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Research article

Coupled fixed point results for non-expansive mapping and its applications to the fractional order HIV/AIDS model

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Abstract: In this work, we develop certain coupled fixed-point results in Banach space by utilizing the Krasnosel'skii's expansive-type fixed-point results. Additionally, we establish existence and uniqueness conditions for coupled fixed points. Applying these results, we investigate the fractional-order HIV/AIDS model, demonstrating the existence of solutions. Our results contribute to the understanding of complex systems and non linear dynamics.

Keywords: coupled fixed point; measure of non-compactness; convex power condensing operator; HIV/AIDS model
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

The investigation of fixed points (in short, FP) for mappings satisfying contractive conditions has been the focus of intense research activity. It is widely used in various areas, such as fractal image decoding, parameterized estimation problems, nonlinear and adaptive control systems, and recurrent network convergence. The first result in FP theory is the Banach contraction principle (in short, BCP) [3], which is a well-known result in fixed-point theory offering a wide range of applications. This principle ensures both the existence and uniqueness of FP for self-mappings on complete metric spaces. By changing the conditions on the mapping or by extending the underlying set's properties, mathematicians have expanded the BCP [12–15].

In certain cases, the resulting mapping is not contractive but rather another kind of mapping, like an expansive or non-expansive one. The investigation of expansive mappings is a fascinating research area of FP theory. Wang et al. [26] introduced the idea of expansive mapping and established several FP theorems in complete metric spaces. Many authors have since explored expansive mappings and their fixed-point results [6, 9–11, 17]. The collection of references presents various developments in the theory of fixed points for expansion and contractive mappings. Kang et al. explored foundational results on fixed points for expansion mappings, establishing key conditions under which such points exist. Further advancements by Kang et al. expanded these results, introducing new theorems that refined the understanding of expansion mappings and their fixed-point behavior. More recently, Kang and Fang introduced novel contractive conditions in Menger PbM-metric spaces, offering new coupled fixed-point results that broaden the applicability of fixed-point theory. While Douek et al.'s work is unrelated to fixed-point theory, it offers valuable insights into the immunopathogenesis of AIDS, reflecting interdisciplinary research diversity.

Both the BCP and the results of Wang et al. [26] are ineffective in determining the FP of the operator equation for the sum of a contraction/expansion and compact operator. A helpful result for determining the FP in such a case is the Krasnosel'skii FP theorem. This pleasant finding has several potential uses. The FP theorem of Krasnosel'skii has experienced so many generalizations and modifications. Burton [5] improved Krasnosel'skii's FP theorem and applied it to integral equations, stability theory, and cover cases where Krasnosel'skii's FP theorem fails. Many researchers generalized Krasnosel'skii's and Burton's FP theorems. However, it should be noted that they only investigated the contraction map. The existence of FP for the sum of expansive and compact operators has been studied in several papers, as surveyed in the relevant literature. Motivated by this, Xiang and Yuan [28] obtained some FP theorems for determining the FP of the sum of a compact and expansion operator.

The coupled fixed-point (in short, CFP) theory, developed by Guo and Lakhsmikantham [8], is a helpful approach for determining the FP of a system of sums of a compact and contraction/expansion operator. The application potential of CFP theorems has attracted the attention of many renowned researchers. Coupled fixed-point theory helps us better understand complicated biological processes and enhances our mathematical abilities. Notably, HIV/AIDS is a perfect illustration of the complicated dynamics that are present in biological systems. In the context of the critical impact that HIV/AIDS has on global health, ongoing research efforts remain essential to fighting this epidemic and improving the lives of millions worldwide.

Since the latter part of the 20th century, millions of individuals worldwide have been infected with HIV/AIDS, which has become a global epidemic [25]. Similarly, for more epidemic diseases, see [29–31]. HIV is a virus that targets $CD^{+}4$ T cells, which are immune system cells that provide a barrier against viral infection [16]. The primary route for HIV transmission is bodily fluids. Additionally, common modes of transmission include sharing syringes and needles, unprotected sexual contact, and transmission from mother to child during delivery or breastfeeding. Therefore, HIV can gradually reduce immunity, making it more challenging for the body to fight against infections and illnesses. Due to the negative impacts of this virus on healthcare systems, public health, communities, and advancements in medical treatment and research, many researchers have worked to control the virus, prevent its spread, and enable those who are infected with HIV to live a healthier life, enhance their overall health, and lower the risk of transmission to others. HIV is a retrovirus that causes AIDS [27]. HIV infects, destroys, and reduces $CD^{+}4$ T cells, thereby lowering immune system defense [7]. The body gets much more highly responsive towards infections and gradually loses its defense. One of today's most serious and life-threatening diseases is AIDS. According to UNAIDS's 2020 annual review, around the world, 38 million people had HIV in 2019, 1.7 million people contracted the virus for the first time, and 690 thousand people died from AIDS-related diseases. Despite significant progress in handling the disease, no vaccine for HIV has ever been found. Researchers have devoted considerable effort over the last two decades to developing mathematical models that play a crucial role in understanding HIV-related disease control and prevention. These mathematical models typically describe the dynamics between HIV viruses and uninfected $CD^{+}4$ cells, as well as the effects of drug treatment on infected cells. The author of [18] extends the HIV model presented in [1, 24] by incorporating those who are infected but not yet infectious, using conformable derivative with fractional order in the Liouville-Caputo derivative sense as

$$\begin{cases} {}_{0}^{\mathrm{q}}D_t^{\omega}S = \rho - \iota(I + \upsilon_c C + \upsilon_A \mathcal{A})S - dS, \\ {}_{0}^{\mathrm{q}}D_t^{\omega}E = \iota\mu(I + \upsilon_c C + \upsilon_A \mathcal{A})S - (\sigma + d)E, \\ {}_{0}^{\mathrm{q}}D_t^{\omega}I = (1-\mu)\iota(I + \upsilon_c C + \upsilon_A \mathcal{A})S + \sigma E - (\tau_1 + \tau_2 + d)I + \lambda C + \theta\mathcal{A}, \\ {}_{0}^{\mathrm{q}}D_t^{\omega}C = \tau_1 I - (\lambda + d)C, \\ {}_{0}^{\mathrm{q}}D_t^{\omega}\mathcal{A} = \tau_2 I - (d + \theta + d_1)\mathcal{A}, \end{cases} \tag{1.1}$$

with initial conditions, $S(t_0) = S_0$, $E(t_0) = E_0$, $I(t_0) = I_0$, $C(t_0) = C_0$, and $\mathcal{A}(t_0) = \mathcal{A}_0$ and $S(t)$, $E(t)$, $I(t)$, $C(t)$, and $A(t)$ are number of susceptible individuals, the number of exposed but not yet infectious individuals, the number of individuals under care or treatment, and the number of individuals in the AIDS stage. The parameters used in the model above are given in Table 1.

Table 1. An overview of the parameters used in model (1.1).

