AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(6): 15069–15087.
DOI: 10.3934/math.2025676
Received: 21 March 2025
Revised: 27 May 2025
Accepted: 10 June 2025
Published: 30 June 2025

Research article

Uniquely identifying vertices and edges of a heptagonal snake graph using distance formulas

Fozia Bashir Farooq[1], Furqan Ahmad[2], Muhammad Kamran Jamil[2], Aisha Javed[3,*] and Nouf Abdulrahman Alqahtani[1]

[1] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Saudi Arabia

[2] Department of Mathematics, Riphah International University, Lahore, Pakistan

[3] Abdus Salam School of Mathematical Sciences, GCU, Lahore, Pakistan

* **Correspondence:** Email: aaishajaved@gmail.com.

Abstract: Consider a city with a network of roads where surveillance cameras are positioned at key intersections. To uniquely determine the position of any car, the minimal number of cameras needed is determined by using the metric dimension of the network. Similarly, if we desire to differentiate each individual road segment using cameras, we depend on the edge metric dimension. These concepts are essential in applications such as transportation planning, autonomous navigation and network monitoring. Let $G = (V(G), E(G))$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$, a subset $R \subseteq V(G)$ is called a resolving set, if each vertex of G has unique distance with respect to R. The cardinality of the smallest resolving set is called the metric dimension, denoted as $dim(G)$. Similarly, a set of vertices $R \subseteq V(G)$ is called an edge resolving set if for every pair of distinct edges $e_1, e_2 \in E(G)$, there exists a vertex $v \in R$ such that $d(e_1, v) \neq d(e_2, v)$, where $d(e, v)$ is defined as $d(e = uw, v) = \min\{d(u, v), d(w, v)\}$. The edge metric dimension is defined as the number of elements in the smallest edge resolving set. Finding these dimensions is an NP-hard problem, making efficient computations precious. In this article, we explored the metric dimension, edge metric dimension, partition dimension and edge partition dimension of the heptagonal snake graph (HPS_k).

Keywords: metric dimension; resolving set; edge metric dimension; edge resolving set; partition dimension; partition resolving set; edge partition dimension; edge partition resolving set; heptagonal snake graph
Mathematics Subject Classification: 05C12, 05C90

1. Introduction

Graph theory is a branch of discrete mathematics that focuses on the studies of structures (*graphs*), which are composed of nodes (also called vertices) and edges (also called links) connecting them. Formally, a graph $G = (V(G), E(G))$ consists of a set $V(G)$ of nodes (*vertices*) and a set $E(G)$ of edges (*links*), in which each edge represents a relationship between two nodes. One of the key concepts in graph theory is distance, which measures how far apart two elements are within the graph. The distance between two nodes u and w, denoted as $d(u, w)$, is the length of the shortest path connecting nodes p and q, where the length is determined by the number of edges or weighted values if the graph is weighted. Similarly, the distance between an edge and a node is defined as the shortest distance from the vertex to any point on the edge, often calculated as the minimum of the distances between the node and the edge's endpoints. Formally, the distance between an edge $e = uw$ and a node p, denoted $d(e, p)$, is defined as the minimum of the distances from the endpoints of the edge to the node, mathematically defined as $d(e = uw, p) = min\{d(u, p), d(w, p)\}$, where u and w are the endpoints of the edge e. Graph theory relies heavily on distance, which is useful in centrality measurements, clustering and shortest path algorithms, as well as determining the efficiency of network connections. It finds extensive usage in contexts where maximizing distance may result in reduced costs and enhanced performance, including communication systems, transportation networks and social networks [1]. Graph theory performs a vital role in numerous fields [2], including social sciences (modeling relationships and influence in social networks) [3], biology (analyzing molecular structures and ecological networks) [4] and computer science (network routing and data structures) [5]. Its capability to model complicated structures and optimize solutions makes it an essential tool in both theoretical studies and real-world applications. Application of graph theory in other areas such as transmission networks, communication network design, computing systems and distributive architectures are present in [6–8]. Applications of graph theory in architectures are explored in [9]. Abbreviations used in our article are listed in Table 1.

Table 1. List of abbreviations.

Abbreviation	Full term
iff	if and only if
RS	Resolving set
MD	Metric dimension
ERS	Edge resolving set
EMD	Edge metric dimension
PRS	Partition resolving set
PD	Partition dimension
$EPRS$	Edge partition resolving set
EPD	Edge partition dimension
HPS_k	Heptagonal snake graph

In graph theory, the concept of a resolving set (RS) was first introduced by Slater in 1975 [10, 11]. Harary and Melter independently studied the same concept under the name "location number" in 1976, focusing on applications in network location problems [12]. An RS is a subset of vertices that uniquely determines the position of each vertex in the graph based on distances. Formally, given a connected graph $G = (V(G), E(G))$, a subset $R \subseteq V(G)$ is called an RS if, for every pair of different vertices $u, w \in V(G)$, there exists at least one vertex $s \in R$ such that $d(s, u) \neq d(s, w)$. This ensures that every vertex within the graph has a unique distance-based representation with respect to R. The metric dimension (MD) of a graph is denoted as $dim(G)$ and is defined as the minimum number of vertices required to resolve the vertex set of the graph. The concepts of resolving sets play an important role in numerous applications, including chemistry (for molecular structure identification), in robotics, where they assist in robot positioning and mapping of environments, as well as network topology, where they help in efficient navigation and localization of nodes. Additionally, they are used in combinatorial optimization and even in cybersecurity for detecting anomalies in network structures. Understanding resolving sets helps in optimizing various processes where precise identification of elements in a network is necessary. The importance of observability in high-dimensional systems has been explored in various contexts, such as in sensor selection for swarm systems [13].

The concept of an edge resolving set (ERS) was introduced by Kelenc et al. in 2018 as an extension of the classical concept of an RS in graph theory. While the concept of resolving sets focuses on uniquely identifying vertices of the graph, the ERS shifts this perspective to edges, ensuring that each edge of the graph has a unique distance-based representation with respect to a selected subset of vertices. Formally, a set of vertices $R \subseteq V(G)$ is called an ERS if for every pair of distinct edges $e_1, e_2 \in E(G)$, there exists a vertex $v \in R$ such that $d(e_1, v) \neq d(e_2, v)$, where $d(e, v)$ is defined as $d(e = uw, v) = \min\{d(u, v), d(w, v)\}$. The edge metric dimension (EMD) of a graph G is denoted as $edim(G)$ and defined as the smallest cardinality of the ERS [14]. The concept of an ERS plays a crucial role in biological networks where it aids in understanding structural properties of molecular graphs, in transportation systems for optimizing routes and identifying critical pathways and in network security by enhancing fault detection and monitoring. The MD of the line graph and subdivision of line graphs are explored in [15]. The MD and EMD of the starphene graph and applications of MD and EMD of the starphene graph in electronics are discussed in [16]. The MD and EMD of the Jahangir graph and heptagonal circular ladder graphs are discussed in [17, 18]. NP-hardness and computational complexity of the resolvability parameters are explored in [19–21]. The researchers are motivated by numerous practical applications of the MD in daily life, which has been the subject of extensive research. The MD is extensively employed in a variety of scientific disciplines, including computer networks [22], robot navigation [23], location problems, sonar and coastguard Loran [10], pharmaceutical chemistry [20], combinatorial optimization [24], weighing problems [25], and image processing. For additional information, we refer the reader to [26, 27]. The partition dimension (PD) also has a wide range of real-world applications, including the popular Djokovic-Winkler relation [28], the procedure of verifying and discovering a network [29] and strategy, decoding and coding of mastermind games [30]. There are numerous applications to investigate, and we recommend exploring [12,20,31]. Applications of MD in digital geometry are discussed in [32,33]. To explore more results on MD and EMD of different structures, see Table 2 below.

In graph theory, a partition resolving set (PRS) in a connected graph $G = (V, E)$ is a partition of the vertex set V that uniquely identifies every vertex based on its distances to these partitions. Formally, let $\Phi = \{X_1, X_2, \ldots, X_k\}$ be a partition of V. The distance vector of a vertex $m \in V$ with respect to Φ is defined as $r(m|\Phi) = (d(m, X_1), d(m, X_2), \ldots, d(m, X_k))$, where $d(m, X_i) = min\{d(m, x)|x \in X_i\}$ represents the shortest distance from m to any vertex in X_i. The partition Φ is referred to as a PRS if each vertex in G has a unique distance vector, ensuring that for any two different vertices $m, n \in V$, we have $r(m|\Phi) \neq (n|\Phi)$. The number of elements in the smallest PRS is called the PD, denoted as $pdim(G)$ [19]. PRSs have numerous applications in different areas. In robotics and navigation, it helps in localization and path planning. In network topology, it is used to identify nodes and optimize routing uniquely. It also has applications in chemistry, molecular chemistry and transportation networks for efficient monitoring, identification and resource optimization.

Table 2. Metric dimension and edge metric dimension of various structures.

