AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(9): 19994–20009.
DOI: 10.3934/math.2025893
Received: 15 April 2025
Revised: 21 August 2025
Accepted: 25 August 2025
Published: 01 September 2025

Research article

ω-left approximation dimensions under stable equivalences of adjoint type

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Abstract: In this paper, we study invariant properties of ω-left approximation dimensions of modules under the stable equivalences of adjoint type. As applications, we prove that Wakamatsu tilting modules, Wakamatsu tilting conjectures, relative n-torsionfree modules, and generalized Gorenstein dimensions of modules are preserved under those equivalences.

Keywords: ω-left approximation dimension; faithful dimension; Wakamatsu tilting module; Wakamatsu tilting conjecture; stable equivalence of adjoint type
Mathematics Subject Classification: 16E30, 18G25

1. Introduction

Let Λ be an Artinian algebra, and let ω and X be finitely generated left Λ-modules. There is a complex

$$\eta: \quad 0 \to X \xrightarrow{f_1} \omega_1 \xrightarrow{f_2} \omega_2 \xrightarrow{f_3} \cdots \xrightarrow{f_i} \omega_i \xrightarrow{f_{i+1}} \cdots$$

with each $\omega_i \in \mathrm{add}\omega$, where $\mathrm{add}\omega$ is the subclass of Λ-modules consisting of all modules isomorphic to direct summands of finite copies of ω, such that $\mathrm{Im} f_i \hookrightarrow \omega_i$ is a left $\mathrm{add}\omega$-approximation of $\mathrm{Im} f_i$, for all i. Let η_n denote the truncated complex ending ω_n obtained from η. Then X is said to have ω-left approximation dimension n, denoted by $\mathrm{l.app}_{\omega}(X) = n$, if n is the largest positive integer such that η_n is exact. If η is exact, then X is said to have infinite ω-left approximation dimension, denoted by $\mathrm{l.app}\,_{\omega}(X) = \infty$. The ω-left approximation dimension of ${}_{\Lambda}\Lambda$ is just the faithful dimension of ω defined by Buan and Solberg in [9], denoted by $\mathrm{fadim}_{\Lambda}\omega$, which is used to describe the number of non-isomorphism indecomposable complements of an almost cotilting module.

The notion of ω-left approximation dimensions of modules was introduced by Huang [15], which plays a very important role in homological algebra and relative homological algebra. It is well known that Wakamatsu tilting modules, (relative) torsionfree modules and modules having generalized

Gorenstein dimension zero with respect to a Wakamatsu tilting Λ-module are characterized in terms of left approximation dimensions (see [5, 9, 14, 15]).

In studying the representation theory of finite groups, Broué [8] introduced the concept of stable equivalences of Morita type, which is a special case of stable equivalences. Surprisingly, to date, every example of stable equivalences of Morita type of Artinian algebras we known has a Frobenius functor. This type of stable equivalence of Morita type associated with a Frobenius pair is referred to by Xi [29] as a stable equivalence of adjoint type. It is well know that two stably equivalent of adjoint type Artinian algebras share many interesting invariants, such as the rigidity dimension, the extension dimension, stable Grothendieck groups, the Gorenstein projective dimension of modules, the tilting module, the Gorenstein projective module, and so on (see [10, 19, 22, 23, 28–30] for details).

The first part of the present paper is devoted to investigating some transfer properties of ω-left approximation dimensions of modules under stable equivalences of adjoint type of Artinian algebras. One of our main results is the following theorem.

Theorem A. *(Theorem 3.4) Let* Λ *and* Γ *be stably equivalent of adjoint type Artinian algebras induced by bimodules* ${}_{\Lambda}M_{\Gamma}$ *and* ${}_{\Gamma}N_{\Lambda}$*, and let n be a positive integer, and let* ω *be a* Λ*-module satisfying* $\mathrm{Ext}^1_{\Lambda}(\omega,\omega) = 0$ *and* $\mathrm{fadim}_{\Lambda}\omega \geq \mathrm{n}+1$*. For a* Λ*-module X, we have*

(1) $\mathrm{l.app}_{\omega}X = n$ *if and only if* $\mathrm{l.app}_{M\otimes_{\Gamma}N\otimes_{\Lambda}\omega}X = n$*;*

(2) $\mathrm{l.app}_{\omega}X = n$ *if and only if* $\mathrm{l.app}_{M\otimes_{\Gamma}N\otimes_{\Lambda}\omega}(M\otimes_{\Gamma}N\otimes_{\Lambda}X) = n$*;*

(3) $\mathrm{l.app}_{\omega}X = n$ *if and only if* $\mathrm{l.app}_{N\otimes_{\Lambda}\omega}(N\otimes X) = n$*.*

It is well known that tilting modules play a central role in the tilting theory. The classical concept of tilting modules was introduced by Brenner and Butler in [7], and Miyashita [20] extended this notion to modules of finite projective dimension. Wakamatsu [25] further generalized the concept of tilting modules, allowing for modules of infinite projective dimension. These generalized tilting modules are commonly referred to as Wakamatsu tilting modules, following the established terminology in [13]. The Wakamatsu tilting conjecture, posed by Beligiannis and Reiten in [6, Chapter III], states that a Wakamatsu tilting module with finite projective dimension is a tilting module. This conjecture is significant in the representation theory of Artinian algebras and is closely related to several homological conjectures, such as the finitistic dimension conjecture, the Nakamaya conjecture conjecture, the Gorenstein symmetry conjecture, and so on (see [11, 21, 26, 27] for details). Li and Sun [17] proved that stable equivalences of adjoint type preserve the partial tilting modules. Using Theorem A, we will investigate invariance properties of Wakamatsu tilting modules that of relative n-torsionfree modules and obtain the following results.

Theorem B. *(Theorem 4.3) Let* Λ *and* Γ *be Artinian algebras such that there exists a stable equivalence of adjoint type between them. Then* Λ *satisfies the Wakamatsu tilting conjecture if and only if* Γ *does.*

Theorem C. *(Theorem 5.1) Let* Λ *and* Γ *be stably equivalent of adjoint type induced by bimodules* ${}_{\Lambda}M_{\Gamma}$ *and* ${}_{\Gamma}N_{\Lambda}$*, and n a positive integer. Suppose that* ω *is a* Λ*-module satisfying* $\mathrm{Ext}^1_{\Lambda}(\omega,\omega) = 0$ *and* $\mathrm{fadim}_{\Lambda}\omega \geq \mathrm{n}+2$*. For a* Λ*-module X, we have*

(1) *X is an* ω*-n- torsionfree* Λ*-module if and only if* $N\otimes_{\Lambda}X$ *is an* $N\otimes_{\Lambda}\omega$*-n-torsionfree* Γ*-module;*

(2) *X is an* ω*-n- torsionfree* Λ*-module if and only if* $M\otimes_{\Gamma}N\otimes_{\Lambda}X$ *is an* ω*-n-torsionfree* Λ*-module.*

The paper is organized as follows. In Section 2, we provide preliminary definitions and results. Sections 3–5 are devoted to the proofs of Theorems A–C.

2. Preliminaries

In this section, we recall some notations and collect some fundamental results. Throughout this paper, all rings are Artinian algebras over a commutative Artinian ring R, all modules are finitely generated left Λ-modules. Let Λ be an Artinian algebra and ω a Λ-module. We use $\operatorname{mod}\Lambda$ to denote the category consisting of all finitely generated Λ-modules and use $\operatorname{add}\omega$ to denote the full subcategory of $\operatorname{mod}\Lambda$ consisting of all modules isomorphic to direct summands of finite copies of ω. And we denote by $\operatorname{gen}\omega$ the full subcategory of $\operatorname{mod}\Lambda$ having as objects those modules X such that there is an epimorphism $\omega_0 \to X$ with $\omega_0 \in \operatorname{add}\omega$.