Symbols	Interpretations
p	Recruitment ratio
ι	Contact ratio
d_1	Death ratio due to AIDS
d	Natural mortality ratio
μ	A portion of susceptible enter to E
υ_c, υ_A	Relative infectiousness ratios
σ	Rate of progression from exposed to infected class
τ_1	Treatment ratio
τ_2	Rate at which infected individuals progress to the AIDS stage
θ	Rate of recovery from AIDS stage due to treatment
λ	Failure rate of treatment

Fractional calculus has diverse applications in modeling, engineering, biology, oncology and dynamical systems, as discussed in [19, 20]. The Mittag-Leffler function and generalized fractional derivatives have found wide-ranging applications across various scientific and engineering disciplines due to their ability to model systems with memory and hereditary properties [21, 22]. The species diversity of microbiomes is a cutting-edge concept in metagenomic research. In [23], the authors discussed a multifractal analysis for metagenomic research. The fractional conformable derivative, while effective in capturing some aspects of the system's dynamics, is not what we are particularly interested in utilizing the Atangana-Baleanu-Caputo (ABC) fractional derivative will enhance our model's ability to capture complex dynamics. The ABC derivative, a powerful tool in fractional calculus, enables a more comprehensive description of real-world phenomena. The proposed method offers several advantages, including the effective modeling of memory effects in dynamic systems through ABC derivative, which capture nonlocal phenomena. Additionally, the CF approach provides flexibility in dealing with expansive mappings, enabling the analysis of complex systems. However, some limitations arise, such as the limited availability of numerical schemes for ABC derivatives and the reliance on compactness assumptions, which may restrict applicability in certain contexts. Despite these challenges, the method presents a promising framework for investigating non-linear dynamics. The study of CF in Banach spaces is crucial for understanding complex systems. The proposed technique, utilizing Krasnosel'skii's expansive-type fixed-point results, offers advantages in modeling nonlinear dynamics and capturing memory effects, making it particularly relevant for applications like the fractional-order HIV/AIDS model.

Inspired by the ABC-fractional derivative and the HIV/AIDS-model, we prolong the HIV/AIDS-model (1.1) to the ABC derivative with fractional order as

$$\begin{cases} {}_0^{\mathrm{ABC}}D_t^{\omega} S = \rho - \iota(I + \upsilon_c C + \upsilon_A \mathcal{A})S - dS, \\ {}_0^{\mathrm{ABC}}D_t^{\omega} E = \iota\mu(I + \upsilon_c C + \upsilon_A \mathcal{A})S - (\sigma + d)E, \\ {}_0^{\mathrm{ABC}}D_t^{\omega} I = (1-\mu)\iota(I + \upsilon_c C + \upsilon_A \mathcal{A})S + \sigma E - (\tau_1 + \tau_2 + d)I + \lambda C + \theta \mathcal{A}, \\ {}_0^{\mathrm{ABC}}D_t^{\omega} C = \tau_1 I - (\lambda + d)C, \\ {}_0^{\mathrm{ABC}}D_t^{\omega} \mathcal{A} = \tau_2 I - (d + \theta + d_1)\mathcal{A}, \end{cases} \tag{1.2}$$

having identical initial conditions as stated in model (1.1), where ${}_0^{\mathrm{ABC}}D_t^{\omega}$ is the ABC derivative with fractional order. Furthermore, the current work aims to illustrate specific CFP results by using the results of Xiang and Yuan [28] and applying the established results to the HIV/AIDS model (1.2).

2. Preliminaries

In this part, we review some essential notions, facts, definitions, and basic results that will serve as a foundation for our established results. In the next section, for simplicity, let $\mathfrak{E} = (\mathfrak{E}, \|.\|)$ and $\mathscr{X} = (\mathscr{X}, d)$ denote the Banach space (in short, BS) and complete metric space, respectively. Further, $\mathfrak{T}$ and $\mathfrak{S}$ are functions from $\mathfrak{I}$ to $\mathfrak{E}$ such that

(H_1) $\mathfrak{S}$ is continuous, $\mathfrak{S}(\mathfrak{I})$ contained within a compact subset of $\mathfrak{E}$;
(H_2) $\mathfrak{T}$ an is expansive map;
(H_3) $\varkappa \in \mathfrak{S}(\mathfrak{I}) \Longrightarrow \mathfrak{T}(\mathfrak{I}) + \varkappa \supset \mathfrak{I}$, where $\mathfrak{T}(\mathfrak{I}) + \varkappa = \{y + \varkappa \mid y \in \mathfrak{T}(\mathfrak{I})\}$;
(H_4) any $\varkappa \in \mathfrak{S}(\mathfrak{I}) \Longrightarrow \mathfrak{I} + \varkappa \subset \mathfrak{T}(\mathfrak{I}) \subset \mathfrak{I}$, where $\mathfrak{I} + \varkappa = \{y + \varkappa \mid y \in \mathfrak{I}\}$;

(H_5) $\mathfrak{S}(\mathfrak{I}) \subset (I-\mathfrak{T})(\mathfrak{E})$ and $[\varkappa = \mathfrak{T}\varkappa + \mathfrak{S}y, y \in \mathfrak{I}] \Longrightarrow \varkappa \in \mathfrak{I}$(*or* $\mathfrak{S}(\mathfrak{I}) \subset (I-\mathfrak{T})(\mathfrak{I})$).

Definition 2.1. *[8] Consider a non-empty set $\mathfrak{P}$ and a mapping $G : \mathfrak{P}\times\mathfrak{P} \to \mathfrak{P}$. Then G has a coupled fixed point $(\varkappa, y) \in \mathfrak{P}\times\mathfrak{P}$, if the following conditions hold: $G(\varkappa, y) = \varkappa$ and $G(y, \varkappa) = y$.*

Example 2.1. The mappings $G_1, G_2 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by $G_1(\varkappa, y) = \varkappa y$ and $G_2(\varkappa, y) = \varkappa + (\varkappa - y)^2$, for all $\varkappa, y \in \mathbb{R}$ have coupled fixed points $(0, 0)$ and $(1, 1)$.

Example 2.2. The mapping $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by $G(x, y) = \frac{xy+1}{x+y+2}$, for all $\varkappa, y \in \mathbb{R}$ has coupled fixed points $\left(-1+\sqrt{2}, -1+\sqrt{2}\right)$ and $\left(-1-\sqrt{2}, -1-\sqrt{2}\right)$.

Definition 2.2. *[26] The mapping $\mathfrak{T} : \mathscr{D} \subset \mathscr{X} \longrightarrow \mathscr{X}$ is expansive, if we can find $h > 1$ such that*

$$d(\mathfrak{T}\varkappa, \mathfrak{T}y) \geq h d(\varkappa, y), \ \forall \varkappa, y \in \mathscr{D}. \tag{2.1}$$

Definition 2.3. *[2] The fractional order ABC derivative is defined by*

$${}_{0}^{ABC}D_{a^+}^{\omega}(g(\mathtt{t})) = \frac{\mathcal{N}(\omega)}{1-\omega}\int_a^{\mathtt{t}} g'(y)E_{\omega}\left(\frac{\omega}{\omega-1}(\mathtt{t}-y)^{\omega}\right)\mathrm{d}y, \tag{2.2}$$

where $\omega \in [0, 1]$, $\mathtt{t} \in [a, b]$, g is a derivable function on $[a, b]$ such that $g' \in L^1(a, b)$, $\mathcal{N}(\omega)$ is a normalization function with $\mathcal{N}(0) = \mathcal{N}(1) = 1$, and $E_{\omega} = E_{\omega}(-\mathtt{t}^{\omega}) = \sum_{k=0}^{\infty}\frac{(-\mathtt{t})^{\omega k}}{\Gamma(\omega k+1)}$, where E_{ω} is the generalized Mittag-Leffler function.

