AIMS Mathematics

AIMS Mathematics, 10(7): 15108–15130.
DOI: 10.3934/math.2025678
Received: 05 February 2025
Revised: 29 May 2025
Accepted: 04 June 2025
Published: 01 July 2025

https://www.aimspress.com/journal/Math

Research article

Fractional order Thau-Luenberger observer for fractional order Takagi-Sugeno dynamical systems under uncertain nonmeasurable variables and unknown inputs

Abdelghani Djeddi[1], Ahmad Taher Azar[2,3], Saim Ahmed[2,3,*], Jalel Dib[1], Nashwa Ahmad Kamal[4] and Zeeshan Haider[2,3]

[1] Department of Electrical Engineering, Echahid Cheikh Larbi Tebessi University, Tebessa, Algeria
[2] College of Computer and Information Sciences, Prince Sultan University, Riyadh, Saudi Arabia
[3] Automated Systems and Computing Lab (ASCL), Prince Sultan University, Riyadh, Saudi Arabia
[4] Faculty of Engineering, Cairo University, Giza, Egypt

* **Correspondence:** Email: sahmed@psu.edu.sa.

Abstract: This paper proposed a novel state estimation framework for nonlinear systems described by fractional-order (FO) and Takagi-Sugeno (TS) fuzzy models, targeting critical challenges associated with large modeling uncertainties. The key innovation was in the development of two complementary observer structures that estimated system states when premise variables were either measurable or non-measurable, and in the presence of unknown inputs and uncertain dynamics. The proposed methodology addressed uncertainties affecting system matrices, input matrices, and unknown input transmission matrices. It introduced a fractional-order Thau-Luenberger observer (FO-TLO) for systems with measurable premise variables, and a dedicated observer adapted to the case of non-measurable premise variables. Both configurations utilized Lyapunov-based stability theory and linear matrix inequality (LMI) methods to ensure robust, asymptotic, and theoretically guaranteed convergence of the state estimation error in the presence of time-varying and bounded uncertainties. The framework extended the observer design to a broader class of FO-TS systems, and it offered effective tools for fault diagnosis, system monitoring, and robust control in the presence of uncertain environments.

Keywords: fractional-order Takagi-Sugeno model; fractional-order Thau-Luenberger observer; non-measurable premise variables; unknown inputs; asymptotic stability; linear matrix inequalities

Mathematics Subject Classification: 34A08, 93B07, 93C10, 93C42, 93D09

1. Introduction

Dynamic system control and modeling are very difficult problems in modern engineering due to natural nonlinearities, uncertainties, and complexities of real processes. Integer-order conventional calculus cannot describe key system characteristics adequately, for instance, memory effects, long-range dependencies, and nonlocal interactions [1,2].

Fractional-order systems (FOS), which are characterized by the existence of non-integer-order derivatives, represent a flexible and powerful paradigm of modeling phenomena. FOSs were shown to simulate viscoelastic materials, electrochemical processes, and thermal dynamics more accurately via the successful modeling of memory effects and hereditary properties [3,4]. Applications of their work have also extended to industrial settings, where they have strengthened process models and made them more realistic [5].

Recent advances in fractional-order synchronization and robust control, for example, adaptive sliding mode control based projective synchronization of uncertain fractional reaction-diffusion systems [6], Fractional proportional-integral-derivative (FPID) controller for interconnected power systems [7], adaptive supervisory fractional-order schemes for wind turbines [8]; or Fractional sliding mode control (FSMC)-based global Mittag-Leffler synchronization of neural networks [9,10], demonstrate the universal applicability of fractional calculus in addressing complex dynamic behaviors. Such synchronization-related strategies, although not necessarily observer-design focused, do have some elements in common with the present work, such as the utilization of fractional operators, Lyapunov-based methodologies, and model uncertainty robustness.

Other observer-based approaches have also been proposed for nonlinear systems under the integer-order framework. For instance, an adaptive sliding-mode observer combined with a fixed-time backstepping controller has been proposed for robotic systems subject to model uncertainties and actuator saturation [11]. High-gain observers with sliding-mode control have also been applied for voltage regulation in power converters and the control of permanent-magnet synchronous motors under parameter uncertainties [12, 13]. These strategies, while effective, are generally confined to integer-order models.

Takagi-Sugeno (TS) fuzzy models have emerged as powerful tools for the representation and control of nonlinear systems. By combining local linear approximations through fuzzy logic, they provide both interpretability and effectiveness in a wide range of applications.

In conjunction with fractional-order dynamics, the TS model results in the Fractional-Order Takagi-Sugeno (FO-TS) paradigm that enhances the capability to represent complex, memory-driven nonlinear phenomena [14,15]. Novel simulation-based confirmations have borne testimony to the efficacy of FO-TS systems to tackle high-dimensional control challenges [16, 17].

Despite these improvements, strong observer design for FO-TS systems remains a severe issue. Structural uncertainties, nonmeasurable premise variables, and unknown disturbances make state estimation complicated. Traditional designs have a tendency to be very much based on idealistic presumptions, i.e., full premise variable and input availability [18,19].

This paper proposes a double and complementary observer synthesis strategy for uncertain FO-TS systems. The first observer is designed for measurable premise variables (MPV) systems, adjusting standard formulations to the extent of taking into account unknown inputs and uncertain transmission matrices [20]. The second observer addresses the more complex case of nonmeasurable premise variables (NPV) systems, generalizing earlier work through an H∞ stabilization method using linear matrix inequalities (LMI) methods [18, 21].

Unlike recent publications like [22], which deals with adaptive observers for fractional-order neural systems, or the approaches in [23–25], which consider observer or controller synthesis under fractional dynamics with bounded disturbances or time delays, this work introduces a new dual observer scheme that can deal with both measurable and nonmeasurable premise variables in TS fuzzy fractional-order systems. Furthermore, our method incorporates unknown inputs and bounded uncertainties explicitly and provides a unified and robust observer design through Lyapunov stability theory and LMI optimization. This combined formulation renders our contribution different from the existing techniques by providing guaranteed asymptotic convergence and by improving applicability to more realistic, uncertain, and high-dimensional nonlinear systems.

The proposed method integrates both observer schemes into a unified and complementary framework that surmounts significant challenges in fractional-order nonlinear systems, including: (i) measurable and nonmeasurable premise variables, (ii) unknown inputs and uncertain transmission matrices, (iii) parametric and structural uncertainties, and (iv) complex multi-input multi-output (MIMO) system arrangements. In contrast to earlier methods limited to ideal single-input single-output (SISO) environments, the proposed framework supports realistic applications in high-dimensional systems under strong uncertainty.

A general strength of the paper is the strict integration of Lyapunov fractional-order stability theory with LMI-based optimization for convex techniques to formally guarantee assurances for asymptotic convergence of state estimation error against bounded as well as time-varying uncertainties. Solid evidence for the effectiveness and correctness of designed observers has been built with extensive numerical testing in proving excellent accuracy, better robustness, and high rates of convergence on typical operating conditions.

To validate the effectiveness and generality of the developed approach, comprehensive simulations are performed for fractional-order MIMO systems. The results recognize the fast convergence, robustness toward modeling errors, and high estimation precision of the constructed observers.

The rest of this paper is organized as follows. Section 2 presents the mathematical preliminaries and the FO-TS system formulation. Section 3 presents the observer synthesis methods. Section 4 provides numerical simulations and compares the performance of the synthesized observers. Section 5 concludes the paper and provides avenues for future research.

Throughout the paper, we employ the following convenient notation: X denotes a symmetric positive definite matrix, X^T is the transpose of X, $I_M = \{1, \cdots, M\}$and $\| \, . \, \|$ is the Euclidean norm when applied to vectors and the spectral norm when applied to matrices.

2. Basic preliminaries and system description

2.1. Basic preliminaries

2.1.1. Types of uncertainties

The fundamental assumption of having an accurate model of the process is rarely validated in practice. Discrepancies almost always exist between the model and the real process; these are referred to as system uncertainties. Due to the inherent simplifications involved in the mathematical modeling process and the unavoidable measurement errors encountered in engineering practice, it is crucial to incorporate uncertainties into the analysis and design of control systems. Model uncertainties generally arise from three primary sources [27]:

a) Incomplete knowledge of the system: Uncertainties emerge when the system's underlying dynamics are not fully understood, often due to limited information or assumptions made during modeling.
b) Insufficient model precision: A model may lack the necessary precision required to accurately capture the system's behavior, which can lead to discrepancies in performance and response.
c) Simplifications based on assumed physical phenomena: Models often rely on simplifications or idealizations of certain physical phenomena, which may not fully represent all of the system's characteristics, particularly under varying operating conditions.