Structures	References
Nanotube	Sikander [34]
Cellulose network	Imran [35]
Kayak paddle graph	Ahmad [36]
Starphene structure	Ahmad [16]
Convex polytopes	Imran [37]
Hypercube	Beardon [38]
Crystal cubic carbon structure	Zhang [39]
1-Pentagonal carbon nanocone networks	Hussaiin [40]
H-Naphtalenic and $VC5C7$ nanotube networks	Siddiqui [41]
Polycyclic aromatic hydrocarbons structure	Azeem [42]
Convex polytopes structure	Ahsan [43]
k-Multiwheel graph	Bataineh [44]
Families of trees	Adawiyah [45]
Generalized Petersen graphs	Kartelj [46]
Windmill graphs	Sharma [47]
Möbius networks	Nadeem [48]
Patched network	Bukhari [49]

The idea of an edge partition resolving set ($EPRS$) is a generalization of the PRS idea, applied to edges rather than vertices. Given a linked graph $G = (V, E)$, an edge partition $\Phi = \{Y_1, Y_2, \ldots, Y_k\}$ is a partition of the vertex set V into disjoint subsets. For each edge $e \in E$, the distance vector with respect to Φ is described as $r(e|\Phi) = (d(e, Y_1), d(e, Y_2), \ldots, d(e, Yk))$, where $d(e, Y_i) = min\{d(e, s)|e \in Y_i\}$ represents the shortest distance from edge e to any vertex in Y_i. The partition Φ is referred to as an $EPRS$ if every edge in G has a unique distance vector, ensuring that for any two different edges $e_1, e_2 \in E$, we have $r(e_1|\Phi) \neq (e_2|\Phi)$. The EPD of G is denoted as $epdim(G)$ and defined as the smallest cardinality $EPRS$ [50]. The concept of $EPRS$ has applications in transportation systems, network design and fault-tolerant communication, where distinguishing edges efficiently can enhance connectivity and optimization techniques. Distance-based parameters in graph theory, such as MD and its variants, play a crucial role in broader topics like observability, fault-tolerant detection and dynamic networks. In observability, these parameters help determine the minimum number of nodes required to uniquely identify the state of a system. For fault-tolerant detection, they ensure that even with node or link failures, critical information about network states can still be recovered. In dynamic networks, distance-based measures support efficient tracking and adaptation by providing compact yet informative node representations, enhancing real-time monitoring and control. To explore other parameters of resolvability, see Table 3.

Table 3. Parameters of resolvability.

Parameters of resolvability	References
Local metric dimension	Okamoto et al. [51]
Local edge metric dimension	Adawiyah et al. [52]
Dominant metric dimension	Susilowati et al. [53]
Dominant edge metric dimension	Siddiqui et al. [54]
Dominant mixed metric dimension	Alfarisi et al. [55]
Multiset dimension	Vetrik et al. [56]
Local multiset dimension	Dafik et al. [57]
Fractional metric dimension	Mathew et al. [58]
Local fractional metric dimension	Benish et al. [59]
Strong metric dimension	Oellermann et al. [60]
Fractional strong metric dimension	E. Yi et al. [61]

2. Main results

In the section, we compute the MD, EMD, PD and edge partition dimension (EPD) of the heptagonal snake graph (HPS_k).

In Figure 1, the graph is a heptagonal snake graph. The heptagonal graph is represented by HPS_k, where $k \geq 1$ (see Figure 1). In HPS_k, the number of 2-degree and 4-degree vertices are $6k$, $k-1$, respectively.

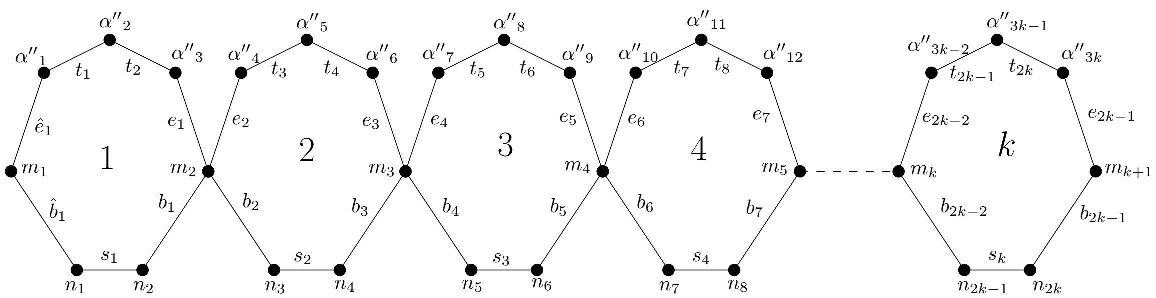

Figure 1. Heptagonal snake graph HPS_k.

The vertex and edge set of the HPS_k are given below:

$$
\begin{aligned}
V(HPS_k) &= \{\alpha''_i; 1 \leq i \leq 3k\} \cup \{m_i; 1 \leq i \leq k+1\} \cup \{n_i; 1 \leq i \leq 2k\},\\
E(HPS_k) &= \{\alpha''_i\alpha''_{i+1} = t_i; 1 \leq i \leq 2k\} \cup \{n_i n_{i+1} = s_i; 1 \leq i \leq k\}\\
&\cup \{\alpha''_i m_i = \hat{e}_i; i = 1\} \cup \{m_i n_i = \hat{b}_i; i = 1\}\\
&\cup \{\alpha''_i m_j = e_p; i = 3, 6, 9, \ldots, 3k; 2 \leq j \leq k+1; p = 1, 3, 5, \ldots, 2k-1\}\\
&\cup \{\alpha''_i m_j = e_p; i = 4, 7, 10, \ldots, 3k-2; 2 \leq j \leq k; p = 2, 4, 6, \ldots, 2k-2; k \geq 2\}\\
&\cup \{n_i m_j = b_p; i = 2, 4, 6, \ldots, 2k; 2 \leq j \leq k+1; p = 1, 3, 5, \ldots, 2k-1\}\\
&\cup \{n_i m_j = b_p; i = 3, 5, 7, \ldots, 2k-1; 2 \leq j \leq k; p = 2, 4, 6, \ldots, 2k-2; k \geq 2\}.
\end{aligned}
$$

The order and size of the HPS_k are given below:

$$|V(HPS_k)| = 6k + 1, \quad |E(HPS_k)| = 7k.$$

Theorem 2.1. *Let HPS_k be the graph of the heptagonal snake graph with $k \geq 2$. Then, $dim(HPS_k) = 2$.*

Proof. To prove that the MD of the HPS_k is 2, let $R = \{m_1, m_{k+1}\}$ be a proper subset of the vertex set of the HPS_k. To show that R is the RS of the heptagonal snake graph, we will show that each vertex of the vertex set of the HPS_k has a unique distance with respect to $R = \{m_1, m_{k+1}\}$. The distance formulas below show the unique representation of each vertex of the heptagonal snake graph with respect to $R = \{m_1, m_{k+1}\}$.

$$r(\alpha''_i|m_1) = \; i, \quad \text{for } 1 \leq i \leq 3k,$$
$$r(m_i|m_1) = \; 3i - 3, \quad \text{for } 1 \leq i \leq k+1,$$
$$r(n_i|m_1) = \begin{cases} \frac{3i-1}{2}, & \text{for } i = 1, 3, 5 \ldots, 2k-1, \\ \frac{3i-2}{2}, & \text{for } i = 2, 4, 6 \ldots, 2k, \end{cases}$$
$$r(\alpha''_i|m_{k+1}) = \; -i + 3k + 1, \quad \text{for } 1 \leq i \leq 3k,$$
$$r(m_i|m_{k+1}) = \; -3i + 3k + 3, \quad \text{for } 1 \leq i \leq k+1,$$
$$r(n_i|m_{k+1}) = \begin{cases} \frac{-3i+6k+1}{2}, & \text{for } i = 1, 3, 5 \ldots, 2k-1, \\ \frac{-3i+6k+2}{2}, & \text{for } i = 2, 4, 6 \ldots, 2k. \end{cases}$$

Let s and s' be two arbitrary vertices of HPS_k, and $R = \{m_1, m_{k+1}\} \subseteq V(HPS_k)$. To show that R uniquely represents vertices of the HPS_k, we have the following cases.

Case 1: Let s and s' be two vertices of the HPS_k. If $d(s, m_1) = d(s', m_1)$ and $d(s, m_{k+1}) = d(s', m_{k+1})$, then $s = s'$.

Subcase 1: Let $s = \{\alpha''_i\}$, $1 \leq i \leq 3k$, and let $s' = \{\alpha''_{i'}\}$, $1 \leq i' \leq 3k$. Then the distances of s and s' with respect to R are $d(s, m_1) = i$, $d(s, m_{k+1}) = -i + 3k + 1$, $d(s', m_1) = i'$ and $d(s', m_{k+1}) = -i' + 3k + 1$. Since we take $d(s, m_1) = d(s', m_1)$ and $d(s, m_{k+1}) = d(s', m_{k+1})$, therefore, $i = i'$ and $-i + 3k + 1 = -i' + 3k + 1$. By solving two equations, we get $i = i'$. Hence, $s = s'$.

Subcase 2: Let $s = \{m_i\}$, $1 \leq i \leq k+1$, and let $s' = \{m_{i'}\}$, $1 \leq i' \leq k+1$. Then the distances of s and s' with respect to R are $d(s, m_1) = 3i - 3$, $d(s, m_{k+1}) = -3i + 3k + 3$, $d(s', m_1) = 3i' - 3$ and $d(s', m_{k+1}) = -3i' + 3k + 3$. Since we take $d(s, m_1) = d(s', m_1)$ and $d(s, m_{k+1}) = d(s', m_{k+1})$, therefore, $3i - 3 = 3i' - 3$ and $-3i + 3k + 3 = -3i' + 3k + 3$. By solve $3i - 3 = 3i' - 3$, we get $i = i'$. Hence, $s = s'$.