Lemma 2.1. *[1] The equivalent integral form of*

$${}^{ABC}D_{\mathtt{t}}^{\omega}\psi(\mathtt{t}) = \xi(\mathtt{t}), \ \ \psi(0) = \psi_0 \tag{2.3}$$

is

$$\psi(\mathtt{t}) = \psi_0 + \frac{1-\omega}{N(\omega)}\xi(\mathtt{t}) + \frac{\omega}{N(\omega)\Gamma(\omega)}\int_0^{\mathtt{t}}(\mathtt{t}-y)^{\omega-1}\xi(y)dy. \tag{2.4}$$

The relationship between the ABC fractional order derivative and the associated AB integral is

$$\left({}^{AB}I_a^{\omega}\right)\left({}^{ABC}D_a^{\omega}\right)f(\mathtt{t}) = f(\mathtt{t}) - f(a).$$

To establish the coupled fixed-point theorems, we need the following results taken from [28]. Throughout the manuscript, $\mathfrak{I}$ represents a nonempty, convex, and closed subset of $\mathfrak{E}$.

Lemma 2.2. *Suppose $\mathscr{D}$ is a closed subset of $\mathscr{X}$. Assume that the mapping $\mathfrak{T} : \mathscr{D} \to \mathscr{X}$ is expansive with $\mathfrak{T}(\mathscr{D}) \supset \mathscr{D}$. Then there exists a unique fixed point $\varkappa^* \in \mathscr{D}$.*

Lemma 2.3. *If (H_1) is satisfied by the map $\mathfrak{S} : \mathfrak{I} \to \mathfrak{I}$. Then $\mathfrak{S}$ possesses at least one fixed point in $\mathfrak{I}$.*

Lemma 2.4. *Assume that (H_1)–(H_3) hold for the functions $\mathfrak{T}$ and $\mathfrak{S}$ from $\mathfrak{I}$ to $\mathfrak{E}$. Then there exists a point $\varkappa^* \in \mathfrak{I}$ with $\mathfrak{S}\varkappa^* + \mathfrak{T}\varkappa^* = \varkappa^*$.*

Lemma 2.5. *Assume that (H_1), (H_2), and (H_4) hold for the functions $\mathfrak{T}$ and $\mathfrak{S}$ from $\mathfrak{I}$ to $\mathfrak{E}$. Then there exists a point $\varkappa^* \in \mathfrak{I}$ with $\mathfrak{T}\circ(I-\mathfrak{S})\varkappa^* = \varkappa^*$.*

Lemma 2.6. *Assume that (H_1), (H_2), and (H_5) hold for the functions $\mathfrak{T}$ and $\mathfrak{S}$ from $\mathfrak{I}$ to $\mathfrak{E}$. Then there exists a point $\varkappa^* \in \mathfrak{I}$ with $\mathfrak{S}\varkappa^* + Tx^* = \varkappa^*$.*

Lemma 2.7. *[4] Let $\widetilde{\mathfrak{E}} = \mathfrak{E}\times\mathfrak{E}$. Then $\widetilde{\mathfrak{E}}$ is a Banach spaces concerning the norm defined by*

$$\|(\varrho, \mu)\| = \|\varrho\| + \|\mu\|, \ \ \textit{for all} \ \ (\varrho, \mu) \in \widetilde{\mathfrak{E}}.$$

3. Fixed point results

The intention of this section is to utilize the results of Xiang and Yuan [28] to establish coupled FP results in Banach space. Let $\widetilde{\mathfrak{E}} = \mathfrak{E} \times \mathfrak{E}$ and $\widetilde{\mathfrak{S}} = \mathfrak{S} \times \mathfrak{S}$. Define the sum and scalar multiplication on $\widetilde{\mathfrak{E}}$ as follows:

$$(\varrho_1, \mu_1) + (\varrho_2, \mu_2) = (\varrho_1 + \varrho_2, \mu_1 + \mu_2), \quad \text{for all } (\varrho_1, \mu_1), (\varrho_2, \mu_2) \in \widetilde{\mathfrak{E}}, \tag{3.1}$$

and

$$\theta(\varrho, \mu) = (\theta\varrho, \theta\mu), \quad \text{for all } \theta \in \mathbb{R} \textit{ and } (\varrho, \mu) \in \widetilde{\mathfrak{E}}. \tag{3.2}$$

Then $\widetilde{\mathfrak{E}}$ is a vector space on $\mathbb{R}$. Also, define $\widetilde{\mathfrak{T}} : \widetilde{\mathfrak{E}} \to \widetilde{\mathfrak{E}}$, $\widetilde{\mathfrak{S}} : \widetilde{\mathfrak{E}} \to \widetilde{\mathfrak{E}}$, $\widetilde{\mathfrak{S}} : \widetilde{\mathfrak{E}} \to \widetilde{\mathfrak{E}}$, and $\mathscr{T} : \mathfrak{E} \times \mathfrak{I} \to \mathfrak{E}$ by

$$\begin{aligned} \widetilde{\mathfrak{T}}(\varrho, \mu) &= (\mathfrak{T}\varrho, \mathfrak{T}\mu), \\ \widetilde{\mathfrak{S}}(\varrho, \mu) &= (\mathfrak{S}\mu, \mathfrak{S}\varrho), \end{aligned} \tag{3.3}$$

and

$$\mathscr{T}(\varrho, \mu) = \mathfrak{T}\varrho + \mathfrak{S}\mu, \tag{3.4}$$

respectively. Now, since

$$\begin{aligned} (\mathscr{T}(\varrho, \mu), \mathscr{T}(\mu, \varrho)) &= (\mathfrak{T}\varrho + \mathfrak{S}\mu, \mathfrak{T}\mu + \mathfrak{S}\varrho) \\ &= (\mathfrak{T}\varrho + \mathfrak{T}\mu) + (\mathfrak{S}\varrho + \mathfrak{S}\mu) \\ &= \widetilde{\mathfrak{T}}(\varrho, \mu) + \widetilde{\mathfrak{S}}(\varrho, \mu). \end{aligned} \tag{3.5}$$

Thus, to prove that $\mathscr{T}(\varrho, \mu)$ has one or more coupled FP in $\mathfrak{E} \times \mathfrak{E}$, it is necessary that $\widetilde{\mathfrak{T}}(\varrho, \mu) + \widetilde{\mathfrak{S}}(\varrho, \mu)$ has at least one FP in $\mathfrak{E} \times \mathfrak{E}$. Utilizing Lemma 2.4, we prove the following coupled FP theorem:

Theorem 3.1. *Suppose that (H_1)–(H_3) hold for $\mathfrak{T}$ and $\mathfrak{S}$, which are functions from $\mathfrak{I}$ to $\mathfrak{E}$. Then, there exists a point $\widetilde{\varkappa^*} \in \widetilde{\mathfrak{I}}$ with $\widetilde{\mathfrak{S}\varkappa^*} + \widetilde{\mathfrak{T}\varkappa^*} = \widetilde{\varkappa^*}$.*