A homomorphism $f : X \to \omega_0$ with $\omega_0 \in \operatorname{add}\omega$ is called a left $\operatorname{add}\omega$-approximation of X, if $\operatorname{Hom}_\Lambda(\omega_0, -) \to \operatorname{Hom}_\Lambda(X, -)$ is exact in $\operatorname{add}\omega$. And f is called a left minimal $\operatorname{add}\omega$-approximation of X if it is also left minimal, that is, $h \in \operatorname{End}X$ is an automorphism whenever $fh = f$ (see [3, 4]).

Lemma 2.1. *Let M and ω be Λ-modules and $M = M_1 \oplus M_2$.*

(1) *Suppose that $f : M \to \omega_0$ is a left $\operatorname{add}\omega$-approximation of M; then there exists a left minimal $\operatorname{add}\omega$-approximation $g : M_1 \to \omega_1$, such that $\omega_1 \in \operatorname{add}\omega_0$.*

(2) $\operatorname{l.app}_\omega M_1 \geq \operatorname{l.app}_\omega M$.

Proof. (1) See [4, P7, Theorem 2.2].

(2) It follows directly from the definition of the ω-left approximation dimension of M and (1). □

We recall the definition of a tilting module and that of a Wakamatsu tilting module. Let n be a positive integer. Recall from [7] that a Λ-module ω is said to be a n-tilting Λ-module if the following conditions are satisfied. (1) ω is self-orthogonal, that is, $\operatorname{Ext}_\Lambda^{\geq 1}(\omega, \omega) = 0$; (2) $\operatorname{pd}_\Lambda \omega = n < \infty$; (3) there exists an exact sequence $0 \to_\Lambda \Lambda \to \omega_0 \to \omega_1 \to \cdots \to \omega_n \to 0$ with $\omega_i \in \operatorname{add}\omega$ for $0 \leq i \leq n$. A Λ-module ω is said to be a tilting module it is an n-tilting Λ-module for some positive integer n. Recall from [25] that a Λ-module ω is called a Wakamatsu tilting module if it is self-orthogonal and $\operatorname{fadim}_\Lambda \omega = \infty$. And Beligiannis and Reiten in [6, Chapter III] proposed the following conjecture.

Wakamatsu tilting conjecture: *Let Λ be an Artinian algebra. Suppose that ω is a Wakamatsu tilting Λ-module with $\operatorname{pd}_\Lambda \omega < \infty$; then ω is a tilting Λ-module.*

Let ω be a Λ-module and n a positive integer. The notion of an ω-n-torsionfree module was introduced by Huang in [15] as a non-trivial generalization of the notion of a n-torsionfree module defined in [1]. We refer the reader to [15] for the original definition; we shall use the following characterization, which is also proved in [15].

Definition 2.1. Let ω be a Λ-module with $\operatorname{fadim}_\Lambda \omega \geq$ n+2. A Λ-module X is said to be ω-n-torsionfree, if $\operatorname{l.app}_\omega X = n$.

In case ${}_\Lambda\omega = {}_\Lambda \Lambda$, an ω-n-torsionfree module defined above is just an n-torsionfree module defined in [1]. We use $\mathcal{T}_\omega^n(\Lambda)$ to denote the subcategory of $\operatorname{mod}\Lambda$ consisting of all ω-n-torsionfree Λ-modules.

Let ω be a Wakamatsu tilting Λ-module with the endomorphism algebra $\Pi = \operatorname{End}_\Lambda \omega$. Recall from [2] that a Λ-module X is said to have generalized Gorenstein dimension zero with respect to ω, denoted by $\operatorname{G-dim}_\omega X = 0$, if the following data is satisfied. (1) X is ω-reflexive, that is, the evaluation map $\sigma_X : X \to \operatorname{Hom}_{\Pi^o}(\operatorname{Hom}_\Lambda(X, \omega), \omega)$ via $\sigma_X(x)(f) = f(x)$, for any $f \in \operatorname{Hom}_\Lambda(X, \omega)$ and $x \in X$, is an isomorphism; (2) $\operatorname{Ext}_\Lambda^i(X, \omega) = 0 = \operatorname{Ext}_{\Pi^{op}}^i(\operatorname{Hom}_\Lambda(X, \omega), \omega)$ for any $i \geq 1$. We denote by $\mathcal{G}_\omega(\Lambda)$ the subcategory of $\operatorname{mod}\Lambda$ consisting of all modules having generalized Gorenstein dimension zero with respect to ω. In case ${}_\Lambda\Lambda =_\Lambda \omega$, a Λ-module G having generalized Gorenstein dimension zero with respect to ω is just a Λ-module having Gorenstein dimension zero defined by Auslander in [1]. Following the terminology of Enoch and Jenda, a module having Gorenstein dimension zero is called Gorenstein projective [12]. According to [5, Lemma 5.1], a Λ-module X has generalized Gorenstein dimension zero with respect to ω if and only if X is a ω-∞-torsionfree Λ-module with $\mathrm{Ext}_\Lambda^{\geq 1}(X, \omega) = 0$.

Definition 2.2. ([18, Definition 6.1] and [28, Definition 3.2]) Let Λ and Γ be two Artinian algebras, and let ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ be finitely generated projective as one-sided modules. Λ and Γ are said to be symmetrically separably equivalent induced by bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$, if there exist bimodules ${}_\Lambda P_\Lambda$ and ${}_\Gamma Q_\Gamma$ and bimodule isomorphisms

$${}_\Lambda M \otimes_\Gamma N_\Lambda \cong {}_\Lambda\Lambda_\Lambda \oplus {}_\Lambda P_\Lambda \qquad \textit{and} \qquad {}_\Gamma(N \otimes_\Lambda M)_\Gamma \cong {}_\Gamma\Gamma_\Gamma \oplus_\Gamma Q_\Gamma$$

such that $(N \otimes_\Lambda -, M \otimes_\Gamma -)$ and $(M \otimes_\Gamma -, N \otimes_\Lambda -)$ are adjoint pairs.

Furthermore, if P is a projective Λ-bimodule and Q is a projective Γ-bimodule, respectively, then Λ and Γ are said to be stably equivalent of adjoint type.

Lemma 2.2. *[19, Lemma 2.2] Let Λ and Γ be stably equivalent of adjoint type defined as Definition 2.3. Then*

(1) ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$ *are projective generators as one-sided modules;*

(2) *for a Λ-module X, $P \otimes_\Lambda X$ is a projective Λ-module;*

(3) $N \otimes_\Lambda -$ *and* $M \otimes_\Gamma -$ *are exact functors and take projective modules to projective modules;*

(4) $(M \otimes_\Gamma -) \circ (N \otimes_\Lambda -) \to \mathrm{Id}_{\mathrm{Mod}\Lambda} \oplus (P \otimes_\Lambda -)$ *and* $(N \otimes_\Lambda -) \circ (M \otimes_\Gamma -) \to \mathrm{Id}_{\mathrm{Mod}\Gamma} \oplus (Q \otimes_\Lambda -)$ *are natural isomorphisms.*

3. ω-left approximation dimensions of modules

In this section, we will investigate the invariant properties of ω-left approximation dimensions under stable equivalences of adjoint type. We begin with the following easy observation.