Subcase 3: Let $s = \{n_i\}$, $i = 1, 3, 5, \ldots, 2k-1$, and let $s' = \{n_{i'}\}$, $i' = 1, 3, 5, \ldots, 2k-1$. Then the distances of s and s' with respect to R are $d(s, m_1) = \frac{3i-1}{2}$, $d(s, m_{k+1}) = \frac{-3i+6k+1}{2}$, $d(s', m_1) = \frac{3i'-1}{2}$ and $d(s', m_{k+1}) = \frac{-3i'+6k+1}{2}$. Since we consider $d(s, m_1) = d(s', m_1)$ and $d(s, m_{k+1}) = d(s', m_{k+1})$, therefore, $\frac{3i-1}{2} = \frac{3i'-1}{2}$ and $\frac{-3i+6k+1}{2} = \frac{-3i'+6k+1}{2}$. By solving two equations, we get $i = i'$. Hence, $s = s'$.

Subcase 4: Let $s = \{n_i\}$, $i = 2, 4, 6, \ldots, 2k$, and let $s' = \{n_{i'}\}$, $i' = 2, 4, 6, \ldots, 2k$. Then the distances of s and s' with respect to R are $d(s, m_1) = \frac{3i-2}{2}$, $d(s, m_{k+1}) = \frac{-3i+6k+2}{2}$, $d(s', m_1) = \frac{3i'-2}{2}$ and $d(s', m_{k+1}) = \frac{-3i'+6k+2}{2}$. Since we consider $d(s, m_1) = d(s', m_1)$ and $d(s, m_{k+1}) = d(s', m_{k+1})$, therefore, $\frac{3i-2}{2} = \frac{3i'-2}{2}$ and $\frac{-3i+6k+2}{2} = \frac{-3i'+6k+2}{2}$. By solving two linear equations, we get $i = i'$. Hence, $s = s'$.

Subcase 5: Let $s = \{\alpha''_i\}$, $1 \leq i \leq 3k$, and then the distances of s with respect to R are $d(s, m_1) = i$ and $d(s, m_k + 1) = -i + 3k + 1$. Let $s' = \{m_{i'}\}$, $1 \leq i' \leq k+1$ with respect to R are $d(s', m_1) = 3i' - 3$ and $d(s', m_k + 1) = -3i' + 3k + 3$. Since we consider $d(s, m_1) = d(s', m_1)$ and $d(s, m_{k+1}) = d(s', m_{k+1})$,

therefore, $i = 3i' - 3$ and $-i + 3k + 1 = -3i' + 3k + 3$. By solving two linear equations, we get $i' = \frac{-1}{6}$. As i and i' are integers, this case is impossible. Hence, $d(s, m_1) \neq d(s', m_1)$ and $d(s, m_{k+1}) \neq d(s', m_{k+1})$.
Subcase 6: Let $s = \{\alpha''_i\}$, $1 \leq i \leq 3k$, and then the distances of s with respect to R are $d(s, m_1) = i$ and $d(s, m_k + 1) = -i + 3k + 1$. Let $s' = \{n_{i'}\}$, $i' = 1, 3, 5, \ldots 2k - 1$, and then the distances of s' with respect to R are $d(s', m_1) = \frac{3i'-1}{2}$ and $d(s', m_k + 1) = \frac{-3i'+6k+1}{2}$. Since we consider $d(s, m_1) = d(s', m_1)$ and $d(s, m_{k+1}) = d(s', m_{k+1})$, therefore, $i = \frac{3i'-1}{2}$ and $-i + 3k + 1 = \frac{-3i'+6k+1}{2}$. By solving two linear equations, we get $-1 = 0$. This case is impossible.
Subcase 7: Let $s = \{\alpha''_i\}$, $1 \leq i \leq 3k$, and then the distances of s with respect to R are $d(s, m_1) = i$ and $d(s, m_k + 1) = -i + 3k + 1$. Let $s' = \{n_{i'}\}$, $i' = 2, 4, 6, \ldots 2k$, and then the distances of s' with respect to R are $d(s', m_1) = \frac{3i'-2}{2}$ and $d(s', m_k + 1) = \frac{-3i'+6k+2}{2}$. Since we consider $d(s, m_1) = d(s', m_1)$ and $d(s, m_{k+1}) = d(s', m_{k+1})$, therefore, $i = \frac{3i'-2}{2}$ and $-i + 3k + 1 = \frac{-3i'+6k+2}{2}$. By solving two linear equations, we get $-1 = 0$. This case is impossible.
Subcase 8: Let $s = \{m_i\}$, $1 \leq i \leq k + 1$, and then the distances of s with respect to R are $d(s, m_1) = 3i-3$ and $d(s, m_k + 1) = -3i + 3k + 3$. Let $s' = \{n_{i'}\}$, $i' = 2, 4, 6, \ldots 2k$, and then the distances of s' with respect to R are $d(s', m_1) = \frac{3i'-2}{2}$ and $d(s', m_k + 1) = \frac{-3i'+6k+2}{2}$. Since we consider $d(s, m_1) = d(s', m_1)$ and $d(s, m_{k+1}) = d(s', m_{k+1})$, therefore, $3i - 3 = \frac{3i'-2}{2}$ and $-3i + 3k + 3 = \frac{-3i'+6k+2}{2}$. By solving two linear equations, we get $i' = 2i - \frac{4}{3}$. This case is impossible.
Subcase 9: Let $s = \{m_i\}$, $1 \leq i \leq k + 1$, and then the distances of s with respect to R are $d(s, m_1) = 3i-3$ and $d(s, m_k + 1) = -3i + 3k + 3$. Let $s' = \{n_{i'}\}$, $i' = 1, 3, 5, \ldots 2k - 1$, and then the distances of s' with respect to R are $d(s', m_1) = \frac{3i'-1}{2}$ and $d(s', m_k + 1) = \frac{-3i'+6k+1}{2}$. Since we consider $d(s, m_1) = d(s', m_1)$ and $d(s, m_{k+1}) = d(s', m_{k+1})$, therefore, $3i - 3 = \frac{3i'-1}{2}$ and $-3i + 3k + 3 = \frac{-3i'+6k+1}{2}$. By solving two linear equations, we get $i' = 2i - \frac{5}{3}$. This case is impossible.

From the above cases, we showed that each vertex of the HPS_k has a unique distance with respect to R. Therefore, $R = \{m_1, m_{k+1}\}$ is the RS of the HPS_k. Hence, the RS of the HPS_k is 2, which means $dim(HPS_k) \leq 2$. Now to show that $dim(HPS_k) \geq 2$, let the MD of the heptagonal snake graph be 1. Harary and Melter [12] showed that $dim(G) = 1$, iff $G = P_n$, which contradicts our assumption. Therefore, $R = \{m_1, m_{k+1}\}$ is the smallest cardinality RS of the HPS_k.

Hence, the outcome is $dim(HPS_k) = 2$.

The unique distances of the vertices of HPS_3 (see Figure 2) are presented in Table 4.

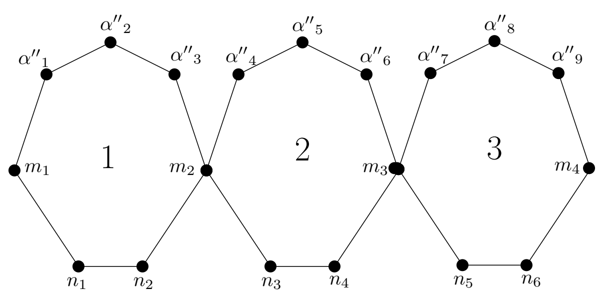

Figure 2. Heptagonal snake graph HPS_3.

Table 4. Unique representation of vertices of HPS_3 with respect to $R = \{m_1, m_4\}$.

