that extend previously studied two- and three-dimensional models. The system incorporates a power-type nonlinearity parameter $q > 0$, and our analysis focuses on the boundedness of solutions and the local asymptotic stability of equilibrium points. By generalizing known results to higher dimensions, we provide a deeper understanding of the local dynamics and structural properties of such systems.

Keywords: system of difference equations; local asymptotic stability; boundedness; oscillation; stability
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1. Introduction

Difference equations have attracted considerable attention due to their wide range of applications in modeling real-world phenomena in fields such as biology, economics, engineering, and computer science. Discrete dynamical systems offer valuable insight into the long-term behavior of processes that evolve in a stepwise manner over time. In particular, rational difference equations - nonlinear relations involving ratios of variables - often display complex dynamics such as stability, periodicity, and bifurcations. Understanding the qualitative behavior of such systems, especially in higher dimensions, is crucial for both theoretical developments and practical applications, notably in areas such as algorithm analyses, discrete system modeling, and data processing within computer science.

Over the past decade, significant attention has been devoted to the study of nonlinear rational difference equations and their systems, particularly in the context of periodicity, asymptotic behavior, and global stability. Several researchers have advanced the theory of rational and nonlinear difference systems. Elsayed and Ibrahim [2] studied periodicity and solutions of nonlinear systems, while Elsayed analyzed further properties of second-order rational systems in [3, 4]. Gümüş and Soykan [5] investigated multidimensional systems, and Gümüş later focused on stability [6], periodicity [7], and delayed systems [8]. Halim and collaborators connected such systems with classical sequences: Halim, Khelifa, and Berkal [9] used Lucas numbers, Halim and Rabago [10] applied Padovan numbers, Halim and Bayram [11] employed Fibonacci sequences, Halim [12] studied Fibonacci-type solutions, and Khelifa and Halim [19] obtained general solution forms. Further contributions include solvability results via generalized Fibonacci sequences by Hamioud, Dekkar, and Touafek [13], and the works of Kara and Yazlik, who examined higher-order nonlinear systems [14], variable-coefficient systems with Touafek and Akrour [15], and Padovan-type solutions in later studies [16, 17]. More recently, Kaouache, Fečkan, Halim, and Khelifa [18], Touafek [23, 24], and Yazlik with co-authors [25, 26] extended the theoretical framework of such systems. These studies formed the foundation upon which the present work builds by considering generalized p-dimensional systems with power-type nonlinearities and cyclic interactions.

In what follows, we present a more detailed review of the relevant literature and highlight the key contributions that motivated our work.

In 2000, Papaschinopoulos and Schinas [22] studied a coupled system of second-order rational difference equations given by the following:

$$^{(1)}x_{n+1} = \alpha + \frac{^{(1)}x_{n-1}}{^{(2)}x_n}, \quad ^{(2)}x_{n+1} = \alpha + \frac{^{(2)}x_{n-1}}{^{(1)}x_n}, \tag{1.1}$$

where α is a positive parameter, and the sequences are assumed to be positive. They analyzed basic qualitative properties such as boundedness and asymptotic behavior.

Later, in 2015, Bao [1] generalized this system by introducing a power-type nonlinearity parameter $q > 0$, and investigated the boundedness, oscillation, and local stability of the following system:

$$^{(1)}x_{n+1} = \alpha + \frac{^{(1)}x^q_{n-1}}{^{(2)}x^q_n}, \quad ^{(2)}x_{n+1} = \alpha + \frac{^{(2)}x^q_{n-1}}{^{(1)}x^q_n}. \tag{1.2}$$

In 2018, Okumuş and Yüksel [21] extended the investigation to a three-dimensional system that involved a cyclic interaction between the components. They studied the persistence, periodicity, and global stability of the following model:

$$\begin{cases} ^{(1)}x_{n+1} &= \alpha + \dfrac{^{(1)}x_{n-1}}{^{(2)}x_n}, \\ ^{(2)}x_{n+1} &= \alpha + \dfrac{^{(2)}x_{n-1}}{^{(3)}x_n}, \\ ^{(3)}x_{n+1} &= \alpha + \dfrac{^{(3)}x_{n-1}}{^{(1)}x_n}, \end{cases} \tag{1.3}$$

and revealed complex dynamical behaviors depending on the initial values and parameter choices.

Most recently, in 2025 [27], Zhang et al. studied the oscillation and local stability properties of the power-type nonlinear extension of the previous three-dimensional model as follows:

$$\begin{cases} {}^{(1)}x_{n+1} = \alpha + \dfrac{{}^{(1)}x_{n-1}^{q}}{{}^{(2)}x_{n}^{q}}, \\ {}^{(2)}x_{n+1} = \alpha + \dfrac{{}^{(2)}x_{n-1}^{q}}{{}^{(3)}x_{n}^{q}}, \\ {}^{(3)}x_{n+1} = \alpha + \dfrac{{}^{(3)}x_{n-1}^{q}}{{}^{(1)}x_{n}^{q}}. \end{cases} \tag{1.4}$$

In this paper, we aim to generalize the previously studied systems (1.1)–(1.4) to a higher-dimensional setting. Building upon the results obtained in two- and three-dimensional cases, we consider a p-dimensional nonlinear rational difference system with cyclic interactions among the components. Our goal is to analyze the qualitative behavior of solutions, including oscillation, persistence, and stability properties. This generalization not only extends the scope of existing models, but also reveals richer dynamics and deeper mathematical structures inherent in higher-dimensional systems.

More precisely, we study the following p-dimensional nonlinear rational difference system:

$$\begin{cases} {}^{(1)}x_{n+1} = \alpha + \dfrac{{}^{(1)}x_{n-1}^{q}}{{}^{(2)}x_{n}^{q}} \\ {}^{(2)}x_{n+1} = \alpha + \dfrac{{}^{(2)}x_{n-1}^{q}}{{}^{(3)}x_{n}^{q}} \\ \quad\vdots \\ {}^{(p)}x_{n+1} = \alpha + \dfrac{{}^{(p)}x_{n-1}^{q}}{{}^{(1)}x_{n}^{q}} \end{cases} \tag{1.5}$$

where α is a nonnegative constant, and ${}^{(j)}x_{-1}, {}^{(j)}x_0, j = 1, 2, \ldots, p$ are positive real numbers.