Lemma 3.1. *Let Λ and Γ be stably equivalent of adjoint type induced by bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ and n a positive integer. For Λ-modules X and Y, there exists an isomorphism*

$$\mathrm{Ext}_\Gamma^n(N \otimes_\Lambda X, N \otimes_\Lambda Y) \cong \mathrm{Ext}_\Lambda^n(X, Y).$$

Proof. By the definition of a stable equivalence of adjoint type, there exist an adjoint pair $(M \otimes_\Gamma -, N \otimes_\Lambda -)$ and a Λ-bimodule isomorphism $M \otimes_\Gamma N \cong \Lambda \oplus P$, where P is a projective Λ-bimodule. By Lemma 2.4(4), we have

$$\begin{aligned}\mathrm{Ext}_\Gamma^n(N \otimes_\Lambda X, N \otimes_\Lambda Y) &\cong \mathrm{Ext}_\Lambda^n(M \otimes_\Gamma N \otimes_\Lambda X, Y)\\ &\cong \mathrm{Ext}_\Lambda^n(X, Y) \oplus \mathrm{Ext}_\Lambda^n(P \otimes_\Lambda X, Y)\\ &\cong \mathrm{Ext}_\Lambda^n(X, Y),\end{aligned}$$

because $P \otimes_\Lambda X$ is a projective Λ-module by Lemma 2.4(2). □

Proposition 3.1. *Let Λ and Γ be stably equivalent of adjoint type induced by bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$, and let ω be a Λ-module satisfying $\mathrm{Ext}_\Lambda^1(\omega, \omega) = 0$. For a Λ-module X, we have*

$$\mathrm{l.app}_{N\otimes_\Lambda\omega}(N \otimes X) \geq \mathrm{l.app}_\omega X.$$

Proof. Without loss of generality, we assume that $\mathrm{l.app}_{\omega}X = n$. Then there exists an exact sequence

$$0 \to X \xrightarrow{f_1} \omega_1 \xrightarrow{f_2} \omega_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} \omega_n \qquad (3.1)$$

with each $\omega_i \in \mathrm{add}\omega$, such that $\mathrm{Im}f_i \hookrightarrow \omega_i$ is a left $\mathrm{add}\omega$-approximation of mf_i, for $1 \leq i \leq n$. Define $T_i = \mathrm{Im}f_i$, $T_{n+1} = \mathrm{Coker}(f_n)$, and $g_i : \mathrm{Im}f_i \hookrightarrow \omega_i$ for any $1 \leq i \leq n$. Combining these facts, we obtain the following exact sequence

$$0 \to T_i \xrightarrow{g_i} \omega_i \to T_{i+1} \to 0 \qquad (3.2i)$$

where g_i is a left $\mathrm{add}\omega$-approximation of T_i, for any $1 \leq i \leq n$. Noting that $\mathrm{Ext}^1_\Lambda(\omega, \omega) = 0$, then one gets that $\mathrm{Ext}^1_\Lambda(T_{i+1}, \omega) = 0$ for any $0 \leq i \leq n$, by applying the exact functor $\mathrm{Hom}_\Lambda(-, \omega)$ to the sequence (3.2i).

Applying the exact functor $N \otimes_\Lambda -$ to (3.1) yields an exact sequence in $\mathrm{mod}\,\Gamma$:

$$0 \to N \otimes_\Lambda X \xrightarrow{N\otimes_\Lambda f_1} N \otimes_\Lambda \omega_1 \xrightarrow{N\otimes_\Lambda f_2} N \otimes_\Lambda \omega_2 \xrightarrow{N\otimes_\Lambda f_3} \cdots \xrightarrow{N\otimes_\Lambda f_n} N \otimes_\Lambda \omega_n \qquad (3.3)$$

with $N \otimes_\Lambda \omega_i \in \mathrm{add}(N \otimes_\Lambda \omega)$ and $\mathrm{Im}(N \otimes_\Lambda f_i) \cong N \otimes_\Lambda T_i$ for each $1 \leq i \leq n$. and $\mathrm{Coker}(N \otimes_\Lambda f_n) \cong N \otimes_\Lambda T_{n+1}$.

According to Lemma 3.1, it follows that $\mathrm{Ext}^1_\Gamma(N \otimes_\Lambda T_{i+1}, N \otimes_\Lambda \omega) \cong \mathrm{Ext}^1_\Lambda(T_{i+1}, \omega) = 0$, for any $0 \leq i \leq n$. Thus, we obtain that $\mathrm{Im}(N \otimes_\Lambda f_i) \hookrightarrow N \otimes_\Lambda \omega_i$ is a left $\mathrm{add}(N \otimes_\Lambda \omega)$ -approximation of $\mathrm{Im}(N \otimes_\Lambda f_i)$. This means $\mathrm{l.app}_{N\otimes_\Lambda\omega} N \otimes X \geq \mathrm{l.app}_\omega X$. □

One of the application of Proposition 3.2 is the following.

Corollary 3.1. *Let* Λ *and* Γ *be stably equivalent of adjoint type induced by bimodules* ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$, *and let* ω *be a* Λ*-module with* $\mathrm{Ext}^1_\Lambda(\omega, \omega) = 0$. *Then we have*

$$\mathrm{fadim}_\Gamma(\mathrm{N} \otimes_\Lambda \omega) \geq \mathrm{fadim}_\Lambda \omega.$$

Proof. By Lemma 2.4(1), we have $\mathrm{add}_\Gamma N = \mathrm{add}_\Gamma \Gamma$. Then, it is easy to see that $\mathrm{l.app}_{N\otimes_\Lambda\omega}\Gamma = \mathrm{l.app}_{N\otimes_\Lambda\omega}N$. Thus, by the definition of faithful dimensions of modules and Proposition 3.2, one gets $\mathrm{fadim}_\Gamma(\mathrm{N} \otimes_\Lambda \omega) = \mathrm{l.app}_{\mathrm{N}\otimes_\Lambda\omega}\Gamma \geq \mathrm{l.app}_\omega\Lambda = \mathrm{fadim}_\Lambda\omega$. □

The following is the main result of this section.

Theorem 3.1. *Let* Λ *and* Γ *be stably equivalent of adjoint type induced by bimodules* ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$, *and let n be a positive integer, and let* ω *be a* Λ*-module satisfying* $\mathrm{Ext}^1_\Lambda(\omega, \omega) = 0$ *and* $\mathrm{fadim}_\Lambda\omega \geq \mathrm{n}+1$. *For a* Λ*-module X, we have*

(1) $\mathrm{l.app}_\omega X = n$ *if and only if* $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega} X = n$;

(2) $\mathrm{l.app}_\omega X = n$ *if and only if* $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(M \otimes_\Gamma N \otimes_\Lambda X) = n$;

(3) $\mathrm{l.app}_\omega X = n$ *if and only if* $\mathrm{l.app}_{N\otimes_\Lambda\omega}(N \otimes X) = n$.

To prove this theorem, we first need to establish the following lemma.

Lemma 3.2. *Let* Λ *and* Γ *be stably equivalent of adjoint type induced by bimodules* ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$, *and let n be a positive integer, and let* ω *be a* Λ*-module satisfying* $\mathrm{Ext}^1_\Lambda(\omega, \omega) = 0$ *and* $\mathrm{fadim}_\Lambda\omega \geq \mathrm{n}$. *For a* Λ*-module X, we have*

(1) $\mathrm{l.app}_\omega X \geq n$ *if and only if* $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega} X \geq n$;

(2) $\mathrm{l.app}_\omega X \geq n$ *if and only if* $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(M \otimes_\Gamma N \otimes_\Lambda X) \geq n$;

(3) $\mathrm{l.app}_\omega X \geq n$ *if and only if* $\mathrm{l.app}_{N\otimes_\Lambda\omega}(N \otimes X) \geq n$.

Proof. According to Corollary 3.3, one obtains $\mathrm{fadim}_\Gamma(\mathrm{N} \otimes_\Lambda \omega) \geq \mathrm{n}$ and $\mathrm{fadim}_\Lambda(\mathrm{M} \otimes_\Gamma \mathrm{N} \otimes_\Lambda \omega) \geq \mathrm{n}$.

