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Research article

Abstract Volterra integro-differential inclusions with multiple variables

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Abstract: In this paper, we investigated abstract Volterra integro-differential inclusions with multiple variables and abstract partial fractional differential inclusions with multiple variables. We also introduced and analyzed several new classes of multidimensional (F, G, C)-resolvent operator families in sequentially complete locally convex spaces and provided certain applications.

Keywords: abstract Volterra integro-differential inclusions; abstract functional Volterra integro-differential inclusions; abstract functional partial fractional differential inclusions; multidimensional vector-valued Laplace transform; multidimensional (F, G, C)-resolvent operator families
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1. Introduction and preliminaries

The class of semigroups over topological monoids provides a proper generalization of the class of one-parameter strongly continuous semigroups of bounded linear operators. The class of strongly continuous semigroups defined on the set $[0, +\infty)^n$, which are usually called multiparameter semigroups (this class of semigroups was introduced by E. Hille in 1944; see [1, 2] for more details in this direction), is a special subclass of the general class of strongly continuous semigroups defined on topological monoids. The precise definition goes as follows: If $(X, \| \cdot \|)$ is a Banach space and $(M, +)$ is a topological monoid with the neutral element 0, then by a semigroup defined over a monoid M, we mean any operator-valued function $T : M \to L(X)$, where $L(X)$ denotes the Banach space of all bounded linear operators on X, such that $T(0) = I$ and $T(t + s) = T(t)T(s)$ for all $t,\ s \in M$. A semigroup $(T(t))_{t\in M}$ is called strongly continuous if the mapping $t \mapsto T(t)x$, $t \in M$, is strongly continuous at $t = 0$.

If $(e_1, ..., e_n)$ is the standard basis of $\mathbb{R}^n$, $M = [0, +\infty)^n$ and $(T(t))_{t\in M}$ is strongly continuous, let $T_i(s) := T(se_i)$, $s \geq 0$, be the corresponding one-parameter strongly continuous semigroup with the integral generator A_i ($i \in \mathbb{N}_n$). Then we have

$$T(t_1, ..., t_n) = T_1(t_1) \cdot ... \cdot T_n(t_n), \quad (t_1, ..., t_n) \in [0, +\infty)^n,$$

and it is said that the tuple $(A_1, A_2, \cdot\cdot\cdot, A_n)$ is the infinitesimal generator of $(T(t))_{t\in[0,+\infty)^n}$. If $I := [0, T_1] \times [0, T_2] \times \cdot\cdot\cdot \times [0, T_n]$ for some $(T_1, T_2, \cdot\cdot\cdot, T_n) \in (0, +\infty)^n$, then the well-posedness of the following n-parameter abstract Cauchy problem

$$(ACP) : \begin{cases} u \in C(I : X) \cap C^1(I^\circ : X), \\ u_{t_i}(t) = A_i u(t) + F_i(t), \ t \in I^\circ, \ 1 \leq i \leq n, \\ u(0) = x, \quad x \in \bigcap_{i\in\mathbb{N}_n} D(A_i), \end{cases}$$

has been analyzed by many authors. Concretely, if $n = 2$, $F_1 = F_2 = F$ and

$$F_{t_1}(t_1, t_2) - F_{t_2}(t_1, t_2) = (A_1 - A_2)F(t_1, t_2), \quad t_1, \ t_2 > 0,$$

then the following formula for a solution of (ACP) has been proposed by Khanehgir et al. [3]:

$$u(t_1, t_2) = T(t_1, t_2)x + \int_0^{t_1} T(t_1 - t, t_2)F(t, 0)\, dt + \int_0^{t_2} T(0, t - t_2)F(t_1, 0)\, dt,$$

for any $t_1, \ t_2 > 0$.

The analysis of abstract Volterra integro-differential inclusions is an important part of study within the functional analysis, which finds many applications in physics, engineering, mechanics, thermo-viscoelasticity, and mathematical biology. To clarify the existence and uniqueness of solutions to abstract Volterra integro-differential inclusions, we use the vector-valued Laplace transform, the vector-valued Fourier transform, some results from fixed point theory, as well as some numerical techniques and approximations. It would be very difficult to summarize here all applications of abstract fractional integro-differential inclusions, which are usually solved by converting them into the equivalent abstract Volterra integro-differential inclusions (see, e.g., the research article [4], where the authors have recently studied the compact finite difference scheme for solving the fractional Black-Scholes option pricing model, and the research article [5], where the authors have recently studied a fourth-order time-fractional sub-diffusion model). For more details about these subjects, we refer the reader to the research monograph [6] and the list of references quoted therein.

The class of n-parameter cosine operator functions and the class of n-parameter fractional solution operator families have not been defined and well explored in the existing literature. Moreover, the abstract Volterra integro-differential inclusions with multiple variables and the abstract partial fractional differential inclusions with multiple variables have not received so much attention up to now. These facts have strongly influenced us to write this paper (we really hope that this paper will inflence many other mathematicians working in the field of applied functional analysis to find some applications of theoretical findings established here to the real-world phenomena). The structure and main ideas of this research article can be described as follows:

After fixing the notation and terminology used throughout the paper, we recall the basic definitions and results about the multidimensional vector-valued Laplace transform in Subsection 1.1. In Section 2, we introduce and analyze several new classes of multidimensional (F, G, C)-resolvent operator families; cf. [7] for the first steps in this direction. Suppose that $m \in \mathbb{N}$, $\omega_l \in \mathbb{R}$ for

$1 \leq l \leq n$, $\Omega = \{\lambda_1 \in \mathbb{C} \,|\, \Re\lambda_1 > \omega_1\} \times ... \times \{\lambda_n \in \mathbb{C} \,|\, \Re\lambda_n > \omega_n\}$, D is a discrete subset of Ω, $F : \Omega \setminus D \to \mathbb{C}$, $F_l : \Omega \setminus D \to \mathbb{C}$, $G_l : \Omega \setminus D \to \mathbb{C}$ for $1 \leq l \leq m$, $P : [L(X)]^m \to L(X)$, and $P_1 : [\mathrm{MLO}(X)]^m \to \mathrm{MLO}(X)$, where $\mathrm{MLO}(X)$ stands for the set of all multivalued linear operators on X. In Definition 2.1, we introduce the classes of mild $(F, G_l, C_1, \mathcal{A}_l, P_1, D)_{1\leq l\leq m}$-regularized existence families, mild (F, G, C_2, D)-regularized uniqueness families, and $(F_l, G_l, C_l, \mathcal{A}_l, P, D)_{1\leq l\leq m}$-regularized resolvent families. The class of holomorphic multidimensional (F, G, C)-regularized resolvent families is introduced in Definition 2.3 and after that analyzed in Theorem 2.4.

In Subsection 2.1, we consider multidimensional (F, G, C)-resolvent operator families of the form $R(t) = R_1(t_1) \cdot ... \cdot R_n(t_n)$, where $t = (t_1, ..., t_n) \in [0, +\infty)^n$ and $(R_j(t_j))_{t_j\geq 0}$ is a strongly continuous operator family; cf. Propositions 2.5 and 2.6 for some results obtained in this direction. If $(R_j(t_j))_{t_j\geq 0}$ is not a fractionally integrated C-semigroup (cf. [6] for the notion), this investigation seems to be completely new.

In Subsection 2.2, we provide some applications of multidimensional (F, G, C)-resolvent operator families to the abstract Volterra integro-differential inclusions with multiple variables and the abstract partial fractional integro-differential inclusions with multiple variables. Here, we use the multidimensional Caputo fractional derivaties, only. For example, if $n = 2$, $\alpha_j > 0$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, and $\mathcal{A}_j$ is a closed subgenerator of an exponentially equicontinuous (g_{α_j}, C_j)-regularized resolvent family $(R_j(t))_{t\geq 0}$, $j = 1, 2$, then we prove the existence of solutions to the following abstract partial fractional differential inclusion

$$\mathbf{D}_{t_1}^{\alpha_1}\mathbf{D}_{t_2}^{\alpha_2}u(t_1, t_2) \in \mathcal{A}_1\mathcal{A}_2 u(t_1, t_2), \quad t_1 \geq 0,\ t_2 \geq 0, \tag{1.1}$$

and the abstract partial fractional differential inclusion

$$\alpha\mathbf{D}_{t_1}^{\alpha_1}u(t_1, t_2) + \beta\mathbf{D}_{t_2}^{\alpha_2}u(t_1, t_2) \in \Big[\alpha\mathcal{A}_1 + \beta\mathcal{A}_2\Big]u(t_1, t_2), \quad t_1 \geq 0,\ t_2 \geq 0, \tag{1.2}$$

under certain reasonable assumptions, which are practically very simply verified. Here, $\mathbf{D}_{t_i}^{\alpha_i}u$ and $\mathbf{D}_{t_1}^{\alpha_1}\mathbf{D}_{t_2}^{\alpha_2}u$ denote the Caputo fractional derivative of order α_i with respect to the variable t_i, $i = 1, 2$ and the mixed Caputo fractional derivative of order (α_1, α_2) with respect to the variables t_1 and t_2. In contrast with all existing research studies of the abstract n-parameter abstract Cauchy problem (ACP), we do not consider here the tuple (A_1, A_2) as the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^2}$, where $\alpha_1 = \alpha_2 = 1$, but the product A_1A_2 of the associated strongly coninuous semigroups $(T_i(s))_{s\geq 0}$, $i = 1, 2$. In our approach, we use the degenerate solution operator families subgenerated by multivalued linear operators $\mathcal{A}_1$ and $\mathcal{A}_2$, as well (for further information about the initial conditions subjected to the problems (1.1) and (1.2), we refer the reader to [8]). We will consider the fractional analogues of the n-parameter abstract Cauchy problem (ACP) somewhere else.

Further on, if $a_1 \in L^1_{loc}([0, \tau_1))$, $a_2 \in L^1_{loc}([0, \tau_2))$, $x \in X$, $a(t_1, t_2) = a_1(t_1) \cdot a_2(t_2)$, $t \in [0, +\infty)^2$, and $(R_j(s))_{s\in[0,\tau_j)}$ is a (local) (a_j, k_j)-regularized C_j-resolvent family with a closed subgenerator $\mathcal{A}_j$, $j = 1, 2$, then the function $u(t_1, t_2) := R_1(t_1)R_2(t_2)x$, $0 \leq t_1 < \tau_1$, $0 \leq t_2 < \tau_2$, is a solution of the following abstract Volterra integral inclusion with multiple variables:

$$\begin{aligned} &u(t_1, t_2) - k_1(t_1) \cdot [k_1(0)]^{-1}u(0, t_2) - k_2(t_2) \cdot [k_2(0)]^{-1}u(t_1, 0) + k_1(t_1)k_2(t_2)C_2C_1x \\ &\in \mathcal{A}_2\mathcal{A}_1 \int_0^{t_1}\int_0^{t_2} a(t_1 - s_1, t_2 - s_2)u(s_1, s_2)\, ds_1\, ds_2, \quad 0 \leq t_1 < \tau_1,\ 0 \leq t_2 < \tau_2. \end{aligned}$$

We also perceive that the function $u(t_1,t_2) := c_1R_1(t_1)x + c_2R_2(t_2)y,\ t_1 \in [0,\tau_1),\ t_2 \in [0,\tau_2)$ is a solution of the following abstract Volterra integral inclusion with multiple variables:

$$u(t_1,t_2) - c_1k_1(t_1)C_1x - c_2k_2(t_2)C_2y \in \mathcal{A}_1 \int_0^{t_1} a_1(t_1-s_1)u(s_1,t_2)\,ds_1 + \mathcal{A}_2 \int_0^{t_2} a_2(t_2-s_2)u(t_1,s_2)\,ds_2,$$

for any $0 \leq t_1 < \tau_1$ and $0 \leq t_2 < \tau_2$.

We continue our exposition by considering some classes of the abstract functional Volterra integro-differential inclusions with multiple variables; cf. Section 3 for more details. First of all, we analyze here the following abstract functional Volterra integro-differential inclusion with multiple variables:

$$Bu(t) \in Cf(t) + \sum_{i=1}^{m} \mathcal{A}_i\Big(a_i *_0 u(\cdot + \tau_i)\Big)(t),\ t \in [0,+\infty)^n;\ u(t) = Cu_0(t),\ t \in \Omega_0, \tag{1.3}$$

where $m \in \mathbb{N}$, $t = (t_1,...,t_n) \in [0,+\infty)^n$, $\tau_j = (\tau_j^1,...,\tau_j^n) \in [0,+\infty)^n$ for $1 \leq j \leq m$, $C \in L(X)$ is injective, Ω_0 is the region

$$\Big[[0,+\infty)^n \setminus \Big([\tau_1^1,+\infty) \times ... \times [\tau_1^n,+\infty)\Big)\Big] \bigcup ... \bigcup \Big[[0,+\infty)^n \setminus \Big([\tau_n^1,+\infty) \times ... \times [\tau_n^n,+\infty)\Big)\Big],$$

$f : [0,+\infty)^n \to X$ is a Laplace transformable function, $\mathcal{A}_i$ is a closed MLO on X, and $a_i(\cdot)$ is a locally integrable scalar-valued function defined for $t \in [0,+\infty)^n$ ($1 \leq i \leq m$). It can be simply shown that, if $h_j = \max\{\tau_i^j : 1 \leq i \leq m\}$ for $1 \leq j \leq n$, then

$$\Omega_0 = [0,+\infty)^n \setminus \Big([h_1,+\infty) \times ... \times [h_n,+\infty)\Big). \tag{1.4}$$

The case in which $\tau_1 = ... = \tau_m = 0$ is not excluded from our analysis; then we consider the problem

$$Bu(t) \in Cf(t) + \sum_{i=1}^{m} \mathcal{A}_i(a_i *_0 u)(t),\ t \in [0,+\infty)^n$$

without any initial condition, since $\Omega_0 = \emptyset$. If $n = 1$ and $\mathcal{A}_i \equiv \mathcal{A}$ for all $1 \leq i \leq m$, then the problem (1.3) has recently been considered in [7]. Unfortunately, there exist many real situations in which we cannot apply the multidimensional vector-valued Laplace transform in the deeper study of problem (1.3); cf. [7, Example 3(i)] and Example 3.14(i) below.

The main structural results of Section 3, which also contains a great number of useful remarks, are Theorems 3.2, 3.5, 3.9, and 3.13. In Definition 3.10, we introduce the notion of an exponentially equicontinuous, k-convoluted C-solution operator family for (1.3) and (1.4); cf. also Theorem 3.11. Several illustrative applications of the established results are presented in Examples 3.12 and 3.14. In Subsection 3.1, we briefly examine the possibility of application of the double vector-valued Laplace transform in the study of the following abstract functional partial fractional differential inclusions:

$$\mathbf{D}_{t_1}^{\alpha_1}\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2) \in \mathcal{A}u(t_1+a,t_2+b) + f(t_1,t_2), \quad t_1 \geq 0,\ t_2 \geq 0,$$

and

$$\mathbf{D}_{t_1}^{\alpha_1}\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2) \in \mathcal{A}(Pu)(t_1,t_2) + Cf(t_1,t_2), \quad t_1 \geq 0,\ t_2 \geq 0,$$

where $a \geq 0$, $b \geq 0$, $ab \neq 0$, $Pu(t_1,t_2) := u(t_1 - a, t_2 - b)$, $t_1 \geq a$, $t_2 \geq b$, and $Pu(t_1,t_2) := 0$, otherwise. In the final section of the paper, we provide some observations and final remarks about the introduced notion.

Notation and preliminaries. If Y is a Hausdorff sequentially complete locally convex space over the field of complex numbers, then we simply write that Y is an SCLCS. If X is likewise an SCLCS, then the abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of X; cf. [9] for more details about topological vector spaces and locally convex spaces. For further information concerning multivalued linear operators in sequentially complete locally convex spaces (MLOs) and solution operator families subgenerated by them, we refer the reader to [6].

The Gamma function will be denoted by $\Gamma(\cdot)$ and the principal branch will be always used to take powers. Define $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ and $0^\zeta := 0$ ($\zeta > 0$, $t > 0$). If $0 < \alpha \leq \pi$, then we define $\Sigma_\alpha := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \alpha\}$. Given the numbers $s \in \mathbb{R}$ and $m \in \mathbb{N}$, we set $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$ and $\mathbb{N}_m := \{1,...,m\}$. Unless stated otherwise, we will always assume henceforth that X is an SCLCS and $n \in \mathbb{N}$; by I we denote the identity operator on X. By $\Re z$ we denote the real part of a complex number z.