For the analysis and design of control systems, uncertainties are typically classified into two categories: structured and unstructured uncertainties [27]. Structured uncertainties arise from known and quantifiable variations in system parameters, while unstructured uncertainties are more unpredictable and difficult to quantify, often due to the complexity or variability in real-world conditions.

2.1.2. Parametric uncertainties

Uncertainty arises from linearizing a nonlinear system at a fixed operating point and can be classified into two categories.

a) Bounded norm uncertainty: The uncertainties in permissible parameters are modeled as described in [26]:

$$\Delta A(t) = M_A F(t)_A N_A, \tag{2.1}$$

where M_A and N_A are constant matrices and $F_A(t)$ represents an unknown time-varying real-valued matrix that satisfies the inequality:

$$F_A^T(t) F_A(t) \leq I, \forall t \geq 0. \tag{2.2}$$

b) Combinations of linear uncertainty: The following model is used to represent this type of parametric uncertainty:

$$\Delta A(t) = \sum_{i=1}^{r} N_i \alpha_i(t), \tag{2.3}$$

where N_i represents known matrices and $\alpha_i(t)$ denotes bounded uncertain parameters, as defined in [26]:

$$|\alpha_i(t)| \leq \bar{\alpha}_i, \forall i \in \{1, \cdots, r\}, \tag{2.4}$$

where $\bar{\alpha}_i > 0$, and N_i is expressed as:

$$N_i = p_i(t)q_i(t)^T, \tag{2.5}$$

where $p_i(t)$ and $q_i(t)$ are matrices of appropriate dimensions.

2.1.3. Nonparametric uncertainties

Nonparametric uncertainties are associated with unmodeled dynamics or system nonlinearity. These parameter uncertainties are modeled as described in [26]:

$$|\Delta A_i| \leq M_i, M_i = \left[m_{ij}\right]_k; m_{ij} \geq 0, \tag{2.6}$$

where $|.|$ represents the input module and M_i represents a known positive constant matrix.

The term unstructured is justified by the fact that certain parameters of the matrices A_i vary (within a range), and only global information about these variations is available. These uncertainties account for neglected dynamics and nonlinearities in the model [28].

2.2. Definition of the fractional-order operator

Definition 2.1. (Caputo fractional derivative):

Let $x(t)$ be a continuously differentiable function in the interval $[0, T]$. The Caputo fractional derivative of order $\alpha \in (0,1)$ is defined as:

$$D^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{dx(\tau)}{d\tau} \cdot \frac{1}{(t-\tau)^{\alpha}} d\tau, \tag{2.7}$$

where $\Gamma(\cdot)$ is the Euler gamma function, and it is defined as:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, for\ x > 0. \tag{2.8}$$

The Caputo definition is particularly helpful in the engineering field, as it allows us to specify initial conditions in terms of integer-order derivatives that are more intuitive and physical. This makes the Caputo derivative extremely suitable for the modeling and observer design of dynamical systems, especially in control theory and in fault detection systems.

In the present study, the Caputo derivative is employed to represent fractional-order dynamics in nonlinear systems since it is consistent with the conventional initial conditions and can reflect the memory and hereditary properties, which are important for the dynamics of complex real-world systems to be described more accurately.

We consider the affine nonlinear fractional-order system given in [19] as the foundation to derive the TS fuzzy approximation model designed in the subsequent subsection.

2.3. FO-TS system description

Consider the following general nonlinear fractional-order system, as described in [19]:

$$\begin{cases} {}_{t_0}D_t^{\alpha}x(t) = f(x,u), \\ y(t) = g(x,u), \end{cases} \tag{2.9}$$

where ${}_{t_0}D_t^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$, $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input vector, and $f(\cdot)$ represents a nonlinear function of the states and inputs.

To approximate such nonlinear dynamics, a TS fuzzy model can be constructed as a convex combination of linear sub-models, each corresponding to a fuzzy rule:

$$i\text{: } IF\ z_1(t)\ is\ F_{i1}\ AND\ \cdots\ AND\ z_P(t)\ is\ F_{IP}\ THEN\ {}_{t_0}D_t^{\alpha} = A_i x(t) + B_i u(t), y(t) = C_i x(t)\ , \tag{2.10}$$

where $z_j(t)$ are the premise variables (measurable or nonmeasurable states, inputs, or external signals), F_{ij} are fuzzy sets, and A_i, B_i, C_i, are constant matrices associated with the i^{th} local model. The global FO-TS model is obtained by weighting these local models through normalized membership functions:

$$\begin{cases} {}_{t_0}D_t^{\alpha}x(t) = \sum_{i=1}^{M} h_i(\xi(t))[A_i x(t) + B_i u(t) + d_i], \\ y(t) = \sum_{i=1}^{M} h_i(\xi(t)) C_i x(t). \end{cases} \tag{2.11}$$

The normalized weights $h_i(\xi(t))$ are computed as:

$$h_i(z(t)) = \frac{\prod_{j=1}^{p} \mu_{F_{ij}}\left(z_j(t)\right)}{\sum_{k=1}^{M} \prod_{j=1}^{p} \mu_{F_{kj}}\left(z_j(t)\right)}, \tag{2.12}$$

where $\mu_{F_{ij}}\left(z_j(t)\right) \in [0,1]$ is the degree of membership of $z_j(t)$ in fuzzy set F_{ij}, $d_i \in R^q$ is a matrix that depends on the operating point, and $\sum_{i=1}^{M} h_i(\xi(t)) = 1$ ensures the convex sum property.

This modeling structure provides a key theoretical advantage: it allows the generalization of classical control and estimation techniques developed for linear systems—such as Lyapunov-based stability analysis and LMI-based observer design—to nonlinear systems in a structured and mathematically rigorous way.

In practical settings, the fuzzy rules are derived either by:

- Local linearization of a known nonlinear model around selected operating points,
- Sector nonlinearity transformation applied to system nonlinearities,
- Or by using system identification methods with real data to estimate local linear models and construct membership functions.

Premise variables $z_j(t)$ are typically selected from measurable values such as states, inputs, or scheduling variables. Membership functions are then defined based on human experience or data-fitting to ensure an appropriate and smooth interpolation among models.

This approach enables the FO-TS framework to estimate a broad category of nonlinear dynamics with great accuracy and makes the framework very effective for robust observer design in partially observable and uncertain conditions.

Here, we are concerned with FO-TS models with uncertain dynamics and unknown input. The known premise variables and unknown input model can be described as:

$$\begin{cases} {}_{t_0}D_t^{\alpha}x(t) = \sum_{i=1}^{M}\mu_i(\xi)(A_ix(t) + B_iu(t) + R_i\bar{u}(t)), \\ y(t) = Cx(t), \end{cases} \tag{2.13}$$

where $x(t) \in R^n$ represents the state vector, $u(t) \in R^m$ denotes control input, $D_i \in R^q$ is a matrix that depends on the operating point, $y(t)$ shows the system's output, and $A_i \in R^{n\times n}$, $B_i \in R^{n\times n_u}$, and $C_i \in R^{n\times n_y}$ are known matrices. Weighting functions are represented by $h_i(\xi(t))$ depending on the variables $\xi(t)$, which can be measurable or nonmeasurable variables; it can be an outside signal as well.

The challenge addressed in this paper is to estimate both the state vector and the unknown input using only the available measurements $y(t)$ and known inputs $u(t)$. The approach enables the construction of robust observers for complex fractional-order systems affected by nonmeasurable premise variables and unknown disturbances.

3. Main results

3.1. State estimation of uncertain FO-TS systems with MPV and unknown inputs

This section focuses on state estimation for TS systems with uncertainties and MPV in the presence of unknown inputs. The study addresses the challenge of accurately estimating the states of nonlinear systems modeled using the fractional-order T-S framework, where uncertainties impact the system matrices, input matrices, and unknown input transmission matrices. The proposed FO-TLO aims to deal with such intricacies by employing advanced stability analysis with LMI formulations and Lyapunov-based methods. The robust observer achieves asymptotic convergence of state estimation errors and accommodates time-varying uncertainties and bounded unknown inputs.