Vertices	m_1	m_4
α''_1	1	9
α''_2	2	8
α''_3	3	7
α''_4	4	6
α''_5	5	5
α''_6	6	4
α''_7	7	3
α''_8	8	2
α''_9	9	1
m_1	0	9
m_2	3	6
m_3	6	3
m_4	9	0
n_1	1	8
n_2	2	7
n_3	4	5
n_4	5	4
n_5	7	2
n_6	8	1

Theorem 2.2. *Let HPS_k be the graph of the heptagonal snake graph with $k = 2$. Then, $edim(HPS_k) = 3$.*

Proof. To prove that the EMD of the HPS_k for $k = 2$ is 3, let $R = \{\alpha''_1, \alpha''_4, n_3\}$ be a proper subset of the vertex set of the HPS_k. To show that R is the ERS of the HPS_k, for this we will show that each edge of the edge set of the HPS_k has a unique distance with respect to $R = \{\alpha''_1, \alpha''_4, n_3\}$. The following are unique distances of the edges of the HPS_k for $k = 2$ with respect to $R = \{\alpha''_1, \alpha''_4, n_3\}$.

$$r(t_i|\alpha''_1) = \begin{cases} 2i-2, & \text{for } i = 1,3, \\ 2i-3, & \text{for } i = 2,4, \end{cases}$$

$$r(b_i|\alpha''_1) = \begin{cases} i+2, & \text{for } i = 1,3, \\ i+1, & \text{for } i = 2, \end{cases}$$

$$r(\hat{b}_i|\alpha''_1) = i, \quad \text{for } i = 1,$$

$$r(\hat{e}_i|\alpha''_1) = i-1, \quad \text{for } i = 1,$$

$$r(s_i|\alpha''_1) = 3i-1, \quad \text{for } i = 1,2,$$

$$r(t_i|\alpha''_4) = \begin{cases} \frac{-3i+9}{2}, & \text{for } i = 1,3, \\ \frac{-i+6}{2}, & \text{for } i = 2,4, \end{cases}$$

$$r(b_i|\alpha''_4) = \begin{cases} i, & \text{for } i = 1,3, \\ i, & \text{for } i = 2, \end{cases}$$

$$r(\hat{b}_i|\alpha''_4) = i+2, \quad \text{for } i = 1,$$

$$r(\hat{e}_i|\alpha''_4) = i+3, \quad \text{for } i = 1,$$

$$r(s_i|\alpha''_4) = \ i, \quad \text{for } i = 1, 2,$$

$$r(t_i|n_3) = \begin{cases} \frac{-i+7}{2}, & \text{for } i = 1, 3, \\ \frac{i+2}{2}, & \text{for } i = 2, 4, \end{cases}$$

$$r(b_i|n_3) = \begin{cases} 1, & \text{for } i = 1, 3, \\ 0, & \text{for } i = 2, \end{cases}$$

$$r(\hat{b}_i|n_3) = \ i + 2, \quad \text{for } i = 1,$$

$$r(\hat{e}_i|n_3) = \ i + 3, \quad \text{for } i = 1,$$

$$r(s_i|n_3) = \ -2i + 4, \quad \text{for } i = 1, 2.$$

From the above distance formulas, we will show that each edge of the HPS_k has unique distance with respect to $R = \{\alpha''_1, \alpha''_4, n_3\}$. Therefore, $R = \{\alpha''_1, \alpha''_4, n_3\}$ is the ERS of the HPS_k for $k = 2$. Hence, the cardinality of the ERS of the HPS_k is 3, which means $edim(HPS_k) \leq 3$. Now to show that $dim(DSC_m) \geq 3$, let the EMD of the heptagonal snake graph be 2. Then we have the following contrary cases.

Case 1: Let $R' = \{\alpha''_1, \alpha''_2, \alpha''_3, m_1, m_2, n_1, n_2\} \subseteq V(HPS_2)$, and if we take any subset of cardinality 2 from R', then $r(e_2|R) = r(b_2|R)$, which contradicts our assumption.

Case 2: Let $R'' = \{\alpha''_4, \alpha''_5, \alpha''_6, m_2, m_3, n_3, n_4\} \subseteq V(HPS_2)$, and if we take any subset of cardinality 2 from R'', then $r(e_1|R) = r(b_1|R)$, which contradicts our assumption.

Case 3: Let $R'' = \{\alpha''_i, \alpha''_4, \alpha''_5, \alpha''_6\} \subseteq V(HPS_2)$, where $i = 1, 2, 3$, and if we take any subset of cardinality 2 from R'', then $r(b_1|R) = r(b_2|R)$.

Case 4: Let $R'' = \{\alpha''_i, n_3, n_4, m_3\} \subseteq V(HPS_2)$, where $i = 1, 2, 3$, and if we take any subset of cardinality 2 from R'', then $r(b_1|R) = r(e_2|R)$.

Case 5: Let $R'' = \{n_i, \alpha''_4, \alpha''_5, \alpha''_6\} \subseteq V(HPS_2)$, where $i = 1, 2$, and if we take any subset of cardinality 2 from R'', then $r(b_2|R) = r(e_2|R)$.

Case 6: Let $R'' = \{n_i, n_3, n_4, m_3\} \subseteq V(HPS_2)$, where $i = 1, 2$, and if we take any subset of cardinality 2 from R'', then $r(e_1|R) = r(e_2|R)$.

Case 7: Let $R'' = \{m_1, \alpha''_4, \alpha''_5, \alpha''_6\} \subseteq V(HPS_2)$, and if we take any subset of cardinality 2 from R'', then $r(b_2|R) = r(e_1|R)$.

Case 8: Let $R'' = \{m_1, n_3, n_4, m_3\} \subseteq V(HPS_2)$, and if we take any subset of cardinality 2 from R'', then $r(e_1|R) = r(e_2|R)$.

The above cases contradicts our assumption. Therefore, $R = \{\alpha''_1, \alpha''_4, n_3\}$ is the smallest cardinality ERS of HPS_k for $k = 2$. Hence, the outcome is $edim(HPS_k) = 3$. The unique representation of edges of HPS_k for $k = 2$ (Figure 3) are presented in Table 5. □

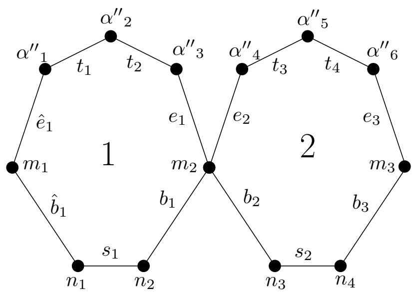

Figure 3. Heptagonal snake graph HPS_2.

Table 5. The unique distances of edges of the HPS_k for $k = 2$ with respect to $R = \{\alpha''_1, \alpha''_4, n_3\}$.

Edges	m_1	m_4	n_3
t_1	0	3	3
t_2	1	2	2
t_3	4	0	2
t_4	5	1	3
b_1	3	1	1
b_2	3	1	0
b_3	5	3	1
e_1	2	1	1
e_2	3	0	1
e_3	6	2	2
s_1	2	2	2
s_2	4	2	2
$\hat{e}_1$	0	4	4
$\hat{b}_1$	1	3	3

Theorem 2.3. *Let HPS_k be the graph of the heptagonal snake graph with $k \geq 3$. Then, $edim(HPS_k) = k$.*

Proof. To prove that the EMD of the HPS_k is k, let $R_e = \{\alpha''_i; i = 1, 4, 7, \ldots, 3k-2\}$ be a proper subset of the vertex set $V(HPS_k)$ of the HPS_k. To show that R_e is the ERS of the HPS_k, we will show that each edge of the HPS_k has a unique distance with respect to $R_e = \{\alpha''_i; i = 1, 4, 7, \ldots, 3k-2\}$. The following are the generalized unique representations of each edge of HPS_k with respect to $R_e = \{\alpha''_i; i = 1, 4, 7, \ldots, 3k-2\}$.

$$r(t_i|R_e) = \begin{cases} (0, 3, 6, 9, \ldots, 3k-3), & \text{for } i = 1, \\ (1, 2, 5, 8, \ldots, 3k-4), & \text{for } i = 2, \\ (\frac{3i}{2}-1, \frac{3i}{2}-4, \frac{3i}{2}-7, \ldots, 8, 5, 1, & \text{for } i = 4, 6, 8, \ldots, 2k, \\ \quad 2, 5, 8, \ldots, \frac{-3i}{2}+3k-4, \frac{-3i}{2}+3k-1), & \\ (\frac{3i-1}{2}, \frac{3i-1}{2}-3, \frac{3i-1}{2}-6, \ldots, 10, 7, 4, & \text{for } i = 3, 5, 7, \ldots, 2k-1, \\ \quad 0, 3, 6, \ldots, \frac{-3i+6k-3}{2}), & \end{cases}$$

$$r(e_i|R_e) = \begin{cases} (\frac{3i}{2}, \frac{3i}{2}-3, \frac{3i}{2}-6, \ldots, 6, 3, 0, & \text{for } i = 4, 6, 8, \ldots, 2k, \\ \quad 4, 7, 10, \ldots, \frac{-3i}{2}+3k-5, \frac{-3i}{2}+3k-2), & \\ (\frac{3i+3}{2}, \frac{3i+3}{2}-3, \frac{3i+3}{2}-6, \ldots, 6, 2, 1, & \text{for } i = 3, 5, 7, \ldots, 2k-1, \\ \quad 4, 7, 10, \ldots, \frac{-3i+6k-7}{2}), & \end{cases}$$

$$r(\hat{e}_i|R_e) = (0, 4, 7, 10 \ldots, 3k-2), \quad \text{for } i = 1,$$

$$r(\hat{b}_i|R_e) = (1, 3, 6, 9 \ldots, 3k-3), \quad \text{for } i = 1,$$

$$r(b_i|R_e)=\begin{cases}(3,1,4,7,\ldots,3k{-}5), & \text{for } i=1,\\ (\frac{2i+5}{3},\frac{2i+5}{3}{-}3,\frac{2i+5}{3}{-}6,\ldots,3,1, & \text{for } i=2,4,6,\ldots,2k{-}2,\\ \;3,6,9,\ldots,\frac{-3i}{2}{+}3k{-}3), & \\ (\frac{3i+1}{2},\frac{3i+1}{2}{-}3,\frac{3i+1}{2}{-}6,\ldots,3,1, & \text{for } i=3,5,7,\ldots,2k{-}1,\\ \;4,7,10,\ldots,\frac{-3i+6k-7}{2}), & \end{cases}$$

$$r(s_i|R_e)=\begin{cases}(2,2,5,8,\ldots,3k{-}4), & \text{for } i=1,\\ (3i{-}2,\;3i{-}5,\;3i{-}8,\ldots,4,2,2, & \text{for } i=2,3,4,\ldots,k.\\ \;5,8,\ldots,\;-3i{+}3k{-}4,\;-3i{+}3k{-}1), & \end{cases}$$

Let r and r' be two arbitrary edges of HPS_k and $R_e = \{\alpha''_i; i = 1,4,7,\ldots,3k-2\} \subseteq V(HPS_k)$. To show that R_e uniquely represents edges of the HPS_k, then we have the following cases.