2. Main result

In this section, we investigate the qualitative dynamics of system (1.5), thereby primarily focusing on the local stability of its unique positive equilibrium, the long-term behavior of its solutions, and their its oscillatory nature under various conditions. Using linearization techniques and an eigenvalue analysis of the associated Jacobian matrix, we establish criteria for local asymptotic stability and instability. Moreover, we examine conditions under which the solutions exhibit oscillatory behavior or divergence, thereby providing a comprehensive understanding of the global behavior of the system. The results presented extend previous findings on rational difference systems to a broader p-dimensional context.

In what follows, we present one of the fundamental results of this paper concerning the qualitative behavior of the system. Specifically, we establish conditions under which the unique positive equilibrium point is locally asymptotically stable or unstable. This result plays a central role in understanding the dynamics of the system and serves as a foundation for the subsequent analysis.

Theorem 2.1. *For system* (1.5)*,*

(i) *Assume that* $\alpha > 2q - 1$*; then, the unique positive equilibrium* $\left(\overline{{}^{(1)}x}, \overline{{}^{(2)}x}, \ldots, \overline{{}^{(p)}x}\right) = (\alpha + 1, \alpha + 1, \ldots, \alpha + 1)$ *of system* (1.5) *is locally asymptotically stable.*

(ii) *Assume that* $0 < \alpha < 2q - 1$*; then, the unique positive equilibrium is unstable.*

Proof. **(i)** The linearized equation of system (1.5) about the equilibrium point $\left(\overline{{}^{(1)}x}, \overline{{}^{(2)}x}, \dots, \overline{{}^{(p)}x}\right)$ is as follows:

$$X_{n+1} = BX_n$$

where $X_n = ({}^{(1)}x_n, {}^{(1)}x_{n-1}, {}^{(2)}x_n, {}^{(2)}x_{n-1}, \dots, {}^{(p)}x_n, {}^{(p)}x_{n-1})^t$, and $B = (b_{ij})$, $1 \le i, j \le 2p$ is an $(2p) \times (2p)$ matrix such that

$$B = \begin{pmatrix} 0 & \frac{q}{\alpha+1} & -\frac{q}{\alpha+1} & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q}{\alpha+1} & -\frac{q}{\alpha+1} & \dots & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & & \dots & 0 & \frac{q}{\alpha+1} & -\frac{q}{\alpha+1} & 0 \\ 0 & 0 & 0 & 0 & \dots & \ddots & 0 & 1 & 0 & 0 & 0 \\ -\frac{q}{\alpha+1} & 0 & 0 & 0 & \dots & & & \dots & 0 & 0 & \frac{q}{\alpha+1} \\ 0 & 0 & 0 & 0 & \dots & & \dots & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_{2p}$ denote the eigenvalues of matrix B and let

$$D = diag(d_1, d_2, \dots, d_{2p})$$

be a diagonal matrix, where $d_1 = d_3 = d_5 = \dots = d_{2p-1} = 1$ and $d_k = d_{2p} = 1 - k\varepsilon$ for $k \in \{1, 2, \dots, p\}$. Since $\alpha > 1$, we can take a positive number ε such that

$$0 < \varepsilon < \frac{1}{2p}\left(1 - \frac{q}{\alpha + 1 - q}\right). \tag{2.1}$$

It is clear that D is an invertible matrix. By computing DBD^{-1}, we obtain the following:

$$= \begin{pmatrix} 0 & \frac{qd_1d_2^{-1}}{\alpha+1} & -\frac{qd_1d_3^{-1}}{\alpha+1} & 0 & 0 & \dots & 0 & 0 \\ d_2d_1^{-1} & 0 & 0 & 0 & 0 & \dots & \dots 0 & 0 \\ 0 & 0 & 0 & \frac{qd_3d_4^{-1}}{\alpha+1} & -\frac{qd_3d_5^{-1}}{\alpha+1} & \dots & \dots 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & & \dots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & & \dots 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \ddots & d_{2p-2}d_{2p-3}^{-1} & 0 \\ -\frac{qd_{2p-1}d_1^{-1}}{\alpha+1} & 0 & 0 & 0 & \dots & & \dots & \frac{qd_{2p-1}d_{2p}^{-1}}{\alpha+1} \\ 0 & 0 & 0 & 0 & \dots & & \dots & 0 \end{pmatrix}.$$

One can easily see that

$$d_2d_1^{-1} < 1, d_4d_3^{-1} < 1, \dots, d_{2p}d_{2p-1}^{-1} < 1.$$

Moreover, based on (2.1), it follows that

$$\frac{qd_1d_2^{-1}}{\alpha + 1} + \frac{qd_1d_3^{-1}}{\alpha + 1} = \frac{q}{\alpha + 1}\left(1 + \frac{1}{d_2}\right) = \frac{q}{\alpha + 1}\left(1 + \frac{1}{1 - 2\varepsilon}\right) < 1.$$

$$\begin{aligned}
\frac{qd_3d_4^{-1}}{\alpha+1} + \frac{qd_4d_5^{-1}}{\alpha+1} &= \frac{q}{\alpha+1}\left(1+\frac{1}{d_4}\right) = \frac{q}{\alpha+1}\left(1+\frac{1}{1-4\varepsilon}\right) < 1.\\
&\vdots\\
\frac{qd_{2p-1}d_{2p}^{-1}}{\alpha+1} + \frac{qd_{2p}d_1^{-1}}{\alpha+1} &= \frac{q}{\alpha+1}\left(1+\frac{1}{d_{2p}}\right) = \frac{q}{\alpha+1}\left(1+\frac{1}{1-2p\varepsilon}\right) < 1.
\end{aligned}$$