3.1.1. Uncertain FO-TS system with MPV and unknown inputs

We consider the uncertain system in the form of a fractional order TS model in the presence of unknown inputs:

$$\begin{cases} {}_{t_0}D_t^{\alpha}x(t) = \sum_{i=1}^{M}h_i(z(t))\,((A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t) + (R_i + \Delta R_i)q(t)) \\ y(t) = Cx(t). \end{cases} \tag{3.1}$$

Taking into account uncertainties in the matrices C and D can also be considered.

In this model, $x(t) \in R^n$ represents the state, known inputs denoted by $u(t) \in R^m$, $q(t) \in R^q$ represents the unknown inputs and $y \in R^p$ is the measurable output. The $q(t)$ is bounded unknown input with $\|q(t)\| < \delta$, and $\delta > 0$.

For the i^{th} local model, $A_i \in R^{n\times n}$ denotes the state matrix, $B_i \in R^{n\times m}$ corresponds to the known input matrix, $C \in R^{p\times n}$ is the output matrix, and $R_i \in R^{n\times q}$ represents the transmission matrices for the unknown inputs. Finally, $z(t)$ is the decision vector depending on the known measured state input variables.

The terms ΔA_i, ΔB_i, and ΔR_i represent modeling errors, uncertainties in the known inputs, and uncertainties in the unknown inputs of the system, respectively. It is worth noting that $\Delta A_i \in$

$R^{n\times n}$, $\Delta B_i \in R^{n\times m}$, and $\Delta R_i \in R^{n\times q}$ are also assumed to be bounded, such that $\|\Delta A_i(t)\| < \lambda_{1i}$, $\|\Delta B_i(t)\| < \lambda_{2i}$, and $\|\Delta R_i(t)\| < \lambda_{3i}$, where λ_{ji} are positive scalars and $\|.\|$ represents the Euclidean norm.

3.1.2. FO-TLO Synthesis for uncertain FO-TS system with MPV and unknown inputs

For the observer design, it is assumed that the local models are locally observable, meaning that all pairs (A_i, C) are observable [28].

The proposed observer for the FO-TS model (3.1) is of the form:

$$\begin{cases} {}_{t_0}D_t^{\alpha}\hat{x}(t) = \sum_{i=1}^{M} h_i\big(z(t)\big)\big[A_i\hat{x}(t) + B_iu(t) + L_i\big(y(t) - \hat{y}(t)\big) + R_i\varepsilon_i(t) + \sigma_i(t)\big], \\ \hat{y}(t) = C\hat{x}(t), \end{cases} \tag{3.2}$$

where $\hat{x}(t) \in R^n$ represents the estimation of x, and $\hat{y}(t) \in R^p$ represents the output of the fractional order Thau-Luenberger observer. $L_i \in R^{n\times m}$ represents the gain of the local observer, and $\sigma_i \in R^n$ are terms added to the structure of each local observer to compensate for uncertainties.

ε_i and σ_i represent auxiliary compensation terms embedded within each local observer to dynamically mitigate the effects of system uncertainties and the influence of unknown input matrices. Their purpose is to reinforce the robustness of the observer and to ensure that the estimation error asymptotically converges to zero, despite the presence of bounded and time-varying disturbances. The gains L_i, ε_i, and σ_i must be determined to ensure that the state estimation error goes to the origin**.**

If the pairs (A_i, C) are observable, and if there exist matrices $L_i \in R^{n\times p}$ such that $\bar{A}_i = A_i - L_iC$ have stable eigenvalues, and if there exist Lyapunov pairs (P, Q_i) and matrices F_i, it satisfies the following structural constraints:

$$\begin{cases} \bar{A}_iP + P\bar{A}_i = -Q_i, i \in 1, \ldots, M, \\ C^TF_i^T = PR_i. \end{cases} \tag{3.3}$$

The state estimation error is expressed as:

$$e(t) = x(t) - \hat{x}(t). \tag{3.4}$$

The output error is expressed as:

$$r(t) = y(t) - \hat{y}(t) = C(x(t) - \hat{x}(t)) = Ce(t). \tag{3.5}$$

The following describes the state estimation error's dynamics:

$${}_{t_0}D_t^{\alpha}e(t) \le \sum_{i=1}^{M} h_i(z(t))[(A_i - L_iC)e(t) + \Delta A_ix(t) + \Delta B_iu(t) + \Delta R_iq(t) - \sigma_i(t)]. \tag{3.6}$$

Theorem 3.1: *Suppose there exist a symmetric and positive definite matrix P and matrices L_i such that:*

$$\bar{A}_i^TP + P\bar{A}_i + (\alpha_1^{-1} + \alpha_3^{-1} + \alpha_4^{-1})P^2 + \alpha_1(1 + \alpha_2^{-1})\lambda_{1i}^2I < 0, \forall i, j \in \{1, \cdots, M\}, \tag{3.7}$$

where $\bar{A}_i = A_i - L_iC$. Then, the state estimation error of the fractional order Thau-Luenberger observer (3.2) converges asymptotically to zero if the following conditions are satisfied:

$$\begin{cases} si \quad r \neq 0 \begin{cases} \varepsilon_i = \delta \frac{F_i r}{\|F_i r\|}, \\ \sigma_i = (\alpha_1(1+\alpha_2)\lambda_{1i}^2 \hat{x}^T \hat{x} + \alpha_3 \lambda_{2i}^2 \|u\|^2 + \alpha_4 \lambda_{3i}^2 \|q\|^2) \frac{P^{-1}C^T r}{2r^T r}, \end{cases} \\ si \quad r = 0 \begin{cases} \varepsilon_i = 0, \\ \sigma_i = 0. \end{cases} \end{cases} \tag{3.8}$$

Proof. We can use the Schur complement, as demonstrated by [28], to establish numerous observers' asymptotic convergence (3.2).

Consider the Lyapunov function that follows:

$$V(t) = e(t)^T P e(t), \tag{3.9}$$

where P denotes a symmetric positive definite matrix. Its dynamics can be expressed as follows:

$$_{t_0}D_t^{\alpha} V(t) \leq _{t_0}D_t^{\alpha} e(t)^T P e(t) + e(t)^T P _{t_0}D_t^{\alpha} e(t). \tag{3.10}$$

Substituting Eqs (3.6) and (3.10), this yields:

$$_{t_0}D_t^{\alpha} V \leq \sum_{i=1}^{M} h_i(z)[e^T(\bar{A}_i^T P + P\bar{A}_i)e + 2e^T P(\Delta A_i x + \Delta B_i u + \Delta R_i q) + 2e^T P(R_i q - R_i \varepsilon_i - \sigma_i)]. \tag{3.11}$$

When $r \neq 0$, by replacing ε_i, using their respective expressions from Eq (3.8), and applying the Schur complement [28] along with the second Eq in (3.3), it can be observed that:

$$\begin{aligned} 2e^T P \Delta A_i x &= x^T \Delta A_i P e + e^T P \Delta A_i x \\ &\leq \alpha_1 x^T \Delta A_i^T \Delta A_i x + \alpha_1^{-1} e^T P^2 e \\ &\leq \alpha_1 \lambda_{1i}^2 (\hat{x}^T \hat{x} + e^T e) + \alpha_1 \lambda_{1i}^2 (\hat{x}^T e + e^T \hat{x}) + \alpha_1^{-1} e^T P^2 e \\ &\leq \alpha_1 (1+\alpha_2) \lambda_{1i}^2 \hat{x}^T \hat{x} + e^T \alpha_1 (1+\alpha_2^{-1}) \lambda_{1i}^2 e + \alpha_1^{-1} e^T P^2 e, \end{aligned}$$

$$\begin{aligned} 2e^T P \Delta B_i u &= u^T \Delta B_i P e + e^T P \Delta B_i u \\ &\leq \alpha_3 u^T \Delta B_i^T \Delta B_i u + \alpha_3^{-1} e^T P^2 e + \\ &\leq \alpha_3 \lambda_{2i}^2 \|u\|^2 + \alpha_3^{-1} e^T P^2 e, \end{aligned}$$

$$\begin{aligned} 2e^T P \Delta R_i q &= e^T P \Delta R_i q + q^T \Delta R_i P e \\ &\leq \alpha_4^{-1} e^T P^2 e + \alpha_4 q^T \Delta R_i^T \Delta R_i q \\ &\leq \alpha_4 \lambda_{3i}^2 \|q\|^2 + \alpha_4^{-1} e^T P^2 e, \end{aligned}$$

$$\begin{aligned} 2e^T P(R_i q - R_i \varepsilon_i) &= 2e^T C^T F_i^T q - 2e^T C^T F_i^T \varepsilon_i \\ &= 2r^T F_i^T q - 2\delta r^T F_i^T \frac{F_i r}{\|F_i r\|} \\ &= 2r^T F_i^T q - 2\delta \|F_i r\| \\ &\leq 2\|q\| \|F_i r\| - 2\delta \|F_i r\| < 0, \end{aligned}$$

$$2e^T P\sigma_i = 2(\alpha_1(1+\alpha_2)\lambda_{1i}^2 \hat{x}^T\hat{x} + \alpha_3\lambda_{2i}^2\|u\|^2 + \alpha_4\lambda_{3i}^2\|q\|^2)\frac{P^{-1}C^T r}{2r^T r}$$
$$= \alpha_1(1+\alpha_2)\lambda_{1i}^2 \hat{x}^T\hat{x} + \alpha_3\lambda_{2i}^2\|u\|^2 + \alpha_4\lambda_{3i}^2\|q\|^2,$$

with:

$$\alpha_5 = \alpha_1(1+\alpha_2),\ \alpha_6 = \alpha_1(1+\alpha_2^{-1}). \tag{3.12}$$