If $a \in L^1_{loc}([0,\infty)^n)$ and $u \in L^1_{loc}([0,\infty)^n : X)$, then we define

$$(a *_0 u)(t) := \int_0^{t_1} \cdot ... \cdot \int_0^{t_n} a(t_1 - s_1, ..., t_n - s_n)u(s_1,...,s_n)\, ds_1 ... ds_n,$$

for any $t = (t_1,...,t_n) \in [0,\infty)^n$. The convolution product $*_0$ introduced in the last formula is sometimes called the convolution product of Faltung. In general SCLCSs, the convolution product $*_0$ is well-defined if $a \in C([0,\infty)^n)$ and $f \in L^1_{loc}([0,\infty)^n : X)$, or $a \in L^1_{loc}([0,\infty)^n)$ and $f \in C([0,\infty)^n : X)$, when we have $a *_0 f \in C([0,\infty)^n : X)$. In Fréchet spaces, the convolution product $*_0$ is well-defined if $a \in L^1_{loc}([0,\infty)^n)$ and $f \in L^1_{loc}([0,\infty)^n : X)$, when we have $a *_0 f \in L^1_{loc}([0,\infty)^n : X)$.

Suppose, finally, that $\alpha > 0$, $m = \lceil \alpha \rceil$, and $I = [0,T)$, $I = [0,T]$, or $I = [0,+\infty)$ for some real number $T > 0$. Then the Caputo fractional derivative $\mathbf{D}_t^\alpha u(\cdot)$ is defined for those functions $u \in C^{m-1}(I : X)$ for which $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} u_k g_{k+1}) \in C^m(I : X)$, by

$$\mathbf{D}_t^\alpha u(\cdot) := \frac{d^m}{dt^m}\Big[g_{m-\alpha} * \Big(u - \sum_{k=0}^{m-1} u_k g_{k+1}\Big)\Big].$$

If $\omega \geq 0$ and for each seminorm $p \in \circledast$ there exists a real number $M_p > 0$ such that $p(u(t)) + p(\mathbf{D}_t^\alpha u(t)) \leq M_p \exp(\omega t)$, $t \geq 0$, and then the Laplace transform of function $\mathbf{D}_t^\alpha u(\cdot)$ can be computed by the formula

$$\int_0^\infty e^{-\lambda t}\mathbf{D}_t^\alpha u(t)\, dt = \lambda^\alpha \hat{u}(\lambda) - \sum_{k=0}^{m-1} u^{(k)}(0)\lambda^{\alpha-1-k}, \quad \Re\lambda > \omega. \tag{1.5}$$

If $0 < T_j < +\infty$ and $I_j = [0,T_j)$, $I_j = [0,T_j]$ or $I_j = [0,+\infty)$ for $1 \leq j \leq n$, set $I := I_1 \times I_2 \times ... \times I_n$. Assuming that $u : I \to X$ is a locally integrable function, $\alpha_j \geq 0$ for all $j \in \mathbb{N}_n$ and $\alpha := (\alpha_1,...,\alpha_n)$, and we define

$$\mathbf{D}_C^\alpha u(t_1,...,t_n) := \Big[\mathbf{D}_{t_1}^{\alpha_1}\Big(\mathbf{D}_{t_2}^{\alpha_2}...\Big(\mathbf{D}_{t_n}^{\alpha_n}u(\cdot,...,\cdot)\Big)...\Big)\Big](t_1,...,t_n), \tag{1.6}$$

for a.e. $(t_1, ..., t_n) \in I$, provided that the right-hand side of (1.6) is well-defined. We call $(\mathbf{D}_C^{\alpha} u)(\cdot)$ the multi-dimensional Caputo fractional derivative of $u(\cdot)$.

Suppose that X is a Fréchet space, $I = I_1 \times I_2$ is a rectangle in $\mathbb{R}^2$, and $F : I \to X$ is a Lebesgue measurable function. If

$$\int_I p(f(t,s))\,dt\,ds < +\infty, \quad p \in \circledast,$$

then the Fubini theorem says that the repeated integrals

$$\int_{I_2}\int_{I_1} f(t,s)\,dt\,ds \quad \text{and} \quad \int_{I_1}\int_{I_2} f(t,s)\,ds\,dt$$

exist, they are equal, and they coincide with the double integral $\int_I f(t,s)\,dt\,ds$. In general SCLCSs, we can only prove that the existence of integral $\int_I f(t,s)\,dt\,ds$ and the repeated integrals $\int_{I_2}\int_{I_1} f(t,s)\,dt\,ds$ and $\int_{I_1}\int_{I_2} f(t,s)\,ds\,dt$ implies their equality.

1.1. Multidimensional vector-valued Laplace transform

The analysis of the double Laplace transform starts probably with the works of Bernstein [10], Jaeger [11] (1939–1941), and Amerio [12] (1940). We need to recall the following notion from [8]:

Definition 1.1. Suppose that X is an SCLCS, $f : [0,+\infty)^n \to X$ is a locally integrable function, and $(\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$. If

$$\begin{aligned} F(\lambda_1, ..., \lambda_n) &:= \lim_{t_1\to+\infty;...;t_n\to+\infty} \int_0^{t_1} ... \int_0^{t_n} e^{-\lambda_1 s_1 - ... - \lambda_n s_n} f(s_1, ..., s_n)\,ds_1 ... ds_n \\ &:= \int_0^{+\infty} ... \int_0^{+\infty} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} f(t_1, ..., t_n)\,dt_1 ... dt_n \end{aligned}$$

exists for the topology of X, i.e., if for each $\epsilon > 0$ and $p \in \circledast$ there exists a real number $M > 0$ such that the assumptions $t_j > M$ for all $j \in \mathbb{N}_n$ imply

$$p\biggl(\int_0^{t_1} ... \int_0^{t_n} e^{-\lambda_1 s_1 - ... - \lambda_n s_n} f(s_1, ..., s_n)\,ds_1 ... ds_n - F(\lambda_1, ..., \lambda_n)\biggr) < \epsilon,$$

then we say that the Laplace integral $(\mathcal{L}f)(\lambda_1, ..., \lambda_n) := \hat{f}(\lambda_1, ..., \lambda_n) := F(\lambda_1, ..., \lambda_n)$ exists. We define the region of convergence of Laplace integral $\Omega(f)$ by

$$\Omega(f) := \Bigl\{(\lambda_1, ..., \lambda_n) \in \mathbb{C}^n : F(\lambda_1, ..., \lambda_n) \text{ exists}\Bigr\}.$$

Furthermore, if for each seminorm $p \in \circledast$ we have

$$\int_0^{+\infty} ... \int_0^{+\infty} p\Bigl(e^{-\lambda_1 t_1 - ... - \lambda_n t_n} f(t_1, ..., t_n)\Bigr)\,dt_1 ... dt_n < +\infty,$$

then we say that the Laplace integral $F(\lambda_1, ..., \lambda_n)$ converges absolutely. We define the region of absolute convergence of Laplace integral $\Omega_{abs}(f)$ by

$$\Omega_{abs}(f) := \Bigl\{(\lambda_1, ..., \lambda_n) \in \mathbb{C}^n : F(\lambda_1, ..., \lambda_n) \text{ converges absolutely}\Bigr\}.$$

Finally, we define $\Omega_b(f)$ as the set of all tuples $(\lambda_1, ...\lambda_n) \in \mathbb{C}^n$ such that the set

$$\left\{\int_0^{t_1}\int_0^{t_2}...\int_0^{t_n} e^{-\lambda_1 s_1-\lambda_2 s_2-...-\lambda_n s_n} f(s_1, s_2, ..., s_n)\, ds_1\, ds_2... ds_n : t_1 \geq 0, ..., t_n \geq 0\right\}$$

is bounded in X.

The following statements hold:

(i) If $(\lambda_1^0, ..., \lambda_n^0) \in \Omega_{abs}(f)$, $(\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$, and $\Re\lambda_j > \Re\lambda_j^0$ for $1 \leq j \leq n$, then $(\lambda_1, ..., \lambda_n) \in \Omega_{abs}(f)$.
(ii) If $(\lambda_1^0, ..., \lambda_n^0) \in \Omega_b(f)$, $(\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$, and $\Re\lambda_j > \Re\lambda_j^0$ for $1 \leq j \leq n$, then $(\lambda_1, ..., \lambda_n) \in \Omega_b(f) \cap \Omega(f)$.
(iii)

$$\bigcup_{(\lambda_1^0,...,\lambda_n^0)\in\Omega_{abs}(f)} \Big[\{\lambda_1 \in \mathbb{C} : \Re\lambda_1 > \Re\lambda_1^0\} \times ... \times \{\lambda_n \in \mathbb{C} : \Re\lambda_n > \Re\lambda_n^0\}\Big] \subseteq \Omega_{abs}(f).$$

(iv)

$$\bigcup_{(\lambda_1^0,...,\lambda_n^0)\in\Omega_b(f)} \Big[\{\lambda_1 \in \mathbb{C} : \Re\lambda_1 > \Re\lambda_1^0\} \times ... \times \{\lambda_n \in \mathbb{C} : \Re\lambda_n > \Re\lambda_n^0\}\Big] \subseteq \Omega(f) \cap \Omega_b(f).$$

If $f : [0, +\infty)^n \to X$ is locally integrable, $\emptyset \neq \Omega \subseteq \Omega(f) \cap \Omega_b(f)$ is open and $(\lambda_1, ..., \lambda_n) \in \Omega$, then the mapping $F : \Omega \to X$ is holomorphic (for the basic information concerning holomorphic vector-valued functions of several variables, we refer the reader to the research article [13] by Kruse and the list of references quoted in [8, 13]). A locally integrable function $f : [0, +\infty)^n \to X$ is said to be Laplace transformable if there exist real numbers $\omega_i \in \mathbb{R}$ $(1 \leq i \leq n)$ such that $\{\lambda_1 \in \mathbb{C} : \Re\lambda_1 > \omega_1\} \times ... \times \{\lambda_n \in \mathbb{C} : \Re\lambda_n > \omega_n\} \subseteq \Omega(f)$.

We need the following properties of the multidimensional vector-valued Laplace transform (cf. [8] for more details):

Lemma 1.2. *Suppose that $f : [0, +\infty)^n \to X$ is a locally integrable function. Then the following holds:*

(i) Suppose that $\mathcal{A}$ is a closed MLO *between X and Y, $g : [0, +\infty)^n \to Y$ is locally integrable, and $g(t_1, ..., t_n) \in \mathcal{A}f(t_1, ..., t_n)$ for a.e. $(t_1, ..., t_n) \in [0, +\infty)^n$. If $(\lambda_1, ..., \lambda_n) \in \Omega(f) \cap \Omega(g)$, then*

$$(\mathcal{L}g)(\lambda_1, ..., \lambda_n) \in \mathcal{A}(\mathcal{L}f)(\lambda_1, ..., \lambda_n).$$

(ii) Suppose that $f : [0, +\infty)^n \to X$ is a locally integrable function, $(\lambda_1^0, ..., \lambda_n^0) \in \Omega_b(f)$, and there exist real numbers $a_1 > \Re\lambda_1^0, ..., a_n > \Re\lambda_n^0$ such that $(\mathcal{L}f)(\lambda_1, ..., \lambda_n) = 0$ for $\lambda_1 > a_1, ..., \lambda_n > a_n$. Then, for every seminorm $p \in \circledast$, there exists a Lebesgue measurable set $N_p \subseteq [0, +\infty)^n$ such that $m(N_p) = 0$ and $p(f(t_1, ..., t_n)) = 0$ for all $(t_1, ..., t_n) \in [0, +\infty)^n \setminus N_p$. In particular, if X is a Fréchet space, then $f(t_1, ..., t_n) = 0$ for a.e. $(t_1, ..., t_n) \in [0, +\infty)^n$.

(iii) Suppose that $\mathcal{A} : X \to P(Y)$ is an MLO, *$\mathcal{A}$ is $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed, $f_1 : [0, +\infty)^n \to X$ and $f_2 : [0, +\infty)^n \to X$ are locally integrable functions, $(\lambda_1^0, ..., \lambda_n^0) \in \Omega_b(f_1) \cap \Omega_b(f_2)$, and there exist real numbers $a_1 > \Re\lambda_1^0, ..., a_n > \Re\lambda_n^0$ such that $((\mathcal{L}f_1)(\lambda_1, ..., \lambda_n), (\mathcal{L}f_2)(\lambda_1, ..., \lambda_n)) \in \mathcal{A}$ for $\lambda_1 > a_1, ..., \lambda_n > a_n$. Then $(f_1(t_1, ..., t_n), f_2(t_1, ..., t_n)) \in \mathcal{A}$ for any $(t_1, ...t_n) \in [0, +\infty)^n$ which is a point of continuity of both functions $f_1(\cdot)$ and $f_2(\cdot)$.*

(iv) *Suppose that* $\omega_1 \geq 0,\ \ldots,\ \omega_n \geq 0$, $\epsilon_1 > 0,\ \ldots,\ \epsilon_n > 0$, $F : \{\lambda \in \mathbb{C} : \Re\lambda > \omega_1\} \times \ldots \times \{\lambda \in \mathbb{C} : \Re\lambda > \omega_n\} \rightarrow X$ *is an analytic function and for each seminorm* $p \in \circledast$ *there exists a finite real constant* $M_p > 0$ *such that*

$$p\Big(F(\lambda_1, \ldots, \lambda_n)\Big) \leq M_p |\lambda_1|^{-1-\epsilon_1} \cdot \ldots \cdot |\lambda_n|^{-1-\epsilon_n}, \quad \Re\lambda_j > \omega_j \ (1 \leq j \leq n).$$

Then there exists a continuous function $f : [0, +\infty)^n \rightarrow X$ *such that for each seminorm* $p \in \circledast$ *there exists a finite real constant* $M_p' > 0$ *such that*

$$p(f(t_1, \ldots, t_n)) \leq M_p' \Big[t_1^{\epsilon_1} e^{\omega_1 t_1} \cdot \ldots \cdot t_n^{\epsilon_n} e^{\omega_n t_n}\Big] \text{ for all } t_1 \geq 0,\ \ldots,\ t_n \geq 0 \tag{1.7}$$

and $F(\lambda_1, \ldots, \lambda_n) = (\mathcal{L}f)(\lambda_1, \ldots, \lambda_n)$ *converges absolutely for* $\Re\lambda_j > \omega_j$ $(1 \leq j \leq n)$.

(v) *Suppose that* $a \in L^1_{loc}([0, \infty)^n)$, $f \in L^1_{loc}([0, \infty)^n : X)$, *and* $(\lambda_1, \ldots, \lambda_n) \in \Omega_{abs}(a) \cap \Omega_{abs}(f)$.

(a) *Let X be a Fréchet space. Then* $(a *_0 f)(\cdot) \in L^1_{loc}([0, +\infty)^n)$ *and*

$$\Big(\mathcal{L}(a *_0 f)\Big)(\lambda_1, \ldots, \lambda_n) = (\mathcal{L}a)(\lambda_1, \ldots, \lambda_n) \cdot (\mathcal{L}f)(\lambda_1, \ldots, \lambda_n). \tag{1.8}$$

(b) *Suppose, in addition, that* $a \in C([0, \infty)^n)$ *or* $f \in C([0, \infty)^n : X)$. *Then we have* $a *_0 f \in L^1_{loc}([0, \infty)^n : X)$ *and* (1.8).