It is well known that B shares the same eigenvalues as DBD^{-1}; therefore, we have the following:

$$\begin{aligned}
\max|\lambda_i| &\leq \|DBD^{-1}\|_\infty\\
&= \max\left\{ d_2d_1^{-1}, d_4d_3^{-1}, \ldots, d_{2p}d_{2p-1}^{-1}, \tfrac{q}{\alpha+1}\left(1+\tfrac{1}{d_2}\right), \ldots, \tfrac{q}{\alpha+1}\left(1+\tfrac{1}{d_{2p}}\right) \right\}\\
&< 1.
\end{aligned}$$

All eigenvalues of B are located within the unit disk. By applying Theorem 2.4 ([20]), it follows that the unique positive equilibrium $\left(\overline{{}^{(1)}x}, \overline{{}^{(2)}x}, \ldots, \overline{{}^{(p)}x}\right) = (\alpha+1, \alpha+1, \ldots, \alpha+1)$ is locally asymptotically stable.

(ii) The validity of this statement is supported by the reasoning developed in the proof of (i).

□

Among the core results of this paper, the following theorem plays a crucial role in understanding the long-term dynamics of the system, thereby providing precise conditions under which the system's solutions exhibit alternating behaviors between divergence and convergence. The result emphasizes the importance of initial values and the influence of the parameter α in shaping the system's evolution. Additionally, this theorem demonstrates how specific patterns emerge in the sequence of solutions under certain assumptions.

Theorem 2.2. *Let* $\alpha \in (0,1)$*, and suppose that* $\left({}^{(1)}x_n, {}^{(2)}x_n, \ldots, {}^{(p)}x_n\right)$ *is a positive solution of* (1.5). *Under these conditions, the following results are established:*

(i) *If*

$$ {}^{(i)}x_{-1} \in (0,1), \quad {}^{(i)}x_0 \in \left(\frac{1}{(1-\alpha)^{1/q}}, +\infty\right), \quad i = 1,2,\ldots,p, \tag{2.2}$$

then

$$\lim_{n\to+\infty} {}^{(i)}x_{2n} = \infty, \quad \lim_{n\to+\infty} {}^{(i)}x_{2n+1} = \alpha, \quad i = 1,2,\ldots,p.$$

(ii) *If*

$$ {}^{(i)}x_{-1}\left(\frac{1}{(1-\alpha)^{1/q}}, +\infty\right), \quad {}^{(i)}x_0 \in (0,1), \quad i = 1,2,\ldots,p, \tag{2.3}$$

then

$$\lim_{n\to+\infty} {}^{(i)}x_{2n} = \alpha, \quad \lim_{n\to+\infty} {}^{(i)}x_{2n+1} = \infty, \quad i = 1,2,\ldots,p.$$

Proof. **(i)** Given that $\alpha \in (0,1)$, it follows that $(1-\alpha)^2 < 1$, and $1/(1-\alpha) > 1+\alpha$, which, in turn, implies the following:

$$\alpha < {}^{(i)}x_1 = \alpha + \frac{{}^{(i)}x_{-1}^q}{{}^{(i+1)}x_0^q} \leq \alpha + \frac{1}{{}^{(i+1)}x_0^q} \leq 1, \quad i = 1,2,\ldots,p-1 \tag{2.4}$$

$$\alpha < {}^{(p)}x_1 = \alpha + \frac{{}^{(p)}x_{-1}^q}{{}^{(1)}x_0^q} \leq \alpha + \frac{1}{{}^{(1)}x_0^q} \leq 1. \tag{2.5}$$

Therefore,

$$\left({}^{(1)}x_1, {}^{(2)}x_1, \ldots, {}^{(p)}x_1\right) \in (\alpha, 1] \times (\alpha, 1] \times \ldots \times (\alpha, 1]. \tag{2.6}$$

Analogously, we obtain the following:

$${}^{(i)}x_2 = \alpha + \frac{{}^{(i)}x_0^q}{{}^{(i+1)}x_1^q} \geq \alpha + {}^{(i)}x_0^q, \quad i = 1, 2, \ldots, p-1 \tag{2.7}$$

$${}^{(p)}x_2 = \alpha + \frac{{}^{(p)}x_0^q}{{}^{(1)}x_1^q} \geq \alpha + {}^{(p)}x_0^q. \tag{2.8}$$

Moreover, for $i = 1, 2, \ldots, p-1$,

$$\alpha < {}^{(i)}x_3 = \alpha + \frac{{}^{(i)}x_1^q}{{}^{(i+1)}x_2^q} \leq \alpha + \frac{1}{{}^{(i+1)}x_2^q} \leq \alpha + \frac{1}{(\alpha + {}^{(i+1)}x_0^q)^q} \leq \alpha + \frac{1}{\alpha + {}^{(i+1)}x_0^q} \leq \alpha + 1 - \alpha = 1, \tag{2.9}$$

$$\alpha < {}^{(p)}x_3 = \alpha + \frac{{}^{(p)}x_1^q}{{}^{(1)}x_2^q} \leq \alpha + \frac{1}{{}^{(1)}x_2^q} \leq \alpha + \frac{1}{(\alpha + {}^{(1)}x_0^q)^q} \leq \alpha + \frac{1}{\alpha + {}^{(1)}x_0^q} \leq \alpha + 1 - \alpha = 1. \tag{2.10}$$

Therefore,

$$\left({}^{(1)}x_3, {}^{(2)}x_3, \ldots, {}^{(p)}x_3\right) \in (0, \alpha] \times (0, \alpha] \times \ldots \times (0, \alpha]. \tag{2.11}$$

Likewise, and in the same fashion, one derives the following:

$${}^{(i)}x_4 = \alpha + \frac{{}^{(i)}x_2^q}{{}^{(i+1)}x_3^q} \geq \alpha + (\alpha + {}^{(i)}x_0^q)^q \geq \alpha + (\alpha + {}^{(i)}x_0^q) = 2\alpha + {}^{(i)}x_0^q, \quad i = 1, 2, \ldots, p-1 \tag{2.12}$$

$${}^{(p)}x_4 = \alpha + \frac{{}^{(p)}x_2^q}{{}^{(1)}x_3^q} \geq \alpha + (\alpha + {}^{(p)}x_0^q)^q \geq \alpha + (\alpha + {}^{(i)}x_0^q) = 2\alpha + {}^{(p)}x_0^q. \tag{2.13}$$