After simplification, we obtain:

$$t_0 D_t^{\alpha} V < \sum_{i=1}^{M} \mu_i(z)[e^T(\bar{A}_i^T P + P\bar{A}_i + \alpha_7 P^2 + \alpha_6\lambda_{1i}^2 I)e]. \tag{3.13}$$

Additionally, we get the same outcome when $r = 0$.

It can be observed that the asymptotic convergence of the multiple observers is guaranteed as long as the righthand side of inequality (3.13) remains negative. Consequently, the state estimation error asymptotically converges to zero if conditions (3.8) and inequalities (3.7) are satisfied.

We note that inequalities (3.7) depend on two unknown matrices, ψ and L_i. Therefore, the LMI formulation can only be applied after linearizing these inequalities [28], where the variable substitution technique is utilized.

Consider the following variable substitution:

$$W_i = PL_i. \tag{3.14}$$

By substituting Eq (3.14) into Eq (3.13), we obtain:

$$\begin{cases} A_i^T P + PA_i - C^T W_i^T - W_i C + \alpha_7 P^2 + \alpha_6\lambda_{1i}^2 I < 0, \\ C^T F_i^T = PR_i, i \in 1,\ldots,M. \end{cases} \tag{3.15}$$

Using the Schur complement, as discussed in [28], the following formulation is derived from (3.15):

$$\begin{cases} \begin{bmatrix} A_i^T P + PA_i - C^T W_i^T - W_i C + \alpha_6\lambda_{1i}^2 I & P \\ P & -\alpha_7^{-1} I \end{bmatrix} < 0, \\ C^T F_i^T = PR_i, i \in 1,\ldots,M. \end{cases} \tag{3.16}$$

The solution to these constraints, P and L_i, enable the calculation of the observer's gains $L_i = P^{-1}W_i$.

Note that in the case where all inputs are known and if the matrix is $D = 0$, the system (3.1) can be rewritten as:

$$\begin{cases} t_0 D_t^{\alpha} x(t) = \sum_{i=1}^{M} h_i(z(t))\left((A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t)\right), \\ y(t) = Cx(t). \end{cases} \tag{3.17}$$

3.1.3. Simulation analysis and results discussion

In this section, the proposed approach for state estimation of the model is applied to a MIMO system with two inputs and three outputs. The input vector is $u(t) = [u_1(t) \quad u_2(t)]^T$.

The proposed FO-TS model is described as:

$$\begin{cases} {}_{t_0}D_t^{\alpha} x = \sum_{i=1}^{M} \mu_i(\xi)[A_i x + B_i u + R_i \bar{u} + D_i], \\ y = Cx, \end{cases} \tag{3.18}$$

where the numerical values of the matrices A_i, B_i, R_i, C, and D_i are as follows:

$$A_1 = 10^{-3}\begin{bmatrix} -18.5 & 0 & 18.5 \\ 0 & -20.9 & 15.0 \\ 18.5 & 15.0 & -33.5 \end{bmatrix},\ A_2 = 10^{-3}\begin{bmatrix} -22.1 & 0 & 22.1 \\ 0 & -23.3 & 17.6 \\ 22.1 & 17.6 & -39.5 \end{bmatrix},$$

$$B_1 = B_2 = \frac{1}{A}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\quad C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},\quad R_1 = 10^{-3}\begin{bmatrix} -0.57 \\ -0.46 \\ -0.52 \end{bmatrix},\ R_2 = 10^{-3}\begin{bmatrix} -0.57 \\ -0.50 \\ -0.54 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -0.225 \\ -0.089 \\ 0.005 \end{bmatrix},\ D_2 = \begin{bmatrix} -0.182 \\ -0.141 \\ 0.003 \end{bmatrix}.$$

To convey how well the suggested method works, let us assume that the parameters of the FO-TS model (3.18) are subject to bounded uncertainties, as indicated by the following equations:

$$\begin{cases} {}_{t_0}D_t^{\alpha} x = \sum_{i=1}^{2} \mu_i(\xi)[(A_i + \Delta A_i)x + (B_i + \Delta B_i)u + (R_i + \Delta R_i)\bar{u}], \\ y = Cx. \end{cases} \tag{3.19}$$

The model uncertainties are $\|\Delta A_i\| < \lambda_1$, $\|\Delta B_i\| < \lambda_2$, and $\|\Delta R_i\| < \lambda_3$ satisfy the conditions $\lambda_1 = \lambda_2 = \lambda_3 = 0.2$.

The simulation results are conducted with $\|r(t)\| < \varepsilon$ and $\varepsilon = 10^{-3}$.

The proposed fractional order Thau-Luenberger observer, which estimates the state vector of the FO-TS model (3.19), is described by:

$$\begin{cases} {}_{t_0}D_t^{\alpha} \hat{x} = \sum_{i=1}^{2} \mu_i(\xi)[A_i \hat{x} + B_i u + L_i(y - \hat{y}) + R_i \varepsilon_i + \sigma_i], \\ \hat{y} = C\hat{x}, \end{cases} \tag{3.20}$$

with:

$$\begin{cases} (A_i - L_i C)^T \psi + \psi(A_i - L_i C) + \alpha_7 \psi^2 + \alpha_6 \lambda_{1i}^2 I < 0, \\ C^T F_i^T = \psi R_i, i \in 1, \ldots, M, \end{cases} \tag{3.21}$$

Remark 1. *Once the output estimation error $r(t)$ goes to origin ($\|r\| < \varepsilon$), to prevent unbounded growth in the amplitude of $\sigma_i(t)$ and $\varepsilon_i(t)$, then $\sigma_i(t)$ and $\varepsilon_i(t)$ are set to zero. In this case, $r(t)$ converges to a small vicinity around 0, based on the suitable value of ε.*

Figure 1 illustrates the convergence of estimated states $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ to the actual states (x_1, x_2, x_3) over time. The estimation errors gradually diminish to a small neighborhood around zero, with minor deviations near the origin attributed to the chosen initial conditions.

Figure 1 shows the performance of the proposed observer in estimating the FO-TS system's states under uncertain conditions. The figure depicts the dynamic development of the state estimate error over time, revealing important details about the observer's accuracy and convergence. Initially, the estimation error demonstrates transient behavior as the observer adjusts to the system's dynamics and uncertainties. However, as time progresses, the error diminishes and converges asymptotically to zero, reflecting the robustness and stability of the proposed method. This convergence is achieved despite unknown inputs and time-varying uncertainties, highlighting the effectiveness of the advanced stability analysis techniques, such as LMI formulations, employed in the observer design.