(vi) *Suppose* $h_j \geq 0$ *for* $1 \leq j \leq n$,

$$f_h(t_1, \ldots, t_n) := f(t_1 + h_1, \ldots, t_n + h_n), \quad t_1 \geq 0, \ldots,\ t_n \geq 0,$$

and $H := [h_1, +\infty) \times \ldots \times [h_n, +\infty)$. *If* $(\lambda_1, \ldots, \lambda_n) \in \Omega_{abs}(f)$, *then* $(\lambda_1, \ldots, \lambda_n) \in \Omega_{abs}(f_h)$ *and*

$$(\mathcal{L}f_h)(\lambda_1, \ldots, \lambda_n) = e^{\lambda_1 h_1 + \ldots + \lambda_n h_n} (\mathcal{L}f)(\lambda_1, \ldots, \lambda_n)$$
$$- e^{\lambda_1 h_1 + \ldots + \lambda_n h_n} \int_{[0,+\infty)^n \setminus H} e^{-\lambda_1 t_1 - \ldots - \lambda_n t_n} f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n.$$

(vii) *Suppose* $h_j \geq 0$ *for* $1 \leq j \leq n$,

$$f_{h-}(t_1, \ldots, t_n) := f(t_1 - h_1, \ldots, t_n - h_n), \quad t_1 \geq h_1, \ldots,\ t_n \geq h_n,$$

and

$$f_{h-}(t_1, \ldots, t_n) := 0 \quad \text{if there exists } j \in \mathbb{N}_n \text{ such that } t_j < h_j.$$

Then $(\lambda_1, \ldots, \lambda_n) \in \Omega_{abs}(f)$, *resp.* $(\lambda_1, \ldots, \lambda_n) \in \Omega(f)$, *if and only if* $(\lambda_1, \ldots, \lambda_n) \in \Omega_{abs}(f_{h-})$, *resp.* $(\lambda_1, \ldots, \lambda_n) \in \Omega(f_{h-})$. *If this is the case, then we have*

$$(\mathcal{L}f_{h-})(\lambda_1, \ldots, \lambda_n) = e^{-\lambda_1 h_1 - \ldots - \lambda_n h_n} (\mathcal{L}f)(\lambda_1, \ldots, \lambda_n).$$

Lemma 1.3. *Suppose that* $\omega_j \in \mathbb{R}$ *and* $\alpha_j \in (0, \pi/2]$ *for all* $j \in \mathbb{N}_n$ *as well as that* $F : (\omega_1, +\infty) \times \ldots \times (\omega_n, +\infty) \rightarrow X$ *is a given function. Then the following assertions are equivalent:*

(i) *There exists a holomorphic function* $f : \Sigma_{\alpha_1} \times \ldots \times \Sigma_{\alpha_n} \rightarrow X$ *such that* $(\mathcal{L}f)(\lambda_1, \ldots, \lambda_n) = F(\lambda_1, \ldots, \lambda_n)$ *for* $\lambda_1 > \omega_1, \ldots,\ \lambda_n > \omega_n$ *and, for every* $\gamma_1 \in (0, \alpha_1), \ldots,\ \gamma_n \in (0, \alpha_n)$, *the set* $\{e^{-\omega_1 z_1 - \ldots - z_n \omega_n} f(z_1, \ldots, z_n) : (z_1, \ldots, z_n) \in \Sigma_{\gamma_1} \times \ldots \times \Sigma_{\gamma_n}\}$ *is bounded in X.*

(ii) There exists a holomorphic function $\tilde{F} : (\omega_1 + \Sigma_{(\pi/2)+\alpha_1}) \times ... \times (\omega_n + \Sigma_{(\pi/2)+\alpha_n}) \rightarrow X$ *such that* $\tilde{F}(\lambda_1, ..., \lambda_n) = F(\lambda_1, ..., \lambda_n)$ *for* $\lambda_1 > \omega_1, ..., \lambda_n > \omega_n$ *and, for every* $\gamma_1 \in (0, \alpha_1), ..., \gamma_n \in (0, \alpha_n)$, *the set* $\{(\lambda_1 - \omega_1) \cdot ... \cdot (\lambda_n - \omega_n)\tilde{F}(\lambda_1, ..., \lambda_n) : (\lambda_1, ..., \lambda_n) \in (\omega_1 + \Sigma_{(\pi/2)+\gamma_1}) \times ... \times (\omega_n + \Sigma_{(\pi/2)+\gamma_n})\}$ *is bounded in X.*

If this is the case, then we have the following:

(a) $\lim_{(t_1,...,t_n)\rightarrow(0+,...,0+)} f(t_1, ..., t_n) = x$ *if and only if* $\lim_{\lambda_1\rightarrow+\infty;...;\lambda_n\rightarrow+\infty}[\lambda_1 \cdot ... \cdot \lambda_n \cdot F(\lambda_1, ..., \lambda_n)] = x$; *in this case, for any angles* $\beta_j \in (0, \alpha_j)$ *(*$1 \leq j \leq n$*), we have* $\lim_{(z_1,...,z_n)\rightarrow(0,...,0);(z_1,...,z_n)\in\Sigma_{\beta_1}\times...\times\Sigma_{\beta_n}} f(z_1, ..., z_n) = x$.

2. Definition and basic properties of multidimensional (F, G, C)-resolvent operator families

In this section, we introduce and analyze various classes of multidimensional (F, G, C)-resolvent operator families in SCLCSs. Unless stated otherwise, we will always assume henceforth that $m \in \mathbb{N}$, $\mathcal{A}_l : X \rightarrow P(X)$ is an MLO for $1 \leq l \leq m$, $C_1 \in L(Y, X)$, and $C_2 \in L(X)$ is injective. If $(\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ and $(t_1, ..., t_n) \in [0, +\infty)^n$, then we simply write $\lambda = (\lambda_1, ..., \lambda_n)$ and $t = (t_1, ..., t_n)$.

The following notion extends the notion introduced in [7, Definition 1]:

Definition 2.1. Suppose that $\omega_l \in \mathbb{R}$ for $1 \leq l \leq n$, $\Omega = \{\lambda_1 \in \mathbb{C} \mid \Re\lambda_1 > \omega_1\} \times ... \times \{\lambda_n \in \mathbb{C} \mid \Re\lambda_n > \omega_n\}$, D is a discrete subset of Ω, $F : \Omega \setminus D \rightarrow \mathbb{C}$ $F_l : \Omega \setminus D \rightarrow \mathbb{C}$, $G_l : \Omega \setminus D \rightarrow \mathbb{C}$ for $1 \leq l \leq m$, $P : [L(X)]^m \rightarrow L(X)$, and $P_1 : [\mathrm{MLO}(X)]^m \rightarrow \mathrm{MLO}(X)$.

(i) A strongly continuous operator family $(R_1(t_1, ..., t_n))_{t_1\geq0;...;t_n\geq0} \subseteq L(Y, X)$ is said to be a mild $(F, G_l, C_1, \mathcal{A}_l, P_1, D)_{1\leq l\leq m}$-regularized existence family if for each $y \in Y$ we have $\Omega \subseteq \Omega(R_1(\cdot)y)$ and

$$F(\lambda)C_1y \in P_1\Big(G_1(\lambda) - \mathcal{A}_1, ..., G_m(\lambda) - \mathcal{A}_m\Big) \times \int_0^{+\infty} ... \int_0^{+\infty} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} R_1(t_1, ..., t_n)y \, dt_1 ... dt_n, \; y \in Y, \; \lambda \in \Omega \setminus D. \tag{2.1}$$

(ii) Let $F : \Omega \setminus D \rightarrow \mathbb{C}$ and $G : \Omega \setminus D \rightarrow \mathbb{C}$. A strongly continuous operator family $(R_2(t_1, ..., t_n))_{t_1\geq0;...;t_n\geq0} \subseteq L(X)$ is said to be a mild (F, G, C_2, D)-regularized uniqueness family with a subgenerator $\mathcal{A}$ if for each $x \in D(\mathcal{A}) \cup R(\mathcal{A})$ we have $\Omega \subseteq \Omega(R_2(\cdot)x) \cap \Omega(R_2(\cdot)y)$ and

$$F(\lambda)C_2y = G(\lambda) \int_0^{+\infty} ... \int_0^{+\infty} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} R_2(t_1, ..., t_n)x \, dt_1 ... dt_n - \int_0^{+\infty} ... \int_0^{+\infty} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} R_2(t_1, ..., t_n)y \, dt_1 ... dt_n \tag{2.2}$$

for any $(x, y) \in \mathcal{A}$ and $\lambda \in \Omega \setminus D$.

(iii) Let $C_l \in L(X)$ be injective and $C_l\mathcal{A}_l \subseteq \mathcal{A}_lC_l$ for $1 \leq l \leq m$. A strongly continuous operator family $(R(t_1, ..., t_n))_{t_1\geq0;...;t_n\geq0} \subseteq L(X)$ is said to be an $(F_l, G_l, C_l, \mathcal{A}_l, P, D)_{1\leq l\leq m}$-regularized resolvent family if for each $x \in X$ we have $\Omega \subseteq \Omega(R(\cdot)x)$, $G_l(\lambda) \in \rho_{C_l}(\mathcal{A}_l)$ for $1 \leq l \leq m$, $\lambda \in \Omega \setminus D$, and

$$P\Big(F_1(\lambda)(G_1(\lambda) - \mathcal{A}_1)^{-1}C_1, ..., F_m(\lambda)(G_m(\lambda) - \mathcal{A}_m)^{-1}C_m\Big)x = \int_0^{+\infty} ... \int_0^{+\infty} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} R(t_1, ..., t_n)x \, dt_1 ... dt_n, \; x \in X, \; \lambda \in \Omega \setminus D. \tag{2.3}$$

In the case that $n = m = 1$, $P : [L(X)] \rightarrow L(X)$ and P_1 : MLO$(X) \rightarrow$ MLO(X) are the identity mappings, then any mild $(F, G_1, C_1, \mathcal{A}_1, P_1)$-regularized existence family, resp. $(F_1, G_1, C_1, \mathcal{A}_1, P)$-regularized resolvent family, is also said to be a mild (F, G_1, C_1)-regularized existence family with a subgenerator $\mathcal{A}_1$, resp. an (F_1, G_1, C_1)-regularized resolvent family with a subgenerator $\mathcal{A}_1$. If $D = \emptyset$, then we omit the term "D" from the notation.

We can also introduce the corresponding classes of multidimensional (F, G, C)-resolvent operator families which are locally integrable at zero.

Remark 2.2. In [7, Proposition 1], we have observed that the notion introduced in [7, Definition 1] generalizes the notion of an exponentially equicontinuous, mild (a, k)-regularized C_1-existence family, the notion of an exponentially equicontinuous, mild (a, k)-regularized C_2-uniqueness family, and the notion of an exponentially equicontinuous, mild (a, k)-regularized C-resolvent family, provided that the functions $|a|(\cdot)$ and $k(\cdot)$ are Laplace transformable as well as that $(\mathcal{L}a)(\lambda) \neq 0$ for some $\lambda \in \mathbb{C}$ such that $\Re\lambda > \max(0, \mathrm{abs}(|a|), \mathrm{abs}(k))$; cf. [6] for the notion. Setting $D := \{\lambda \in \mathbb{C} : \Re\lambda > \omega,\ (\mathcal{L}a)(\lambda) \cdot (\mathcal{L}k)(\lambda) = 0\}$, the notion introduced in Definition 2.1 fully generalizes these concepts, with $m = n = 1$ and $P,\ P_1$ being the identity mappings; cf. [6, Definition 3.2.1, Theorems 3.2.4 and 3.2.5].

Although the case $m = n$ may seem the most important from the previous research studies of multiparameter strongly continuous semigroups, the case in which $m \neq n$ also deserves attention. For example, if $m = 1$ and $n > 1$, then we can furnish some applications of the multidimensional (a, k)-regularized C-resolvent solution operator families to the abstract Volterra integro-differential inclusion with multiple variables:

$$\begin{aligned} Bu(t_1, ..., t_n) \in \mathcal{A} \int_0^{t_1} ... \int_0^{t_n} a(t_1 - s_1, ..., t_n - s_n) u(s_1, ..., s_n)\, ds_1 ... ds_n \\ + Cf(t_1, ..., t_n), \quad t_1 \in [0, \tau_1),\ ...,\ t_n \in [0, \tau_n), \end{aligned}$$

where $a \in L^1_{loc}([0, +\infty)^n)$, $0 < \tau \leq +\infty$, $0 < \tau_j \leq +\infty$ for $1 \leq j \leq n$, $\mathcal{A}$ is a closed MLO in X, B is a closed linear operator on X and the operator $C \in L(X)$ is injective. The notion introduced in Definition 2.1[(i) and (iii)], with $m = 1$ and $n > 1$, generalizes the notion of an exponentially equicontinuous (a, k)-regularized C_1-existence family and the notion of an exponentially equicontinuous (a, k)-regularized C-resolvent family, while the notion introduced in Definition 2.1(ii), with $m = 1$ and $n > 1$, generalizes the notion of an exponentially equicontinuous (a, k)-regularized C_2-uniqueness family in the case that $\Omega \subseteq \Omega(R_2(\cdot)x) \cap \Omega_{abs}(R_2(\cdot)y) \cap \Omega_{abs}(a)$ for all $(x, y) \in \mathcal{A}$.

Keeping in mind Lemma 1.2(v), we can simply reformulate [6, Proposition 3.2.3] in our new framework. We proceed by introducing the following notion:

Definition 2.3. Suppose that $\alpha_j \in (0, \pi]$ for all $j \in \mathbb{N}_n$ and $\Sigma_{\alpha_1,...,\alpha_n} := \Sigma_{\alpha_1} \times ... \times \Sigma_{\alpha_n}$. Then we say that a global $(F_l, G_l, C_l, \mathcal{A}_l, P, D)_{1 \leq l \leq m}$-regularized resolvent family $(R(t_1, ..., t_n))_{t_1 \geq 0; ...; t_n \geq 0} \subseteq L(X)$ is a holomorphic $(F_l, G_l, C_l, \mathcal{A}_l, P, D)_{1 \leq l \leq m}$-regularized resolvent family of type $(\alpha_1, ..., \alpha_n)$ if there exists a function $\mathbf{R} : \Sigma_{\alpha_1,...,\alpha_n} \rightarrow L(X)$ such that, for every $x \in X$, the function $\mathbf{R}(\cdot)x : \Sigma_{\alpha_1,...,\alpha_n} \rightarrow X$ is holomorphic, $\mathbf{R}(t_1, ..., t_n) = R(t_1, ..., t_n)$ for all $t_1 > 0, ..., t_n > 0$, and for each number $\beta_1 \in (0, \alpha_1), ..., \beta_n \in (0, \alpha_n)$, we have

$$\lim_{(z_1,...,z_n) \rightarrow (0,...,0); (z_1,...,z_n) \in \Sigma_{\beta_1,...,\beta_n}} \mathbf{R}(z_1, ..., z_n)x = R(0, ..., 0)x, \quad x \in X.$$

Furthermore, we say that a global $(F_l, G_l, C_l, \mathcal{A}_l, P, D)_{1\leq l\leq m}$-regularized resolvent family $(R(t_1, ..., t_n))_{t_1\geq 0;...;t_n\geq 0} \subseteq L(X)$ is an exponentially equicontinuous (equicontinuous), holomorphic $(F_l, G_l, C_l, \mathcal{A}_l, P, D)_{1\leq l\leq m}$-regularized resolvent family of type $(\alpha_1, ..., \alpha_n)$ if for each numbers $\beta_1 \in (0, \alpha_1), ..., \beta_n \in (0, \alpha_n)$, there exist real numbers $\omega_1, ..., \omega_n$ ($\omega_1 = 0, ..., \omega_n = 0$) such that the operator family $\{e^{-\omega_1 z_1 - ... -\omega_n z_n}\mathbf{R}(z_1, ..., z_n) : (z_1, ..., z_n) \in \Sigma_{\beta_1,...,\beta_n}\}$ is equicontiunuous. In order to avoid confusion, we will identify $\mathbf{R}(\cdot)$ and $R(\cdot)$ henceforth.

As an immediate consequence of Lemma 1.3, we have the following result:

Theorem 2.4. *Suppose that $m \in \mathbb{N}$, $\mathcal{A}_l : X \to P(X)$ is an* MLO *for $1 \leq l \leq m$, $C_l \in L(X)$ is injective, $C_l\mathcal{A}_l \subseteq \mathcal{A}_l C_l$ for $1 \leq l \leq m$, $P : [L(X)]^m \to L(X)$, $\omega_j \in \mathbb{R}$ ($\omega_j = 0$) and $\alpha_j \in (0, \pi/2]$ for all $j \in \mathbb{N}_n$, $\Omega = \{\lambda_1 \in \mathbb{C} \,|\, \Re\lambda_1 > \omega_1\} \times ... \times \{\lambda_n \in \mathbb{C} \,|\, \Re\lambda_n > \omega_n\}$, D is a discrete subset of Ω, $F_l : \Omega \setminus D \to \mathbb{C}$, $G_l : \Omega \setminus D \to \mathbb{C}$ for $1 \leq l \leq m$, $G_l(\lambda) \in \rho_{C_l}(\mathcal{A}_l)$ for $1 \leq l \leq m$, $\lambda \in \Omega \setminus D$, and the following conditions hold:*

(i) There exists a strongly holomorphic function $H : \Omega \to L(X)$ such that

$$H(\lambda_1, ..., \lambda_n) = P\Big(F_1(\lambda)(G_1(\lambda) - \mathcal{A}_1)^{-1}C_1, ..., F_m(\lambda)(G_m(\lambda) - \mathcal{A}_m)^{-1}C_m\Big),$$

for any $\lambda \in \Omega \setminus D$.