Through mathematical induction, we establish that

$$\begin{cases} \left({}^{(1)}x_{2n}, {}^{(2)}x_{2n}, \ldots, {}^{(p)}x_{2n}\right) \in [n\alpha + {}^{(1)}x_0^q, +\infty) \times \ldots \times [n\alpha + {}^{(p)}x_0^q, +\infty) \\ \left({}^{(1)}x_{2n+1}, {}^{(2)}x_{2n+1}, \ldots, {}^{(p)}x_{2n+1}\right) \in (\alpha, 1] \times (\alpha, 1] \times \ldots \times (\alpha, 1] \end{cases}. \tag{2.14}$$

Hence, we conclude that

$$\lim_{n \to +\infty} {}^{(i)}x_{2n} = \infty, \quad i = 1, 2, \ldots, p,$$

and

$$\lim_{n \to +\infty} {}^{(i)}x_{2n+1} = \alpha + \lim_{n \to +\infty} \frac{{}^{(i)}x_{2n-1}^q}{{}^{(i+1)}x_{2n}^q} = \alpha, \quad i = 1, 2, \ldots, p-1.$$

(ii) The proof of **(ii)** proceeds by analogous techniques to those used in **(i)**; therefore, it is omitted to avoid redundancy.

□

The next result investigates the oscillatory nature of the system's solutions, thereby identifying conditions under which each component of the solution does not settle to a constant value but instead fluctuates indefinitely. This behavior provides a deeper insight into the dynamic complexity of the system. The theorem lays the groundwork for understanding when and how oscillations arise based on the initial data.

Theorem 2.3. *Suppose that* $\left({}^{(1)}x_n, {}^{(2)}x_n, \ldots, {}^{(p)}x_n\right)$ *is a positive solution of* (1.5). *By assuming that at least one* $s \geq 0$, *the following conditions are fulfilled:*

$$ {}^{(i)}x_{s-1} < \alpha + 1 \leq {}^{(i)}x_s, \quad i = 1, 2, \ldots, p, \tag{2.15}$$

or

$$ {}^{(i)}x_{s-1} > \alpha + 1 \geq {}^{(i)}x_s, \quad i = 1, 2, \ldots, p. \tag{2.16}$$

Then, the solution $\left({}^{(1)}x_n, {}^{(2)}x_n, \ldots, {}^{(p)}x_n\right)$ *is oscillatory in all of its coordinate components.*

Proof. Provided that condition (2.15) is satisfied, it follows that

$$ {}^{(i)}x_{s+1} = \alpha + \frac{{}^{(i)}x_{s-1}^q}{{}^{(i+1)}x_s^q} < \alpha + 1, \quad i = 1, 2, \ldots, p, \tag{2.17}$$

$$ {}^{(p)}x_{s+1} = \alpha + \frac{{}^{(p)}x_{s-1}^q}{{}^{(1)}x_s^q} < \alpha + 1, \tag{2.18}$$

and

$$ {}^{(i)}x_{s+2} = \alpha + \frac{{}^{(i)}x_s^q}{{}^{(i+1)}x_{s+1}^q} > \alpha + 1, \quad i = 1, 2, \ldots, p, \tag{2.19}$$

$$ {}^{(p)}x_{s+2} = \alpha + \frac{{}^{(p)}x_s^q}{{}^{(1)}x_{s+1}^q} > \alpha + 1. \tag{2.20}$$

From (2.17)–(2.20), it can be concluded that

$$ {}^{(p)}x_{s+1} < \alpha + 1 \leq {}^{(p)}x_{s+2}, \quad i = 1, 2, \ldots, p. \tag{2.21}$$

Provided that condition (2.16) is satisfied, it follows that

$$ {}^{(i)}x_{s+1} = \alpha + \frac{{}^{(i)}x_{s-1}^q}{{}^{(i+1)}x_s^q} > \alpha + 1, \quad i = 1, 2, \ldots, p, \tag{2.22}$$

$$ {}^{(p)}x_{s+1} = \alpha + \frac{{}^{(p)}x_{s-1}^q}{{}^{(1)}x_s^q} > \alpha + 1, \tag{2.23}$$

and

$$ {}^{(i)}x_{s+2} = \alpha + \frac{{}^{(i)}x_s^q}{{}^{(i+1)}x_{s+1}^q} < \alpha + 1, \quad i = 1, 2, \ldots, p, \tag{2.24}$$

$$ {}^{(p)}x_{s+2} = \alpha + \frac{{}^{(p)}x_s^q}{{}^{(1)}x_{s+1}^q} < \alpha + 1. \tag{2.25}$$

From (2.22)–(2.25), it can be concluded that

$$ {}^{(p)}x_{s+2} < \alpha + 1 < {}^{(p)}x_{s+1}, \quad i = 1, 2, \ldots, p. \tag{2.26}$$

This implies that the solution $\left({}^{(1)}x_n, {}^{(2)}x_n, \ldots, {}^{(p)}x_n\right)$ is oscillatory in all of its coordinate components. □

The following theorem provides additional conditions under which the system exhibits oscillatory behaviors in all components, thereby further illustrating how initial orderings of values can lead to persistent fluctuations.