(1) Suppose that $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega} X \geq n$; then there exists an exact sequence:

$$0 \to X \xrightarrow{f_1} Y_1 \xrightarrow{f_2} Y_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} Y_n$$

with each $Y_i \in \mathrm{add}(M \otimes_\Gamma N \otimes_\Lambda \omega)$, such that each $\mathrm{Im} f_i \to Y_i$ is a left $\mathrm{add}(M \otimes_\Gamma N \otimes_\Lambda \omega)$-approximation of $\mathrm{Im} f_i$. By Lemma 2.4(4), there is a Λ-module isomorphism $M \otimes_\Gamma N \otimes_\Lambda \omega \cong \omega \oplus P \otimes_\Lambda \omega$, where P is a projective Λ-bimodule. By Lemma 2.4(2), $P \otimes_\Lambda \omega$ is a projective Λ-module. Since $\mathrm{fadim}_\Lambda \omega \geq \mathrm{n}$ and each $Y_i \in \mathrm{add}(\omega \oplus P \otimes_\Lambda \omega)$ by assumption, for any $1 \leq Y_i \leq n$, there exists an exact sequence:

$$0 \to Y_i \xrightarrow{f_1^i} \omega_1^i \xrightarrow{f_2^i} \omega_2^i \xrightarrow{f_3^i} \cdots \xrightarrow{f_n^i} \omega_n^i$$

with each $\omega_j^i \in \mathrm{add}\omega$, such that each $\mathrm{Im} f_j^i \hookrightarrow \omega_j^i$ is a left $\mathrm{add}\omega$-approximation of $\mathrm{Im} f_j^i$. Due to [16, Corollary 3.19], we obtain an exact sequence:

$$0 \to X \xrightarrow{h_1} \omega_1^1 \xrightarrow{h_2} \omega_2^1 \oplus \omega_1^2 \xrightarrow{h_3} \cdots \xrightarrow{h_j} \oplus_{i=1}^j \omega_{j+1-i}^i \xrightarrow{h_{j+1}} \cdots \to\xrightarrow{h_n} \oplus_{i=1}^n \omega_{n+1-i}^i$$

with each $\omega_j^i \in \mathrm{add}\omega$ such that $\mathrm{Im} h_j \hookrightarrow \oplus_{i=1}^j \omega_{j+1-i}^i$ is a left $\mathrm{add}\omega$-approximation of $\mathrm{Im} h_j$. This means $\mathrm{l.app}_\omega X \geq n$.

Conversely, if $\mathrm{l.app}_\omega X \geq n$, then we have $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega} X \geq \mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(M \otimes_\Gamma N \otimes_\Lambda X) \geq \mathrm{l.app}_\omega X \geq n$, where the first inequality holds from Lemma 2.1(2) and the second inequality from Proposition 3.2 twice.

(2) Assume that $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(M \otimes_\Gamma N \otimes_\Lambda X) \geq n$. Since ${}_\Lambda X$ is isomorphic to a direct summand of ${}_\Lambda(M \otimes_\Gamma N \otimes_\Lambda X)$ by Lemma 2.4(4), we obtain $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega} X \geq n$ by Lemma 2.1(2). It follows from (1) that $\mathrm{l.app}_\omega X \geq n$.

Conversely, assume that $\mathrm{l.app}_\omega X \geq n$. This result follows immediately from Proposition 3.2.

(3) By Proposition 3.2, one obtains $\mathrm{l.app}_{N\otimes_\Lambda\omega}(N \otimes_\Lambda X) \geq n$, when $\mathrm{l.app}_\omega(X) \geq n$.

Conversely, assume that $\mathrm{l.app}_{N\otimes_\Lambda\omega}(N \otimes_\Lambda X) \geq n$. Then we get $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(M \otimes_\Gamma N \otimes_\Lambda X) \geq n$, by Proposition 3.2. Thus, by (2), we have $\mathrm{l.app}_\omega(X) \geq n$. □

The proof of Theorem 3.4.

Since $\mathrm{fadim}_\Lambda \omega \geq \mathrm{n}+1$, by Corollary 3.3, one gets $\mathrm{fadim}_\Gamma(\mathrm{N}\otimes_\Lambda\omega) \geq \mathrm{n}+1$ and $\mathrm{fadim}_\Lambda(\mathrm{M}\otimes_\Gamma\mathrm{N}\otimes_\Lambda\omega) \geq \mathrm{n} + 1$.

(1) Assume that $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega} X = n$; we have $\mathrm{l.app}_\omega X \geq n$ by Lemma 3.5(1). If $\mathrm{l.app}_\omega X \neq n$, then $\mathrm{l.app}_\omega X \geq n + 1$. Since $\mathrm{fadim}_\Lambda \omega \geq \mathrm{n} + 1$ by assumption, one has $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega} X \geq n + 1$ by Lemma 3.4(1). This leads to a contradiction. Hence, one has $\mathrm{l.app}_\omega X = n$.

Conversely, assume that $\mathrm{l.app}_\omega X = n$. By Proposition 3.2 twice, one gets $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(M \otimes_\Gamma N \otimes_\Lambda X) \geq n$. Since ${}_\Lambda X$ is isomorphic to a direct summand of ${}_\Lambda(M \otimes_\Gamma N \otimes_\Lambda X)$ by Lemma 2.4(4), $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega} X \geq n$ by Lemma 2.1(2). And hence we have $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega} X = n$. If not, then one has $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega} X \geq n + 1$. Noting that $\mathrm{fadim}_\Lambda \omega \geq \mathrm{n} + 1$, then we obtain $\mathrm{l.app}_\omega X \geq n + 1$ by Lemma 3.5(1), which leads to a contradiction!

(2) By Proposition 3.2 and Lemma 2.1(2), we have $\mathrm{l.app}_\omega X \leq \mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(M \otimes_\Gamma N \otimes_\Lambda X) \leq \mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(X)$. The result follows directly by Lemma 3.5(2).

(3) Assume that $\mathrm{l.app}_\omega X = n$. Then we have $\mathrm{l.app}_{N\otimes_\Lambda\omega}(N \otimes X) \geq n$ by Proposition 3.2. If $\mathrm{l.app}_{N\otimes_\Lambda\omega}(N \otimes X) \neq n$. By Proposition 3.2 again, we have $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(M \otimes_\Gamma N \otimes_\Lambda X) \geq$

$\mathrm{l.app}_{N\otimes_\Lambda\omega}(N\otimes_\Lambda X)\geq n+1$. By Lemma 3.5(1), one has $\mathrm{l.app}_\omega X\geq n+1$. This leads to a contradiction. Hence we have $\mathrm{l.app}_{N\otimes_\Lambda\omega}(N\otimes X)=n$.

Conversely, assume that $\mathrm{l.app}_{N\otimes_\Lambda\omega}(N\otimes_\Lambda X)=n$. Then we have $\mathrm{l.app}_\omega X\leq n$ by Proposition 3.2. On the other hand, by Proposition 3.2 again, $\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(M\otimes_\Gamma N\otimes_\Lambda X)\geq \mathrm{l.app}_{N\otimes_\Lambda\omega}(N\otimes_\Lambda X)=n$. And hence, by Lemma 3.5(2), we have $\mathrm{l.app}_\omega X\geq n$. And so the result follows.

The next corollary follows directly from Theorem 3.4.

Corollary 3.2. *Let Λ and Γ be stably equivalent of adjoint type induced by bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$, and let ω be a Λ-module satisfying $\mathrm{Ext}^1_\Lambda(\omega,\omega)=0$ and $\mathrm{fadim}_\Lambda\omega=\infty$. For a Λ-module X, we have*

$$\mathrm{l.app}_\omega X=\mathrm{l.app}_{N\otimes_\Lambda\omega}(N\otimes_\Lambda X)=\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(M\otimes_\Gamma N\otimes_\Lambda X)=\mathrm{l.app}_{M\otimes_\Gamma N\otimes_\Lambda\omega}X.$$

4. Wakamatsu tilting conjectures

In this section, we will give some applications of results in Section 3 and further prove that the Wakamatsu tilting conjecture holds true under stable equivalences of adjoint type.

Proposition 4.1. *Let Λ and Γ be stably equivalent of adjoint type induced by bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$, and let ω be a Λ-module. If ω is a Wakamatsu tilting Λ-module, then $N\otimes_\Lambda\omega$ is a Wakamatsu tilting Γ-module.*

Proof. This follows directly from Lemma 3.1 and Corollary 3.3. □

The following lemma is due to Sun and Zhao.