(ii) There exists a strongly holomorphic function $F : (\omega_1 + \Sigma_{(\pi/2)+\alpha_1}) \times ... \times (\omega_n + \Sigma_{(\pi/2)+\alpha_n}) \to L(X)$ such that

$$F(\lambda_1, ..., \lambda_n) = H(\lambda_1, ..., \lambda_n)$$

for $\Re\lambda_1 > \omega_1, ..., \Re\lambda_n > \omega_n$ and, for every $\gamma_1 \in (0, \alpha_1), ..., \gamma_n \in (0, \alpha_n)$, the set $\{(\lambda_1 - \omega_1) \cdot ... \cdot (\lambda_n - \omega_n)F(\lambda_1, ..., \lambda_n) : (\lambda_1, ..., \lambda_n) \in (\omega_1 + \Sigma_{(\pi/2)+\gamma_1}) \times ... \times (\omega_n + \Sigma_{(\pi/2)+\gamma_n})\}$ is bounded in X.

(iii) There exists an operator $W \in L(X)$ such that, for every $x \in X$, we have

$$\lim_{\lambda_1\to+\infty;...;\lambda_n\to+\infty}\Big[\lambda_1 \cdot ... \cdot \lambda_n \cdot F(\lambda_1, ..., \lambda_n)x\Big] = Wx.$$

Then there exists an exponentially equicontinuous (equicontinuous), holomorphic $(F_l, G_l, C_l, \mathcal{A}_l, P, D)_{1\leq l\leq m}$-regularized resolvent family of type $(\alpha_1, ..., \alpha_n)$.

2.1. Separation of variables

In this subsection, we consider multidimensional (F, G, C)-resolvent operator families of the form $R(t) = R_1(t_1)\cdot ... \cdot R_n(t_n)$, where $t = (t_1, ..., t_n) \in [0, +\infty)^n$ and $(R_j(t_j))_{t_j\geq 0}$ is a strongly continuous operator family. The case in which $m = n$ will be dominant here.

We will first state and prove the following results (cf. also [7, Example 2, Theorems 1 and 2] for some results and observations established in the one-dimensional setting):

Proposition 2.5. *(i) Let $(R_l(s))_{s\geq 0} \subseteq L(X)$ be a global (a_l, k_l)-regularized C_l-resolvent family such that there exists $\omega_l^0 \geq 0$ satisfying that the family $\{e^{-\omega_l^0 s}R_l(s) : s \geq 0\}$ is equicontinuous, $\omega_l \geq \max(\omega_l^0, \mathrm{abs}(|a_l|), \mathrm{abs}(k_l))$, and $(\int_0^t a_l(t - s)R_l(s)x, R_l(t)x - k_l(t)C_l x) \in \mathcal{A}_l$, $t \geq 0$, $x \in X$ ($1 \leq l \leq n$). Define*

$$R(t) := R_1(t_1) \cdot ... \cdot R_n(t_n), \quad t = (t_1, ..., t_n) \in [0, +\infty)^n. \tag{2.4}$$

Then $(R(t_1,...,t_n))_{t_1\geq 0;...;t_n\geq 0} \subseteq L(X)$ *is a global* $(F_l, G_l, C_l, \mathcal{A}_l, P, D)_{1\leq l\leq n}$*-regularized resolvent family, where the mapping* $P : [L(X)]^n \to L(X)$ *is given by* $P(T_1 \cdot ... \cdot T_n) := T_1 \cdot ... \cdot T_n$ *for all* $T_1,...,T_n \in L(X)$*,* $D := \prod_{1\leq l\leq n}\{\lambda_l \in \mathbb{C} : \Re\lambda > \omega_l,\ (\mathcal{L}a_l)(\lambda_l)\cdot(\mathcal{L}k_l)(\lambda_l)\ \neq 0\}$*, and*

$$F_l(\lambda) = \frac{\tilde{k}_l(\lambda_l)}{\tilde{a}_l(\lambda_l)}\left(\frac{1}{\tilde{a}_l(\lambda_l)} - \mathcal{A}_l\right)^{-1} C_l, \quad 1 \leq l \leq n,\ \lambda \in \Omega \setminus D.$$

(ii) Let $(R_l(s))_{s\geq 0} \subseteq L(X)$ *be a global* (F_l, G_l, C_l, D_l)*-regularized resolvent family with a subgenerator* $\mathcal{A}_l$*, where* $\omega_l \in \mathbb{R}$ *is such that the family* $\{e^{-\omega_l s}R_l(s) : s \geq 0\}$ *is equicontinuous,* $\Omega_l = \{\lambda_l \in \mathbb{C} : \Re\lambda_l > \omega_l\}$*, and* D_l *is a discrete subset of* Ω_l $(1 \leq l \leq n)$*. Define* $R(\cdot)$ *through* (2.4)*,* $\Omega := \Omega_1 \times ... \times \Omega_n$*,* $D := D_1 \times ... \times D_n$*, and* $P : [L(X)]^n \to L(X)$ *as in part* (i)*. Then* $(R(t_1,...,t_n))_{t_1\geq 0;...;t_n\geq 0} \subseteq L(X)$ *is an* $(F_l, G_l, C_l, \mathcal{A}_l, P, D)_{1\leq l\leq n}$*-regularized resolvent family.*

Proof. By [6, Theorem 3.2.5], we have

$$\tilde{k}_l(\lambda)(I - \tilde{a}_l(\lambda)\mathcal{A}_l)^{-1}C_l x = \int_0^\infty e^{-\lambda t}R_l(t)x\,dt,\ x \in X,\ \Re\lambda > \omega_l,\ \tilde{a}_l(\lambda)\tilde{k}_l(\lambda) \neq 0$$

and

$$\left\{\frac{1}{\tilde{a}_l(\lambda)} : \Re\lambda > \omega_l,\ \tilde{k}_l(\lambda)\tilde{a}_l(\lambda) \neq 0\right\} \subseteq \rho_{C_l}(\mathcal{A}_l).$$

Then the proof of (i) simply follows by applying the Fubini theorem. The proof of (ii) can be given in a similar manner. □

Proposition 2.6. *Suppose that* $X = Y$*.*

(i) Let $(R_l^1(t))_{t\geq 0}$ *be a mild* (a_l, k_l)*-regularized* C_1^l*-existence family with a closed subgenerator* $\mathcal{A}_l$*, and let the family* $\{e^{-\omega_l t}R_l^1(s) : s \geq 0\}$ *be equicontinuous for some real number* $\omega_l \geq \max(\operatorname{abs}(|a_l|), \operatorname{abs}(k_l))$ $(1 \leq l \leq n)$*. Define*

$$R_1(t) := R_1^1(t_1)\cdot ... \cdot R_n^1(t_n), \quad t = (t_1,...,t_n) \in [0,+\infty)^n.$$

Then $(R_1(t_1,...,t_n))_{t_1\geq 0;...;t_n\geq 0} \subseteq L(X)$ *is a mild* $(F, G_l, C_1, \mathcal{A}_l, P_1, D)_{1\leq l\leq m}$*-regularized existence family, where the mapping* $P_1 : [\mathrm{MLO}(X)]^n \to \mathrm{MLO}(X)$ *is given by* $P_1(\mathcal{B}_1 \cdot ... \cdot \mathcal{B}_n) := \mathcal{B}_1 \cdot ... \cdot \mathcal{B}_n$ *for all* $\mathcal{B}_1,...,\mathcal{B}_n \in \mathrm{MLO}(X)$*,* $\Omega_l := \{\lambda_l \in \mathbb{C} : \Re\lambda_l > \omega_l\}$*,* $D_l := \{\lambda_l \in \mathbb{C} : \Re\lambda > \omega_l,\ (\mathcal{L}a_l)(\lambda_l)\cdot(\mathcal{L}k_l)(\lambda_l)\ \neq 0\}$ $(1 \leq l \leq n)$*,* $D := D_1 \times ... \times D_n$*,*

$$G_l(\lambda) = \left(\frac{1}{\tilde{a}_l(\lambda_l)} - \mathcal{A}_l\right), \quad 1 \leq l \leq n,\ \lambda \in \Omega \setminus D,$$

$$F(\lambda) := \prod_{l=1}^n \frac{\tilde{k}_l(\lambda_l)}{\tilde{a}_l(\lambda_l)}, \quad \lambda \in \Omega \setminus D,$$

and $C_1 := \prod_{l=1}^n C_l^1$*.*

(ii) Let $(R_l(s))_{s\geq 0} \subseteq L(X)$ *be a global* (F_l, G_l, C_l, D_l)*-regularized resolvent family with a subgenerator* $\mathcal{A}_l$*, where* $\omega_l \in \mathbb{R}$ *is such that the family* $\{e^{-\omega_l s}R_l(s) : s \geq 0\}$ *is equicontinuous,* $\Omega_l = \{\lambda_l \in \mathbb{C} : \Re\lambda_l > \omega_l\}$*, and* D_l *is a discrete subset of* Ω_l $(1 \leq l \leq n)$*. Define* $R(\cdot)$ *through* (2.4)*,* $\Omega := \Omega_1 \times ... \times \Omega_n$*,* $D := D_1 \times ... \times D_n$*, and* $P : [L(X)]^n \to L(X)$ *as in part* (i)*. Then* $(R(t_1,...,t_n))_{t_1\geq 0;...;t_n\geq 0} \subseteq L(X)$ *is an* $(F_l, G_l, C_l, \mathcal{A}_l, P, D)_{1\leq l\leq n}$*-regularized resolvent family.*

Proof. Due to [6, Theorem 3.2.4(ii)], for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_l$ and $\tilde{a}_l(\lambda)\tilde{k}_l(\lambda) \neq 0$, one has $R(C_1^l) \subseteq R(I - \tilde{a}_l(\lambda)\mathcal{A}_l)$ and

$$\tilde{k}_l(\lambda)C_1^l x \in (I - \tilde{a}_l(\lambda)\mathcal{A}_l)\int_0^\infty e^{-\lambda t}R_l^1(t)x\,dt, \quad x \in X.$$

Since $\mathcal{A}_n$ is closed, we can apply the Fubini theorem, the above equality with $l = n$, and [6, Theorem 1.2.3] to show that

$$\begin{aligned}
&\int_0^{+\infty}\dots\int_0^{+\infty} e^{-\lambda_1 t_1-\dots-\lambda_n t_n}R_1(t_1,\dots,t_n)x\,dt_1\dots dt_n\\
&=\int_0^{+\infty}\dots\int_0^{+\infty} e^{-\lambda_1 t_1-\dots-\lambda_{n-1} t_{n-1}}R_1^1(t_1)\cdot\dots\cdot R_{n-1}^1(t_{n-1})\\
&\quad\times\left[\int_0^{+\infty} e^{-\lambda_n t_n}R_n^1(t_n)x\,dt_n\right]dt_1\dots dt_{n-1},
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{\tilde{k}_n(\lambda_n)}{\tilde{a}_n(\lambda_n)}\int_0^{+\infty}\dots\int_0^{+\infty} e^{-\lambda_1 t_1-\dots-\lambda_{n-1} t_{n-1}}R_1^1(t_1)\cdot\dots\cdot R_{n-1}^1(t_{n-1})C_n^1 x\,dt_1\dots dt_{n-1}\\
&\in\left(\frac{1}{\tilde{a}_n(\lambda_n)}-\mathcal{A}_n\right)R(t)x,
\end{aligned}$$

for any $x \in X$ and $\lambda \in \Omega \setminus D$. Then the first part follows by induction and the second part can be deduced in a similar manner. □

The above results can be also formulated for the class of $(F_l, G_l, C_l, \mathcal{A}_l, P)_{1\le l\le n}$-regularized resolvent families and the class of mild $(F, G_l, C_1, \mathcal{A}_l, P_1)_{1\le l\le n}$-regularized existence families. Further on, we are obliged to say that the notion introduced in Definition 2.1(ii) is only an unsatisfactory attempt to extend the notion introduced in [7, Definition 1(ii)] in a proper way; concerning this issue, we would like to emphasize that we have only one MLO in Definition 2.1(ii) as well as that it would be very difficult to constitute this notion for two or more MLOs, as the following illustrative example shows:

Example 2.7. Suppose that $n = 2$ and $(R_l(t_l))_{t_l\ge 0}$ is a mild (F_l, G_l, C_l^2)-regularized uniqueness family with a subgenerator $\mathcal{A}_l$, where $l = 1, 2$, $(x, y) \in \mathcal{A}_2$ and $(C_2^2 y, z) \in \mathcal{A}_1$. Using the Fubini theorem, we get

$$\begin{aligned}
&G_2(\lambda_2)\int_0^{+\infty}\int_0^{+\infty} e^{-\lambda_1 t_1-\lambda_2 t_2}R(t)x\,dt_1\,dt_2\\
&=F_2(\lambda_2)\int_0^{+\infty} e^{-\lambda_1 t_1}R_1^2(t_1)C_2^2 y_1\,dt_1-\int_0^{+\infty}\int_0^{+\infty} e^{-\lambda_1 t_1-\lambda_2 t_2}R(t)y\,dt_1\,dt_2,
\end{aligned}$$

for any $\lambda \in D$. Multiplying both sides of this equality by $G_1(\lambda_1)$, we get

$$\begin{aligned}
&G_2(\lambda_2)\left[F_1(\lambda_1)C_1^2 z+\int_0^{+\infty} e^{-\lambda_1 t_1}R_1^2(t_1)z\,dt_1\right]\\
&\qquad=G_1(\lambda_1)G_2(\lambda_2)\int_0^{+\infty}\int_0^{+\infty} e^{-\lambda_1 t_1-\lambda_2 t_2}R(t)x\,dt_1\,dt_2\\
&\qquad+G_1(\lambda_1)\int_0^{+\infty}\int_0^{+\infty} e^{-\lambda_1 t_1-\lambda_2 t_2}R(t)y\,dt_1\,dt_2, \quad \lambda \in D,
\end{aligned}$$

which cannot be easily interpreted in any reasonable sense.

2.2. Some applications to the abstract Volterra integro-differential inclusions with multiple variables

In this subsection, we will first continue our recent analyses of abstract partial fractional differential inclusions with Caputo derivatives; cf. parts [1–4] below.

Suppose that $n = 2$, $\alpha_j > 0$ and $\mathcal{A}_j$ is a closed subgenerator of an exponentially equicontinuous (g_{α_j}, C_j)-regularized resolvent family $(R_j(t))_{t\geq 0}$, $j = 1, 2$. The similar conclusions can be given in the case that the operator $\mathcal{A}_j$ is a closed subgenerator of a (local) (g_{α_j}, C_j)-regularized resolvent family $(R_j(t))_{t\in[0,\tau_j)}$, where $0 < \tau_j \leq +\infty$ for $j = 1, 2$ (cf. parts [3 and 4] below):
1. Assume that $x \in D(\mathcal{A}_2)$, $R(\mathcal{A}_2) \subseteq D(\mathcal{A}_1)$, and $R_1(t)\mathcal{A}_2 \subseteq \mathcal{A}_2 R_1(t)$ for all $t \geq 0$. Define

$$u(t_1, t_2) := R_1(t_1)R_2(t_2)x, \quad t_1 \geq 0,\ t_2 \geq 0.$$

Due to [6, Proposition 3.2.15(i)], we have $\mathbf{D}_{t_2}^{\alpha_2} u(t_1, t_2) \in R_1(t_1)\mathcal{A}_2[R_2(t_2)x]$, $t_1 \geq 0$, $t_2 \geq 0$, so that for each $t_2 \geq 0$ there exists an element $y(t_2) \in \mathcal{A}_2 R_2(t_2)x$ such that $\mathbf{D}_{t_2}^{\alpha_2} u(t_1, t_2) = R_1(t_1)y(t_2)$, $t_1 \geq 0$, $t_2 \geq 0$. Since $R(\mathcal{A}_2) \subseteq D(\mathcal{A}_1)$ and $R_1(t)\mathcal{A}_2 \subseteq \mathcal{A}_2 R_1(t)$ for all $t \geq 0$, we have that the mixed partial fractional derivative $\mathbf{D}_{t_1}^{\alpha_1}\mathbf{D}_{t_2}^{\alpha_2} u(t_1, t_2)$ is well-defined for all $t_1 \geq 0$ and $t_2 \geq 0$, as well as that

$$\begin{aligned}\mathbf{D}_{t_1}^{\alpha_1}\mathbf{D}_{t_2}^{\alpha_2} u(t_1, t_2) &\in \mathcal{A}_1 R_1(t_1)y(t_2) \subseteq \mathcal{A}_1 R_1(t_1)\mathcal{A}_2 R_2(t_2)x \\ &\subseteq \mathcal{A}_1\mathcal{A}_2 R_1(t_1)R_2(t_2)x = \mathcal{A}_1\mathcal{A}_2 u(t_1, t_2), \quad t_1 \geq 0,\ t_2 \geq 0.\end{aligned}$$