Theorem 2.4. *Suppose that* $\left({}^{(1)}x_n, {}^{(2)}x_n, \ldots, {}^{(p)}x_n\right)$ *is a positive solution of* (1.5)*. By assuming that at least one* $s \geq 0$*, the following conditions are fulfilled:*

$$\alpha + 1 < {}^{(1)}x_{-1} < {}^{(2)}x_{-1} < \ldots < {}^{(p)}x_{-1} < {}^{(1)}x_0 < {}^{(2)}x_0 < \ldots < {}^{(p)}x_0, \tag{2.27}$$

or

$$\alpha + 1 < {}^{(p)}x_0 < {}^{(p-1)}x_0 < \ldots < {}^{(1)}x_0 < {}^{(p)}x_{-1} < {}^{(p-1)}x_{-1} < \ldots < {}^{(1)}x_{-1}. \tag{2.28}$$

Then, the solution $\left({}^{(1)}x_n, {}^{(2)}x_n, \ldots, {}^{(p)}x_n\right)$ *is oscillatory in all of its coordinate components.*

Proof. Provided that condition (2.27) is satisfied, it follows that

$${}^{(i)}x_1 = \alpha + \frac{{}^{(i)}x_{-1}^q}{{}^{(i+1)}x_0^q} < \alpha + 1, \quad i = 1, 2, \ldots, p, \tag{2.29}$$

$${}^{(p)}x_1 = \alpha + \frac{{}^{(p)}x_{-1}^q}{{}^{(1)}x_0^q} < \alpha + 1. \tag{2.30}$$

In view of (2.29) and (2.30), we conclude that

$${}^{(i)}x_2 = \alpha + \frac{{}^{(i)}x_0^q}{{}^{(i+1)}x_1^q} > \alpha + 1, \quad i = 1, 2, \ldots, p, \tag{2.31}$$

$${}^{(p)}x_2 = \alpha + \frac{{}^{(p)}x_0^q}{{}^{(1)}x_1^q} > \alpha + 1. \tag{2.32}$$

Assume that the following statement holds for $n = k$ as the induction hypothesis:

$${}^{(i)}x_{2k-1} < \alpha + 1, \quad {}^{(i)}x_{2k} > \alpha + 1, \quad i = 1, 2, \ldots, p. \tag{2.33}$$

Now, for $n = k + 1$, we obtain

$${}^{(i)}x_{2(k+1)-1} = \alpha + \frac{{}^{(i)}x_{2k-1}^q}{{}^{(i+1)}x_{2k}^q} < \alpha + 1, \quad i = 1, 2, \ldots, p, \tag{2.34}$$

and

$${}^{(i)}x_{2(k+1)} = \alpha + \frac{{}^{(i)}x_{2k}^q}{{}^{(i+1)}x_{2k+1}^q} > \alpha + 1, \quad i = 1, 2, \ldots, p. \tag{2.35}$$

From (2.34) and (2.35), it follows that the solution $\left({}^{(1)}x_n, {}^{(2)}x_n, \ldots, {}^{(p)}x_n\right)$ is oscillatory in all of its coordinate components.

The theorem can be proven in the same manner to the case where condition (2.28) is satisfied. □

This final result establishes the boundedness and persistence of the system's solutions under a specific condition on the parameter α, thereby ensuring It ensures that solutions remain positive and do not diverge.

Persistence is a key concept in the study of dynamical systems because it guarantees that the variables of interest remain bounded away from zero and infinity over time, which is essential for the long-term viability and stability of the modeled process. For instance, in biological and ecological models, persistence implies that populations do not go extinct, while in economic models, it means that quantities do not collapse or grow without bound.

Definition 2.1. *We say that system* (1.5) *is persistent if there are positive constants m and M such that every positive solution* $\left({}^{(1)}x_n, {}^{(2)}x_n, \ldots, {}^{(p)}x_n\right)$ *to the system satisfies the following inequalities:*

$$m \leq {}^{(1)}x_n \leq M,$$

for $j = 1, 2, \ldots, p$ *and a sufficiently large n.*

Theorem 2.5. *Let* $\left({}^{(1)}x_n, {}^{(2)}x_n, \ldots, {}^{(p)}x_n\right)$ *be a positive solution of system* (1.5)*. If* $\alpha^q > 1$*, then the solution is bounded and persists.*

Proof. From System (1.5), it follows that for all $n \geq 1$,

$$ {}^{(i)}x_n \geq \alpha, \quad i = 1, 2, \ldots, p. \tag{2.36}$$

Moreover, we observe that

$$\begin{aligned} {}^{(i)}x_n &= \alpha + \frac{{}^{(i)}x_{n-2}^q}{{}^{(i+1)}x_{n-1}^q} \leq \alpha + \frac{{}^{(i)}x_{n-2}^q}{\alpha^q} \leq \alpha + \frac{\alpha}{\alpha^q} + \frac{1}{\alpha^{2q}}{}^{(i)}x_{n-4}^q \\ &\leq \alpha + \frac{\alpha}{\alpha^q} + \frac{\alpha}{\alpha^{2q}} + \frac{1}{\alpha^{3q}}{}^{(i)}x_{n-6}^q \\ &\leq \alpha + \frac{\alpha}{\alpha^q} + \frac{\alpha}{\alpha^{2q}} + \cdots + \frac{\alpha}{\alpha^{(k-1)q}} + \frac{1}{\alpha^{kq}}{}^{(i)}x_{n-2k}^q \\ &\leq \begin{cases} \frac{\alpha^{q+1}}{\alpha^q-1} + {}^{(i)}x_0^q, & n = 2k \\ \frac{\alpha^{q+1}}{\alpha^q-1} + {}^{(i)}x_{-1}^q, & n = 2k-1 \end{cases}. \end{aligned} \tag{2.37}$$

For $i = 1, 2, \ldots, p$,

$$M = \max\left\{\frac{\alpha^{q+1}}{\alpha^q - 1}, {}^{(i)}x_0, {}^{(i)}x_{-1}\right\}. \tag{2.38}$$

Therefore, based on steps (2.35) to (2.38), we obtain the following:

$$\alpha \leq {}^{(i)}x_n \leq M, \quad n \in \mathbb{N}, \quad i = 1, 2, \ldots, p.$$

This completes the proof of the Theorem. □

3. Numerical examples

Example 3.1. *Utilize system* (1.5)*, where the parameters and initial conditions are chosen as follows:*