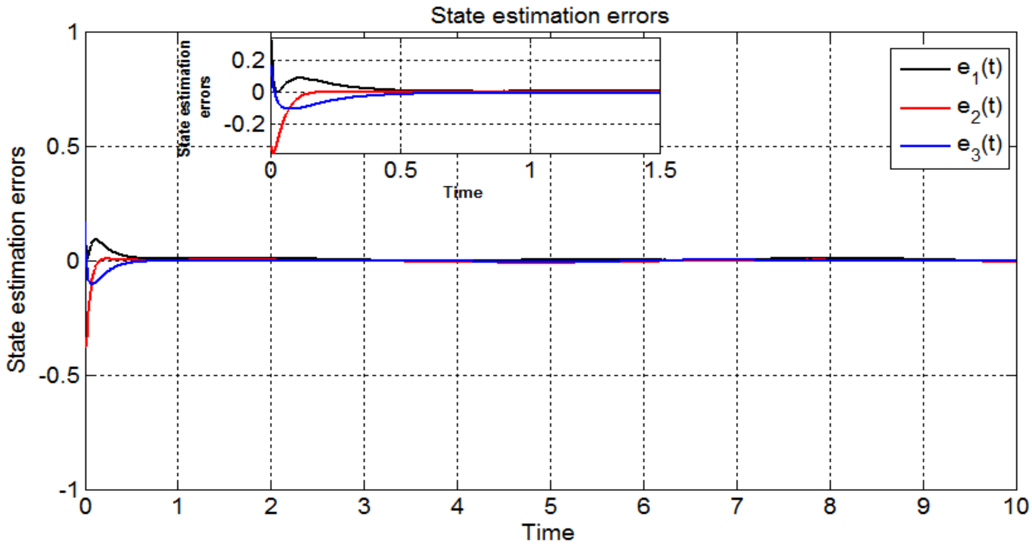

Figure 1. State estimation error.

Figure 2 presents the evolution of the output estimation errors $r_1(t)$ and $r_2(t)$ over time. It demonstrates that both errors converge to a small neighborhood around zero, validating the effectiveness of the proposed observer. The transient behavior at the beginning reflects the adjustment phase, influenced by the initial conditions.

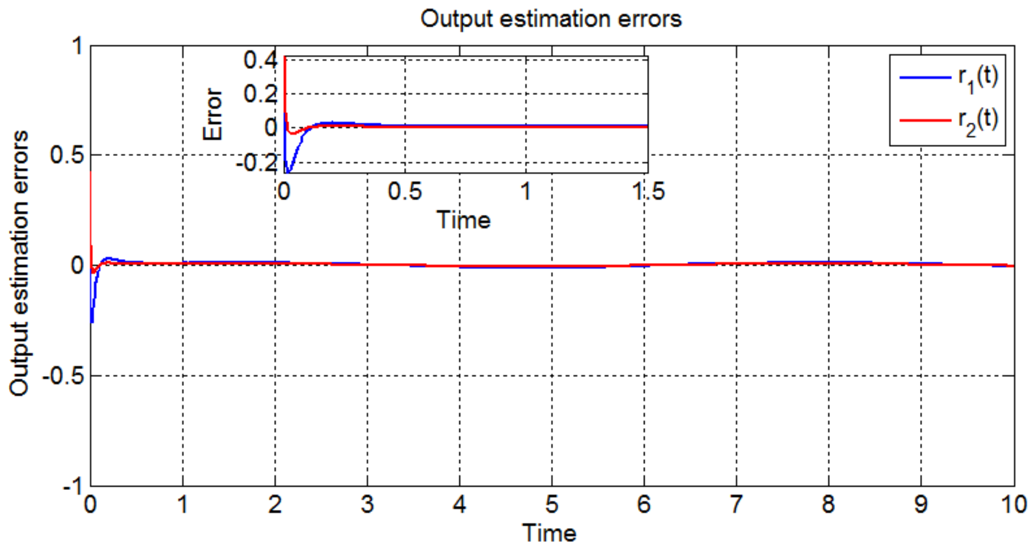

Figure 2. Output estimation errors.

3.2. State estimation of uncertain FO-TS systems with NPV and unknown inputs

State estimation in systems under uncertainties, NPVs, as well as unknown inputs, pose significant challenges in control and observation design. These factors can lead to inaccuracies in modeling and performance degradation if not properly addressed. This section explores robust state estimation methods for T-S systems with NPV under bounded uncertainties, time-varying nonmeasurable premise variables, and unknown inputs. The goal is to ensure accurate state estimation while accommodating these complexities in the system's behavior.

3.2.1. Uncertain FO-TS system with NPV and unknown inputs

This section addresses the state estimation of T-S systems with uncertain NPV. The considered uncertainties are bounded and time-varying. The system (3.22) is assumed to be stable.

$$\begin{cases} {}_{t_0}D_t^{\alpha} x(t) = \sum_{i=1}^{M} \mu_i(\xi(t))\left[\left(A_i + \Delta A_i(t)\right)x(t) + \left(B_i + \Delta B_i(t)\right)u(t) + \left(R_i + \Delta R_i(t)\right)\bar{u}(t)\right], \\ y(t) = Cx(t), \end{cases} \tag{3.22}$$

where

$$\Delta A_i(t) = M_A^i F_A(t) I_A^i, \tag{3.23}$$

$$\Delta B_i(t) = M_B^i F_B(t) I_B^i, \tag{3.24}$$

$$\Delta R_i(t) = M_R^i F_R(t) I_R^i, \tag{3.25}$$

with

$$F_A^T(t) F_A(t) \le I, \forall t, \tag{3.26}$$

$$F_B^T(t) F_B(t) \le I, \forall t, \tag{3.27}$$

$$F_R^T(t) F_R(t) \le I, \forall t, \tag{3.28}$$

Hypothesis 2.1. *We consider that the following assumptions are satisfied:*

- ***H1:*** *The input-to-state stability of the system is verified.*
- ***H2:*** *The system input is bounded:* $|u(t)| < \rho$ *with* $\rho > 0$.
- ***H3:*** *Disturbances are bounded.*

Under these assumptions, the system (3.22) can be rewritten in the following form:

$$\begin{cases} {}_{t_0}D_t^{\alpha} x(t) = \sum_{i=1}^{M} \mu_i(\hat{x}(t))\left[\left(A_i + \Delta A_i(t)\right)x(t) + \left(B_i + \Delta B_i(t)\right)u(t) + \left(R_i + \Delta R_i(t)\right)\bar{u}(t) + \omega(t)\right], \\ y(t) = Cx(t), \end{cases} \tag{3.29}$$

where

$$\omega(t) = \sum_{i=1}^{M} (\mu_i(x) - \mu_i(\hat{x}))[(A_i + \Delta A_i)x + (B_i + \Delta B_i)u + (R_i + \Delta R_i)\bar{u}]. \tag{3.30}$$

3.2.2. Fractional order Thau-Luenberger observer (FO-TLO) synthesis of uncertain FO-TS System with NPV and unknown inputs

The suggested observer is expressed as:

$$\begin{cases} {}_{t_0}D_t^{\alpha}\hat{x}(t) = \sum_{i=1}^{M}\mu_i\big(\hat{x}(t)\big)\big[A_i\hat{x}(t) + B_iu(t) + L_i\big(y(t) - \hat{y}(t)\big)\big], \\ \hat{y}(t) = C\hat{x}(t). \end{cases} \tag{3.31}$$

The estimation error between Eqs (3.29) and (3.31) is defined as:

$$e(t) = x(t) - \hat{x}(t). \tag{3.32}$$

The output estimation error between Eqs (3.29) and (3.31) is defined as:

$$r(t) = y(t) - \hat{y}(t) = C(x(t) - \hat{x}(t)). \tag{3.33}$$

Its dynamics are given by:

$${}_{t_0}D_t^{\alpha}e(t) \leq \sum_{i=1}^{M}\mu_i(\hat{x})\big((A_i - L_iC)e(t) + \Delta A_ix(t) + \Delta B_iu(t) + \Delta R_i\bar{u}(t) + \omega(t)\big). \tag{3.34}$$

Equation (3.34) can be expressed as:

$${}_{t_0}D_t^{\alpha}e(t) \leq \sum_{i=1}^{M}\mu_i(\hat{x})(\bar{A}_ie(t) + \bar{R}_i\bar{\omega}(t)), \tag{3.35}$$

where

$$\bar{A}_i = A_i - L_iC,\ \bar{\omega}(t) = [\omega(t)^T \quad x(t)^T \quad u(t)^T \quad \bar{u}(t)^T]^T,\ \bar{R}_i = [I \quad \Delta A_i(t) \quad \Delta B_i(t) \quad \Delta R_i(t)]. \tag{3.36}$$

Based on hypotheses H1–H3, the state vector $x(t)$ and the term $\bar{\omega}(t)$ are bounded.