Proof. First, we have to show that $\widetilde{\mathfrak{S}}$ is continuous on $\widetilde{\mathfrak{I}}$. Let $\widetilde{\varkappa}_n=(\varkappa_1^n, \varkappa_2^n)$ be a sequence in $\widetilde{\mathfrak{I}}$ converging to a point $\widetilde{\varkappa} = (\varkappa_1, \varkappa_2) \in \widetilde{\mathfrak{I}}$. Since $\widetilde{\mathfrak{S}}$ is continuous, we have

$$\begin{aligned} \lim_{n\to\infty} \widetilde{\mathfrak{S}}\widetilde{\varkappa}_n &= \lim_{n\to\infty} \widetilde{\mathfrak{S}}(\varkappa_1^n, \varkappa_2^n) = \lim_{n\to\infty} (\mathfrak{S}\varkappa_1^n, \mathfrak{S}\varkappa_2^n) \\ &= (\mathfrak{S}\varkappa_1, \mathfrak{S}\varkappa_2) = \widetilde{\mathfrak{S}}(\varkappa_1, \varkappa_2) \\ &= \widetilde{\mathfrak{S}}\widetilde{\varkappa}. \end{aligned}$$

Thus, $\widetilde{\mathfrak{S}}$ is continuous on $\widetilde{\mathfrak{I}}$. Next, we have to show that the mapping $\widetilde{\mathfrak{T}} : \widetilde{\mathfrak{I}} \to \widetilde{\mathfrak{E}}$ is expansive. For this, if $\widetilde{\varkappa}, \widetilde{y} \in \widetilde{\mathfrak{E}}$ with $\widetilde{\varkappa} = (\varkappa_1, \varkappa_2)$ and $\widetilde{y} = (y_1, y_2)$. Then

$$\begin{aligned} \|\widetilde{\mathfrak{T}}\widetilde{\varkappa} - \widetilde{\mathfrak{T}}\widetilde{y}\| &= \|\widetilde{\mathfrak{T}}(\varkappa_1, \varkappa_2) - \widetilde{\mathfrak{T}}(y_1, y_2)\| \\ &= \|(\mathfrak{T}\varkappa_1, \mathfrak{T}\varkappa_2) - (\mathfrak{T}y_1, \mathfrak{T}y_2)\| \\ &= \|\mathfrak{T}\varkappa_1 - \mathfrak{T}y_1\| + \|\mathfrak{T}\varkappa_2 - \mathfrak{T}y_2\|. \end{aligned}$$

But $\mathfrak{T}$ is expensive, so that

$$\begin{aligned}\|\widetilde{\mathfrak{T}}\widetilde{\varkappa}-\widetilde{\mathfrak{T}}\widetilde{y}\| &\geq h\|\varkappa_1-y_1\|+h\|\varkappa_2-y_2\|\\ &= h\|(\varkappa_1,\varkappa_2)-(y_1,y_2)\|\\ &= h\|\widetilde{\varkappa}-\widetilde{y}\|.\end{aligned}$$

Thus, the mapping $\widetilde{\mathfrak{T}}$ is expansive. Finally, let $\widetilde{\varkappa}\in\widetilde{\mathfrak{S}(\mathfrak{J})}$. Then we have to show that $\widetilde{\mathfrak{T}(\mathfrak{J})}+\widetilde{\varkappa}\supset\widetilde{\mathfrak{J}}$, where $\widetilde{\mathfrak{T}(\mathfrak{J})}+\widetilde{\varkappa}=\{\widetilde{y}+\widetilde{\varkappa}\mid\widetilde{y}\in\widetilde{\mathfrak{T}(\mathfrak{J})}\}$. For this, since $(\varkappa_1,\varkappa_2)=\widetilde{\varkappa}\in\widetilde{\mathfrak{S}(\mathfrak{J})}$ implies that $\varkappa_1,\varkappa_2\in\mathfrak{S}(\mathfrak{J})$ and thus by condition (H_3), we have

$$\begin{aligned}&\mathfrak{T}(\mathfrak{J})+\varkappa_1\supset\mathfrak{J}\text{ and }\mathfrak{T}(\mathfrak{J})+\varkappa_2\supset\mathfrak{J}\\ \Longrightarrow\ &\{y_1+\varkappa_1: y_1\in\mathfrak{T}(\mathfrak{J})\}\supset\mathfrak{J}\text{ and }\{y_2+\varkappa_2: y_2\in\mathfrak{T}(\mathfrak{J})\}\supset\mathfrak{J}\\ \Longrightarrow\ &\{(y_1,y_2)+(\varkappa_1,\varkappa_2):(y_1,y_2)\in\widetilde{\mathfrak{T}(\mathfrak{J})}\}\supset\widetilde{\mathfrak{J}}\\ \Longrightarrow\ &\{\widetilde{y}+\widetilde{\varkappa}:\widetilde{y}=(y_1,y_2)\in\widetilde{\mathfrak{T}(\mathfrak{J})}\}\supset\widetilde{\mathfrak{J}}\\ \Longrightarrow\ &\widetilde{\mathfrak{T}(\mathfrak{J})}+\widetilde{\varkappa}\supset\widetilde{\mathfrak{J}}.\end{aligned}$$

Consequently, by Lemma 2.3, $\widetilde{\mathfrak{S}}$ possesses at least one fixed point in $\widetilde{\mathfrak{J}}$, that is we can find $\widetilde{\varkappa^*}\in\widetilde{\mathfrak{J}}$, such that $\widetilde{\mathfrak{S}\varkappa^*}+\widetilde{\mathfrak{T}\varkappa^*}=\widetilde{\varkappa^*}$. □

Remark 3.1. *It is clear that if $\mathfrak{J}$ is a nonempty, closed, bounded, and convex subset of $\mathfrak{E}$, then condition (H_1) can be equivalently replaced by the requirement that $\mathfrak{S}$ is completely continuous (i.e., continuous and compact).*

Utilizing Lemma 2.5, we prove the following coupled FP theorem:

Theorem 3.2. *Assume that (H_1), (H_2), and (H_4) hold for $\mathfrak{T}$ and $\mathfrak{S}$, which are functions from $\mathfrak{J}$ to $\mathfrak{E}$. Then there exists a point $\widetilde{\varkappa^*}\in\widetilde{\mathfrak{J}}$ with $\widetilde{\mathfrak{T}}\circ(I-\widetilde{\mathfrak{S}})\widetilde{\varkappa^*}=\widetilde{\varkappa^*}$.*

Proof. First, we have to show that $\widetilde{\mathfrak{S}}$ is continuous on $\widetilde{\mathfrak{J}}$. Let $\widetilde{\varkappa}_n=(\varkappa_1^n,\varkappa_2^n)$ be a sequence in $\widetilde{\mathfrak{J}}$ converging to a point $\widetilde{\varkappa}=(\varkappa_1,\varkappa_2)\in\widetilde{\mathfrak{J}}$, since $\widetilde{\mathfrak{S}}$ is continuous, we have

$$\begin{aligned}\lim_{n\to\infty}\widetilde{\mathfrak{S}}\widetilde{\varkappa}_n &= \lim_{n\to\infty}\widetilde{\mathfrak{S}}(\varkappa_1^n,\varkappa_2^n)=\lim_{n\to\infty}(\mathfrak{S}\varkappa_1^n,\mathfrak{S}\varkappa_2^n)\\ &=(\mathfrak{S}\varkappa_1,\mathfrak{S}\varkappa_2)=\widetilde{\mathfrak{S}}(\varkappa_1,\varkappa_2)\\ &=\widetilde{\mathfrak{S}}\widetilde{\varkappa}.\end{aligned}$$