- $\alpha = 7$, $p = 10$ *and* $q = 3$.
- *Initial conditions:*

$$^{(1)}x_0 = 2.25, \quad ^{(1)}x_1 = 3.90, \quad ^{(2)}x_0 = 3.37, \quad ^{(2)}x_1 = 3.34, \quad ^{(3)}x_0 = 3.13,$$
$$^{(3)}x_1 = 2.20, \quad ^{(4)}x_0 = 2.99, \quad ^{(4)}x_1 = 3.31, \quad ^{(5)}x_0 = 1.66, \quad ^{(5)}x_1 = 2.72,$$
$$^{(6)}x_0 = 1.67, \quad ^{(6)}x_1 = 3.32, \quad ^{(7)}x_0 = 1.56, \quad ^{(7)}x_1 = 2.76, \quad ^{(8)}x_0 = 3.66,$$
$$^{(8)}x_1 = 2.88, \quad ^{(9)}x_0 = 1.83, \quad ^{(9)}x_1 = 3.46, \quad ^{(10)}x_0 = 3.15, \quad ^{(10)}x_1 = 3.16.$$

Then, we obtain the following system

$$\begin{cases} ^{(1)}x_{n+1} & = & 7 + \dfrac{^{(1)}x_{n-1}^3}{^{(2)}x_n^3} \\ ^{(2)}x_{n+1} & = & 7 + \dfrac{^{(2)}x_{n-1}^3}{^{(3)}x_n^3} \\ & \vdots & \\ ^{(10)}x_{n+1} & = & 7 + \dfrac{^{(10)}x_{n-1}^3}{^{(1)}x_n^3} \end{cases} . \tag{3.1}$$

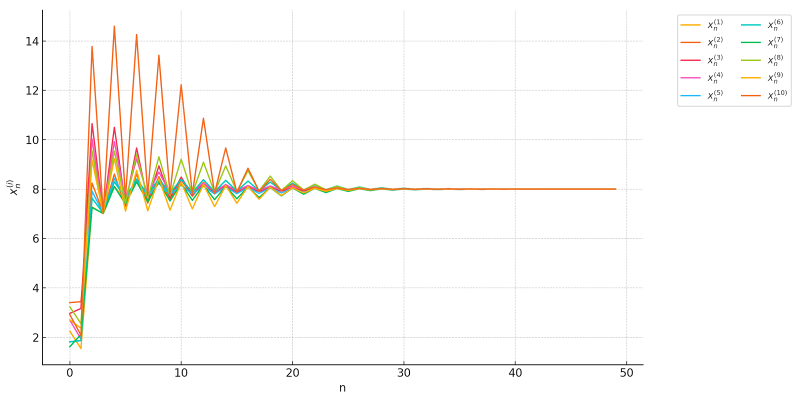

Figure 1. Plot of the numerical solution of the system (3.1).

Figure 1 illustrates the numerical solution of system (3.1). According to Theorem (2.1) **(i)**, the unique positive equilibrium of the system is locally asymptotically stable if $\alpha > 2q - 1$. In this case, since $2q - 1 = 5$, the condition is exactly met, thus positioning the system at the threshold of stability. The plotted trajectories demonstrate convergence toward the same positive value from the distinct initial conditions, thus visually supporting the theoretical result. The behavior seen in the graph confirms that the equilibrium $^{(i)}x = 8, i = 1, 2, \ldots, 10$ acts as an attractor, which aligns with the stability condition outlined in Theorem (2.1) .

Example 3.2. *Utilize system* (1.5)*, where the parameters and initial conditions are chosen as follows:*

- $\alpha = 1.5$, $p = 10$ *and* $q = 1.27$.
- *Initial conditions:*

$$^{(1)}x_{-1}=1.000,\quad ^{(1)}x_0=1.050,\quad ^{(2)}x_{-1}=1.111,\quad ^{(2)}x_0=1.161,\quad ^{(3)}x_{-1}=1.222,$$
$$^{(3)}x_0=1.272,\quad ^{(4)}x_{-1}=1.333,\quad ^{(4)}x_0=1.383,\quad ^{(5)}x_{-1}=1.444,\quad ^{(5)}x_0=1.494,$$
$$^{(6)}x_{-1}=1.556,\quad ^{(6)}x_0=1.605,\quad ^{(7)}x_{-1}=1.667,\quad ^{(7)}x_0=1.716,\quad ^{(8)}x_{-1}=1.778,$$
$$^{(8)}x_0=1.827,\quad ^{(9)}x_{-1}=1.889,\quad ^{(9)}x_0=1.938,\quad ^{(10)}x_{-1}=2.000,\quad ^{(10)}x_0=2.050.$$

Then, we obtain the following system:

$$\begin{cases} ^{(1)}x_{n+1} &= 1.5+\dfrac{^{(1)}x_{n-1}^{1.27}}{^{(2)}x_n^{1.27}} \\ ^{(2)}x_{n+1} &= 1.5+\dfrac{^{(2)}x_{n-1}^{1.27}}{^{(3)}x_n^{1.27}} \\ &\vdots \\ ^{(10)}x_{n+1} &= 1.5+\dfrac{^{(10)}x_{n-1}^{1.27}}{^{(1)}x_n^{1.27}} \end{cases}. \tag{3.2}$$

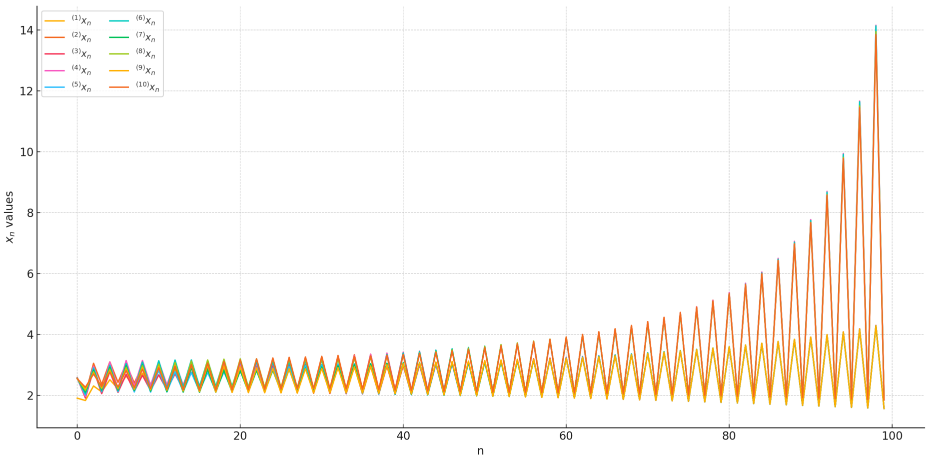

Figure 2. Plot of the numerical solution of the system (3.2).