Theorem 3.2: *Error in state estimation between the FO-TLO (3.31) and the uncertain FO-TS system with NPV (3.29) converges asymptotically to 0. Furthermore,* $\bar{\gamma}$ *defines the bounds of the gain* L_2 *of the transfer from* $\bar{\omega}(t)$ *to* $e(t)$*, provided there exist matrices* $X = X^T > 0$*,* K_i *and* $\bar{\gamma} > 0$*,* $\varepsilon_1 > 0$ *, and* $\varepsilon_2 > 0$ *that solve the optimization problem (3.37) under the LMI constraints (3.38) for* $i \in \{1, \cdots, M\}$.

$$\min_{X, K_i, \varepsilon_1, \varepsilon_2, \bar{\gamma}} \quad \bar{\gamma} \tag{3.37}$$

under the following constraints $i \in \{1, \cdots, M\}$

$$\begin{bmatrix} \phi_{1i} & X & 0 & 0 & 0 & XM_A & XM_B & XM_R \\ X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_{1i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varphi_{2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varphi_{3i} & 0 & 0 & 0 \\ M_A^TX & 0 & 0 & 0 & 0 & -\lambda_3I & 0 & 0 \\ M_B^TX & 0 & 0 & 0 & 0 & 0 & -\lambda_4I & 0 \\ M_R^TX & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_5I \end{bmatrix} < 0, \tag{3.38}$$

where,

$$\phi_1 = A_i^T X + XA_i - C^T L_i^T X - XL_i C + I, \tag{3.39}$$

$$\varphi_{1i} = -\gamma^2 I + \lambda_3 I_A^T I_A, \tag{3.40}$$

$$\varphi_{2i} = -\gamma^2 I + \lambda_4 I_B^T I_B, \tag{3.41}$$

$$\varphi_{3i} = -\gamma^2 I + \lambda_5 I_R^T I_R. \tag{3.42}$$

The observer gains, and the attenuation level γ *for the transfer of uncertainties to the estimation error are given by:*

$$L_i = X^{-1} K_i, \tag{3.43}$$

$$\gamma = \sqrt{\bar{\gamma}}. \tag{3.44}$$

Proof. To prove Theorem (3.2), the quadratic Lyapunov function that follows is used:

$$V(e(t)) = e(t)^T X e(t), X = X^T > 0. \tag{3.45}$$

The fractional derivative can be expressed as:

$$_{t_0}D_t^\alpha V(e(t)) \leq {}_{t_0}D_t^\alpha e(t)^T X e(t) + e(t)^T X {}_{t_0}D_t^\alpha e(t). \tag{3.46}$$

Using the state estimation error from Eq (3.35), we obtain:

$$_{t_0}D_t^\alpha V(e(t)) \leq \sum_{i=1}^M h_i(\hat{x}(t)) \left(e^T \bar{A}_i^T X e + e^T X \bar{A}_i e + \bar{\omega}^T \bar{R}_i^T X e + e^T X \bar{R}_i \bar{\omega}\right). \tag{3.47}$$

If the following condition is met, the gain L_2 of the transfer from $\bar{\omega}(t)$ to $e(t)$ is bounded by γ and the state estimate error converges to zero:

$$_{t_0}D_t^\alpha V(e(t)) + e(t)^T e(t) - \gamma^2 \bar{\omega}(t)^T \bar{\omega}(t) < 0. \tag{3.48}$$

By substituting Eq $\dot{V}(e(t))$ (3.47) into Eq (3.48), we obtain:

$$\sum_{i=1}^M h_i(\hat{x}(t)) \left(e^T \bar{A}_i^T X e + e^T X \bar{A}_i e + \bar{\omega}^T \bar{R}_i^T X e + e^T X \bar{R}_i \bar{\omega} + e^T e \quad - \gamma^2 \bar{\omega}^T \bar{\omega}\right) < 0. \tag{3.49}$$

The matrix form of inequality (3.49) yields

$$\sum_{i=1}^M h_i(\hat{x}(t)) \begin{bmatrix} e(t) \\ \bar{\omega}(t) \end{bmatrix}^T \begin{bmatrix} \bar{A}_i^T X + X\bar{A}_i + I & X\bar{R}_i \\ \bar{R}_i^T X & -\gamma^2 I \end{bmatrix} \begin{bmatrix} e(t) \\ \bar{\omega}(t) \end{bmatrix} < 0. \tag{3.50}$$

Given the property of the activation functions h_i, inequality (3.50) is negative if:

$$\begin{bmatrix} \bar{A}_i^T X + X\bar{A}_i + I & X\bar{R}_i \\ \bar{R}_i^T X & -\gamma^2 I \end{bmatrix} < 0, \ \forall i, j \in \{1, \cdots, M\}. \tag{3.51}$$

By substituting $\bar{A}_i = A_i - L_i C$, we obtain:

$$\begin{bmatrix} \phi_{1i} & X\bar{R}_i \\ \bar{R}_i^T X & -\gamma^2 I \end{bmatrix} < 0,\ \forall i,j \in \{1,\cdots,M\}, \tag{3.52}$$

where,

$$\phi_{1i} = A_i^T X + XA_i - C^T L_i^T X - XL_i^T C + I. \tag{3.53}$$

Using the definitions of the matrices $\bar{R}_i$, we obtain:

$$\begin{bmatrix} \phi_{1i} & X & X\Delta A_i(t) & X\Delta B_i(t) & X\Delta R_i(t) \\ X & -\gamma^2 I & 0 & 0 & 0 \\ \Delta A_i(t)^T X & 0 & -\gamma^2 I & 0 & 0 \\ \Delta B_i(t)^T X & 0 & 0 & -\gamma^2 I & 0 \\ \Delta R_i(t)^T X & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0. \tag{3.54}$$

To solve the matrix inequality (3.52), a transformation of inequality (3.54) is performed to separate the constant terms from the time-varying terms. Using Schur's complement [28], we then obtain:

$$\begin{bmatrix} \phi_{1i} & X & 0 & 0 & 0 \\ X & -\gamma^2 I & 0 & 0 & 0 \\ 0 & 0 & -\gamma^2 I & 0 & 0 \\ 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} + Q(t)^T + Q(t) < 0, \tag{3.55}$$

where,

$$Q(t) = \begin{bmatrix} 0 & 0 & X\Delta A_i(t) & X\Delta B_i(t) & X\Delta R_i(t) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.56}$$

Given the definitions of $\Delta A_i(t)$, $\Delta B_i(t)$ and $\Delta R_i(t)$, the matrix $Q(t)$ is expressed as:

$$Q(t) = \begin{bmatrix} 0 & 0 & XM_A & XM_B & XM_R \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F_A(t)I_A & 0 & 0 \\ 0 & 0 & 0 & F_B(t)I_B & 0 \\ 0 & 0 & 0 & 0 & F_R(t)I_R \end{bmatrix}. \tag{3.57}$$

Lemma 3.1. For all matrices X and Y of suitable dimensions, the following property is:

$$M^T N + MN^T < M^T \Phi^{-1} M + N\Phi \quad N^T, \Phi > 0.$$

Using Lemma 3.1, and by choosing the matrix $Q(t)$ in the following form:

$$\Sigma = diag(\lambda_1 I, \lambda_2 I, \lambda_3 I, \lambda_4 I, \lambda_5 I), \lambda_i > 0\ \forall i = 1,\cdots,5. \tag{3.58}$$

We obtain:

$$Q(t)^T + Q(t) < \begin{bmatrix} 0 & 0 & XM_A & XM_B & XM_R \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Sigma^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ M_A^T X & 0 & 0 & 0 & 0 \\ M_B^T X & 0 & 0 & 0 & 0 \\ M_R^T X & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_A^T F_A^T(t) & 0 & 0 \\ 0 & 0 & 0 & I_B^T F_B^T(t) & 0 \\ 0 & 0 & 0 & 0 & I_R^T F_R^T(t) \end{bmatrix} \Sigma \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F_A(t) I_A & 0 & 0 \\ 0 & 0 & 0 & F_B(t) I_B & 0 \\ 0 & 0 & 0 & 0 & F_R(t) I_R \end{bmatrix}, \quad (3.59)$$

After calculations and using the properties of the terms $F_A(t)$, $F_B(t)$ and $F_R(t)$, we obtain:

$$Q(t)^T + Q(t) < \begin{bmatrix} \phi_{2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 I_A^T I_A & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 I_B^T I_B & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 I_R^T I_R \end{bmatrix}, \quad (3.60)$$

where

$$\phi_{2i} = \lambda_3^{-1} X M_A M_A^T X + \lambda_4^{-1} X M_B M_B^T X + \lambda_5^{-1} X M_E M_E^T X. \quad (3.61)$$