Thus, $\widetilde{\mathfrak{S}}$ is continuous on $\widetilde{\mathfrak{J}}$. Next, we have to show that the mapping $\widetilde{\mathfrak{T}}:\widetilde{\mathfrak{J}}\to\widetilde{\mathfrak{E}}$ is expansive, that is,

$$\begin{aligned}\|\widetilde{\mathfrak{T}}\widetilde{\varkappa}-\widetilde{\mathfrak{T}}\widetilde{y}\| &= \|\widetilde{\mathfrak{T}}(\varkappa_1,\varkappa_2)-\widetilde{\mathfrak{T}}(y_1,y_2)\|\\ &=\|(\mathfrak{T}\varkappa_1,\mathfrak{T}\varkappa_2)-(\mathfrak{T}y_1,\mathfrak{T}y_2)\|\\ &=\|\mathfrak{T}\varkappa_1-\mathfrak{T}y_1\|+\|\mathfrak{T}\varkappa_2-\mathfrak{T}y_2\|\\ &\geq h\|\varkappa_1-y_1\|+h\|\varkappa_2-y_2\|\\ &=h\|(\varkappa_1,\varkappa_2)-(y_1,y_2)\|\\ &=h\|\widetilde{\varkappa}-\widetilde{y}\|.\end{aligned}$$

Thus, the mapping $\widetilde{\mathfrak{T}}$ is expansive. Finally, for any $\widetilde{\varkappa} \in \widetilde{\mathfrak{S}(\mathfrak{I})}$, we have to show that $\widetilde{\mathfrak{I}} + \widetilde{\varkappa} \subset \widetilde{\mathfrak{T}(\mathfrak{I})} \subset \widetilde{\mathfrak{I}}$, where $\widetilde{\mathfrak{I}} + \widetilde{\varkappa} = \{\widetilde{y} + \widetilde{\varkappa} \mid \widetilde{y} \in \widetilde{\mathfrak{I}}\}$. For this, for any $(\varkappa_1, \varkappa_2) = \widetilde{\varkappa} \in \widetilde{\mathfrak{S}(\mathfrak{I})}$ and by Theorem 2.3, we have

$$\begin{aligned}
&\mathfrak{I} + \varkappa_1 \subset \mathfrak{I} \text{ and } \mathfrak{I} + \varkappa_2 \subset \mathfrak{I} \\
\Longrightarrow &\{y_1 + \varkappa_1 : y_1 \in \mathfrak{I}\} \subset \mathfrak{I} \text{ and } \{y_2 + \varkappa_2 : y_2 \in \mathfrak{I}\} \subset \mathfrak{I} \\
\Longrightarrow &\{(y_1, y_2) + (\varkappa_1, \varkappa_2) : (y_1, y_2) \in \widetilde{\mathfrak{I}})\} \subset \widetilde{\mathfrak{I}} \\
\Longrightarrow &\{\widetilde{y} + \widetilde{\varkappa} : \widetilde{y} = (y_1, y_2) \in \widetilde{\mathfrak{I}}\} \subset \widetilde{\mathfrak{I}} \\
\Longrightarrow &\widetilde{\mathfrak{I}} + \widetilde{\varkappa} \subset \widetilde{\mathfrak{I}}.
\end{aligned}$$

Thus by Lemma 2.3, there exists a point $\widetilde{\varkappa^*} \in \widetilde{\mathfrak{I}}$ such that $\widetilde{\mathfrak{T}^{-1}\varkappa^*} + \widetilde{\mathfrak{S}\varkappa^*} = \widetilde{\varkappa^*}$. □

Utilizing Lemma 2.6, we prove the following coupled FP theorem:

Theorem 3.3. *Assume that* (H_1), (H_2)*, and* (H_5) *hold for* $\mathfrak{T}$ *and* $\mathfrak{S}$*, which are functions from* $\mathfrak{I}$ *to* $\mathfrak{E}$*. Then there exists a point* $\widetilde{\varkappa^*} \in \widetilde{\mathfrak{I}}$ *with* $\widetilde{\mathfrak{S}\varkappa^*} + \widetilde{\mathfrak{T}\varkappa^*} = \widetilde{\varkappa^*}$.

Proof. First, we have to show that $\widetilde{\mathfrak{S}}$ is continuous on $\widetilde{\mathfrak{I}}$. Let $\widetilde{\varkappa}_n = (\varkappa_1^n, \varkappa_2^n)$ be a sequence in $\widetilde{\mathfrak{I}}$ converging to a point $\widetilde{\varkappa} = (\varkappa_1, \varkappa_2) \in \widetilde{\mathfrak{I}}$, since $\widetilde{\mathfrak{S}}$ is continuous, we have

$$\begin{aligned}
\lim_{n\to\infty} \widetilde{\mathfrak{S}}\widetilde{\varkappa}_n &= \lim_{n\to\infty} \widetilde{\mathfrak{S}}(\varkappa_1^n, \varkappa_2^n) = \lim_{n\to\infty} (\mathfrak{S}\varkappa_1^n, \mathfrak{S}\varkappa_2^n) \\
&= (\mathfrak{S}\varkappa_1, \mathfrak{S}\varkappa_2) = \widetilde{\mathfrak{S}}(\varkappa_1, \varkappa_2) \\
&= \widetilde{\mathfrak{S}}\widetilde{\varkappa}.
\end{aligned}$$

Thus, $\widetilde{\mathfrak{S}}$ is continuous on $\widetilde{\mathfrak{I}}$. Next, we have to show that the mapping $\widetilde{\mathfrak{T}} : \widetilde{\mathfrak{I}} \to \widetilde{\mathfrak{E}}$ is expansive, that is,

$$\begin{aligned}
\|\widetilde{\mathfrak{T}}\widetilde{\varkappa} - \widetilde{\mathfrak{T}}\widetilde{y}\| &= \|\widetilde{\mathfrak{T}}(\varkappa_1, \varkappa_2) - \widetilde{\mathfrak{T}}(y_1, y_2)\| \\
&= \|\mathfrak{T}\varkappa_1 - \mathfrak{T}y_1\| + \|\mathfrak{T}\varkappa_2 - \mathfrak{T}y_2\| \\
&\geq h\|\varkappa_1 - y_1\| + h\|\varkappa_2 - y_2\| \\
&= h\|\varkappa_1 - y_1, \varkappa_2 - y_2\| \\
&= h\|(\varkappa_1, \varkappa_2) - (y_1, y_2)\| \\
&= h\|\widetilde{\varkappa} - \widetilde{y}\|.
\end{aligned}$$