Lemma 4.1. *[24, Lemma 3.6] Let Λ and Γ be stably equivalent of adjoint type induced by bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$. If ω is a tilting Λ-module, then $N\otimes_\Lambda\omega$ is a tilting Γ-module.*

Lemma 4.2. *Let ω be a Wakamatsu tilting Λ-module with an exact sequence*

$$0\to\Lambda\xrightarrow{g_0}\omega_0\xrightarrow{g_1}\omega_1\xrightarrow{g_2}\cdots\xrightarrow{g_{n-1}}\omega_{n-1}\xrightarrow{g_n}\omega_n\to\cdots\qquad(4.1)$$

such that $\mathrm{Im}g_i\hookrightarrow\omega_i$ is a left $\mathrm{add}\omega$-approximation of $\mathrm{Im}g_i$ for all $i\geq 0$. And let P be a projective Λ-module with each $\mathrm{Ext}^1_\Lambda(\mathrm{Im}g_i,P)=0$ for all i. If $\omega\oplus P$ is a tilting Λ-module, then ω is so.

Proof. Assume that $\omega\oplus P$ is an n-tilting Λ-module. Then we have $\mathrm{pd}_\Lambda\omega=\mathrm{pd}_\Lambda(\omega\oplus P)=n$ and a long exact sequence

$$0\to\Lambda\xrightarrow{f_0}T_0\xrightarrow{f_1}T_1\xrightarrow{f_2}\cdots\xrightarrow{f_{n-1}}T_{n-1}\xrightarrow{f_n}T_n\to 0\qquad(4.2)$$

with each $T_i\in\mathrm{add}(\omega\oplus P)$ for $0\leq i<n$ and $T_n\in\mathrm{add}\omega$, because P is a projective Λ-module. Define $L_i=\mathrm{Im}(f_i)$ for all i. Then $L_n=T_n$. Since $\mathrm{Ext}^{\geq1}_\Lambda(\omega\oplus P,\omega\oplus P)=0$, we have $\mathrm{Ext}^1_\Lambda(L_i,\omega\oplus P)=0$ by dimension shifting, for $1\leq i\leq n$. This implies that $L_i\hookrightarrow T_i$ is a left $\mathrm{add}(\omega\oplus P)$ -approximation of $\mathrm{Im}f_i$, for all $1\leq i<n$.

Since ω is a Wakamatsu tilting Λ-module with the exact sequence (4.1) by assumption, we can write $K_i=\mathrm{Im}g_i$ in the sequence (4.1), for all i. Since $\mathrm{Ext}^1_\Lambda(\omega,\omega)=0$, we have $\mathrm{Ext}^1_\Lambda(K_i,\omega)=0$ for all i. Noting that $\mathrm{Ext}^{\geq1}_\Lambda(K_i,P)=0$ by assumption, one gets $\mathrm{Ext}^{\geq1}_\Lambda(K_i,\omega\oplus P)=0$ for any i.

Since $f_0 : \Lambda \to T_0$ is a left $\mathrm{add}(\omega \oplus P)$-approximation of Λ and $\omega_0 \in \mathrm{add}\omega \subset \mathrm{add}(\omega \oplus P)$, for $g_0 : \Lambda \to \omega_0$, there exists $\alpha_0 : T_0 \to \omega_0$ such that $g_0 = \alpha_0 f_0$. This yields the following commutative diagram with exact rows

$$\begin{CD}
0 @>>> \Lambda @>{f_0}>> T_0 @>>> L_1 @>>> 0\\
@. @| @V{\alpha_0}VV @VVV @.\\
0 @>>> \Lambda @>{g_0}>> \omega_0 @>>> K_1 @>>> 0
\end{CD}$$

which is both a pullback and a pushout. Thus, one gets an exact sequence

$$0 \to T_0 \to \omega_0 \oplus L_1 \to K_1 \to 0,$$

which splits, because $\mathrm{Ext}^1_\Lambda(K_1, \omega \oplus P) = 0$ and $T_0 \in \mathrm{add}(\omega \oplus P)$. Thus, we have $K_1 \in \mathrm{add}(\omega \oplus L_1)$.

Decompose K_1 as $K_1 = \omega_0' \oplus L_1'$ with $L_1' \in \mathrm{add}L_1$ and $\omega_0' \in \mathrm{add}\omega/\mathrm{add}L_1$, where $\mathrm{add}\omega/\mathrm{add}L_1$ is the subclass of $\mathrm{add}\omega$ consisting of all modules without nonzero direct summands that lie in $\mathrm{add}L_1$. By assumption and by Lemma 2.1, there exists an exact sequence $0 \to L_1' \xrightarrow{\gamma_1} \omega_1' \xrightarrow{\beta} H_2 \to 0$ with $\omega_1' \in \mathrm{add}\omega$ such that γ_1 is a left minimal $\mathrm{add}\omega$-approximation of L_1'. Hence, we have the following commutative diagram with exact rows

$$\begin{CD}
0 @>>> L_1' @>{\gamma_1}>> \omega_1' @>>> H_2 @>>> 0\\
@. @V{i}VV @V{t}VV @V{\omega}VV @.\\
0 @>>> K_1 @>{g_1}>> \omega_1 @>>> K_2 @>>> 0\\
@. @V{p}VV @V{s}VV @V{\nu}VV @.\\
0 @>>> L_1' @>{\gamma_1}>> \omega_1' @>>> H_2 @>>> 0
\end{CD}$$

with $pi = \mathrm{Id}_{L_1'}$, one gets st as an isomorphism, because γ_1 is left minimal. It follows that $\nu\omega$ is an isomorphism. Thus, we obtain that p, s and ν are split epic. Therefore, there exists an exact commutative diagram with split columns

$$\begin{CD}
@. 0 @. 0 @. 0 @.\\
@. @VVV @VVV @VVV @.\\
0 @>>> \omega_1' @>>> \mathrm{Ker}s @>>> \mathrm{Ker}\nu @>>> 0\\
@. @VVV @VVV @VVV @.\\
0 @>>> K_1 @>>> \omega_1 @>>> K_2 @>>> 0\\
@. @V{p}VV @V{s}VV @V{\nu}VV @.\\
0 @>>> L_1' @>>> \omega_1' @>>> H_2 @>>> 0\\
@. @VVV @VVV @VVV @.\\
@. 0 @. 0 @. 0 @.
\end{CD} \tag{4.3}$$

Thus, we have $K_2 \cong H_2 \oplus \mathrm{Ker}\nu$ and $\mathrm{Ker}s \in \mathrm{add}\omega$. Noting that $\mathrm{Ext}^1_\Lambda(K_2, \omega) = 0$ and $\omega_1' \in \mathrm{add}\omega$, one gets the top row in commutative diagram (4.3) split. This implies that $\mathrm{Ker}\nu \in \mathrm{add}\omega$.

On the other hand, since $L_1' \in \text{add}L_1$, by Lemma 2.1(1) and assumption, we obtain an exact sequence

$$0 \to L_1' \xrightarrow{\beta_1} T_1' \to L_2' \to 0$$

such that β_1 is a minimal left add$(\omega \oplus P)$-approximation of L_1' and $L_2' \in \text{add}L_2$. Notice that $\omega_1' \in \text{add}\omega \subset \text{add}(\omega \oplus P)$; we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1' & \xrightarrow{\beta_1} & T_1' & \longrightarrow & L_2' & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\alpha_1}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_1' & \xrightarrow{\gamma_1} & \omega_1' & \longrightarrow & H_2 & \longrightarrow & 0 \end{array}$$

which is a pullback as well as a pushout. Then, we obtain an exact sequence

$$0 \to T_1' \to \omega_1' \oplus L_2' \to H_2 \to 0 \qquad (4.4)$$

Since $H_2 \in \text{add}K_2$, $T_2' \in \text{add}(\omega \oplus P_1)$ and $\text{Ext}^1_\Lambda(K_2, \omega \oplus P_1) = 0$, the exact sequence (4.4) splits. So, $H_2 \in \text{add}(\omega_1' \oplus L_2') \subset \text{add}(\omega \oplus L_2)$. Thus, $K_2 \cong H_2 \oplus \text{Ker}\nu \in \text{add}(\omega \oplus L_2)$.