Case 1: Let r and r' be two edges of the HPS_k. If $d(r|R_e) = d(r'|R_e)$, then $r = r'$.

Subcase 1: Let $r = \{t_i\}$, $i = 2k$ and let $r' = \{t_{i'}\}$, $i' = 2k$, and then the distances of r and r' with respect to R_e are $d(r|R_e) = \frac{3i}{2} - 1$, $d(r'|R_e) = \frac{3i'}{2} - 1$. Since we take $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i}{2} - 1 = \frac{3i'}{2} - 1$. By solving the linear equations, we get $i = i'$. Hence, $r = r'$.

Subcase 2: Let $r = \{t_i\}$, $i = 2k - 1$ and let $r' = \{t_{i'}\}$, $i' = 2k - 1$, and then the distances of r and r' with respect to R_e are $d(r|R_e) = \frac{3i-1}{2}$, $d(r'|R_e) = \frac{3i'-1}{2}$. Since we take $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i-1}{2} = \frac{3i'-1}{2}$. By solving the linear equations, we get $i = i'$. Hence, $r = r'$.

Subcase 3: Let $r = \{e_i\}$, $i = 2k$ and let $r' = \{e_{i'}\}$, $i' = 2k$, and then the distances of r and r' with respect to R_e are $d(r|R_e) = \frac{3i}{2}$, $d(r'|R_e) = \frac{3i'}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i}{2} = \frac{3i'}{2}$. By solving the linear equations, we get $i = i'$. Hence, $r = r'$.

Subcase 4: Let $r = \{e_i\}$, $i = 2k - 1$ and let $r' = \{e_{i'}\}$, $i' = 2k - 1$, and then the distances of r and r' with respect to R_e are $d(r|R_e) = \frac{3i+3}{2}$, $d(r'|R_e) = \frac{3i'+3}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i+3}{2} = \frac{3i'+3}{2}$. By solving linear equations, we get $i = i'$. Hence, $r = r'$.

Subcase 5: Let $r = \{b_i\}$, $i = 2k - 2$ and let $r' = \{b_{i'}\}$, $i' = 2k - 2$, and then the distances of r and r' with respect to R_e are $d(r|R_e) = \frac{2i+5}{3}$, $d(r'|R_e) = \frac{2i'+5}{3}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{2i+5}{3} = \frac{2i'+5}{3}$. By solving the linear equations, we get $i = i'$. Hence, $r = r'$.

Subcase 6: Let $r = \{b_i\}$, $i = 2k - 1$ and let $r' = \{b_{i'}\}$, $i' = 2k - 1$, and then the distances of r and r' with respect to R_e are $d(r|R_e) = \frac{3i+1}{2}$, $d(r'|R_e) = \frac{3i'+1}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i+1}{2} = \frac{3i'+1}{2}$. By solving linear equations, we get $i = i'$. Hence, $r = r'$.

Subcase 7: Let $r = \{s_i\}$, $i = k$ and let $r' = \{s_{i'}\}$, $i' = k$, and then the distances of r and r' with respect to R_e are $d(r|R_e) = 3i - 2$, $d(r'|R_e) = 3i' - 2$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $3i - 2 = 3i' - 2$. By solving the linear equation, we get $i = i'$. Hence, $r = r'$.

Subcase 8: Let $r = \{s_i\}$, $i = 1$ and let $r' = \{s_{i'}\}$, $i' = 1$, and then the distances of r and r' with respect to R_e are $d(r|R_e) = 3k - 4$, $d(r'|R_e) = 3k - 4$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $3k - 4 = 3k - 4$. As $i = i' = 1$. Hence, $r = r'$.

Subcase 9: Let $r = \{t_i\}$, $i = 2k$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i}{2} - 1$. Let $r' = \{t_{i'}\}$, $i' = 2k - 1$, and then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{3i'-1}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i}{2} - 1 = \frac{3i'-1}{2}$. By solving the equation we get $i = i' + \frac{1}{3}$. Since i and i' are integers. Hence, this case is not possible.

Subcase 10: Let $r = \{t_i\}$, $i = 2k$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i}{2} - 1$. Let $r' = \{e_{i'}\}$, $i' = 2k$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{3i'}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i}{2} - 1 = \frac{3i'}{2}$. By solving the equation we get $i = i' + \frac{2}{3}$. Since i and i' are integers. Hence, this case is not possible.

Subcase 11: Let $r = \{t_i\}$, $i = 2k$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i}{2} - 1$. Let $r' = \{e_{i'}\}$, $i' = 2k - 1$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{3i'+3}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i}{2} - 1 = \frac{3i'+3}{2}$. By solving the equation we get $i = i' + \frac{1}{3}$. Since i and i' are integers. Hence, this case is impossible.

Subcase 12: Let $r = \{t_i\}$, $i = 2k$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i}{2} - 1$. Let $r' = \{b_{i'}\}$, $i' = 2k - 1$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{3i'+1}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i}{2} - 1 = \frac{3i'+1}{2}$. By solving the equation we get $i = i' + \frac{1}{3}$. Since i and i' are positive integers. Hence, this case is impossible.

Subcase 13: Let $r = \{t_i\}$, $i = 2k$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i}{2} - 1$. Let $r' = \{b_{i'}\}$, $i' = 2k - 2$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{2i'+5}{3}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i}{2} - 1 = \frac{2i'+5}{3}$. By solving the equation we get $i = \frac{4i'}{9} + \frac{10}{9}$. Since i and i' are positive integers. Hence, this case is not possible.

Subcase 14: Let $r = \{t_i\}$, $i = 2k$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i}{2} - 1$. Let $r' = \{s_{i'}\}$, $i' = k$, then the distance of r' with respect to R_e is $d(r'|R_e) = 3i' - 2$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i}{2} - 1 = 3i' - 2$. By solving the equation we get $i = 2i' - \frac{2}{3}$. Since i and i' are positive integers. Hence, this case is not possible.

Subcase 15: Let $r = \{t_i\}$, $i = 2k - 1$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i-1}{2}$. Let $r' = \{e_{i'}\}$, $i' = 2k$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{3i'}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i-1}{2} = \frac{3i'}{2}$. By solving the equation we get $i = i' + \frac{1}{3}$. Since i and i' are positive integers, this case is not possible.

Subcase 16: Let $r = \{t_i\}$, $i = 2k - 1$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i-1}{2}$. Let $r' = \{e_{i'}\}$, $i' = 2k - 1$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{3i'+3}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i-1}{2} = \frac{3i'+3}{2}$. By solving the equation we get $i = i' + \frac{4}{3}$. Since i and i' are positive integers, this case is impossible.

Subcase 17: Let $r = \{t_i\}$, $i = 2k - 1$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i-1}{2}$. Let $r' = \{b_{i'}\}$, $i' = 2k - 1$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{3i'+1}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i-1}{2} = \frac{3i'+1}{2}$. By solving the equation we get $i = i' + \frac{2}{3}$. Since i and i' are positive integers, this case is impossible.

Subcase 18: Let $r = \{t_i\}$, $i = 2k - 1$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i-1}{2}$. Let $r' = \{b_{i'}\}$, $i' = 2k - 2$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{2i'+5}{3}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i-1}{2} = \frac{2i'+5}{3}$. By solving the equation we get $i = \frac{4i'}{9} + \frac{7}{9}$. Since i and i' are positive integers, this case is not possible.

Subcase 19: Let $r = \{e_i\}$, $i = 2k$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i}{2}$. Let $r' = \{b_{i'}\}$, $i' = 2k - 1$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{3i'+1}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i}{2} = \frac{3i'+1}{2}$. By solving the equation we get $i = i' + \frac{1}{3}$. Since i and i' are positive integers, this case contradicts our assumption.

Subcase 20: Let $r = \{e_i\}$, $i = 2k$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i}{2}$. Let $r' = \{b_{i'}\}$, $i' = 2k - 2$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{2i'+5}{3}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i}{2} = \frac{2i'+5}{3}$. By solving the equation we get $i = \frac{4i'}{9} + \frac{10}{9}$. Since i and i' are positive integers, this case contradicts our assumption.

Subcase 21: Let $r = \{e_i\}$, $i = 2k$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i}{2}$. Let $r' = \{s_{i'}\}$, $i' = k$, then the distance of r' with respect to R_e is $d(r'|R_e) = 3i' - 2$. Since we consider

$d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i}{2} = 3i' - 2$. By solving the equation we get $i = 2i' - \frac{4}{3}$. Since i and i' are positive integers, this case is impossible.