Thus, the mapping $\widetilde{\mathfrak{T}}$ is expansive. Finally, for any $(\varkappa_1, \varkappa_2) = \widetilde{\varkappa} \in \widetilde{\mathfrak{S}(\mathfrak{I})}$, we have

$$\begin{aligned}
&\varkappa_1 \subset (I - \mathfrak{T})(\mathfrak{I}) \text{ and } \varkappa_2 \subset (I - \mathfrak{T})(\mathfrak{I}) \\
\Longrightarrow &\{\mathfrak{T}_1 + \mathfrak{S}y_1 : y_1 \in \mathfrak{I}\} \subset (I - \mathfrak{T})(\mathfrak{I}) \text{ and } \{\mathfrak{T}\varkappa_2 + \mathfrak{S}y_2 : y_2 \in \mathfrak{I}\} \subset (I - \mathfrak{T})(\mathfrak{I}) \\
\Longrightarrow &\{(\mathfrak{T}\varkappa_1 + \mathfrak{S}y_1, \mathfrak{T}\varkappa_2 + \mathfrak{S}y_2) : (y_1, y_2) \in \widetilde{\mathfrak{I}}\} \subset (I - \widetilde{\mathfrak{T}})\widetilde{(\mathfrak{I})} \\
\Longrightarrow &\{\widetilde{\mathfrak{T}}(\varkappa_1, \varkappa_2) + \widetilde{\mathfrak{S}}(y_1, y_2) : \widetilde{y} = (y_1, y_2) \in \widetilde{\mathfrak{I}}\} \subset (I - \widetilde{\mathfrak{T}})\widetilde{(\mathfrak{I})} \\
\Longrightarrow &\widetilde{\mathfrak{T}}\widetilde{\varkappa} + \widetilde{\mathfrak{S}}\widetilde{y} \subset (I - \widetilde{\mathfrak{T}})\widetilde{(\mathfrak{I})} \\
\Longrightarrow &\widetilde{\varkappa} \in \widetilde{\mathfrak{S}(\mathfrak{I})} \subset (I - \widetilde{\mathfrak{T}})\widetilde{(\mathfrak{I})}.
\end{aligned}$$

Thus by Lemma 2.3, there exists a point $\widetilde{\varkappa^*} \in \widetilde{\mathfrak{I}}$ such that $\widetilde{\mathfrak{S}\varkappa^*} + \widetilde{\mathfrak{T}\varkappa^*} = \widetilde{\varkappa^*}$. □

4. Application

In this section, we utilize our established results to examine the existence of a solution for the HIV/AIDS (1.2) infection model involving a fractional-order ABC derivative. The model (1.2) can be expressed as follows:

$$\begin{cases} {}^{\mathrm{ABC}}_{0}D_t^{\omega} S = Q_1\left(S, E, I, C, \mathcal{A}\right), \\ {}^{\mathrm{ABC}}_{0}D_t^{\omega} E = Q_2\left(S, E, I, C, \mathcal{A}\right), \\ {}^{\mathrm{ABC}}_{0}D_t^{\omega} I = Q_3\left(S, E, I, C, \mathcal{A}\right), \\ {}^{\mathrm{ABC}}_{0}D_t^{\omega} C = Q_4\left(S, E, I, C, \mathcal{A}\right), \\ {}^{\mathrm{ABC}}_{0}D_t^{\omega} \mathcal{A} = Q_5\left(S, E, I, C, \mathcal{A}\right), \end{cases}$$

where

$$\begin{cases} Q_1\left(S, E, I, C, \mathcal{A}\right) = \rho - \iota(I(t) + \upsilon_c C(t) + \upsilon_A \mathcal{A}(t)) S(t) - d S(t), \\ Q_2\left(S, E, I, C, \mathcal{A}\right) = \iota\mu(I(t) + \upsilon_c C(t) + \upsilon_A \mathcal{A}(t)) S(t) - (\sigma + d) E(t), \\ Q_3\left(S, E, I, C, \mathcal{A}\right) = (1 - \mu)\iota(I(t) + \upsilon_c C(t) + \upsilon_A \mathcal{A}(t)) S(t) + \sigma E(t) - (\tau_1 + \tau_2 + d) I(t) + \lambda C(t) + \theta \mathcal{A}(t), \\ Q_4\left(S, E, I, C, \mathcal{A}\right) = \tau_1 I(t) - (\lambda + d) C(t), \\ Q_5\left(S, E, I, C, \mathcal{A}\right) = \tau_2 I(t) - (d + \theta + d_1) \mathcal{A}(t). \end{cases}$$

Let us consider system (1.2) as

$$ {}^{\mathrm{ABC}}D_t^{\omega} \mathrm{u}(t) = \mathfrak{D}(t, \mathrm{u}(t)), \quad \forall t \in I^*, \tag{4.1}$$

with initial condition $\mathrm{u}(0) = \mathrm{u}_0 \geq 0$, where

$$\begin{aligned} \mathrm{u}(t) &= \left(S, E, I, C, \mathcal{A}\right)^T, \\ \mathrm{u}_0 &= \left(S_0, E_0, I_0, C_0, \mathcal{A}_0\right)^T, \\ \mathfrak{D}(t, \mathrm{u}(t)) &= \left(\mathfrak{F}_n(t, S, E, I, C, \mathcal{A})\right)^T, \quad n = 1, 2, 3, 4, 5, \end{aligned} \tag{4.2}$$

$\omega \in [0, 1]$ be any real number, $I^* = [0, \mathcal{T}]$, ${}^{\mathrm{ABC}}D^{\omega}$ be the ABC derivative. In Eq (4.2), the superscript T represents the transpose. By using Lemma 2.1, Eq (4.1) becomes

$$\mathrm{u}(t) = \mathrm{u}_0 + \frac{1 - \omega}{AB(\omega)\Gamma(\omega)} \mathfrak{D}(t, \mathrm{u}(t)) + \frac{1 - \omega}{AB(\omega)\Gamma(\omega)} \int_0^t (t - \varrho)^{1-\omega} \mathfrak{D}(\varrho, \mathrm{u}(\varrho)) d\varrho. \tag{4.3}$$

To proceed further, assume that $\mathrm{u}, \mathrm{v} : I^* \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ are continuous and obey the following conditions:

(H_1) There is a number σ_0 such that

$$|\mathfrak{D}(t, \mathrm{u}(t)) - \mathfrak{D}(t, \mathrm{v}(t))| \geq \frac{AB(\omega)\Gamma(\omega)\sigma_0}{1 - \omega} |\mathrm{u}(t) - \mathrm{v}(t)|, \quad \forall \mathrm{u}, \mathrm{v} \in \mathcal{R}. \tag{4.4}$$

(H_2) For any two constants $\mathscr{P}$ and $\mathscr{Q}$, we have

$$|\mathfrak{D}(\varrho, \mathrm{v}(\varrho))| \leq \mathscr{P}\|\mathrm{v}\| + \mathscr{Q}, \quad \forall \varrho \in I^*. \tag{4.5}$$

(H_3) There exists a $R > 0$ such that

$$R^3 + (\sigma^0 - 1)R + \|p\| \geq \frac{\mathcal{T}^{\omega}}{N(\omega)\Gamma(\omega)} \left[\mathscr{P}\epsilon + \mathscr{Q}\right], \tag{4.6}$$

where $\sigma^0 := \sup_{t\in\mathcal{R}} \sigma(t)$, $\|p\| = \sup_{t\in\mathcal{R}} |p(t)|$, $\sigma_0 := \inf_{t\in\mathcal{R}} > 1$ and $\epsilon > 0$.