We inductively prove that $K_i \in \text{add}(\omega \oplus L_i)$ for $1 \le i \le n$. Therefore, $K_n \in \text{add}\omega$ for $L_n = \omega_n \in \text{add}\omega$. Thus, one gets that ω is a tilting Λ-module as regards. □

Theorem 4.1. *Let Λ and Γ be stably equivalent of adjoint type. Then Λ satisfies the Wakamatsu tilting conjecture if and only if Γ does.*

Proof. Assume that Λ and Γ are stably equivalent of adjoint type induced by bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$. That is, there exist projective bimodules ${}_\Lambda P_\Lambda$ and ${}_\Gamma Q_\Gamma$ and bimodule isomorphisms ${}_\Lambda M \otimes_\Gamma N_\Lambda \cong {}_\Lambda \Lambda_\Lambda \oplus_\Lambda P_\Lambda$ and ${}_\Gamma N \otimes {}_\Lambda M_\Gamma \cong {}_\Gamma \Gamma_\Gamma \oplus {}_\Gamma Q_\Gamma$.

Assume that Γ satisfies the Wakamatsu tilting conjecture. Let ω be a Wakamatsu tilting Λ-module with $\text{pd}_\Lambda \omega < \infty$. By Proposition 4.1, $N \otimes_\Lambda \omega$ is a Wakamatsu tilting Γ-module. Since functor $N \otimes_\Lambda -$ is exact and takes projective Λ-modules to projective Γ-modules by Lemma 2.4(3), we have $\text{pd}_\Gamma(N \otimes_\Lambda \omega) \le \text{pd}_\Lambda \omega < \infty$. Thus, one obtains that $N \otimes_\Lambda \omega$ is a tilting Γ-module by assumption.

According to Lemma 4.2, it follows that $M \otimes_\Gamma N \otimes_\Lambda \omega$ is a tilting Λ-module. By Lemma 2.4(4), there exists a Λ-module isomorphism $M \otimes_\Gamma N \otimes_\Lambda \omega \cong \omega \oplus P \otimes_\Lambda \omega$. Define $P_1 = P \otimes_\Lambda \omega$. Then P_1 is a projective Λ-module by Lemma 2.4(2). By Lemma 4.2 again, one gets $N \otimes_\Lambda (\omega \oplus P_1)$ is a tilting Γ-module, where $N \otimes_\Lambda P_1$ is projective by Lemma 2.4(3). We claim that $N \otimes_\Lambda P_1 \in \text{add}(N \otimes_\Lambda \omega)$. In fact, noting that $N \otimes_\Lambda \omega$ is a tilting Γ-module, then, by [26, Lemma 3.3], we have an exact sequence

$$0 \to N \otimes_\Lambda P_1 \to C \to D \to 0 \qquad (4.5)$$

with $C \in \text{gen}(N \otimes_\Lambda \omega)$ and $D \in \text{add}(N \otimes_\Lambda \omega)$. On the other hand, since $N \otimes_\Lambda \omega$ and $N \otimes_\Lambda (\omega \oplus P_1)$ are tilting Γ-modules, one has $0 = \text{Ext}^{\ge 1}_\Gamma(N \otimes_\Lambda (\omega \oplus P_1), N \otimes_\Lambda (\omega \oplus P_1)) = 0 = \text{Ext}^1_\Gamma(N \otimes_\Lambda \omega, N \otimes_\Lambda \omega)$. It follows that $\text{Ext}^1_\Gamma(N \otimes_\Lambda \omega, N \otimes_\Lambda P_1) = 0$, which implies that the exact sequence (4.5) splits. Therefore, $N \otimes_\Lambda P_1 \in \text{gen}(N \otimes_\Lambda \omega)$. Thus, one gets $N \otimes_\Lambda P_1 \in \text{add}(N \otimes_\Lambda \omega)$ by the projective property of $N \otimes_\Lambda P_1$, and our claim is obtained.

Since ω is a Wakamatsu tilting Λ-module, there exists an exact sequence

$$0 \to \Lambda \xrightarrow{g_0} \omega_0 \xrightarrow{g_1} \omega_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \omega_{n-1} \xrightarrow{g_n} \omega_n \to \cdots$$

with each $\omega_i \in \mathrm{add}\omega$, such that $\mathrm{Img}_i \to \omega_i$ is a left $\mathrm{add}_\Lambda\omega$ of Img_i for all $i \geq 0$. Let $K_i = \mathrm{Img}_i$, for all i. Since $\mathrm{Ext}^1_\Lambda(\omega, \omega) = 0$ in the above sequence, we have $\mathrm{Ext}^1_\Lambda(K_i, \omega) = 0$ for all i. Thanks to Lemma 3.1, it follows that $\mathrm{Ext}^1_\Gamma(N \otimes_\Lambda K_i, N \otimes_\Lambda \omega) \cong \mathrm{Ext}^{\geq 1}_\Lambda(K_i, \omega) = 0$ for each i. Since $N \otimes_\Lambda P_1 \in \mathrm{add}(N \otimes_\Lambda \omega)$, one has $\mathrm{Ext}^{\geq 1}_\Lambda(N \otimes_\Lambda K_i, N \otimes_\Lambda P_1) = 0$. Hence, we obtain $\mathrm{Ext}^{\geq 1}_\Lambda(K_i, P_1) = 0$ by Lemma 3.1 again. Noting that $\omega \oplus P_1$ is a tilting Λ-module, then, by Lemma 4.3, ω is a tilting Λ-module as regards.

Similarly, we can prove that Γ satisfies the Wakamatsu tilting conjecture when Λ does. □

Proposition 4.2. *Let Λ and Γ be stably equivalent of adjoint type induced by bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ and n a positive integer, and let ω be a Wakamatsu tilting Λ-module. If there exists an exact sequence*

$$0 \to \Lambda \to \omega_0 \to \omega_1 \to \cdots \to \omega_n \to 0$$

then $\mathrm{End}_\Lambda\omega$ *and* $\mathrm{End}_\Gamma(N \otimes_\Lambda \omega)$ *are symmetrically separably equivalent.*

Proof. Due to Proposition 4.1 twice, it follows that $M \otimes_\Gamma N \otimes_\Lambda \omega$ is a Wakamatsu tilting Λ-module. By Lemma 2.4(4), there is a Λ-module isomorphism $M \otimes_\Gamma N \otimes_\Lambda \omega \cong \omega \oplus (P \otimes_\Lambda \omega)$, where P is a projective Λ-bimodule. Define $P_1 = P \otimes_\Lambda \omega$. Then P_1 is a projective Λ-module by Lemma 2.4(2). Since $M \otimes_\Gamma N \otimes_\Lambda \omega$ and ω are Wakamatsu tilting Λ-modules, we have $\mathrm{Ext}^1_\Lambda(\omega \oplus P_1, \omega \oplus P_1) = 0 = \mathrm{Ext}^1_\Lambda(\omega, \omega)$. Combining this results, one gets $\mathrm{Ext}^{\geq 1}_\Lambda(\omega, P_1) = 0$. We claim that $P_1 \in \mathrm{add}\omega$. By assumption and by Lemma 2.1(1), it is straightforward to verify that there exists an exact sequence

$$0 \to P_1 \xrightarrow{f_0} \omega'_0 \xrightarrow{f_1} \omega'_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} \omega'_n \to 0$$

with $\omega'_i \in \mathrm{add}\omega$ for all $0 \leq i \leq n$. Let $T_0 = \mathrm{Coker} f_0$. Then, we obtain the following exact sequences

$$0 \to P_1 \xrightarrow{f_0} \omega'_0 \to T_0 \to 0 \qquad (4.6)$$

and

$$0 \to T_0 \to \omega'_1 \to \omega'_2 \to \cdots \to \omega'_n \to 0 \qquad (4.7)$$

with each $\omega'_i \in \mathrm{add}\omega$. Noting that $\mathrm{Ext}^{\geq 1}_\Lambda(\omega, P_1) = 0$, then one gets $\mathrm{Ext}^1_\Lambda(T_0, P_1) \cong \mathrm{Ext}^n_\Lambda(\omega'_n, P_1) = 0$ by dimension shifting. This implies the exact sequence (4.6) is split. Thus, $\omega'_0 \cong P_1 \oplus T_0$. And the claim is proved.