2. Suppose that $x \in D(\mathcal{A}_1)$, $y \in D(\mathcal{A}_2)$, $c_1 \in \mathbb{C}$, $c_2 \in \mathbb{C}$, and $c_1c_2 \neq 0$. Set $u(t_1, t_2) := c_1R_1(t_1)x + c_2R_2(t_2)y$, $t_1 \geq 0$, $t_2 \geq 0$. If

$$0 \in \mathcal{A}_2 R_1(t_1)x \cap \mathcal{A}_1 R_2(t_2)y, \quad t_1 \geq 0,\ t_2 \geq 0, \tag{2.5}$$

then we have

$$\mathbf{D}_{t_1}^{\alpha_1} u(t_1, t_2) \in \mathcal{A}_1[c_1R_1(t_1)x] \subseteq \mathcal{A}_1[c_1R_1(t_1)x + c_2R_2(t_2)y], \quad t_1 \geq 0,\ t_2 \geq 0,$$

and

$$\mathbf{D}_{t_2}^{\alpha_2} u(t_1, t_2) \in \mathcal{A}_2[c_2R_2(t_2)y] \subseteq \mathcal{A}_2[c_1R_1(t_1)x + c_2R_2(t_2)y], \quad t_1 \geq 0,\ t_2 \geq 0,$$

so that, for every $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$,

$$\alpha\mathbf{D}_{t_1}^{\alpha_1} u(t_1, t_2) + \beta\mathbf{D}_{t_2}^{\alpha_2} u(t_1, t_2) \in \Big[\alpha\mathcal{A}_1 + \beta\mathcal{A}_2\Big]u(t_1, t_2), \quad t_1 \geq 0,\ t_2 \geq 0.$$

In the remainder of this subsection, we will consider some classes of the abstract Volterra integro-differential inclusions with multiple variables. We will always assume that $0 < \tau_1 \leq +\infty$, $0 < \tau_2 \leq +\infty$, $(R_j(s))_{s\in[0,\tau_j)}$ is a (local) (a_j, k_j)-regularized C_j-resolvent family with a closed subgenerator $\mathcal{A}_j$, and $(\int_0^{t_j} a_j(t_j - s_j)R_j(s_j)x\,ds_j, R_j(t_j)x - C_jx) \in \mathcal{A}_j$, $x \in X$, $j = 1, 2$.
3. Let $a_1 \in L^1_{loc}([0, \tau_1))$, $a_2 \in L^1_{loc}([0, \tau_2))$, and $a(t_1, t_2) = a_1(t_1) \cdot a_2(t_2)$, $t \in [0, +\infty)^2$. Suppose that $(R_j(s))_{s\in[0,\tau_j)}$ is a (local) (a_j, k_j)-regularized C_j-resolvent family with a closed subgenerator $\mathcal{A}_j$, $j = 1, 2$, as well as $R_1(\cdot)R_2(\cdot) = R_2(\cdot)R_1(\cdot)$ and $R_1(\cdot)C_2 = C_2R_1(\cdot)$. Let $y \in \mathcal{A}_1x$ and $z \in \mathcal{A}_2y$. Then we have $R_1(t_1)x - k_1(t_1)C_1x = \int_0^{t_1} a_1(t_1 - s)R_1(s)y\,ds$, $t_1 \in [0, \tau_1)$, and $R_2(t_2)y - k_2(t_2)C_2y = \int_0^{t_2} a_2(t_2 - s)R_2(s)z\,ds$, $t_2 \in [0, \tau_2)$, so that the Fubini theorem implies

$$\int_0^{t_1}\int_0^{t_2} a(t_1 - s_1, t_2 - s_2)R_1(s_1)R_2(s_2)z\,ds_1\,ds_2$$

$$= \int_0^{t_1}\int_0^{t_2} a_1(t_1-s_1)a_2(t_2-s_2)R_1(s_1)R_2(s_2)z\,ds_1\,ds_2$$
$$= \int_0^{t_1} a_1(t_1-s_1)R_1(s_1)\Bigl[\int_0^{t_2} a_2(t_2-s_2)R_2(s_2)z\,ds_2\Bigr]ds_1$$
$$= \int_0^{t_1} a_1(t_1-s_1)R_1(s_1)\Bigl[R_2(t_2)y-k_2(t_2)C_2y\Bigr]ds_1$$
$$= R_2(t_2)\Bigl[R_1(t_1)x-k_1(t_1)C_1x\Bigr]-k_2(t_2)C_2\Bigl[R_1(t_1)x-k_1(t_1)C_1x\Bigr]$$
$$= R_2(t_2)R_1(t_1)x-k_1(t_1)R_2(t_2)C_1x-k_2(t_2)C_2R_1(t_1)x+k_1(t_1)k_2(t_2)C_2C_1x,$$

for any $t_1 \in [0,\tau_1)$ and $t_2 \in [0,\tau_2)$. Let us assume now that $R_1(\cdot)\mathcal{A}_2 \subseteq \mathcal{A}_2R_1(\cdot)$, $R_2(\cdot)\mathcal{A}_1 \subseteq \mathcal{A}_1R_2(\cdot)$, $C_1R_2(\cdot)=R_2(\cdot)C_1$, and $k_1(0)\cdot k_2(0)\neq 0$. Define $u(t_1,t_2):=R_1(t_1)R_2(t_2)x$, $0\le t_1<\tau_1$, $0\le t_2<\tau_2$. Then we have $k_1(t_1)R_2(t_2)C_1x = k_1(t_1)\cdot[k_1(0)]^{-1}u(0,t_2)$, $k_2(t_2)R_1(t_1)C_2x = k_2(t_2)\cdot[k_2(0)]^{-1}u(t_1,0)$, and

$$R_1(t_1)R_2(t_2)z \in R_1(t_1)\mathcal{A}_2R_2(t_2)y \subseteq \mathcal{A}_2R_1(t_1)R_2(t_2)y \subseteq \mathcal{A}_2\mathcal{A}_1R_1(t_1)R_2(t_2)x,$$

for any $0\le t_1<\tau_1$ and $0\le t_2<\tau_2$. Keeping in mind [6, Theorem 1.2.3] and the closedness of operators $\mathcal{A}_1$ and $\mathcal{A}_2$, there exists a continuous function $u_{12}:[0,\tau_1)\times[0,\tau_2)\to X$ such that $u_{12}(t_1,t_2)\in \mathcal{A}_2\mathcal{A}_1u(t_1,t_2)$, $0\le t_1<\tau_1$, $0\le t_2<\tau_2$, and

$$\int_0^{t_1}\int_0^{t_2} a(t_1-s_1,t_2-s_2)u_{12}(s_1,s_2)\,ds_1\,ds_2$$
$$= u(t_1,t_2)-k_1(t_1)\cdot[k_1(0)]^{-1}u(0,t_2)$$
$$-k_2(t_2)\cdot[k_2(0)]^{-1}u(t_1,0)+k_1(t_1)k_2(t_2)C_2C_1x$$
$$\in \mathcal{A}_2\mathcal{A}_1\int_0^{t_1}\int_0^{t_2} a(t_1-s_1,t_2-s_2)u(s_1,s_2)\,ds_1\,ds_2,\quad 0\le t_1<\tau_1,\ 0\le t_2<\tau_2.$$

4. Let (2.5) hold for $0\le t_1<\tau_1$, $0\le t_2<\tau_2$, $c_1\in\mathbb{C}$, $c_2\in\mathbb{C}$, and $c_1c_2\neq 0$. Set $u(t_1,t_2):=c_1R_1(t_1)x+c_2R_2(t_2)y$, $t_1\in[0,\tau_1)$, $t_2\in[0,\tau_2)$. Then we have

$$c_1\Bigl[R_1(t_1)x-k_1(t_1)C_1x\Bigr] \in \mathcal{A}_1\int_0^{t_1} a_1(t_1-s_1)\Bigl[c_1R_1(s_1)x+c_2R_2(t_2)y\Bigr]ds_1$$

and

$$c_2\Bigl[R_2(t_2)y-k_2(t_2)C_2y\Bigr] \in \mathcal{A}_2\int_0^{t_2} a_2(t_2-s_2)\Bigl[c_1R_1(t_1)x+c_2R_2(s_2)y\Bigr]ds_2,$$

so that

$$u(t_1,t_2)-c_1k_1(t_1)C_1x-c_2k_2(t_2)C_2y$$
$$\in \mathcal{A}_1\int_0^{t_1} a_1(t_1-s_1)u(s_1,t_2)\,ds_1+\mathcal{A}_2\int_0^{t_2} a_2(t_2-s_2)u(t_1,s_2)\,ds_2,$$

for any $0\le t_1<\tau_1$ and $0\le t_2<\tau_2$.

We close this section with the observation that we can similarly analyze the multidimensional (F,G,C)-resolvent operator families of the form $R(t)=R_1(t_1,...,t_n)\cdot ...\cdot R_n(t_1,...t_n)$, where $t=(t_1,...,t_n)\in[0,+\infty)^n$ and $(R_j(t_1,...,t_n))_{t_1\ge 0;...;t_n\ge 0}$ is a multidimensional (a,k)-regularized C-resolvent solution operator family $(1\le j\le n)$.

3. Abstract functional Volterra integro-differential inclusions with multiple variables

In this section, we consider the existence and uniqueness of solutions to the abstract functional Volterra integro-differential inclusion with multiple variables (1.3). For simplicity, we will not conisder here the case in which some of the kernels $a_i(\cdot)$ are certain translations and derivatives of multidimensional Dirac delta (ultra)distributions, as has been done in [7, Section 3] in the one-dimensional setting. Let us only emphasize that, since

$$\delta(t) = \delta(t_1) \cdot ... \cdot \delta(t_n)$$

in a certain sense, we have $(\mathcal{L}(\delta^{(\alpha)}(\cdot - a)))(\lambda) = \lambda^{\alpha} e^{-a\lambda}$, where $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, $a \in \mathbb{R}^n$, $\lambda^{\alpha} = \lambda_1^{\alpha_1} \cdot ... \cdot \lambda_n^{\alpha_n}$, and $a\lambda = a_1\lambda_1 + ... + a_n\lambda_n$. We refer the reader to [14–19] for further information concerning the multidimensional Laplace transform of (ultra)distributions.

In this section, we will always assume the following condition:

(F) $m \in \mathbb{N}$, $\tau_j = (\tau_j^1, ..., \tau_j^n) \in [0, +\infty)^n$ for $1 \leq j \leq m$, $C \in L(X)$ is injective, Ω_0 is given by (1.4), $f : [0, +\infty)^n \to X$ is a Laplace transformable function, B is a closed linear operator on X, $\mathcal{A}_i$ is a closed MLO on X ($1 \leq j \leq m$), $u_0 \in L^1_{loc}(\Omega_0 : X)$, $Cu_0 \in C(\Omega_0)$, and $a_1(t), ..., a_m(t)$ are locally integrable scalar-valued functions defined for $t \in [0, +\infty)^n$.

We will use the following concepts of solutions (cf. also [7, Definition 3] for parts (i)–(iv) in one-dimensional setting; the notion introduced in part (v) is new):

Definition 3.1. Suppose that (F) holds.

(i) By a mild solution of (1.3), we mean any continuous function $u : [0, +\infty)^n \to X$ such that $(a_i *_0 u(\cdot + \tau_i))(t) \in D(\mathcal{A}_i)$ for all $t \in [0, +\infty)^n$, $i \in \mathbb{N}_m$, and

$$Bu(t) \in Cf(t) + \sum_{i=1}^{m} \mathcal{A}_i\Big(a_i *_0 u(\cdot + \tau_i)\Big)(t), \ t \in [0, +\infty)^n.$$

(ii) By an LT-mild solution of (1.3), we mean any Laplace transformable mild solution of (1.3).

(iii) By a strong solution of (1.3), we mean any continuous function $u : [0, +\infty)^n \to X$ such that, for every $i \in \mathbb{N}_m$, there exists a continuous function $u_i : [0, +\infty)^n \to X$ such that $(u(t + \tau_i), u_i(t)) \in \mathcal{A}_i$ for all $t \in [0, +\infty)^n$, $i \in \mathbb{N}_m$, and

$$Bu(t) = Cf(t) + \sum_{i=1}^{m} (a_i *_0 u_i)(t), \ t \in [0, +\infty)^n. \tag{3.1}$$

(iv) By an LT-strong solution of (1.3), we mean any continuous Laplace transformable function $u : [0, +\infty)^n \to X$ such that, for every $i \in \mathbb{N}_m$, there exists a continuous Laplace transformable function $u_i : [0, +\infty)^n \to X$ such that $(u(t + \tau_i), u_i(t)) \in \mathcal{A}_i$ for all $t \in [0, +\infty)^n$, $i \in \mathbb{N}_m$, and (3.1) holds.

(v) By a mild LT-solution of (1.3), we mean any continuous Laplace transformable function $u : [0, +\infty)^n \to X$ such that there exist real numbers $\omega_i \in \mathbb{R}$ $(1 \leq i \leq n)$ satisfying that, for every $\lambda \in \Omega := \{\lambda_1 \in \mathbb{C} : \Re\lambda_1 > \omega_1\} \times ... \times \{\lambda_n \in \mathbb{C} : \Re\lambda_n > \omega_n\}$, we have $\lambda \in \Omega_{abs}(u) \cap \Omega(f)$ and

$$B\hat{u}(\lambda) \in C\hat{f}(\lambda) + \sum_{i=1}^{m} \mathcal{A}_i\bigg(\mathcal{L}\Big(a_i *_0 u(\cdot + \tau_i)\Big)(\lambda)\bigg). \tag{3.2}$$

Any (LT-)strong solution of (1.3) is an (LT-)mild solution of (1.3), any strong LT-solution of problem (1.3) is a strong solution of (1.3), and any mild LT-solution of problem (1.3) is a mild solution of (1.3). Now we will state the following result concerning the uniqueness of LT-strong solutions to (1.3) and the uniqueness of mild LT-solutions to (1.3); observe, however, that it is not so simple to state a satisfactory analogue of this result for LT-mild solutions to (1.3):

Theorem 3.2. *Suppose that* (F) *holds and there exist real numbers* $\omega_i \in \mathbb{R}$ *(*$1 \leq i \leq n$*) such that, for every* $\lambda \in \Omega := \{\lambda_1 \in \mathbb{C} : \Re\lambda_1 > \omega_1\} \times ... \times \{\lambda_n \in \mathbb{C} : \Re\lambda_n > \omega_n\}$ *and* $i \in \mathbb{N}_m$*, we have* $\lambda \in \Omega_{abs}(a_i)$ *and the operator* $B - \sum_{i=1}^{m} \widehat{a_i}(\lambda)e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}\mathcal{A}_i$ *is injective for all* $\lambda \in \Omega$*. Then there exists at most one* LT-*strong solution* $u(\cdot)$ *to* (1.3) *(mild* LT-*solution* $u(\cdot)$ *to* (1.3)*) with the property that* $\Omega \subseteq \Omega(Bu)$ *and* $u(t) = Cu_0(t)$, $t \in \Omega_0$.

Proof. We will provide all details of proof for mild LT-solutions to (1.3); the proof is almost the same for LT-strong solutions to (1.3). Let $u(\cdot)$ and $v(\cdot)$ be two mild LT-solutions to (1.3) such that $\Omega \subseteq \Omega(Bu)\cap\Omega(Bv)$, $u(t) = Cu_0(t)$, $t \in \Omega_0$, and $v(t) = Cu_0(t)$, $t \in \Omega_0$. Set $z(t) := u(t)-v(t)$, $t \in [0,+\infty)^n$. Then $\Omega \subseteq \Omega_{abs}(z) \cap \Omega(Bz)$, $z(t) = 0$ for all $t \in \Omega_0$, and the closedness of B implies $\widehat{Bz}(\lambda) = B\hat{z}(\lambda)$ for all $\lambda \in \Omega$. Now we can apply Lemma 1.2[(v),(vi)] with a view to obtain

$$B\hat{z}(\lambda) \in \sum_{i=1}^{m} \widehat{a_i}(\lambda)e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}\mathcal{A}_i\hat{z}(\lambda), \quad \lambda \in \Omega.$$

Hence, we get

$$0 \in \left[B - \sum_{i=1}^{m} \widehat{a_i}(\lambda)e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}\mathcal{A}_i\right] \cdot \hat{z}(\lambda), \quad \lambda \in \Omega.$$

Since the operator $B - \sum_{i=1}^{m} \widehat{a_i}(\lambda)e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}\mathcal{A}_i$ is injective for all $\lambda \in \Omega$, we get that $\hat{z}(\lambda) = 0$ for all $\lambda \in \Omega$. By Lemma 1.2(ii) and the continuity of $z(\cdot)$, it follows that $z \equiv 0$, as required. □

Remark 3.3. In particular, if there exists an operator $C_1 \in L(X)$ such that $[B - \sum_{i=1}^{m} \widehat{a_i}(\lambda)e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}\mathcal{A}_i]^{-1}C_1 \in L(X)$, then $B - \sum_{i=1}^{m} \widehat{a_i}(\lambda)e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}\mathcal{A}_i$ is injective. To prove this, set $\mathcal{A} := B - \sum_{i=1}^{m} \widehat{a_i}(\lambda)e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}\mathcal{A}_i$. Then we need to prove that the operator $\mathcal{A}^{-1}$ is single-valued; therefore, let us assume that $x \in D(\mathcal{A}^{-1})$ and $\{y,z\} \subseteq \mathcal{A}^{-1}x$. Then $x \in \mathcal{A}y$, $x \in \mathcal{A}z$, $0 \in \mathcal{A}y - \mathcal{A}z = \mathcal{A}(y-z)$, $y - z \in \mathcal{A}^{-1}0 = \mathcal{A}^{-1}C_1 0 = \{0\}$, so that $y = z$.