With the help of Eq (4.3), we can write

$$\begin{cases} \mathrm{u}(t) =\mathrm{u}_0 + \dfrac{1-\omega}{AB(\omega)\Gamma(\omega)}\mathfrak{D}(t,\mathrm{u}(t)) + \dfrac{1-\omega}{AB(\omega)\Gamma(\omega)}\displaystyle\int_0^t (t-\varrho)^{1-\omega}\mathfrak{D}(\varrho,\mathrm{u}(\varrho))d\varrho, \\ \mathrm{v}(t) =\mathrm{v}_0 + \dfrac{1-\omega}{AB(\omega)\Gamma(\omega)}\mathfrak{D}(t,\mathrm{v}(t)) + \dfrac{1-\omega}{AB(\omega)\Gamma(\omega)}\displaystyle\int_0^t (t-\varrho)^{1-\omega}\mathfrak{D}(\varrho,\mathrm{v}(\varrho))d\varrho. \end{cases} \tag{4.7}$$

The following result establishes the existence of a solution.

Theorem 4.1. *The system* (4.1) *has a solution defined on I^*, provided that the conditions* (H_1)–(H_3) *hold.*

Proof. Let $\mathcal{X} = C(t,\mathcal{R})$. Define $\mathcal{M} \subset \mathcal{X}$ by

$$\mathcal{M} = \{\mathrm{u} \in \mathcal{X} : \|\mathrm{u}\| \leq \triangle\}, \tag{4.8}$$

where $\Delta = \max(\triangle_1, \triangle_2)$ with $\triangle_1 \geq \frac{\mathcal{T}^{\omega}}{N(\omega)\Gamma(\omega)} [\mathscr{P}\epsilon + \mathscr{Q}]$, $\triangle_2 \geq R^3 + \sigma^0 R + \|p\|$, $I^* = [0,\mathcal{T}]$ and $\mathrm{u}, \mathrm{v} : I^* \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ are continuous functions. Clearly, $\mathcal{M}$ is a closed, non-empty, convex, and bounded subset of $\mathcal{X}$. Since u is a solution of the system (4.1) if and only if u satisfies the system (4.3). Thus, finding the existence of a solution to system (4.3) is equivalent to finding the existence of a solution to system (4.1). To achieve this, define the operator $\mathcal{M} : \mathcal{X} \to \mathcal{X}$ by

$$\begin{cases} \mathfrak{T}\mathrm{u}(t) =\mathrm{u}_0 + \dfrac{1-\omega}{AB(\omega)\Gamma(\omega)}\mathfrak{D}(t,\mathrm{u}(t)), \\ \mathfrak{S}\mathrm{v}(t) =\dfrac{\omega}{AB(\omega)\Gamma(\omega)}\displaystyle\int_0^t (t-\varrho)^{1-\omega}\mathfrak{D}(\varrho,\mathrm{v}(\varrho))d\varrho. \end{cases} \tag{4.9}$$

Therefore, the system of integral equation (4.7) can be rewritten as the following equivalent system of operator equations:

$$\begin{cases} \mathrm{u}(t) =\mathfrak{T}\mathrm{u}(t) + \mathfrak{S}\mathrm{v}(t), \\ \mathrm{v}(t) =\mathfrak{T}\mathrm{v}(t) + \mathfrak{S}\mathrm{u}(t). \end{cases} \tag{4.10}$$

We will show that the operators $\mathfrak{T}$ and $\mathfrak{S}$ satisfy all the conditions of Theorem 3.1. First, we show that the operator S is continuous and compact on $\mathcal{M}$ into $\mathcal{X}$. Let $\{\mathrm{v}_n\}$ be any sequence in $\mathcal{M}$ converging to a point $\mathrm{v} \in \mathcal{M}$. By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty}\mathfrak{S}\mathrm{v}_n(t) =& \lim_{n\to\infty}\left(\frac{1-\omega}{AB(\omega)\Gamma(\omega)}\int_0^t (t-\varrho)^{1-\omega}\mathfrak{D}(\varrho,\mathrm{v}_n(\varrho))d\varrho\right) \\ =&\frac{1-\omega}{AB(\omega)\Gamma(\omega)}\lim_{n\to\infty}\int_0^t (t-\varrho)^{1-\omega}\mathfrak{D}(\varrho,\mathrm{v}_n(\varrho))d\varrho \\ =&\frac{1-\omega}{AB(\omega)\Gamma(\omega)}\int_0^t (t-\varrho)^{1-\omega}\mathfrak{D}(\varrho,\mathrm{v}(\varrho))d\varrho \\ =&\mathfrak{S}\mathrm{v}(t). \end{aligned} \tag{4.11}$$

Thus, $\mathfrak{S}$ is continuous. Now, to show that $\mathfrak{S}\mathcal{M}$ is compact on $\mathcal{M}$, it is sufficient to show that $\mathfrak{S}$ is a uniformly bounded and equi-continuous set in $\mathcal{M}$. For this, we have

$$\begin{aligned}
|\mathfrak{S}\mathrm{v}(t)| =& \left|\frac{\omega}{N(\omega)\Gamma(\omega)}\int_0^t (t-\varrho)^{\omega-1}[\mathfrak{D}(\varrho,\mathrm{v}(\varrho))]\mathrm{d}\varrho\right| \\
\leq& \frac{\omega}{N(\omega)\Gamma(\omega)}\int_0^t (t-\varrho)^{\omega-1}|\mathfrak{D}(\varrho,\mathrm{v}(\varrho))|\mathrm{d}\varrho \\
\leq& \frac{\omega}{N(\omega)\Gamma(\omega)}\int_0^t (t-\varrho)^{\omega-1}\left[\mathscr{P}\|\mathrm{v}\|+\mathscr{Q}\right]\mathrm{d}\varrho \\
=& \frac{\omega[\mathscr{P}\|\mathrm{v}\|+\mathscr{Q}]}{N(\omega)\Gamma(\omega)}\int_0^t (t-\varrho)^{\omega-1}\mathrm{d}\varrho \\
=& \frac{\mathcal{T}^{\omega}}{N(\omega)\Gamma(\omega)}[\mathscr{P}\|\mathrm{v}\|+\mathscr{Q}].
\end{aligned}$$

Taking the supremum of both sides, we obtain

$$\|\mathfrak{S}\mathrm{v}\| \leq \frac{\mathcal{T}^{\omega}}{N(\omega)\Gamma(\omega)}\left[\mathscr{P}\epsilon+\mathscr{Q}\right] \leq \triangle_1.$$