Consequently, one obtains $M \otimes_\Gamma N \otimes_\Lambda \omega \in \mathrm{add}\omega$. According to [24, Theorem 3.1], it follows that $\mathrm{End}_\Lambda\omega$ and $\mathrm{End}_\Gamma(N \otimes_\Lambda \omega)$ are symmetrically separably equivalent. □

As a consequence of Proposition 4.5, we recover a result of [24].

Corollary 4.1. *Let T be a tilting Λ-module. Then* $\mathrm{End}_\Lambda T$ *and* $\mathrm{End}_\Gamma(N \otimes_\Lambda T)$ *are symmetrically separably equivalent.*

5. Relative torsionfree modules

In this section, we give some applications of results in Section 3 and Section 4 and prove that the stable equivalent of adjoint type Artinian algebra preserves relative torsionfree modules and modules having generalized Gorenstein dimension zero with respect to a Wakamatsu tilting module.

Theorem 5.1. *Let* Λ *and* Γ *be stably equivalent of adjoint type induced by bimodules* ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$ *and let n be a positive integer. Suppose that* ω *is a* Λ*-module satisfying* $\mathrm{Ext}^1_\Lambda(\omega, \omega) = 0$ *and* $\mathrm{fadim}_\Lambda\omega \geq \mathrm{n}+2$. *For a* Λ*-module X, we have*

(1) *X is an* ω*-n- torsionfree* Λ*-module if and only if* $N \otimes_\Lambda X$ *is an* $N \otimes_\Lambda \omega$*-n-torsionfree* Γ*-module;*

(2) *X is an* ω*-n- torsionfree* Λ*-module if and only if* $M \otimes_\Gamma N \otimes_\Lambda X$ *is an* ω*-n-torsionfree* Λ*-module;*

Proof. Since $\mathrm{fadim}_\Lambda\omega \geq \mathrm{n}+2$ by assumption, we have $\mathrm{fadim}_\Gamma(\mathrm{N} \otimes_\Lambda \omega) \geq \mathrm{n}+2$ and $\mathrm{fadim}_\Lambda(\mathrm{M} \otimes_\Gamma \mathrm{N} \otimes_\Lambda \omega) \geq \mathrm{n}+2$ by Corollary 3.3. The result follows directly from Theorem 3.4. □

The next corollary is immediate from Theorem 5.1.

Corollary 5.1. *Let* Λ *and* Γ *be stably equivalent of adjoint type induced by bimodules* ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$, *and let* ω *be a Wakamatsu tilting* Λ*-module and X a* Λ*-module. Then*

(1) *X is an* ω*-*∞*-torsionfree* Λ*-module if and only if* $N \otimes_\Lambda X$ *is an* $N \otimes_\Lambda \omega$*-*∞*-torsionfree* Γ*-module.*

(2) *X is an* ω*-*∞*-torsionfree* Λ*-module if and only if* $M \otimes_\Gamma \otimes_\Lambda X$ *is an* $M \otimes_\Gamma N \otimes_\Lambda \omega$*-*$\infty$*-torsionfree* Γ*-module.*

Recall that a subcategory C of $\mathrm{mod}\,\Lambda$ is called *extension-closed* if the middle terms of any short exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$ are in C, provided the end terms X_1 and X_3 are in C. More detail about the extension closure of the category of modules consisting of relative n-torsionfree modules can be found in [16]. The following gives the transfer of the extension closedness of a subcategory of Λ-modules consisting of ω-n-torsionfree modules.

Proposition 5.1. *Let* Λ *and* Γ *be stably equivalent of adjoint type induced by bimodules* ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$ *and let n be a positive integer, and let* ω *be a* Λ*-modules satisfying* $\mathrm{Ext}^1_\Lambda(\omega, \omega) = 0$ *and* $\mathrm{fadim}_\Lambda\omega \geq \mathrm{n}+2$. *Then* $\mathcal{T}^n_\omega(\Lambda)$ *is closed under extensions if and only if so is* $\mathcal{T}^n_{N\otimes_\Lambda\omega}(\Gamma)$.

Proof. Assume that $\mathcal{T}^n_\omega(\Lambda)$ is closed under extensions, and let

$$0 \to Y_1 \to Y_2 \to Y_3 \to 0 \qquad (5.1)$$

be an exact sequence in $\mathrm{mod}\,\Gamma$, where Y_1 and Y_3 are $N \otimes_\Gamma \omega$-n-torsionfree Γ-modules. Applying $M \otimes_\Gamma -$ to the sequence (5.1) induces a short exact sequence of Λ-modules

$$0 \to M \otimes_\Gamma Y_1 \to M \otimes_\Gamma Y_2 \to M \otimes_\Gamma Y_3 \to 0.$$

By Theorem 5.1 (1), one has $M \otimes_\Gamma Y_1$ and $M \otimes_\Gamma Y_3$ are $M \otimes_\Gamma N \otimes_\Lambda \omega$-$n$-torsionfree Λ-modules. So, by Theorem 5.1(2), one gets $M \otimes_\Gamma Y_1$ and $M \otimes_\Gamma Y_3$ are ω-n-torsionfree Λ-modules. By assumption, we obtain that $M \otimes_\Gamma Y_2$ is a ω-n-torsionfree Λ-module. It follows from Theorem 5.1(1) that $N \otimes_\Lambda M \otimes_\Gamma Y_2$ is an $N \otimes_\Lambda \omega$-n-torsionfree Γ-module. By Theorem 5.1(2) again, we obtain Y_2 which is also an $N \otimes_\Lambda \omega$-n-torsionfree Γ-module as regards.

Similarly, we can prove that $\mathcal{T}^n_\omega(\Lambda)$ is closed under extensions, when $\mathcal{T}^n_{N\otimes_\Lambda\omega}(\Gamma)$ does so. □

Theorem 5.2. *Let* Λ *and* Γ *be stably equivalent of adjoint type induced by bimodules* ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$, *and let* ω *be a Wakamatsu tilting* Λ*-module and X a* Λ*-module. Then*

(1) *X has generalized Gorenstein dimension zero with respect to* ω *if and only if* $N \otimes_\Lambda X$ *has generalized Gorenstein dimension zero with respect to* $N \otimes_\Lambda \omega$.

(2) *X has generalized Gorenstein dimension zero with respect to* ω *if and only if* $M \otimes_\Gamma N \otimes_\omega X$ *has generalized Gorenstein dimension zero with respect to* $M \otimes_\Gamma N \otimes_\Lambda \omega$.

Proof. Since ω is a Wakamatsu tilting Λ-module, by Proposition 4.1, one obtains that $N \otimes_\Lambda \omega$ is a Wakamatsu tilting Γ-module and $M \otimes_\Gamma N \otimes_\Lambda \omega$ is a Wakamatsu tilting Λ-module. According to [5, Lemma 5.1], a Λ-module X has generalized Gorenstein dimension zero with respect to a Wakamatsu tilting Λ-module ω if and only if $\mathrm{Ext}^{\geq 1}_\Lambda(X, \omega) = 0$ and X is an ω-∞-torsionfree Λ-module. This result follows directly from Lemma 3.1 and Theorem 5.1. □

Let ω be a Wakamatsu tilting Λ-module. Recall from that Λ-module X is said to *have generalized Gorenstein dimension with respect to ω less than or equal to n*, denoted by $\text{G-dim}_\omega X \leq n$, if there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to X \to 0$$

with each $G_i \in \mathcal{G}_\omega(\Lambda)$. If n is the least nonnegative integer for which such a sequence exists, then $\text{G-dim}_\omega(X) = n$, and if there is no such n, then $\text{G-dim}_\omega(X) = \infty$.