Remark 3.4. Suppose that $u_1 : [0,+\infty)^n \to X$ and $u_2 : [0,+\infty)^n \to X$ are Laplace transformable functions such that $u_1(t) = Cu_0(t)$ for a.e. $t \in \Omega_0$, $u_2(t) = Cu_0(t)$ for a.e. $t \in \Omega_0$, and there exist real numbers $\omega_i \in \mathbb{R}$ ($1 \leq i \leq n$) satisfying that, for every $\lambda \in \Omega := \{\lambda_1 \in \mathbb{C} : \Re\lambda_1 > \omega_1\} \times ... \times \{\lambda_n \in \mathbb{C} : \Re\lambda_n > \omega_n\}$, we have $\lambda \in \Omega_{abs}(u_1) \cap \Omega_{abs}(u_2) \cap \Omega(f)$ and (3.2) holds with the function $u(\cdot)$ replaced therein with the function $u_j(\cdot)$, $j = 1,2$. Arguing as in the proof of Theorem 3.2, we may conclude that $u_1(t) = u_2(t)$ for any $t \in [0,+\infty)^n$ which is a point continuity of both functions $u_1(\cdot)$ and $u_2(\cdot)$; cf. Lemma 1.2(ii).

Applying the multidimensional vector-valued Laplace transform, Lemma 1.2[(i),(vi)] yields:

$$B\hat{u}(\lambda) \in C\hat{f}(\lambda) + \sum_{i=1}^{m} \widehat{a_i}(\lambda)\mathcal{A}_i\Bigg[e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}\hat{u}(\lambda) - e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}$$

$$\times \int_{[0,+\infty)^n \setminus \left([\tau_i^1,+\infty)\times\ldots\times[\tau_i^n,+\infty)\right)} e^{-\lambda_1 t_1-\ldots-\lambda_n t_n} Cu_0(t_1,\ldots,t_n)\, dt_1\ldots dt_n \Bigg], \tag{3.3}$$

where $u(t) = Cu_0(t),\ t \in \Omega_0$, and Ω_0 is given by (1.4). If $(Cu_0(t), Cu_{0,i}(t)) \in D(\mathcal{A}_i),\ t \in \Omega_0,\ 1 \leq i \leq m$, then (3.4) implies (3.3), where:

$$\left[B - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1\tau_i^1+\ldots+\lambda_n\tau_i^n} \mathcal{A}_i\right] \hat{u}(\lambda) \ni C\hat{f}(\lambda) - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1\tau_i^1+\ldots+\lambda_n\tau_i^n}$$
$$\times \int_{[0,+\infty)^n \setminus \left([\tau_i^1,+\infty)\times\ldots\times[\tau_i^n,+\infty)\right)} e^{-\lambda_1 t_1-\ldots-\lambda_n t_n} Cu_{0,i}(t_1,\ldots,t_n)\, dt_1\ldots dt_n. \tag{3.4}$$

Assuming that the operator $[B - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1\tau_i^1+\ldots+\lambda_n\tau_i^n} \mathcal{A}_i]^{-1} C$ belongs to $L(X)$ for all $\lambda \in \Omega := \{\lambda_1 \in \mathbb{C} : \Re\lambda_1 > \omega_1\} \times \ldots \times \{\lambda_n \in \mathbb{C} : \Re\lambda_n > \omega_n\}$, for some real numbers $\omega_i \in \mathbb{R}$ $(1 \leq i \leq n)$, we get that (3.5) implies (3.4), where:

$$\hat{u}(\lambda) = \left[B - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1\tau_i^1+\ldots+\lambda_n\tau_i^n} \mathcal{A}_i\right]^{-1} C\Bigg\{\hat{f}(\lambda) - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1\tau_i^1+\ldots+\lambda_n\tau_i^n}$$
$$\times \int_{[0,+\infty)^n \setminus \left([\tau_i^1,+\infty)\times\ldots\times[\tau_i^n,+\infty)\right)} e^{-\lambda_1 t_1-\ldots-\lambda_n t_n} u_{0,i}(t_1,\ldots,t_n)\, dt_1\ldots dt_n\Bigg\}. \tag{3.5}$$

Now we will formalize all this and prove the following result:

Theorem 3.5. *Suppose that* (F) *and the following conditions hold:*

(i) There exist real numbers $\omega_i \in \mathbb{R}$ $(1 \leq i \leq n)$ satisfying that $\Omega_{abs}(a_i) \cup \Omega(f) \subseteq \Omega := \{\lambda_1 \in \mathbb{C} : \Re\lambda_1 > \omega_1\} \times \ldots \times \{\lambda_n \in \mathbb{C} : \Re\lambda_n > \omega_n\}$ $(1 \leq i \leq m)$ and $[B - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1\tau_i^1+\ldots+\lambda_n\tau_i^n} \mathcal{A}_i]^{-1} C \in L(X)$ for all $\lambda \in \Omega$;
(ii) $u_{0,i} \in L^1_{loc}(\Omega_0)$, $Cu_{0,i} \in C(\Omega_0)$, and $(Cu_0(t), Cu_{0,i}(t)) \in \mathcal{A}_i,\ t \in \Omega_0,\ 1 \leq i \leq m$;
(iii) $u \in C([0,+\infty)^n : X)$, $\Omega \subseteq \Omega_{abs}(u)$, and (3.5) *holds.*

Then the function $u(\cdot)$ is a mild LT*-solution of* (1.3)*; furthermore, we have the following:*

(a) If $\mathcal{A}_i \equiv \mathcal{A}$ for $1 \leq i \leq m$, $Cf \in C([0,+\infty)^n : X)$, $Bu \in C([0,+\infty)^n : X)$, and $\Omega \subseteq \Omega_b(Bu) \cap \Omega(Bu) \cap \Omega_b(Cf)$, then $u(\cdot)$ is an LT*-mild solution of* (1.3).
(b) If $\mathcal{A}_i = A_i$ is single-valued for $1 \leq i \leq m$, $Cf \in C([0,+\infty)^n : X)$, $Bu \in C([0,+\infty)^n : X)$, $\Omega \subseteq \Omega_b(Bu) \cap \Omega(Bu) \cap \Omega_b(Cf)$, and, for every $i \in \mathbb{N}_m$, there exists a continuous function $u_i : [0,+\infty)^n \to X$ such that $(u(t+\tau_i), u_i(t)) \in \mathcal{A}_i$ for all $t \in [0,+\infty)^n$ and $i \in \mathbb{N}_m$, as well as $\Omega \subseteq \Omega(a_i *_0 u_i) \cap \Omega_b(a_i *_0 u_i)$ for all $i \in \mathbb{N}_m$, then $u(\cdot)$ is an LT*-strong solution of* (1.3).

Proof. Since all necessary requirements are satisfied, we get that (3.5) implies (3.3) and (3.4). Further on, by Lemma 1.2(v), (3.3) implies

$$B\hat{u}(\lambda) \in C\hat{f}(\lambda) + \sum_{i=1}^{m} \widehat{a_i}(\lambda) \mathcal{A}_i \Big(\mathcal{L}u(\cdot + \tau_i)\Big)(\lambda)$$
$$\subseteq C\hat{f}(\lambda) + \sum_{i=1}^{m} \mathcal{A}_i \Big(\widehat{a_i}(\lambda) \mathcal{L}u(\cdot + \tau_i)(\lambda)\Big)$$

$$= C\hat{f}(\lambda) + \sum_{i=1}^{m} \mathcal{A}_i \Big[\mathcal{L}\Big(a_i *_0 u(\cdot + \tau_i) \Big)(\lambda) \Big], \quad \lambda \in \Omega.$$

Therefore, $u(\cdot)$ is a mild LT-solution of (1.3). Let us prove (a). If $\mathcal{A}_i \equiv \mathcal{A}$ for $1 \leq i \leq m$ and all other requirements hold, then we have

$$\Big(\mathcal{L}[Bu(\cdot) - Cf(\cdot)] \Big)(\lambda) \in \mathcal{A}\Bigg(\sum_{i=1}^{m} \Big(a_i *_0 u(\cdot + \tau_i) \Big) \Bigg)(\lambda), \quad \lambda \in \Omega.$$

Since we have assumed that $Cf \in C([0, +\infty)^n : X)$, $Bu \in C([0, +\infty)^n : X)$, and $\Omega \subseteq \Omega_b(Bu) \cap \Omega(Bu) \cap \Omega_b(Cf)$, Lemma 1.2(iii) yields that

$$\Bigg(Bu(t) - Cf(t), \sum_{i=1}^{m} \Big(a_i *_0 u(\cdot + \tau_i) \Big)(t) \Bigg) \in \mathcal{A}, \quad t \in [0, +\infty)^n.$$

This implies that $u(\cdot)$ is an LT-mild solution of (1.3). Let us prove (b). If $\mathcal{A}_i = A_i$ is single-valued for $1 \leq i \leq m$ and all other requirements hold, then we can apply [6, Theorem 1.2.3] two times in order to conclude that

$$A_i\Big(\mathcal{L}\Big[a_i *_0 u(\cdot + \tau_i) \Big](\lambda) \Big) = \Big(\mathcal{L}[a_i *_0 u_i] \Big)(\lambda), \quad \lambda \in \Omega, \ 1 \leq i \leq m.$$

This implies

$$\Big(\mathcal{L}[a_1 *_0 u_1 + ... + a_m *_0 u_m] \Big)(\lambda) = \Big(\mathcal{L}[Bu - Cf] \Big)(\lambda), \quad \lambda \in \Omega.$$

Keeping in mind the prescribed assumptions, it readily follows from Lemma 1.2(iii) that $u(\cdot)$ is an LT-strong solution of (1.3). □

Before proceeding further, let us emphasize the following facts:

Remark 3.6. It is far from being true that the solution $u(\cdot)$, which has been constructed in the previous theorem, satisfies $u(t) = Cu_0(t)$, $t \in \Omega_0$. For example, if $u_0 \equiv u_{0,i} \equiv 0$ on Ω_0, then we have

$$\hat{u}(\lambda) = \Bigg[B - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1 \tau_i^1 + ... + \lambda_n \tau_i^n} \mathcal{A}_i \Bigg]^{-1} C\hat{f}(\lambda), \quad \lambda \in \Omega.$$

Furthermore, if $\mathcal{A}_i = 0$ for $1 \leq i \leq m$ and $B = C = \mathrm{I}$, then we have $u(t) = f(t)$, $t \geq 0$.

Remark 3.7. If there exist real numbers $\omega_1 \geq 0$, ..., $\omega_n \geq 0$, $\epsilon_1 > 0$, ..., $\epsilon_n > 0$ such that the functions $F : \Omega \to X$ and $BF : \Omega \to X$ are analytic, where

$$F(\lambda) := \Bigg[B - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1 \tau_i^1 + ... + \lambda_n \tau_i^n} \mathcal{A}_i \Bigg]^{-1} C \Bigg\{ \hat{f}(\lambda) - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1 \tau_i^1 + ... + \lambda_n \tau_i^n}$$
$$\times \int_{[0,+\infty)^n \setminus \big([\tau_i^1, +\infty) \times ... \times [\tau_i^n, +\infty) \big)} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} u_{0,i}(t_1, ..., t_n)\, dt_1 ... dt_n \Bigg\}, \quad \lambda \in \Omega,$$

and for each seminorm $p \in \circledast$ there exist a finite real constant $M_p > 0$ and a seminorm $q \in \circledast$ such that

$$p\Big(F(\lambda_1, ..., \lambda_n) x \Big) + p\Big(BF(\lambda_1, ..., \lambda_n) x \Big)$$

$$\leq M_p q(x)|\lambda_1|^{-1-\epsilon_1} \cdot ... \cdot |\lambda_n|^{-1-\epsilon_n}, \quad x \in X,\ \Re\lambda_j > \omega_j\ (1 \leq j \leq n),$$

then by Lemma 1.2(iv) there exists a continuous function $u : [0, +\infty)^n \to X$ such that the function $Bu : [0, +\infty)^n \to X$ is also continuous, as well as that for each seminorm $p \in \circledast$ there exists a finite real constant $M'_p > 0$ such that (1.7) holds with the function $f(\cdot)$ replaced therein with the function $u(\cdot)$ $[Bu(\cdot)]$.

The condition on continuity of solutions is a little bit redundant in Definition 3.5 (cf. also Remark 3.6) and, because of that, we would like to propose the following notion (the interested reader may also introduce and analyze the notion of a strong DC-solution of (1.3), as well):

Definition 3.8. Suppose that (F) holds.

(i) By a DCLT-mild solution of (1.3), we mean any Laplace transformable function $v : [0, +\infty)^n \to X$ such that $v(t) = cu_0(t)$ for a.e. $t \in \Omega_0$, for every $i \in \mathbb{N}_m$, we have $(a_i *_0 v(\cdot + \tau_i))(t) \in D(\mathcal{A}_i)$ for a.e. $t \in [0, +\infty)^n$, and

$$Bv(t) \in Cf(t) + \sum_{i=1}^{m} \mathcal{A}_i\Big(a_i *_0 v(\cdot + \tau_i)\Big)(t) \text{ for a.e. } t \in [0, +\infty)^n.$$

(ii) By a DCLT-strong solution of (1.3), we mean any Laplace transformable function $v : [0, +\infty)^n \to X$ such that $v(t) = cu_0(t)$ for a.e. $t \in \Omega_0$ and, for every $i \in \mathbb{N}_m$, there exists a Laplace transformable function $v_i : [0, +\infty)^n \to X$ such that $(v(t + \tau_i), v_i(t)) \in \mathcal{A}_i$ for a.e. $t \in [0, +\infty)^n$, and

$$Bv(t) = Cf(t) + \sum_{i=1}^{m} (a_i *_0 v_i)(t) \text{ for a.e. } t \in [0, +\infty)^n.$$

(iii) By a mild DCLT-solution of (1.3), we mean any Laplace transformable function $v : [0, +\infty)^n \to X$ such that there exist real numbers $\omega_i \in \mathbb{R}$ $(1 \leq i \leq n)$ satisfying that, for every $\lambda \in \Omega := \{\lambda_1 \in \mathbb{C} : \Re\lambda_1 > \omega_1\} \times ... \times \{\lambda_n \in \mathbb{C} : \Re\lambda_n > \omega_n\}$, we have $\lambda \in \Omega_{abs}(v) \cap \Omega(f)$ and

$$B\hat{v}(\lambda) \in C\hat{f}(\lambda) + \sum_{i=1}^{m} \mathcal{A}_i\bigg(\mathcal{L}\Big(a_i *_0 v(\cdot + \tau_i)\Big)(\lambda)\bigg). \tag{3.6}$$

The following existence-type result is new even in the one-dimensional setting, with $\mathcal{A}_i \equiv \mathcal{A}$ for all $i \in \mathbb{N}_m$:

Theorem 3.9. *Suppose that condition* (F) *and conditions* (i)–(iii) *from the formulation of* Theorem 3.5 *hold as well as that for each seminorm $p \in \circledast$ we have*

$$\int_{\Omega_0} p\bigg(e^{-\lambda_1 t_1 - ... - \lambda_n t_n} Cu_0(t_1, ..., t_n)\bigg)\, dt_1 ... dt_n < +\infty. \tag{3.7}$$

Define the function $v : [0, +\infty)^n \to X$ by $v(t) := Cu_0(t)$, if $t \in \Omega_0$, and $v(t) := u(t+h)$, if $t \in [0, +\infty)^n \setminus \Omega_0$. Then $\Omega \subseteq \Omega_{abs}(v)$, any point of discontinuity of function $v(\cdot)$ belongs to Γ_0, and the function $v(\cdot)$ is a mild DCLT-*solution of* (1.3). *Furthermore, we have the following:*

(a) If $\mathcal{A}_i \equiv \mathcal{A}$ *for* $1 \leq i \leq m$, $\Omega \subseteq \Omega_b(Bv) \cap \Omega(Bv) \cap \Omega_b(Cf)$, *there exists a Lebesgue measurable set* N_0 *such that* $m(N_0) = 0$, *the functions* $Bv(\cdot)$, $Cf(\cdot)$, *and* $(a_i *_0 v(\cdot + \tau_i))(\cdot)$ *are continuous on* $[0, +\infty)^n \setminus N_0$, *then* $v(\cdot)$ *is an* DCLT-*mild solution of* (1.3).