Figure 2 illustrates the numerical solution of system (3.2), where the parameters are set to $\alpha = 1.5$ and $q = 1.27$. According to Theorem (2.1), **(ii)**, the unique positive equilibrium of the system is unstable when $0 < \alpha < 2q - 1$. In this case, since $2q - 1 \approx 1.54$, the condition $0 < \alpha = 1.5 < 1.54$ is satisfied, thus fulfilling the second condition of the theorem. The plot shows that the solution trajectories diverge from the equilibrium rather than converging, which visually confirms the theoretical prediction of instability.

Example 3.3. *Utilize system* (1.5)*, where the parameters and initial conditions are chosen as follows:*

- $\alpha = 0.8$*,* $p = 10$ *and* $q = 1$.
- *Initial conditions:*

$$^{(1)}x_{-1}=1.5,\quad ^{(1)}x_0=1.9,\quad ^{(2)}x_{-1}=1.6,\quad ^{(2)}x_0=2.0,\quad ^{(3)}x_{-1}=1.7,$$
$$^{(3)}x_0=2.1,\quad ^{(4)}x_{-1}=1.4,\quad ^{(4)}x_0=1.95,\quad ^{(5)}x_{-1}=1.3,\quad ^{(5)}x_0=2.05,$$
$$^{(6)}x_{-1}=1.2,\quad ^{(6)}x_0=2.2,\quad ^{(7)}x_{-1}=1.0,\quad ^{(7)}x_0=1.85,\quad ^{(8)}x_{-1}=1.6,$$
$$^{(8)}x_0=1.95,\quad ^{(9)}x_{-1}=1.1,\quad ^{(9)}x_0=2.0,\quad ^{(10)}x_{-1}=1.4,\quad ^{(10)}x_0=2.1.$$

Then, we obtain the following system:

$$\begin{cases} {}^{(1)}x_{n+1} &= 0.8 + \dfrac{{}^{(1)}x_{n-1}}{{}^{(2)}x_n} \\ {}^{(2)}x_{n+1} &= 0.8 + \dfrac{{}^{(2)}x_{n-1}}{{}^{(3)}x_n} \\ &\vdots \\ {}^{(10)}x_{n+1} &= 0.8 + \dfrac{{}^{(10)}x_{n-1}}{{}^{(1)}x_n} \end{cases}. \tag{3.3}$$

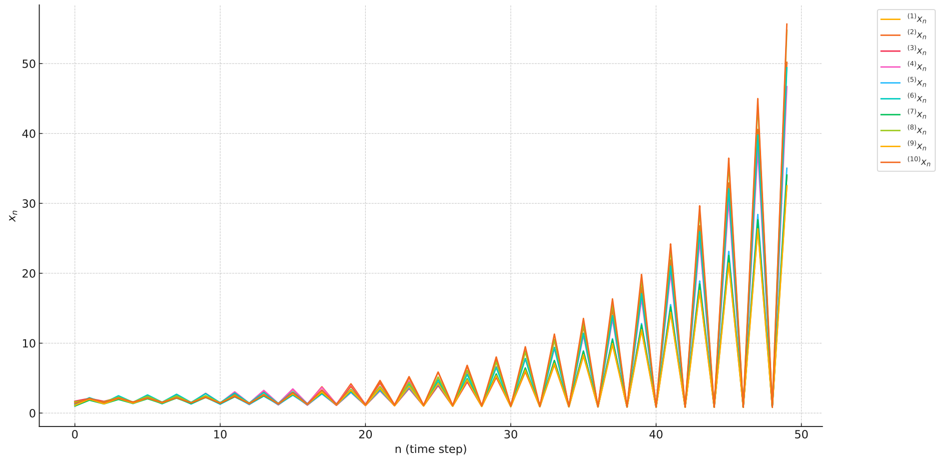

Figure 3. Plot of the numerical solution of the system (3.3).

Figure 3 provides a numerical illustration of the solution of system (3.3), where each coordinate component clearly exhibits oscillatory behavior. The initial conditions applied in this example are chosen to satisfy all the assumptions stated in Theorem (2.3), particularly those related to continuity and sign-changing properties. As a result, the theorem guarantees that the solution is oscillatory in all components. Figure 3, therefore, serves as a concrete numerical validation of Theorem (2.3).

Example 3.4. *Utilize system* (1.5)*, where the parameters and initial conditions are chosen as follows:*

- $\alpha = 2.5$*,* $p = 10$ *and* $q = 1.7$.
- *Initial conditions:*

$$\begin{array}{lllll} {}^{(1)}x_{-1} = 3.6, & {}^{(1)}x_0 = 4.6, & {}^{(2)}x_{-1} = 3.7, & {}^{(2)}x_0 = 4.7, & {}^{(3)}x_{-1} = 3.8, \\ {}^{(3)}x_0 = 4.8, & {}^{(4)}x_{-1} = 3.9, & {}^{(4)}x_0 = 4.9, & {}^{(5)}x_{-1} = 4.0, & {}^{(5)}x_0 = 5.0, \\ {}^{(6)}x_{-1} = 4.1, & {}^{(6)}x_0 = 5.1, & {}^{(7)}x_{-1} = 4.2, & {}^{(7)}x_0 = 5.2, & {}^{(8)}x_{-1} = 4.3, \\ {}^{(8)}x_0 = 5.3, & {}^{(9)}x_{-1} = 4.4, & {}^{(9)}x_0 = 5.4, & {}^{(10)}x_{-1} = 4.5, & {}^{(10)}x_0 = 5.5. \end{array}$$