Proposition 5.2. *Let Λ and Γ be stably equivalent of adjoint type induced by bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$, and let ω be a Wakamatsu tilting Λ-module and X a Λ-module. Then*

$$\text{G-dim}_\omega(X) = \text{G-dim}_{N\otimes_\Lambda\omega}(N \otimes_\Lambda X).$$

Proof. Without loss of generality, we assume that $\text{G-dim}_{N\otimes_\Lambda\omega}(N \otimes_\Lambda X) = n$. Then there is an exact sequence in $\mathrm{mod}\,\Lambda$

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to X \to 0 \qquad (5.2)$$

such that $G_i \in \mathcal{G}_\omega(\Lambda)$ for each $0 \leq i \leq n$. Applying the exact functor $N \otimes_\Lambda -$ to the sequence (5.2) induces an exact sequence of Γ-modules

$$0 \to N \otimes_\Lambda G_n \to N \otimes_\Lambda G_{n-1} \to \cdots \to N \otimes_\Lambda G_1 \to N \otimes_\Lambda G_0 \to N \otimes_\Lambda X \to 0 \qquad (5.3)$$

with each $N \otimes_\Lambda G_i \in \mathcal{G}_{N\otimes_\Lambda\omega}(\Gamma)$. This implies $\text{G-dim}_{N\otimes_\Lambda\omega}(N \otimes_\Lambda X) \leq n$.

From the above step, one obtains $\text{G-dim}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(M \otimes_\Gamma N \otimes_\Lambda X) \leq \text{G-dim}_{N\otimes_\Lambda\omega}(N \otimes_\Lambda X)$. By Lemma 2.4(4), there exists a Λ-module isomorphism $M \otimes_\Gamma N \otimes_\Lambda X \cong X \oplus P \otimes_\Lambda X$, where P is a projective Λ-bimodule. Define $P_X = P \otimes_\Lambda X$. Then P_X is a projective Λ-module by Lemma 2.4(2). We assume that $\text{G-dim}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(M \otimes_\Gamma N \otimes_\Lambda X) = m$. Thus, there exists an exact sequence

$$0 \to V_m \xrightarrow{f_m} V_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \xrightarrow{f_0} X \oplus P_X \to 0 \qquad (5.4)$$

with each $V_i \in \mathcal{G}_{M\otimes_\Gamma N\otimes_\Lambda\omega}(\Lambda)$. By Theorem 5.4(2), one has $V_i \in \mathcal{G}_\omega(\Lambda)$ for any $0 \leq i \leq m$. Taking $T = \ker f_0$, the sequence (5.4) induces the following exact sequences:

$$0 \to T \to V_0 \xrightarrow{f_0} X \oplus P_X \to 0 \qquad (5.5)$$

and

$$0 \to V_m \xrightarrow{f_m} V_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_2} V_1 \to T \to 0 \qquad (5.6)$$

By pullback, we have the following exact commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & L & \longrightarrow & P_X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & V_0 & \longrightarrow & X \oplus P_X & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & X & = & X & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

which induces an exact sequences:

$$0 \to T \oplus P_X \to V_0 \to X \to 0 \qquad (5.7)$$

because P_X is projective. Combining the sequences (5.6) and (5.7), one obtains a long exact sequence

$$0 \to V_m \overset{f_m}{\to} V_{m-1} \overset{f_{m-1}}{\to} \cdots \overset{f_2}{\to} V_1 \oplus P_X \to V_0 \to X \to 0 \qquad (5.8)$$

Noting that P_X is projective, we have $P_X \in \mathcal{G}_\omega(\Lambda)$. Due to [14, Lemma 5.9], it follows $V_1 \oplus P_X \in \mathcal{G}_\omega(\Lambda)$. Thus, we have $\text{G-dim}_\omega(X) \leq m$. We thus prove this proposition by the above discussion. □

In case ${}_\Lambda\omega = {}_\Lambda\Lambda$, the generalized Gorenstein dimension with respect to ω of a Λ-module X is just the Gorenstein projective dimension of X defined by [1], denoted by $G\text{-dim}\, X$. Due to Proposition 4.19, we have the following corollary, which is a result in [19, Corollary 4.5].

Corollary 5.2. *Let Λ and Γ be stably equivalent of adjoint type induced by bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$. Suppose that X is a Λ-module. Then* $\text{G-dim} X \leq n$ *if and only if* $\text{G-dim}(N \otimes_\Lambda X) \leq n$.

We conclude with an example to illustrate our results.

Example: Let k be an algebraically closed field. And let Λ and Γ be finite-dimensional k-algebras given by the following quivers with relations

$$\Lambda \qquad \underset{1}{\cdot} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \underset{2}{\cdot} \quad \text{with relation} \quad \alpha\beta\alpha\beta = 0$$

and

$$\Gamma \qquad \underset{1}{\cdot} \underset{y}{\overset{x}{\rightleftarrows}} \underset{2}{\cdot} \circlearrowleft_z \quad \text{with relation} \quad xy = xz = zy = z^2 - yx = 0.$$

Then, Λ and Γ are stably equivalent of adjoint type induced by bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ (see [19, Example, P581] and [23, Example 1] for details). Note that Λ is a Nakayama algebra, and indecomposable projective and injective Λ-modules are

$$P(1) = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, P(2) = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} = I(2) \quad \text{and} \quad I(1) = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}.$$

and $S(2) = (2)$ is a simple Λ-module. Thus, we obtain the minimal injective resolution of $P(1)$:

$$0 \to P(1) \to I(2) \to I(2) \to I(1) \to 0, \qquad (5.9)$$

and the minimal injective resolution of $S(2)$:

$$0 \to S(2) \to I(2) \to I(1) \to 0. \qquad (5.10)$$

(1) Λ and Γ satisfy the Wakamatsu tilting conjecture.

(2) Let $\omega = I(1) \oplus I(2)$. Then ω is a 2-tilting Λ-module, and $N \otimes_\Lambda \omega$ is a 2-tilting Γ-module.

(3) $S(2)$ has generalized Gorenstein dimension zero with respect to ω, and $N \otimes_\Lambda S(2)$ has generalized Gorenstein dimension zero with respect to $N \otimes_\Lambda \omega$.

Proof. (1) From the sequence (5.9), one gets $\mathrm{id}_\Lambda \Lambda = 2$ and $\mathrm{pd}_\Lambda I(1) = 2$. Similarly, we have $\mathrm{id}\Lambda_\Lambda = 2$. Thus, Λ is a Gorenstein algebra with $\mathrm{id}_\Lambda \Lambda = \mathrm{id}\Lambda_\Lambda = 2$. According to [21, Proposition 3.2], a Wakamatsu tilting Λ-module ω is a tilting module provided $\mathrm{id}_\Lambda \Lambda < \infty$. Thus, one gets Λ satisfying the Wakamatsu tilting conjecture. Therefore, Γ also satisfying the Wakamatsu tilting conjecture by Theorem 4.4.

(2) By the definition of tilting modules, we obtain that ω is a 2-tilting Λ-module, since $P(2)$ is a projective-injective module. And hence, $N \otimes_\Lambda \omega$ is a 2-tilting Γ-module by Lemma 2.2.

(3) Noting that ω is a tilting and injective Λ-module, then, from the sequence (5.10), one gets $\mathrm{l.app}_\omega S(2) = \infty$ and $\mathrm{Ext}_\Lambda^{i \geq 1}(S(2), \omega) = 0$. It follows from [5, Lemma 5.1] that $S(2)$ has generalized Gorenstein dimension zero with respect to ω. By Theorem 5.4, $N \otimes_\Lambda S(2)$ has generalized Gorenstein dimension zero with respect to $N \otimes_\Lambda \omega$. □

6. Conclusions

In this study, we mainly show that ω-left approximation dimensions of modules, Wakamatsu tilting modules, Wakamatsu tilting conjectures, relative n-torsionfree modules, and generalized Gorenstein dimensions of modules are preserved under the stable equivalences of adjoint type. In future work, we will study those invariants hold under symmetric separable equivalences.

Author contributions

Juxiang Sun: Contributed the creative ideas and proof techniques for this paper; Weimin Liu: Consulted the relevant background of the paper and composed the article, encompassing the structure of the article and the modification of grammar. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was financially supported by Key Research Projects Plan of Higher Education Institutions in Henan Province (Grant Nos. 26B110007, 25A160001).

Conflict of interest

The authors declare no competing interests.

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