(b) Suppose that $\mathcal{A}_i = A_i$ *is single-valued for* $1 \leq i \leq m$, *there exists a Lebesgue measurable set* N_0 *such that* $m(N_0) = 0$, $Cf \in C([0, +\infty)^n \setminus N_0 : X)$, $Bv \in C([0, +\infty)^n \setminus N_0 : X)$, $\Omega \subseteq \Omega_b(Bv) \cap \Omega(Bv) \cap \Omega_b(Cf)$, *and, for every* $i \in \mathbb{N}_m$, *there exists a Laplace transformable function* $v_i : [0, +\infty)^n \to X$ *such that* $(v(t + \tau_i), v_i(t)) \in \mathcal{A}_i$ *for all* $t \in [0, +\infty)^n \setminus N_0$, *as well as* $\Omega \subseteq \Omega(a_i *_0 v_i) \cap \Omega_b(a_i *_0 v_i)$ *for all* $i \in \mathbb{N}_m$ *and* $a_i *_0 v_i \in C([0, +\infty)^n \setminus N_0 : X)$ *for all* $i \in \mathbb{N}_m$,*. Then* $v(\cdot)$ *is an* DCLT-*strong solution of* (1.3).

Proof. Since we have assumed (3.7), it is clear that $\Omega \subseteq \Omega_{abs}(v)$. Further on, we have $v(t) = Cu_0(t)$, $t \in \Omega_0$, $Cu_0 \in C(\Omega)$, and $u \in C([0, +\infty)^n)$, so that any point of discontinuity of function $v(\cdot)$ belongs to Γ_0. Now we will prove that Eq (3.6) holds with the function $u(\cdot)$ replaced therein with the function $v(\cdot)$. First of all, let us observe that for each $x \in D(B) \cap D(\mathcal{A}_1) \cap ... \cap D(\mathcal{A}_m)$ one has

$$\left[B - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1 \tau_i^1 + ... + \lambda_n \tau_i^n} \mathcal{A}_i\right]^{-1} \cdot \left[B - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1 \tau_i^1 + ... + \lambda_n \tau_i^n} \mathcal{A}_i\right] Cx = Cx,$$

which implies that

$$\begin{aligned} \hat{u}(\lambda) = & \int_{\Omega_0} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} Cu_0(t_1, ..., t_n)\, dt_1 ... dt_n \\ & + \left[B - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1 \tau_i^1 + ... + \lambda_n \tau_i^n} \mathcal{A}_i\right]^{-1} \\ & \times \Bigg\{ B \int_{\Omega_0} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} Cu_0(t_1, ..., t_n)\, dt_1 ... dt_n + C\hat{f}(\lambda) \qquad (3.8) \\ & - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1 \tau_i^1 + ... + \lambda_n \tau_i^n} \Bigg[\int_{\Omega_0} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} Cu_{0,i}(t_1, ..., t_n)\, dt_1 ... dt_n \\ & - \int_{[0,+\infty)^n \setminus \left([\tau_i^1, +\infty) \times ... \times [\tau_i^n, +\infty)\right)} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} Cu_{0,i}(t_1, ..., t_n)\, dt_1 ... dt_n \Bigg] \Bigg\} \\ := & \int_{\Omega_0} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} Cu_0(t_1, ..., t_n)\, dt_1 ... dt_n + w_L(\lambda). \qquad (3.9) \end{aligned}$$

Arguing as in the proof of Theorem 3.5, Eq (3.6) with the function $u(\cdot)$ replaced therein with the function $v(\cdot)$ is equivalent to (let us recall that $e^{-\lambda h} = e^{-\lambda_1 h_1 - ... - \lambda_n h_n}$):

$$\begin{aligned} & B\left[\int_{\Omega_0} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} Cu_0(t_1, ..., t_n)\, dt_1 ... dt_n + e^{-\lambda h}\Big(\mathcal{L}u(\cdot + h)\Big)(\lambda)\right] \\ & \in C\hat{f}(\lambda) + \sum_{i=1}^{m} \widehat{a_i}(\lambda) \mathcal{A}_i \Bigg[e^{\lambda_1 \tau_i^1 + ... + \lambda_n \tau_i^n} \\ & \times \left\{ \int_{\Omega_0} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} Cu_0(t_1, ..., t_n)\, dt_1 ... dt_n + e^{-\lambda h}\Big(\mathcal{L}u(\cdot + h)\Big)(\lambda) \right\} \end{aligned}$$

$$- e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n} \int_{[0,+\infty)^n \setminus \left([\tau_i^1,+\infty)\times...\times[\tau_i^n,+\infty)\right)} e^{-\lambda_1 t_1-...-\lambda_n t_n} Cu_0(t_1,...,t_n)\,dt_1...\,dt_n\Bigg],$$

for any $\lambda \in \Omega$, i.e., with

$$e^{-\lambda h}\Big(\mathcal{L}u(\cdot + h)\Big)(\lambda) = w_L(\lambda), \quad \lambda \in \Omega,$$

which is true on account of our definition of function $w_L(\cdot)$ in (3.8) and (3.9). Therefore, the function $v(\cdot)$ is a mild DCLT-solution of (1.3). The proofs of (a) and (b) can be deduced in almost the same way as the proofs of corresponding assertions of Theorem 3.5 and are therefore omitted. □

The following notion can be also introduced:

Definition 3.10. Suppose that (F) holds and $k = \delta$ or $k \in C([0,+\infty)^n)$. If there exists a strongly continuous operator family $(R(t))_{t\in[0,+\infty)^n} \subseteq L(X)$ and real numbers $\omega_i \in \mathbb{R}$ $(1 \le i \le n)$ such that, for every $\lambda \in \Omega := \{\lambda_1 \in \mathbb{C} : \Re\lambda_1 > \omega_1\} \times ... \times \{\lambda_n \in \mathbb{C} : \Re\lambda_n > \omega_n\}$, we have $[B - \sum_{i=1}^m \widehat{a_i}(\lambda)e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}\mathcal{A}_i]^{-1}C \in L(X)$, $\lambda \in \Omega(R(\cdot)x) \cap \Omega(k)$, $x \in X$, and

$$\begin{aligned}
&\hat{k}(\lambda)\left[B - \sum_{i=1}^m \widehat{a_i}(\lambda)e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}\mathcal{A}_i\right]^{-1} Cx \\
&= \int_0^{+\infty} ... \int_0^{+\infty} e^{-\lambda_1 t_1-...-\lambda_n t_n} R(t_1,...,t_n)x\,dt_1\,...\,dt_n, \quad \lambda \in \Omega,\ x \in X,
\end{aligned}$$

then it is said that $(R(t))_{t\in[0,+\infty)^n}$ is an exponentially equicontinuous, k-convoluted C-solution operator family for (1.3) and (1.4). Here, $\tilde{k}(\lambda) \equiv 1$ if $k = \delta$.

It is clear that the notion of an exponentially equicontinuous, k-convoluted C-solution operator family for (1.3) and (1.4) is a special case of the notion introduced in Definition 2.1(iii) only if $m = 1$; if this is not the case, then a new class of multidimensional (F, G, C)-solution operator families can be introduced and analyzed by replacing the functions $\hat{k}(\lambda)$ and $\widehat{a_i}(\lambda)e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}$ by general functions $F(\lambda)$ and $G_i(\lambda)$, respectively $(1 \le i \le m)$. We will skip all details regarding this issue here.

Concerning the notion introduced in Definition 3.10, we will only state the following multidimensional extension of [7, Theorem 3(ii)] without proof, which can be left to the interested readers:

Theorem 3.11. *Suppose that* (F) *holds and there exist real numbers $\omega_i \in \mathbb{R}$ $(1 \le i \le n)$ such that, for every $\lambda \in \Omega := \{\lambda_1 \in \mathbb{C} : \Re\lambda_1 > \omega_1\} \times ... \times \{\lambda_n \in \mathbb{C} : \Re\lambda_n > \omega_n\}$, all requirements from Definition 3.10 hold, $(R(t))_{t\in[0,+\infty)^n}$ is an exponentially equicontinuous, k-convoluted C-solution operator family for* (1.3) *and* (1.4)*, and there exists a Laplace transformable function $u_1 : [0,+\infty)^n \to X$ such that*

$$\begin{aligned}
\widehat{u_1}(\lambda) &= \sum_{i=1}^m \widehat{a_i}(\lambda)e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n} \\
&\times \int_{[0,+\infty)^n \setminus \left([\tau_i^1,+\infty)\times...\times[\tau_i^n,+\infty)\right)} e^{-\lambda_1 t_1-...-\lambda_n t_n} Cu_0(t_1,...,t_n)\,dt_1...\,dt_n,
\end{aligned}$$

for any $\lambda \in \Omega$. Then a continuous function $u : [0,+\infty)^n \to X$ is an LT*-strong solution of* (1.3) *if and only if*

$$(k *_0 u)(t) = \Big(R *_0 [f - u_1]\Big)(t), \quad t \in [0,+\infty)^n.$$

Now we will present some applications of Theorems 3.2, 3.5, and 3.9:

Example 3.12. (i) Suppose that $B = \mathrm{I}$, $m = 1$, $a \in L^1_{loc}([0,+\infty)^n)$, and $\mathcal{A}_1$ is a closed subgenerator of an exponentially equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,+\infty)^n}$; cf. also [7, Example 4(ii)]. Then Theorem 3.2 and Theorem 3.5 can be simply applied in the analysis of problem

$$u(t) \in Cf(t) + \mathcal{A}_1\Big(b *_0 u(\cdot + \tau_1)\Big)(t), \quad t \in [0,+\infty)^n,$$

where

$$b(t_1,...,t_n) := a(t_1 - \tau_1^1,...,t_n - \tau_1^n), \quad t_1 \geq \tau_1^1,..., \; t_n \geq \tau_1^n,$$

and

$$b(t_1,...,t_n) := 0 \quad \text{if there exists } j \in \mathbb{N}_n \text{ such that } t_j < \tau_1^j.$$

Keeping in mind Lemma 1.2(vii), we get that

$$\Big[B - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}\mathcal{A}_i\Big]^{-1} C\Big\{\hat{f}(\lambda) - \sum_{i=1}^{m} \widehat{a_i}(\lambda) e^{\lambda_1\tau_i^1+...+\lambda_n\tau_i^n}$$
$$\times \int_{[0,+\infty)^n \setminus ([\tau_i^1,+\infty)\times...\times[\tau_i^n,+\infty))} e^{-\lambda_1 t_1-...-\lambda_n t_n} u_{0,i}(t_1,...,t_n)\,dt_1...\,dt_n\Big\}$$
$$= \Big(\mathrm{I} - \hat{a}(\lambda)\mathcal{A}_1\Big)^{-1} C\Big\{\hat{f}(\lambda) - \widehat{a}(\lambda)$$
$$\times \int_{[0,+\infty)^n \setminus ([\tau_1^1,+\infty)\times...\times[\tau_1^n,+\infty))} e^{-\lambda_1 t_1-...-\lambda_n t_n} u_{0,i}(t_1,...,t_n)\,dt_1...\,dt_n\Big\}, \quad \lambda \in \Omega,$$

and we can make many relevant applications provided that the function

$$\lambda \mapsto \frac{1}{\hat{k}(\lambda)}\Big\{\hat{f}(\lambda) - \widehat{a}(\lambda) \int_{[0,+\infty)^n \setminus ([\tau_1^1,+\infty)\times...\times[\tau_1^n,+\infty))} e^{-\lambda_1 t_1-...-\lambda_n t_n}$$
$$\times u_{0,i}(t_1,...,t_n)\,dt_1...\,dt_n\Big\}, \quad \lambda \in \Omega, \tag{3.10}$$

is well-defined and satisfies all requirements for application of Lemma 1.2(iv) or Lemma 1.3. In particular, if $n = 2$, then Theorem 3.5 implies that $u(t) = Cu_0(t)$, if $0 \leq t_1 < \tau_1^1$ and $0 \leq t_2 < \tau_1^2$, as well as that

$$u(t) \in Cf(t) + \mathcal{A}_1 \int_0^{t_1}\int_0^{t_2} b(s_1,s_2) u(t_1 - s_1 + \tau_1^1, t_2 - s_2 + \tau_1^2)\,ds_1\,ds_2$$
$$= Cf(t) + \mathcal{A}_1 \int_{\tau_1^1}^{t_1}\int_{\tau_1^2}^{t_2} a(s_1 - \tau_1^1, s_2 - \tau_1^2) u(t_1 - s_1 + \tau_1^1, t_2 - s_2 + \tau_1^2)\,ds_1\,ds_2$$
$$= Cf(t) + \mathcal{A}_1 \int_0^{t_1-\tau_1^1}\int_0^{t_2-\tau_1^2} a(s_1,s_2) u(t_1 - s_1, t_2 - s_2)\,ds_1\,ds_2, \; t \in [0,+\infty)^2,$$

where we define $a(\cdot)$ by zero outside the first quadrant.

(ii) In the following part, we will use the operators from [6, Example 2.2.18]; cf. also [7, Examples 1 and 5]. Assume that $s > 1$,

$$X := \Big\{ f \in C^{\infty}[0,1] \; ; \; \|f\| := \sup_{p\geq 0} \frac{\|f^{(p)}\|_{\infty}}{p!^s} < \infty \Big\},$$

and

$$A := -d/ds,\ D(A) := \{f \in X\ ;\ f' \in X,\ f(0) = 0\}.$$

Then $(X, \|\cdot\|)$ is a complex Banach space. For any complex polynomial $P(z)$, we define a closed linear operator $P(A)$ on X in the usual way.

Assume now that $\omega_i \in \mathbb{R}$ $(1 \leq i \leq n)$, $\Omega_{abs}(a_i) \cup \Omega(f) \subseteq \Omega := \{\lambda_1 \in \mathbb{C} : \Re\lambda_1 > \omega_1\} \times ... \times \{\lambda_n \in \mathbb{C} : \Re\lambda_n > \omega_n\}$ $(1 \leq i \leq m)$, $Q_1(z) = \sum_{j=0}^{N_1} a_{j,1} z^j$, $z \in \mathbb{C}$, $a_{N_1,1} \neq 0$ is a complex non-zero polynomial, $Q_2(z) = \sum_{j=0}^{N_2} a_{j,2} z^j$, $z \in \mathbb{C}$, $a_{N_2,2} \neq 0$ is a complex non-zero polynomial, and $N_1 = dg(Q_1) > 1 + dg(Q_2) = 1 + N_2$. Then we know that there exist real numbers $b > 0$, $c > 0$, and $\zeta > 0$ such that

$$\left\|(\lambda Q_2(A) - Q_1(A))^{-1}\right\| = O\left(e^{b|\lambda|^{1/(N_1-N_2)s} + c|\lambda|^{1/(N_1-N_2)}}\right), \quad \lambda \in \mathbb{C},$$

and

$$\left\|Q_2(A)(\lambda Q_2(A) - Q_1(A))^{-1} f\right\| \leq \zeta \|f\| e^{b|\lambda|^{1/(N_1-N_2)s} + c|\lambda|^{1/(N_1-N_2)}}, \tag{3.11}$$

for all $\lambda \in \mathbb{C}$ and $f \in D(Q_2(A))$.