Thus, $\mathfrak{S}$ is uniformly bounded. Next, we show that $\mathfrak{S}$ is equicontinuous. To do this, consider $t_1, t_2 \in [0, I^*]$ with $t_1 < t_2$. Then, we have

$$\begin{aligned}
|\mathfrak{S}\mathrm{v}\,(t_2)-\mathfrak{S}\mathrm{v}\,(t_1)| =& \left|\frac{\omega}{N(\omega)\Gamma(\omega)}\int_0^{t_1}(t_2-\varrho)^{\omega-1}\,\mathfrak{D}(\varrho,\mathrm{v}(\varrho))\mathrm{d}\varrho\right. \\
&+\frac{\omega}{N(\omega)\Gamma(\omega)}\int_{t_1}^{t_2}(t_2-\varrho)^{\omega-1}\,G(\varrho,\mathrm{v}(\varrho))\mathrm{d}\varrho \\
&\left.-\frac{\omega}{N(\omega)\Gamma(\omega)}\int_0^{t_1}(t_1-\varrho)^{\omega-1}\int_0^{t_1}(t_1-\varrho)^{\omega-1}\,\mathfrak{D}(\varrho,\mathrm{v}(\varrho))\mathrm{d}\varrho\right| \\
\leq& \frac{\omega}{N(\omega)\Gamma(\omega)}\int_{t_1}^{t_2}(t_2-\varrho)^{\omega-1}\,|\mathfrak{D}(\varrho,\mathrm{v}(\varrho))|\mathrm{d}\varrho \\
&+\frac{\omega}{N(\omega)\Gamma(\omega)}\int_0^{t_1}\left[(t_2-\varrho)^{\omega-1}-(t_1-\varrho)^{\omega-1}\right]|\mathfrak{D}(\varrho,\mathrm{v}(\varrho))|\mathrm{d}\varrho \\
\leq& \frac{\omega}{N(\omega)\Gamma(\omega)}\int_{t_1}^{t_2}(t_2-\varrho)^{\omega-1}\left[\mathscr{P}\|\mathrm{v}\|+\mathscr{Q}\right]\mathrm{d}\varrho \\
&+\frac{\omega}{N(\omega)\Gamma(\omega)}\int_0^{t_1}\left[(t_2-\varrho)^{\omega-1}-(t_1-\varrho)^{\omega-1}\right]\left[\mathscr{P}\|\mathrm{v}\|+\mathscr{Q}\right]\mathrm{d}\varrho \\
=& \frac{[\mathscr{P}\|\mathrm{v}\|+\mathscr{Q}]\omega}{N(\omega)\Gamma(\omega)}\int_{t_1}^{t_2}(t_2-\varrho)^{\omega-1}\,\mathrm{d}\varrho+\frac{[\mathscr{P}\|\mathrm{v}\|+\mathscr{Q}]\omega}{N(\omega)\Gamma(\omega)}\int_0^{t_2}(t_2-\varrho)^{\omega-1}\,\mathrm{d}\varrho \\
&-\frac{[\mathscr{P}\|\mathrm{v}\|+\mathscr{Q}]\omega}{N(\omega)\Gamma(\omega)}\int_0^{t_1}(t_1-\varrho)^{\omega-1}\,\mathrm{d}\varrho \\
\leq& \frac{[\mathscr{P}\epsilon+\mathscr{Q}]}{N(\omega)\Gamma(\omega)}\left[(t_2-t_1)^{\omega}+(t_1^{\omega}-t_2^{\omega})+(t_2-t_1)^{\omega}\right] \\
=& \frac{[\mathscr{P}\epsilon+\mathscr{Q}]}{N(\omega)\Gamma(\omega)}\Big[2\,(t_2-t_1)^{\omega}+(t_2^{\omega}-t_1^{\omega})\Big].
\end{aligned}$$

Taking the limit as $t_1 \longrightarrow t_2$, we obtain

$$|\mathfrak{S}\mathrm{v}(t_2) - \mathfrak{S}\mathrm{v}(t_1)| \longrightarrow 0.$$

That is, $\mathfrak{S}$ is equicontinuous. The Arzelá-Ascoli theorem suggests that $\mathfrak{S}$ is relatively compact, and hence, it is completely continuous. Next, we show that the operator T satisfies condition A_2 of Theorem 3.1. By using the hypothesis (H_2), we have

$$\begin{aligned}|\mathfrak{T}\mathrm{u}(t) - \mathfrak{T}\mathrm{v}(t)| =& \left|\frac{\omega}{AB(\omega)\Gamma(\omega)}\mathfrak{D}(t,\mathrm{u}(t)) - \frac{\omega}{AB(\omega)\Gamma(\omega)}\mathfrak{D}(t,\mathrm{v}(t))\right| \\ =& \frac{1-\omega}{AB(\omega)\Gamma(\omega)}|\mathfrak{D}(t,\mathrm{u}(t)) - \mathfrak{D}(t,\mathrm{v}(t))| \\ \geq& \sigma_0|\mathrm{u}(t) - \mathrm{v}(t)|.\end{aligned} \tag{4.12}$$

Taking the supremum, we obtain

$$\|\mathfrak{T}\mathrm{u} - \mathfrak{T}\mathrm{v}\| \geq \sigma_0\|\mathrm{u} - \mathrm{v}\|. \tag{4.13}$$

Thus, the condition (A_2) of Theorem 3.1 is satisfied. Last, we need to verify that condition (A_3) of Theorem 3.1 also holds. To do this, fixing any $x_0(t) = (\mathfrak{S}\mathrm{u}_0)(t) \in \mathfrak{S}(\mathfrak{I})$, then for each $x \in \mathfrak{I}$, we need to find a corresponding $y \in \mathfrak{I}$ such that

$$(\mathfrak{T}y)(t) + x_0(t) = \mathrm{u}(t). \tag{4.14}$$

Assuming that (4.14) is true, we may infer that

$$|(\mathfrak{T}y)(t)| \leq \|\mathrm{u}\| + \|x_0\| \leq R + \frac{\mathcal{T}^{\omega}}{N(\omega)\Gamma(\omega)}\left[\mathscr{P}\epsilon + \mathscr{Q}\right]. \tag{4.15}$$

The expression $\mathfrak{T}$ yields that for each $z \in \mathfrak{I}$, we have

$$|(\mathfrak{T}y)(t)| \leq R^3 + \sigma^0 R + \|p\| \leq \triangle_2. \tag{4.16}$$

From H_3, (4.15), and (4.16), we obtain that $z(t) - x_0(t) \in \mathfrak{T}(\mathfrak{I})$. Thus, there exists $y \in \mathfrak{I}$ solving (4.14), i.e., A_3 of Theorem 3.1 holds. Consequently, the system (4.1) has a solution in I^*. □

5. Conclusions

In this work, we developed new coupled fixed-point results in Banach spaces by employing Krasnosel'skii's expansive-type fixed-point theory and established conditions for the existence and uniqueness of coupled fixed points. These theoretical findings were applied to a fractional-order HIV/AIDS model extended via the ABC derivative, demonstrating the existence of solutions and contributing to the analysis of nonlinear and memory-dependent biological systems. While the ABC derivative was used as a representative framework, the developed methodology is general and can be adapted to models involving other fractional or fractal derivatives. This flexibility underscores the broader applicability of the proposed results and their potential relevance to a wide range of complex dynamical systems.

Author contributions

M. B. Zada, M. Sarwar and H. Rashid: Conceptualization, Writing–original draft, Writing–review and editing; K. Abodayeh, S. Chasreecha and T. Sitthiwirattham: Formal analysis, Funding acquisition, Visualization; M. B. Zada, M. Sarwar and K. Abodayeh: Supervision, Methodology, Project administration. The final version has been read and approved by all the authors.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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