Subcase 22: Let $r = \{e_i\}$, $i = 2k-1$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i+3}{2}$. Let $r' = \{b_{i'}\}$, $i' = 2k-2$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{2i'+5}{3}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i+3}{2} = \frac{2i'+5}{3}$. By solving the equation we get $i = \frac{4i'}{9} + \frac{1}{9}$. Since i and i' are positive integers, this case contradicts our supposition.

Subcase 23: Let $r = \{e_i\}$, $i = 2k-1$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i+3}{2}$. Let $r' = \{b_{i'}\}$, $i' = 2k-1$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{3i'+1}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i+3}{2} = \frac{3i'+1}{2}$. By solving the equation we get $i = i' + \frac{2}{3}$. Since i and i' are positive integers, this case is not possible.

Subcase 24: Let $r = \{e_i\}$, $i = 2k-1$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i+3}{2}$. Let $r' = \{s_{i'}\}$, $i' = k$, then the distance of r' with respect to R_e is $d(r'|R_e) = 3i' - 2$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i+3}{2} = 3i' - 2$. By solving the equation we get $i = 2i' - \frac{1}{3}$. Since i and i' are positive integers, this case contradicts our assumption.

Subcase 25: Let $r = \{b_i\}$, $i = 2k-2$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{2i+5}{3}$. Let $r' = \{b_{i'}\}$, $i' = 2k-1$, then the distance of r' with respect to R_e is $d(r'|R_e) = \frac{3i'+1}{2}$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{2i+5}{3} = \frac{3i'+1}{2}$. By solving the equation we get $i = \frac{9i'}{4} - \frac{7}{4}$. Since i and i' are positive integers, this case is impossible.

Subcase 26: Let $r = \{b_i\}$, $i = 2k-2$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{2i+5}{3}$. Let $r' = \{s_{i'}\}$, $i' = k$, then the distance of r' with respect to R_e is $d(r'|R_e) = 3i' - 2$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{2i+5}{3} = 3i' - 2$. By solving the equation we get $i = \frac{9i'}{2} - \frac{11}{2}$. Since i and i' are positive integers, this case contradicts our assumption.

Subcase 27: Let $r = \{b_i\}$, $i = 2k-1$, and then the distance of r with respect R_e is $d(r|R_e) = \frac{3i+1}{2}$. Let $r' = \{s_{i'}\}$, $i' = k$, then the distance of r' with respect to R_e is $d(r'|R_e) = 3i' - 2$. Since we consider $d(r|R_e) = d(r'|R_e)$, therefore, $\frac{3i+1}{2} = 3i' - 2$. By solving the equation we get $i = 2i' - \frac{5}{3}$. Since i and i' are positive integers, this case is not possible.

From above cases we showed that $R_e = \{\alpha''_i; i = 1, 4, 7, \dots, 3k-2\}$ uniquely represents each edge of the HPS_k. Therefore, R_e is the ERS of the HPS_k. Hence, the cardinality of the ERS of the HPS_k is k. Now to prove that the EMD of the HPS_k is k, here, we use the double inequality method. The above generalized distance formulas show that $edim(HPS_k) \leq k$. Now to show that $edim(HPS_k) \geq 2$, let the EMD of the heptagonal snake graph is $k-1$. Then we have the following contrary cases.

Case 1: Let $R_{e_1} = \{\alpha''_i; i = 4, 7, 10, \dots, 3k-2\}$ be a proper subset of vertex set of the HPS_k, and then $|R_{e_1}| = k-1$ but $r(e_1|R_{e_1}) = r(b_1|R_{e_1})$, which contradicts our assumption.

Case 2: Suppose $R = R_e|\alpha''_i$, where $i = 4, 7, 10, \dots, 3k-2$, and then $|R| = k-1$ but $r(e_{\frac{2i-2}{3}}|R) = r(b_{\frac{2i-5}{3}}|R)$, which contradicts our assumption.

Therefore, $R_e = \{\alpha''_i; i = 1, 4, 7, \dots, 3k-2\}$ is the smallest cardinality ERS of HPS_k. Hence, our final result is $edim(HPS_k) = k$. □

Theorem 2.4. *Let HPS_k be the graph of the heptagonal snake graph with $k \geq 2$. Then, $pdim(HPS_k) = 3$.*

Proof. To prove that the PD of the HPS_k is 3, let $R_p = \{R_{p_1}, R_{p_2}, R_{p_3}\}$ be a partition of the vertex set of the HPS_k, where $R_{p_1} = \{m_1\}$, $R_{p_2} = \{m_{k+1}\}$ and $R_{p_3} = V(HPS_k)|\{m_1, m_{k+1}\}$. To show that R_p is the PRS of the HPS_k, for this we will show that each vertex of the HPS_k has a unique distance with respect to $R_p = \{R_{p_1}, R_{p_2}, R_{p_3}\}$. The generalized distance formulas below show the unique representation of each vertex of the HPS_k with respect to R_p.

$$r(\alpha''_i|m_1) = i, \quad \text{for } 1 \le i \le 3k,$$
$$r(m_i|m_1) = 3i-3, \quad \text{for } 1 \le i \le k+1,$$
$$r(n_i|m_1) = \begin{cases} \frac{3i-1}{2}, & \text{for } i = 1,3,5\ldots,2k-1, \\ \frac{3i-2}{2}, & \text{for } i = 2,4,6\ldots,2k, \end{cases}$$
$$r(\alpha''_i|m_{k+1}) = -i+3k+1, \quad \text{for } 1 \le i \le 3k,$$
$$r(m_i|m_{k+1}) = -3i+3k+3, \quad \text{for } 1 \le i \le k+1,$$
$$r(n_i|m_{k+1}) = \begin{cases} \frac{-3i+6k+1}{2}, & \text{for } i = 1,3,5\ldots,2k-1, \\ \frac{-3i+6k+2}{2}, & \text{for } i = 2,4,6\ldots,2k, \end{cases}$$
$$r(m_i|R_{p_3}) = \begin{cases} 1, & \text{for } i = 1,k+1, \\ 0, & \text{otherwise}, \end{cases}$$
$$r(n_i|R_{p_3}) = 0, \quad \text{for each } i,$$
$$r(\alpha''_i|R_{p_3}) = 0, \quad \text{for each } i.$$

From the above generalized distance formulas we showed each vertex of the HPS_k has a unique distance with respect to $R_p = \{R_{p_1}, R_{p_2}, R_{p_3}\}$. Therefore, $R_p = \{R_{p_1}, R_{p_2}, R_{p_3}\}$ is the PRS of the HPS_k. Hence, the cardinality of PRS of the HPS_k is 3, which means $pdim(HPS_k) \le 3$. Now we will show that $pdim(HPS_k) \ge 3$, let the PD of the heptagonal snake graph be 2. Chartrand [19] showed that $pdim(G) = 2$, iff $G = P_n$, which contradicts our assumption. Therefore, $R_p = \{R_{p_1}, R_{p_2}, R_{p_3}\}$ is the smallest cardinality PRS of heptagonal snake graph HPS_k.

Hence, the outcome is $pdim(HPS_k) = 3$. □

Theorem 2.5. *Let HPS_k be the graph of the heptagonal snake graph with $k \ge 3$. Then, $epdim(HPS_k) = k+1$.*

Proof. To prove that the EPD of the HPS_k is $k+1$, let $R_{p_e} = \{R_p = \{R_{p_{e_1}}, R_{p_{e_2}}, R_{p_{e_3}}, \ldots, R_{p_{e_n}}\}, R_{p_{e_{n+1}}}\}$ be a partition of the vertex set $V(HPS_k)$ of the HPS_k, where $R_{p_{e_1}} = \{\alpha''_1\}$, $R_{p_{e_2}} = \{\alpha''_4\}$, $R_{p_{e_3}} = \{\alpha''_7\}\ldots$, $R_{p_{e_n}} = \{\alpha''_{3k-2}\}$ and $R_{p_{e_{n+1}}} = V(HPS_k)|\{\alpha''_1, \alpha''_4, \alpha''_7, \ldots, \alpha''_{3k-2}\}$. To show that R_{p_e} is the $PERS$ of the HPS_k, for this we will show that each edge of the edge set $E(HPS_k)$ of the HPS_k has a unique distance with respect to $R_{p_e} = \{R_p = \{R_{p_{e_1}}, R_{p_{e_2}}, R_{p_{e_3}}, \ldots, R_{p_{e_n}}\}, R_{p_{e_{n+1}}}\}$. The following are the generalized unique distances of each edge of HPS_k with respect to R_{p_e}.