By substituting (3.59) into (3.55), we obtain:

$$\begin{bmatrix} \phi_{1i} & X & 0 & 0 & 0 & XM_A & XM_B & XM_R \\ X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_{1i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varphi_{2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varphi_{3i} & 0 & 0 & 0 \\ M_A^T X & 0 & 0 & 0 & 0 & -\lambda_3 I & 0 & 0 \\ M_B^T X & 0 & 0 & 0 & 0 & 0 & -\lambda_4 I & 0 \\ M_R^T X & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_5 I \end{bmatrix} < 0, \quad (3.62)$$

where

$$\phi_1 = A_i^T X + X A_i - C^T L_i^T X - X L_i C + I, \quad (3.63)$$

$$\varphi_{1i} = -\gamma^2 I + \lambda_3 I_A^T I_A, \quad (3.64)$$

$$\varphi_{2i} = -\gamma^2 I + \lambda_4 I_B^T I_B, \quad (3.65)$$

$$\varphi_{3i} = -\gamma^2 I + \lambda_5 I_R^T I_R. \quad (3.66)$$

To solve inequality (3.61) with existing LMI-solving software, the following variable changes must be used:

$$K_i = X L_i, \quad (3.67)$$

$$\bar{\gamma} = \gamma^2. \quad (3.68)$$

3.2.3. Simulation analysis and results discussion

In this section, the proposed approach for state estimation of the model is applied to a nonlinear system with one input and two outputs. The proposed FO-TS model is described by:

$$\begin{cases} {}_{t_0}D_t^{\alpha} x = \sum_{i=1}^{M} \mu_i(\xi)[A_i x + B_i u + R_i \bar{u}], \\ y = Cx, \end{cases} \tag{3.69}$$

where the values of A_i, B_i, R_i, C and D_i are given as:

$$A_1 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -8 \end{bmatrix}, A_2 = \begin{bmatrix} -3 & 2 & -2 \\ 5 & -3 & 0 \\ 1 & 2 & -4 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 5 \\ 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 3 \\ 2 \\ -7 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, R_1 = R_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

To show the effectiveness of the recommended technique, let us assume that the parameters of the FO-TS model (3.69) are affected by bounded uncertainties as indicated by the following equations:

$$\begin{cases} {}_{t_0}D_t^{\alpha} x = \sum_{i=1}^{2} \mu_i(\xi)[(A_i + \Delta A_i)x + (B_i + \Delta B_i)u + (R_i + \Delta R_i)\bar{u}], \\ y = Cx. \end{cases} \tag{3.70}$$

The proposed fractional order Thau-Luenberger observer, which estimates the state vector of FO-TS dynamics (3.70), is described by:

$$\begin{cases} {}_{t_0}D_t^{\alpha} \hat{x}(t) = \sum_{i=1}^{M} \mu_i(\hat{x}(t))\left[A_i \hat{x}(t) + B_i u(t) + L_i\left(y(t) - \hat{y}(t)\right)\right], \\ \hat{y}(t) = C\hat{x}(t). \end{cases} \tag{3.71}$$

Figure 3 illustrates the state estimation errors over time for a system with three states. Initially, all three states exhibited noticeable errors at $t = 0$, which rapidly decreased, demonstrating the effectiveness of the observer in correcting deviations. The errors converge to near-zero values within a short time, indicating that the proposed observer stabilizes quickly and accurately tracks the system states. After this initial transient period, the estimation errors remain very close to zero, reflecting the observer's stability and consistent performance over time. Additionally, the performance is uniform across all three states, with negligible differences between the errors after convergence, highlighting the balanced and robust behavior of the observer. This rapid convergence and sustained accuracy showcases the reliability of the observer in estimating system states, even in the presence of initial discrepancies.

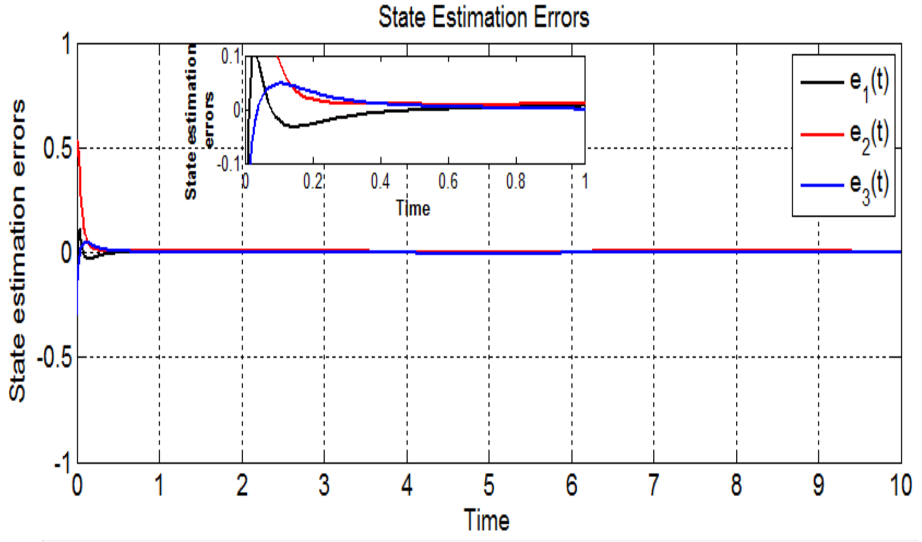

Figure 3. State estimation errors.

Figure (4) shows that both outputs ($r_1(t)$ and $r_2(t)$) initially have errors, with $r_1(t)$ slightly larger than $r_2(t)$. The errors rapidly decay, showcasing the observer's remarkable efficiency in minimizing output estimation inaccuracies. Following this transient phase, both error signals converge and stabilize at zero, exhibiting no significant oscillations or divergence over time. This behavior aids the stability and reliability of the estimation process. In addition, the observer demonstrates consistent performance in both outputs, aiding its balanced and efficient control of system dynamics. Generally, the new approach has been demonstrated to be highly accurate and stable in achieving precise output estimation.

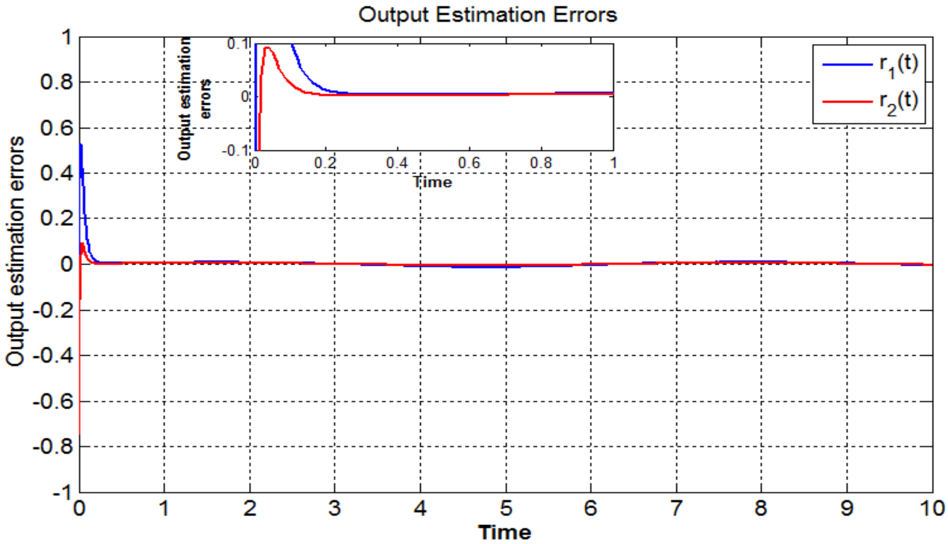

Figure 4. Output estimation errors.

The computational complexity of the proposed method is primarily linked to the solution of LMI constraints during the observer gain design stage. This step is performed offline using convex optimization solvers. Once the gains are computed, the online implementation of the observer requires only a weighted summation of linear observer outputs, which is computationally efficient. For high-dimensional systems, the number of LMIs grows with the state dimension and the number of fuzzy rules, but practical implementations can control this growth by selecting a reduced number of dominant rules. This trade-off allows the framework to be scalable and applicable to real-time systems of moderate to high dimension.

4. Conclusions

In this work, two new observer structures were proposed for state estimation in FO-TS systems to address the challenges related to both nonmeasurable and measurable premise variables, unknown inputs, and bounded uncertainties. The double-observer structure is of high theoretical and applicative interest since it offers an efficient and formal approach to address state estimation for uncertain, sophisticated systems. The observers ensure asymptotic convergence as well as robustness under conditions of unknown time-varying uncertainties, as well as disturbances that are unobserved, ensuring their utility and applicability for real applications. Mathematical rigor and numerical simulation validate the effectiveness and optimality of the observers in the realm of MIMO systems.