Let us consider now the problem

$$Bu(t) \in f(t) + \sum_{i=1}^{m} P_i(A)\Big(b_i *_0 u(\cdot + \tau_i)\Big)(t), \quad t \in [0, +\infty)^n, \tag{3.12}$$

where $P_i(\cdot)$ are complex polynomials $(0 \leq i \leq m)$ such that $B = P_0 = Q_2$ and $\sum_{i=1}^m P_i = Q_1$ as well as that, for every $i \in \mathbb{N}_m$, we have $b_i(t_1, ..., t_n) := a_i(t_1 - \tau_1^1, ..., t_n - \tau_1^n)$, $t_1 \geq \tau_1^1, ..., t_n \geq \tau_1^n$, and $b_i(t_1, ..., t_n) := 0$, if there exists $j \in \mathbb{N}_n$ such that $t_j < \tau_1^j$. Using again Lemma 1.2(vii), we have $\widehat{b_i}(\lambda) e^{\lambda_1 \tau_i^1 + ... + \lambda_n \tau_i^n} = \widehat{a_i}(\lambda)$, $\lambda \in \Omega$, $i \in \mathbb{N}_m$, and one can apply Theorems 3.2 or 3.5 provided that the term

$$\left\|\left[P_0(A) - \sum_{i=1}^{m} \widehat{a_i}(\lambda) P_i(A)\right]^{-1}\right\| + \left\|P_0(A)\left[P_0(A) - \sum_{i=1}^{m} \widehat{a_i}(\lambda) P_i(A)\right]^{-1}\right\|$$

can be majorized on Ω by the constant multiple of the term $\exp(|\lambda_1|^{\sigma_1}) \cdot ... \cdot \exp(|\lambda_n|^{\sigma_n})$, where $\sigma_i \in (0, 1)$ for $1 \leq i \leq n$. This occurs, for example, if $a_i(t) = a(t) = g_{\zeta_1}(t_1) g_{\zeta_2}(t_2)$ for $1 \leq i \leq m$ and $\max(\zeta_1, \zeta_2) < N_1 - N_2$, when we have

$$\begin{aligned}\left\|\left[P_0(A) - \sum_{i=1}^{m} \widehat{a_i}(\lambda) P_i(A)\right]^{-1}\right\| &= \left\|\lambda_1^{\zeta_1} \lambda_2^{\zeta_2} \left[\lambda_1^{\zeta_1} \lambda_2^{\zeta_2} Q_2(A) - Q_1(A)\right]^{-1}\right\| \\ &\leq \left|\lambda_1\right|^{\zeta_1} \left|\lambda_2\right|^{\zeta_2} e^{b\left|\lambda_1\right|^{\zeta_1/(N_1-N_2)s} \left|\lambda_2\right|^{\zeta_2/(N_1-N_2)s} + c\left|\lambda_1\right|^{\zeta_1/(N_1-N_2)} \left|\lambda_2\right|^{\zeta_2/(N_1-N_2)}}, \quad \lambda \in \mathbb{C}^2.\end{aligned}$$

Using the estimate (3.11), we can similarly majorize the term

$$\left\|P_0(A)\left[P_0(A) - \sum_{i=1}^{m} \widehat{a_i}(\lambda) P_i(A)\right]^{-1}\right\|$$

on Ω, provided that the function defined by (3.10) belongs to $D(Q_2(A)) = D(A^{N_2})$.

3.1. The abstract functional partial fractional differential inclusions

In this subsection, we investigate some classes of the abstract functional partial fractional differential inclusions with Caputo variables. We will consider a two-dimensional setting for simplicity.

First of all, we will precisely compute the double Laplace transform of the mixed partial fractional derivative $\mathbf{D}_{t_1}^{\alpha_1}\mathbf{D}_{t_2}^{\alpha_2}u(\cdot,\cdot)$:

Theorem 3.13. *Suppose that $u \in L^1_{loc}([0,+\infty)^2)$, $\omega_i \in \mathbb{R}$, $\alpha_i > 0$, $m_i = \lceil \alpha_i \rceil$, $i = 1, 2$, the mixed partial fractional Caputo derivative $\mathbf{D}_{t_1}^{\alpha_1}\mathbf{D}_{t_2}^{\alpha_2}u(\cdot,\cdot)$ is well-defined and locally integrable, the functions $t_1 \mapsto [u^{(0,k_2)}(t_1,t_2)]_{t_2=0}$, $t_1 \geq 0$, and $t_2 \mapsto [(\partial^{k_1}/\partial t_1^{k_1})\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2)]_{t_1=0}$, $t_2 \geq 0$, are locally integrable for $0 \leq k_1 \leq m_1 - 1$ and $0 \leq k_2 \leq m_2 - 1$, and for each seminorm $p \in \circledast$ there exists a real constant $M_p > 0$ such that*

$$\begin{aligned} &p\Big(\mathbf{D}_{t_1}^{\alpha_1}\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2)\Big) + p\Big(\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2)\Big) + p\Big(u(t_1,t_2)\Big) \\ &+ \sum_{k_2=0}^{m_2-1} p\Big(\Big[u^{(0,k_2)}(t_1,t_2)\Big]_{t_2=0}\Big) + \sum_{k_1=0}^{m_1-1} p\Big(\Big[\frac{\partial^{k_1}}{\partial t_1^{k_1}}\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2)\Big]_{t_1=0}\Big) \\ &\leq M_p \exp(\omega_1 t_1 + \omega_2 t_2), \end{aligned}$$

for any $t_1,\ t_2 \geq 0$. Then we have

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty} e^{-\lambda_1 t_1 - \lambda_2 t_2}\mathbf{D}_{t_1}^{\alpha_1}\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2)\,dt_1\,dt_2 \\ &= \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\int_0^{+\infty}\int_0^{+\infty} e^{-\lambda_1 t_1 - \lambda_2 t_2}u(t_1,t_2)\,dt_1\,dt_2 \\ &- \lambda_1^{\alpha_1}\sum_{k_2=0}^{m_2-1}\lambda_2^{\alpha_2-1-k_2}\int_0^{+\infty} e^{-\lambda_1 t_1}\Big[u^{(0,k_2)}(t_1,t_2)\Big]_{t_2=0}\,dt_1 \\ &- \sum_{k_1=0}^{m_1-1}\lambda_1^{\alpha_1-1-k_1}\int_0^{+\infty} e^{-\lambda_2 t_2}\Big[\frac{\partial^{k_1}}{\partial t_1^{k_1}}\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2)\Big]_{t_1=0}\,dt_2,\ \Re\lambda_1 > \omega_1,\ \Re\lambda_2 > \omega_2. \end{aligned} \tag{3.13}$$

Proof. Since all terms in (3.13) are well-defined, taking the appropriate functionals we may assume w.l.o.g. that $X = \mathbb{C}$. Keeping in mind the given assumptions, the formula (1.5), and the Fubini theorem, the proof follows from the next calculation:

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty} e^{-\lambda_1 t_1 - \lambda_2 t_2}\mathbf{D}_{t_1}^{\alpha_1}\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2)\,dt_1\,dt_2 \\ &= \int_0^{+\infty} e^{-\lambda_2 t_2}\Big\{\lambda_1^{\alpha_1}\int_0^{+\infty} e^{-\lambda_1 t_1}\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2)\,dt_1 - \sum_{k_1=0}^{m_1-1}\lambda_1^{\alpha_1-k_1-1}\Big[\frac{\partial^{k_1}}{\partial t_1^{k_1}}\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2)\Big]_{t_1=0}\Big\}\,dt_2 \\ &= \lambda_1^{\alpha_1}\int_0^{+\infty} e^{-\lambda_1 t_1}\int_0^{+\infty} e^{-\lambda_2 t_2}\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2)\,dt_2\,dt_1 \\ &- \sum_{k_1=0}^{m_1-1}\lambda_1^{\alpha_1-k_1-1}\int_0^{+\infty} e^{-\lambda_2 t_2}\Big[\frac{\partial^{k_1}}{\partial t_1^{k_1}}\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2)\Big]_{t_1=0}\,dt_2 \\ &= \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\int_0^{+\infty}\int_0^{+\infty} e^{-\lambda_1 t_1 - \lambda_2 t_2}u(t_1,t_2)\,dt_1\,dt_2 \end{aligned}$$

$$- \lambda_1^{\alpha_1} \sum_{k_2=0}^{m_2-1} \lambda_2^{\alpha_2-k_2-1} \int_0^{+\infty} e^{-\lambda_1 t_1} \Big[u^{(0,k_2)}(t_1,t_2) \Big]_{t_2=0} \, dt_1$$
$$- \sum_{k_1=0}^{m_1-1} \lambda_1^{\alpha_1-k_1-1} \int_0^{+\infty} e^{-\lambda_2 t_2} \left[\frac{\partial^{k_1}}{\partial t_1^{k_1}} \mathbf{D}_{t_2}^{\alpha_2} u(t_1,t_2) \right]_{t_1=0} dt_2,$$

which simply yields (3.13). □

Now we will examine possible applications of Theorem 3.13 in the study of the simplest forms of abstract functional partial fractional differential equations with Caputo derivatives; cf. also [7, Example 3]:

Example 3.14. (i) Let $\alpha_i > 0$, $m_i = \lceil \alpha_i \rceil$, $i = 1, 2$, and let us consider the following problem:

$$\mathbf{D}_{t_1}^{\alpha_1} \mathbf{D}_{t_2}^{\alpha_2} u(t_1,t_2) \in \mathcal{A} u(t_1+a, t_2+b) + f(t_1,t_2), \quad t_1 \geq 0, \ t_2 \geq 0, \tag{3.14}$$

where $a \geq 0$, $b \geq 0$, and $ab \neq 0$. If $a > 0$ and $b > 0$, then the computation (3.15) carried out below shows that we should accompany the inclusion (3.14) with the initial condition $u(t_1,t_2) = u_0(t_1,t_2)$, where $u_0 \in L^1_{loc}([0,+\infty)^2 \setminus ([a,+\infty) \times [b,+\infty)))$ is a given function since, in this case, the values of terms

$$f_{k_1}(t_1) := \Big[u^{(0,k_2)}(t_1,t_2) \Big]_{t_2=0}, \quad 0 \leq k_1 \leq m_1 - 1,$$

and

$$h_{k_2}(t_2) := \left[\frac{\partial^{k_1}}{\partial t_1^{k_1}} \mathbf{D}_{t_2}^{\alpha_2} u(t_1,t_2) \right]_{t_1=0}, \quad 0 \leq k_2 \leq m_2 - 1,$$

are already determined. If $a > 0$ and $b = 0$, then we should accompany the inclusion (3.14) with the initial condition $u(t_1,t_2) = u_0(t_1,t_2)$, where $u_0 \in L^1_{loc}([0,+\infty)^2 \setminus ([a,+\infty) \times [b,+\infty)))$ is a given function and the initial conditions $f_{k_1}(t_1),\ t_1 \geq a,\ 0 \leq k_1 \leq m_1 - 1$. Finally, if $a = 0$ and $b > 0$, then we should accompany the inclusion (3.14) with the initial condition $u(t_1,t_2) = u_0(t_1,t_2)$, where $u_0 \in L^1_{loc}([0,+\infty)^2 \setminus ([a,+\infty) \times [b,+\infty)))$ is a given function and the initial conditions $h_{k_2}(t_2),\ t_2 \geq b,\ 0 \leq k_1 \leq m_2 - 1$.
We will consider the case where $a > 0$ and $b > 0$ below. If $u(\cdot,\cdot)$, $f(\cdot,\cdot)$, $f_{k_1}(\cdot)$, and $h_{k_2}(\cdot)$ are Laplace transformable functions ($0 \leq k_1 \leq m_1 - 1$; $0 \leq k_2 \leq m_2 - 1$), then there exist $\omega_1 \geq 0$ and $\omega_2 \geq 0$ such that (cf. Lemma 1.2(vi) and Theorem 3.13):

$$\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \hat{u}(\lambda) - \sum_{k_2=0}^{m_2-1} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2-1-k_2} \widehat{h_{k_2}}(\lambda_2) - \sum_{k_1=0}^{m_1-1} \lambda_1^{\alpha_1-1-k_1} \widehat{f_{k_1}}(\lambda_1) - \hat{f}(\lambda)$$
$$\in \mathcal{A} \left[e^{\lambda_1 a + \lambda_2 b} \hat{u}(\lambda) - e^{\lambda_1 a + \lambda_2 b} \int_{[0,+\infty)^2 \setminus ([a,+\infty) \times [b,+\infty))} e^{-\lambda_1 t_1 - \lambda_2 t_2} u_0(t_1,t_2) \, dt_1 \, dt_2 \right], \tag{3.15}$$

for $\Re \lambda_1 > \omega_1$ and $\Re \lambda_2 > \omega_2$. Arguing as in [7, Example 3(i)], it follows that the uniqueness of solutions to (3.14) can be only proved in the case that $\sigma_p(\mathcal{A}) \subseteq \{0\}$, where $\sigma_p(\mathcal{A})$ denotes the point spectrum of $\mathcal{A}$, while the existence of solutions to (3.14) can be proved only if $\mathbb{C} \setminus \{0\} \subseteq \rho_C(\mathcal{A})$, where $\rho_C(\mathcal{A})$ denotes the C-resolvent set of $\mathcal{A}$, since the operator $\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} - e^{\lambda_1 a + \lambda_2 b} \mathcal{A}$ has to be injective for $\Re \lambda_1 > \omega_1$ and $\Re \lambda_2 > \omega_2$, resp., we should have $[\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} - e^{\lambda_1 a + \lambda_2 b} \mathcal{A}]^{-1} C \in L(X)$ for $\Re \lambda_1 > \omega_1$ and $\Re \lambda_2 > \omega_2$.

(ii) There are also many difficulties with regards to the possible applications of double vector-valued Laplace transforms in the study of problem

$$\mathbf{D}_{t_1}^{\alpha_1}\mathbf{D}_{t_2}^{\alpha_2}u(t_1,t_2) \in \mathcal{A}(Pu)(t_1,t_2) + Cf(t_1,t_2), \quad t_1 \geq 0,\ t_2 \geq 0, \tag{3.16}$$

where $a \geq 0$, $b \geq 0$, $ab \neq 0$, $Pu(t_1,t_2) := u(t_1 - a, t_2 - b)$, $t_1 \geq a$, $t_2 \geq b$, and $Pu(t_1,t_2) := 0$, otherwise. Applying Lemma 1.2(vii) and Theorem 3.13, with $f_{k_1}(t_1) \equiv 0,\ 0 \leq k_1 \leq m_1 - 1$, and $h_{k_2}(t_2) \equiv 0,\ 0 \leq k_2 \leq m_2 - 1$, we should have

$$\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\hat{u}(\lambda) - C\hat{f}(\lambda) \in e^{-\lambda_1 a - \lambda_2 b}\mathcal{A}\hat{u}(\lambda), \quad \Re\lambda_1 > \omega_1,\ \Re\lambda_2 > \omega_2,$$

so that the solution $u(\cdot,\cdot)$ of (3.16) is given by

$$\hat{u}(\lambda) = \left[\lambda_1^{\alpha_1}\lambda_2^{\alpha_2} - e^{-\lambda_1 a - \lambda_2 b}\mathcal{A}\right]^{-1}C\hat{f}(\lambda), \quad \Re\lambda_1 > \omega_1,\ \Re\lambda_2 > \omega_2.$$

In this situation, we can construct the solution only in the case that the C-resolvent set of $\mathcal{A}$ contains a set $\mathbb{C} \setminus K$, where K is a compact set. For example, if A is the operator from Example 3.12(ii) and $P(\cdot)$ is any non-zero complex polynomial with $dg(P) \geq 2$, then $\rho(P(A)) = \mathbb{C}$ and there exists a bounded linear operator $C \in L(X)$ such that, for every $r > 0$, there exists a finite real number $M_r \geq 1$ such that $\|(\lambda + P(A))^{-1}C\| \leq M_r/(1 + |\lambda|)$ for $|\lambda| \geq r$. Then we can apply Lemma 1.2(iv) to prove the existence of Laplace transformable solutions to (3.16) in the case that $\alpha_1 > 1$ and $\alpha_2 > 1$.

4. Conclusions

In this paper, we have investigated some classes of abstract (functional) Volterra integro-differential inclusions with multiple variables and abstract (functional) partial fractional differential inclusions with multiple variables. We have also provided some applications of the introduced classes of multidimensional (F, G, C)-resolvent operator families.

Let us finally mention some topics which have not been considered:

1. In [8], we have not analyzed the differential properties of multidimensional vector-valued Laplace transform (cf. also [6, Theorems 3.2.25 and 3.2.26]). We will consider the differential properties of multidimensional (F, G, C)-solution operator families at a later time.

2. We have recently employed the vector-valued Laplace transform to introduce and thoroughly analyze the notion of a generalized Laplace fractional derivative. The abstract fractional differential inclusions with generalized Laplace derivatives are explored with the help of (F, G, C)-regularized resolvent operator families. We will consider the multidimensional generalized Laplace fractional derivatives and abstract fractional differential inclusions with multidimensional generalized Laplace derivatives in a follow-up research study.

3. We will not consider here the multidimensional analogues of problem [7, (14); pp. 13–15].

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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