$$r(t_i|R_p) = \begin{cases} (0,3,6,9,\ldots,3k-3), & \text{for } i = 1, \\ (1,2,5,8,\ldots,3k-4), & \text{for } i = 2, \\ (\frac{3i}{2}-1, \frac{3i}{2}-4, \frac{3i}{2}-7, \ldots, 8,5,1, & \text{for } i = 4,6,8,\ldots,2k, \\ \quad 2,5,8,\ldots, \frac{-3i}{2}+3k-4, \frac{-3i}{2}+3k-1), & \\ (\frac{3i-1}{2}, \frac{3i-1}{2}-3, \frac{3i-1}{2}-6, \ldots, 10,7,4, & \text{for } i = 3,5,7,\ldots,2k-1, \\ \quad 0,3,6,\ldots, \frac{-3i+6k-3}{2}), & \end{cases}$$

$$r(e_i|R_p) = \begin{cases} (\frac{3i}{2}, \frac{3i}{2}-3, \frac{3i}{2}-6, \ldots, 6,3,0, & \text{for } i = 4,6,8,\ldots,2k, \\ \quad 4,7,10,\ldots, \frac{-3i}{2}+3k-5, \frac{-3i}{2}+3k-2), & \\ (\frac{3i+3}{2}, \frac{3i+3}{2}-3, \frac{3i+3}{2}-6, \ldots, 6,2,1, & \text{for } i = 3,5,7,\ldots,2k-1, \\ \quad 4,7,10,\ldots, \frac{-3i+6k-7}{2}), & \end{cases}$$

$$r(\hat{e}_i|R_p) = (0, 4, 7, 10 \ldots, 3k-2), \quad \text{for } i = 1,$$
$$r(\hat{b}_i|R_p) = (1, 3, 6, 9 \ldots, 3k-3), \quad \text{for } i = 1,$$
$$r(b_i|R_p) = \begin{cases} (3, 1, 4, 7, \ldots, 3k-5), & \text{for } i = 1, \\ (\frac{2i+5}{3}, \frac{2i+5}{3} - 3, \frac{2i+5}{3} - 6, \ldots, 3, 1, & \text{for } i = 2, 4, 6, \ldots, 2k-2, \\ 3, 6, 9, \ldots, \frac{-3i}{2} + 3k-3), & \\ (\frac{3i+1}{2}, \frac{3i+1}{2} - 3, \frac{3i+1}{2} - 6, \ldots, 3, 1, & \text{for } i = 3, 5, 7, \ldots, 2k-1, \\ 4, 7, 10, \ldots, \frac{-3i+6k-7}{2}), & \end{cases}$$
$$r(s_i|R_p) = \begin{cases} (2, 2, 5, 8, \ldots, 3k-4), & \text{for } i = 1, \\ (3i-2,\ 3i-5,\ 3i-8, \ldots, 4, 2, 2, & \text{for } i = 2, 3, 4, \ldots, k, \\ 5, 8, \ldots,\ -3i + 3k-4,\ -3i + 3k-1), & \end{cases}$$
$$r(t_i|R_{p_n+1}) = 0, \quad \text{for each } i,$$
$$r(e_i|R_{p_n+1}) = 0, \quad \text{for each } i,$$
$$r(\hat{e}_i|R_{p_n+1}) = 0, \quad \text{for } i = 1,$$
$$r(\hat{b}_i|R_{p_n+1}) = 0, \quad \text{for } i = 1,$$
$$r(b_i|R_{p_n+1}) = 0, \quad \text{for each } i,$$
$$r(s_i|R_{p_n+1}) = 0, \quad \text{for each } i.$$

From the above generalized distance formulas, we showed that each edge of the HPS_k has a unique distance with respect to $R_{p_e} = \{R_p = \{R_{p_{e_1}}, R_{p_{e_2}}, R_{p_{e_3}}, \ldots, R_{p_{e_n}}\}, R_{p_{e_n+1}}\}$. Therefore, $R_{p_e} = \{R_p = \{R_{p_{e_1}}, R_{p_{e_2}}, R_{p_{e_3}}, \ldots, R_{p_{e_n}}\}, R_{p_{e_n+1}}\}$ is the $EPRS$ of the HPS_k. Hence, the cardinality of the $EPRS$ of the HPS_k is $k+1$. Now to prove that the EPD of the HPS_k is $k+1$, in Theorem 2.3, we proved that the EMD of the HPS_k is k. Therefore, $R_{p_e} = \{R_p = \{R_{p_{e_1}}, R_{p_{e_2}}, R_{p_{e_3}}, \ldots, R_{p_{e_n}}\}, R_{p_{e_n+1}}\}$ is the smallest cardinality $EPRS$ of the HPS_k. Hence, our final result is $epdim(HPS_k) = k+1$. □

3. Conclusions

In this article, we analyzed the structural properties of the HPS_k by computing its MD, EMD, PD and EPD. Our findings reveal that the MD of the HPS_k is 2, indicating that two vertices are enough to uniquely determine the location of each vertex of the HPS_k. The EMD was determined to be k, reflecting k number of vertices required for edge-based unique identification for an HPS_k. Further, the PD was found to be 3, highlighting the minimal number of vertex partitions essential to distinguish all vertices uniquely of the HPS_k. Lastly, we computed that the EPD is $k+1$, demonstrating the partitioning complexity for edge identification. These results provide precious insights into resolving properties of the HPS_k, which have potential applications in nanotechnology, network topology and chemistry. In the future, we can explore other parameters of resolvability for the HPS_k, such as mixed metric dimension, strong metric dimension, fault-tolerant metric dimension, fractional metric dimension and multiset dimension. Investigating these additional parameters will further enhance our expertise in the structural complexity and applications of the HPS_k in numerous scientific domains.

Author contributions

F. B. Farooq, F. Ahmad and M. K. Jamil: Conceptualization, Formal analysis, Methodology, Writing–review and editing; A. Javed and N. A. Alqahtani: Formal analysis, Investigation, Writing–original draft, Writing–review and editing. All authors have read and agree to publish the paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2503).

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. D. B. West, *Introduction to graph theory*, Upper Saddle River: Prentice Hall, 2001.
2. A. Prathik, K. Uma, J. Anuradha, An overview of application of graph theory, *Int. J. ChemTech Res.*, **9** (2016), 242–248.
3. F. Harary, R. Z. Norman, Graph theory as a mathematical model in social science, 1953.
4. G. A. Pavlopoulos, M. Secrier, C. N. Moschopoulos, T. G. Soldatos, S. Kossida, J. Aerts, et al., Using graph theory to analyze biological networks, *BioData Min.*, **4** (2011), 1–27. https://doi.org/10.1186/1756-0381-4-10
5. R. P. Singh, Application of graph theory in computer science and engineering, *Int. J. Comput. Appl.*, **104** (2014), 1.
6. D. H. Cai, P. Z. Fan, Q. Y. Zou, Y. Q. Xu, Z. G. Ding, Z. Q. Liu, Active device detection and performance analysis of massive non-orthogonal transmissions in cellular Internet of Things, *Sci. China Inform. Sci.*, **65** (2022), 182301. https://doi.org/10.1007/s11432-021-3328-y
7. S. H. Zheng, C. Shen, X. Chen, Design and analysis of uplink and downlink communications for federated learning, *IEEE J. Sel. Areas Commun.*, **39** (2021), 2150–2167. https://doi.org/10.1109/JSAC.2020.3041388
8. Y. H. Guo, R. Zhao, S. W. Lai, L. S. Fan, X. F. Lei, G. K. Karagiannidis, Distributed machine learning for multiuser mobile edge computing systems, *IEEE J. Sel. Top. Signal Process.*, **16** (2022), 460–473. https://doi.org/10.1109/JSTSP.2022.3140660
9. C. Y. Tan, D. H. Cai, F. Fang, Z. G. Ding, P. Z. Fan, Federated unfolding learning for CSI feedback in distributed edge networks, *IEEE Trans. Commun.*, **73** (2025), 410–424. https://doi.org/10.1109/TCOMM.2024.3429170

10. P. J. Slater, Leaves of trees, In: *Proceeding of the 6th Southeastern Conference on Combinatorics, Graph Theory and Computing*, **14** (1975), 549–559.
11. P. J. Slater, Dominating and reference sets in a graph, *J. Math. Phys. Sci.*, **22** (1988), 445–455.
12. F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combin.*, **2** (1976), 191–195.
13. Q. K. Meng, M. M. Polycarpo, Sensor selection for high-dimensional swarm systems based on observability analysis, In: *2023 31st Mediterranean Conference on Control and Automation (MED)*, 2023, 967–973. https://doi.org/10.1109/MED59994.2023.10185854
14. A. Kelenc, N. Tratnik, I. G. Yero, Uniquely identifying the edges of a graph: the edge metric dimension, *Discrete Appl. Math.*, **251** (2018), 204–220. https://doi.org/10.1016/j.dam.2018.05.052
15. D. J. Klein, E. Yi, A comparison on metric dimension of graphs, line graphs and line graphs of the subdivision graphs, *Eur. J. Pure Appl. Math.*, **5** (2012), 302–316.
16. A. Ahmad, A. N. A. Koam, M. H. F. Siddiqui, M. Azeem, Resolvability of the starphene structure and applications in electronics, *Ain Shams Eng. J.*, **13** (2022), 101587. https://doi.org/10.1016/j.asej.2021.09.014
17. I. Tomescu, I. Javaid, On the metric dimension of the Jahangir graph, *Bull. Math. Soc. Sci. Math. Roumanie*, **50** (2007), 371–376.
18. S. K. Sharma, V. K. Bhat, Metric dimension of heptagonal circular ladder, *Discrete Math. Algorithms Appl.*, **13** (2021), 2050095. https://doi.org/10.1142/S1793830920500950
19. G. Chartrand, E. Salehi, P. Zhang, The partition dimension of a graph, *Aequationes Math.*, **59** (2000), 45–54. https://doi.org/10.1007/PL00000127
20. G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, **105** (2000), 99–113. https://doi.org/10.1016/S0166-218X(00)00198-0
21. M. Hauptmann, R. Schmied, C. Viehmann, Approximation complexity of metric dimension problem, *J. Discrete Algorithms*, **14** (2012), 214–222. https://doi.org/10.1016/j.jda.2011.12.010
22. P. Manuel, B. Rajan, I. Rajasingh, M. C. Monica, On minimum metric dimension of honeycomb networks, *J. Discrete Algorithms*, **6** (2008), 20–27. https://doi.org/10.1016/j.jda.2006.09.002
23. S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, *Discrete Appl. Math.*, **70** (1996), 217–229. https://doi.org/10.1016/0166-218X(95)00106-2
24. A. Sebő, E. Tannier, On metric generators of graphs, *Math. Oper. Res.*, **29** (2004), 383–393. https://doi.org/10.1287/moor.1030.0070
25. S. Söderberg, H. S. Shapiro, A combinatory detection problem, *Amer. Math. Monthly*, **70** (1963), 1066–1070. https://doi.org/10.1080/00029890.1963.11992174
26. M. Perc, J. Gómez-Gardenes, A. Szolnoki, L. M. Floría, Y. Moreno, Evolutionary dynamics of group interactions on structured populations: a review, *J. R. Soc. Interface*, **10** (2013), 20120997. https://doi.org/10.1098/rsif.2012.0997
27. M. Perc, A. Szolnoki, Coevolutionary games–a mini review, *BioSystems*, **99** (2010), 109–125. https://doi.org/10.1016/j.biosystems.2009.10.003