Future work is introduced in the paper with some directions. An extension of the developed observers to handle higher-order nonlinearities and higher-dimensional systems is one of them. Experimental validation of the proposed frameworks on physical systems would also be an important step toward real-time implementation. In addition, integration of these observers into the newest control algorithms, e.g., predictive and adaptive control, would further enhance system robustness and responsiveness in uncertain environments.

Author contributions

Conceptualization: A. D.; Methodology: A. D. and J. D.; Software: A. T. A. and S. A.; Investigation: N. A. K. and Z. H.; Resources: A. D. and J. D.; Data curation: A. T. A. and S. A.; Writing—original draft: A. D., A. T. A., S. A., J. D., N. A. K., and Z. H.; Funding acquisition: A. T. A. and S. A. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This paper is derived from a research grant funded by the Research, Development, and Innovation Authority (RDIA), Kingdom of Saudi Arabia, with grant number 13382-psu-2023-PSNU-R-3-1-EI-. The authors would like to acknowledge the support of Prince Sultan University, Riyadh, Saudi Arabia in paying the article processing charges of this publication. This research is supported by the Automated Systems and Computing Lab (ASCL), Prince Sultan University, Riyadh, Saudi Arabia. In addition, the authors wish to acknowledge the editor and anonymous reviewers for their insightful comments, which have improved the quality of this publication.

Conflict of interest

The authors declare there is no conflict of interest in this paper.

References

1. I. Podlubny, *Fractional differential equations*, Academic Press, 1999.
2. V. E. Tarasov, Discrete map with memory from fractional differential equation of arbitrary positive order, *J. Math. Phys.*, **50** (2009), 122703. https://doi.org/10.1063/1.3272791
3. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.
4. J. Sabatier, O. P. Agrawal, J. A. T. Machado, *Advances in fractional calculus: Theoretical developments and applications in physics and engineering*, Springer, 2007. https://doi.org/10.1007/978-1-4020-6042-7
5. M. Edelman, Fractional maps as maps with power-law memory, In: *Nonlinear dynamics and complexity*, Springer, 2014. https://doi.org/10.1007/978-3-319-02353-3_3
6. Y. Kao, C. Wang, H. Xia, Y. Cao, Projective synchronization for uncertain fractional reaction-diffusion systems via adaptive sliding mode control based on finite-time scheme, In: *Analysis and control for fractional-order systems*, Springer, 2024. https://doi.org/10.1007/978-981-99-6054-5_8
7. B. Meghni, D. Dib, A. T. Azar, S. Ghoudelbourk, A. Saadoun, Robust adaptive supervisory fractional order controller for optimal energy management in wind turbine with battery storage, In: Fractional order control and synchronization of chaotic systems, Springer, 2017. https://doi.org/10.1007/978-3-319-50249-6_6
8. T. S. Gorripotu, H. Samalla, C. J. M. Rao, A. T. Azar, D. Pelusi, TLBO algorithm optimized fractional-order PID controller for AGC of interconnected power system, In: *Soft computing in data analytics*, Singapore: Springer, 2019. https://doi.org/10.1007/978-981-13-0514-6_80
9. Y. Cao, Y. Kao, Z. Wang, X. Yang, J. H. Park, W. Xie, Sliding mode control for uncertain fractional-order reaction-diffusion memristor neural networks with time delays, *Neural Networks*, **178** (2024), 106402. https://doi.org/10.1016/j.neunet.2024.106402
10. Y. Kao, Y. Cao, X. Chen, Global Mittag-Leffler synchronization of coupled delayed fractional reaction-diffusion Cohen-Grossberg neural networks via sliding mode control, *Chaos*, **32** (2022), 113123. https://doi.org/10.1063/5.0102787

11. Z. Liu, J. Liu, O. Zhang, Y. Zhao, W. Chen, Y. Gao, Adaptive disturbance observer-based fixed-time tracking control for uncertain robotic systems, *IEEE T. Ind. Electron.*, **71** (2024), 14823–14831. https://doi.org/10.1109/TIE.2024.3366204
12. J. Liu, X. Yu, J. Wang, Z. Li, W. Wang, C. Yang, Sliding mode control of grid-connected neutral-point-clamped converters via high-Gain observer, *IEEE T. Ind. Electron.*, **69** (2021), 4010–4021. https://doi.org/10.1109/TIE.2021.3070496
13. X. Lin, Z. Zhao, W. Hu, Z. Qiu, H. Liu, H. Yang, Observer-based fixed-time control for permanent-magnet synchronous motors with parameter uncertainties, *IEEE T. Power Electr*, **38** (2023), 4335–4344. https://doi.org/10.1109/TPEL.2022.3226033
14. E. H. Mamdani, Application of fuzzy algorithms for control of simple dynamic plant, *Proc. Inst. Electr. Eng.*, **121** (1974), 1585–1588. https://doi.org/10.1049/piee.1974.0328
15. T. Takagi, M. Sugeno, Fuzzy identification of systems and its applications to modelling and control, *IEEE T. Syst. Man Cyb.*, **SMC-15** (1985), 116–132. https://doi.org/10.1109/TSMC.1985.6313399
16. Q. Liu, B. Hu, W. Liu, J. Li, W. Yu, G. Li, A fractional-order model predictive control strategy with Takagi-Sugeno fuzzy optimization for vehicle active suspension system, *Fractal Fract.*, **8** (2024), 610. https://doi.org/10.3390/fractalfract8100610
17. L. Dami, A. Benzaouia, K. Badie, Stabilization of continuous two-dimensional fractional order positive Takagi–Sugeno fuzzy systems with delays, *J. Control Autom. Electr. Syst.*, **35** (2024), 999–1007. https://doi.org/10.1007/s40313-024-01127-4
18. D. Ichalal, B. Marx, J. Ragot, D. Maquin, State estimation of Takagi-Sugeno systems with non-measurable premise variables, *IET Control Theory A.*, **4** (2010), 897–908. https://doi.org/10.1049/iet-cta.2009.0054
19. A. Djeddi, D. Dib, A. T. Azar, S. Abdelmalek, Fractional order unknown inputs fuzzy observer for Takagi-Sugeno systems with non-measurable premise variables, *Mathematics*, **7** (2019), 984. https://doi.org/10.3390/math7100984
20. A. Djeddi, A. T. Azar, A. Fekik, N. A. Kamal, C. B. Njima, Synthesis of non-integer order uncertain Thau-Luenberger observer for non-integer order multiple model with unknown inputs, *ICCAD*, **2024** (2024), 1–6. https://doi.org/10.1109/ICCAD60883.2024.10553912
21. W. Hamdi, M. Y. Hammoudi, A. Boukhlouf, Observer design for Takagi-Sugeno fuzzy systems with non-measurable premise variables based on differential mean value theorem, *Eng. Proc.*, **58** (2023), 28. https://doi.org/10.3390/ecsa-10-16008
22. H. Xiao, Z. Li, Y. Zhang, X. Liu, Refinement of a Lyapunov-type inequality for a fractional differential equation, *Symmetry*, **16** (2024), 941. https://doi.org/10.3390/sym16080941
23. D. Luo, W. Xue, C. Tomás, Z. Quanxin, Ulam-Hyers stability of Caputo-type fractional fuzzy stochastic differential equations with delay, *Commun. Nonlinear Sci.*, **121** (2023), 107229. https://doi.org/10.1016/j.cnsns.2023.107229
24. D. Luo, M. Tian, Q. Zhu, Some results on finite-time stability of stochastic fractional-order delay differential equations, *Chaos Soliton. Fract.*, **158** (2022), 111996. https://doi.org/10.1016/j.chaos.2022.111996
25. R. Dhayal Q. Zhu, Stability and controllability results of ψ-Hilfer fractional integro-differential systems under the influence of impulses, *Chaos Soliton. Fract.*, **168** (2023), 113105. https://doi.org/10.1016/j.chaos.2023.113105

26. L. O. G. Gloria, *Observateurs des systèmes singuliers incertains : Application au contrôle et au diagnostic,* Université de Lorraine, 2015.
27. S. Aberkane, D. Sauter, J. C. Ponsart, Output feedback robust H∞ control of uncertain active fault control systems via convex analysis, *Int. J. Control*, **81** (2008), 252–263. https://doi.org/10.1080/00207170701535959
28. A. Djeddi, *Diagnostic des systèmes non linéaires à base d'observateurs*, Doctoral thesis, Université Badji Mokhtar, Annaba, 2017.