Then, we obtain the following system:

$$\begin{cases} {}^{(1)}x_{n+1} &= 2.5 + \dfrac{{}^{(1)}x_{n-1}^{1.7}}{{}^{(2)}x_n^{1.7}} \\ {}^{(2)}x_{n+1} &= 2.5 + \dfrac{{}^{(2)}x_{n-1}^{1.7}}{{}^{(3)}x_n^{1.7}} \\ &\vdots \\ {}^{(10)}x_{n+1} &= 2.5 + \dfrac{{}^{(10)}x_{n-1}^{1.7}}{{}^{(1)}x_n^{1.7}} \end{cases} . \tag{3.4}$$

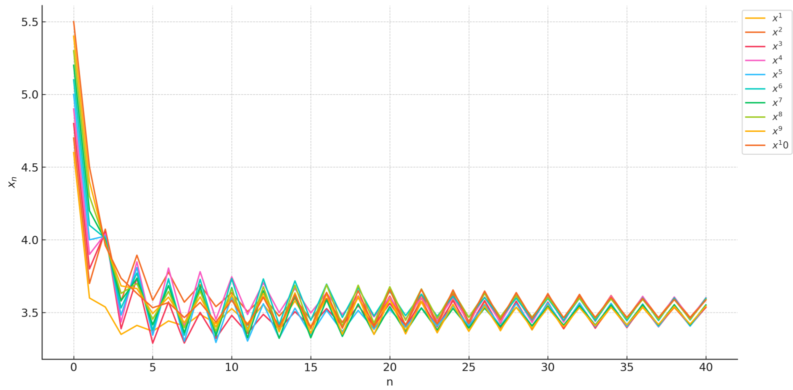

Figure 4. Plot of the numerical solution of the system (3.4).

Figure 4 displays the evolution of the system (3.4). The initial conditions were selected to satisfy the increasing order condition stated in Theorem (2.4), namely the following:

$$\alpha + 1 < {}^{(1)}x_{-1} < {}^{(2)}x_{-1} < \ldots < {}^{(p)}x_{-1} < {}^{(1)}x_0 < {}^{(2)}x_0 < \ldots < {}^{(p)}x_0.$$

These inequalities were ensured by carefully choosing strictly increasing initial values for both $x_{-1}^{(i)}$ and $x_0^{(i)}$, as listed earlier. According to the theorem, such an initial configuration guarantees that the solution is oscillatory in all coordinate components. The plot confirms this theoretical prediction, as each sequence ${}^{(i)}x_n$ clearly exhibits an oscillatory behavior over the time interval considered.

4. Conclusions and several open problems

In this paper, we studied a class of p-dimensional rational difference equations characterized by power-type nonlinearities. Building upon earlier work in lower dimensions, we extended the analysis to more general systems and established results concerning the boundedness of solutions, the local asymptotic stability of equilibria, and the rate at which solutions converge to equilibrium. Here, the theoretical results presented here provide a foundation for the further exploration of high-dimensional discrete dynamical systems, with potential applications in modeling complex processes across various scientific and engineering disciplines. Future research may focus on the global behavior, bifurcation analyses, or the impact of varying the nonlinearity parameter q on the qualitative dynamics of the system.

This study presented a generalization of the systems analyzed in [1, 21, 22, 27], thus offering a broader framework for higher-dimensional cases.

The present findings suggest that this methodology holds significant potential for extension to systems characterized by arbitrary constant parameters, nonautonomous parameters, or systems that involve diverse parameters combined with arbitrary exponents. Accordingly, we outline several key open problems that warrant further investigation by researchers in the field of difference equations.

Open problem 1. Study the dynamical behaviors of the following system of difference equations:

$$\begin{cases} {}^{(1)}x_{n+1} &= \alpha_1 + \dfrac{{}^{(1)}x_{n-m}^{q}}{{}^{(2)}x_{n}^{q}} \\ {}^{(2)}x_{n+1} &= \alpha_2 + \dfrac{{}^{(2)}x_{n-m}^{q}}{{}^{(3)}x_{n}^{q}} \\ &\vdots \\ {}^{(p)}x_{n+1} &= \alpha_p + \dfrac{{}^{(p)}x_{n-m}^{q}}{{}^{(1)}x_{n}^{q}} \end{cases}$$

where α_i, $i = 1, 2, \ldots, p$ are nonnegative constants, and ${}^{(j)}x_{-m}, {}^{(j)}x_{-m+1}, \ldots, {}^{(j)}x_{-1}, {}^{(j)}x_0$, $j = 1, 2, \ldots, p$ are positive real numbers.

Open problem 2. Study the dynamical behaviors of the following system of difference equations:

$$\begin{cases} {}^{(1)}x_{n+1} &= \alpha_n + \dfrac{{}^{(1)}x_{n-m}^{q}}{{}^{(2)}x_{n}^{q}} \\ {}^{(2)}x_{n+1} &= \alpha_n + \dfrac{{}^{(2)}x_{n-m}^{q}}{{}^{(3)}x_{n}^{q}} \\ &\vdots \\ {}^{(p)}x_{n+1} &= \alpha_n + \dfrac{{}^{(p)}x_{n-m}^{q}}{{}^{(1)}x_{n}^{q}} \end{cases}$$

where α_n is a sequence (this sequence can be chosen as convergent, periodic or bounded), and ${}^{(j)}x_{-m}, {}^{(j)}x_{-m+1}, \ldots, {}^{(j)}x_{-1}, {}^{(j)}x_0$, $j = 1, 2, \ldots, p$ are positive real numbers.

Author contributions

Y. Halim: Writing–review & editing, Conceptualization; T. D. Alharbi: Writing–original draft, Methodology. Both authors have been working together in the mathematical development of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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