28. J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, et al., On the metric dimension of cartesian products of graphs, *SIAM J. Discrete Math.*, **21** (2007), 423–441. https://doi.org/10.1137/050641867
29. Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihal'ak, Network discovery and verification, *IEEE J. Sel. Areas Commun.*, **24** (2006), 2168–2181. https://doi.org/10.1109/JSAC.2006.884015
30. V. Chvátal, Mastermind, *Combinatorica*, **3** (1983), 325–329. https://doi.org/10.1007/BF02579188
31. M. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, *J. Biopharm. Statist.*, **3** (1993), 203–236. https://doi.org/10.1080/10543409308835060
32. M. A. Johnson, Browsable structure-activity datasets, *Adv. Mol. Similarity*, **2** (1998), 153–170.
33. R. A. Melter, I. Tomescu, Metric bases in digital geometry, *Comput. Vis. Graph. Image Process.*, **25** (1984), 113–121. https://doi.org/10.1016/0734-189X(84)90051-3
34. A. N. A. Koam, S. Ali, A. Ahmad, M. Azeem, M. K. Jamil, Resolving set and exchange property in nanotube, *AIMS Math.*, **8** (2023), 20305–20323. https://doi.org/10.3934/math.20231035
35. S. Imran, M. K. Siddique, M. Hussain, Computing the upper bounds for the metric dimension of cellulose network, *Appl. Math.*, **19** (2019), 585–605.
36. A. Ahmad, M. Bača, S. Sultan, Computing the metric dimension of kayak paddle graph and cycles with chord, *Proyecciones*, **39** (2020), 287–300. http://dx.doi.org/10.22199/issn.0717-6279-2020-02-0018
37. M. Imran, H. M. A. Siddiqui, Computing the metric dimension of convex polytopes generated by wheel related graphs, *Acta Math. Hungar.*, **149** (2016), 10–30. https://doi.org/10.1007/s10474-016-0606-1
38. A. F. Beardon, Resolving the hypercube, *Discrete Appl. Math.*, **161** (2013), 1882–1887. https://doi.org/10.1016/j.dam.2013.02.012
39. X. J. Zhang, M. Naeem, Metric dimension of crystal cubic carbon structure, *J. Math.*, **2021** (2021), 3438611. https://doi.org/10.1155/2021/3438611
40. Z. Hussain, M. Munir, A. Ahmad, M. Chaudhary, J. A. Khan, I. Ahmed, Metric basis and metric dimension of 1-pentagonal carbon nanocone networks, *Sci. Rep.*, **10** (2020), 19687. https://doi.org/10.1038/s41598-020-76516-1
41. M. K. Siddiqui, M. Imran, Computing the metric and partition dimension of H-Naphtalenic and VC5C7 nanotubes, *J. Optoelectron. Adv. Mater.*, **17** (2015), 790–794.
42. M. Azeem, M. F. Nadeem, Metric-based resolvability of polycyclic aromatic hydrocarbons, *Eur. Phys. J. Plus*, **136** (2021), 1–14. https://doi.org/10.1140/epjp/s13360-021-01399-8
43. M. Ahsan, Z. Zahid, S. Zafar, A. Rafiq, M. S. Sindhu, M. Umar, Computing the edge metric dimension of convex polytopes related graphs, *J. Math. Comput. Sci.*, **22** (2021), 174–188. https://doi.org/10.22436/jmcs.022.02.08
44. M. S. Bataineh, N. Siddiqui, Z. Raza, Edge metric dimension of k multiwheel graph, *Rocky Mountain J. Math.*, **50** (2020), 1175–1180. https://doi.org/10.1216/rmj.2020.50.1175
45. R. Adawiyah, Dafik, R. Alfarisi, R. M. Prihandini, I. H. Agustin, Edge metric dimension on some families of tree, *J. Phys. Conf. Ser.*, **1180** (2019), 012005. https://doi.org/10.1088/1742-6596/1180/1/012005

46. V. Filipović, A. Kartelj, J. Kratica, Edge metric dimension of some generalized Petersen graphs, *Results Math.*, **74** (2019), 1–15. https://doi.org/10.1007/s00025-019-1105-9
47. P. Singh, S. Sharma, S. K. Sharma, V. K. Bhat, Metric dimension and edge metric dimension of windmill graphs, *AIMS Math.*, **6** (2021), 9138–9153. https://doi.org/10.3934/math.2021531
48. B. Deng, M. F. Nadeem, M. Azeem, On the edge metric dimension of different families of Möbius networks, *Math. Probl. Eng.*, **2021** (2021), 6623208. https://doi.org/10.1155/2021/6623208
49. S. Bukhari, M. K. Jamil, M. Azeem, S. Swaray, Patched network and its vertex-edge metric-based dimension, *IEEE Access*, **11** (2023), 4478–4485. https://doi.org/10.1109/ACCESS.2023.3235398
50. D. Kuziak, E. Maritz, T. Vetrìk, I. G. Yero, The edge partition dimension of graphs, *Discrete Math. Lett.*, **12** (2023), 34–39. https://doi.org/10.47443/dml.2023.010
51. F. Okamoto, B. Phinezy, P. Zhang, The local metric dimension of a graph, *Math. Bohem.*, **135** (2010), 239–255. https://doi.org/10.21136/MB.2010.140702
52. R. Adawiyah, Dafik, R. Alfarisi, R. M. Prihandini, I. H. Agustin, M. Venkatachalam, The local edge metric dimension of graph, *J. Phys. Conf. Ser.*, **1543** (2020), 012009. https://doi.org/10.1088/1742-6596/1543/1/012009
53. L. Susilowati, I. Sa'adah, R. Z. Fauziyyah, A. Erfanian, Slamin, The dominant metric dimension of graphs, *Heliyon*, **6** (2020), e03633. https://doi.org/10.1016/j.heliyon.2020.e03633
54. H. M. Ikhlaq, S. Hayat, H. M. A. Siddiqui, Unique identification and domination of edges in a graph: the vertex-edge dominant edge metric dimension, 2022, arXiv: 2211.09327.
55. R. Alfarisi, S. K. S. Husain, L. Susilowati, A. I. Kristiana, Dominant mixed metric dimension of graph, *Stat. Optim. Inform. Comput.*, **12** (2024), 1826–1833. https://doi.org/10.19139/soic-2310-5070-1925
56. R. Simanjuntak, P. Siagian, T. Vetrik, The multiset dimension of graphs, 2017, arXiv: 1711.00225.
57. R. Alfarisi, Y. Q. Lin, J. Ryan, D. Dafik, I. H. Agustin, A note on multiset dimension and local multiset dimension of graphs, *Stat. Optim. Inform. Comput.*, **8** (2020), 890–901. https://doi.org/10.19139/soic-2310-5070-727
58. S. Arumugam, V. Mathew, The fractional metric dimension of graphs, *Discrete Math.*, **312** (2012), 1584–1590. https://doi.org/10.1016/j.disc.2011.05.039
59. H. Benish, M. Murtaza, I. Javaid, The fractional local metric dimension of graphs, 2018, arXiv: 1810.02882.
60. O. R. Oellermann, J. Peters-Fransen, The strong metric dimension of graphs and digraphs, *Discrete Appl. Math.*, **155** (2007), 356–364. https://doi.org/10.1016/j.dam.2006.06.009
61. C. X. Kang, E. Yi, The fractional strong metric dimension of graphs, In: *Combinatorial Optimization and Applications*, Cham: Springer, 2013, 84–95. https://doi.org/10.1007/978-3-319-03780-6_8

AIMS Press