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Research article

Integrating axiomatic and dynamic mechanisms under industrial management situations: game-theoretical analysis

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Abstract: In numerous industrial management situations, multiple departments or units engage in complex, multi-grade interactions that influence overall organizational decisions. This study introduces a new assessing scheme that accounts for units' activity grades within such a situation. Unlike the Shapley value and the equal allocation of nonseparable costs, this assessing scheme remains computationally linear due to its single-pass step-grade decomposition, and it preserves several useful properties under any graded coalition. By applying specific reduction and excess function, several axiomatic and dynamic results are established to characterize the scheme's consistency and stability. These theoretical findings, generated from game-theoretical analysis, offer actionable insights into key participating units, including stakeholder engagement, incentive structures, and coalition formation, thereby enhancing the strategic decision-making processes in real-world industrial management contexts.

Keywords: axiomatic and dynamic result; industrial management; game-theoretical analysis; participating unit
Mathematics Subject Classification: Primary 91A, Secondary 91B

1. Introduction

In recent developments, game-theoretical approaches have been employed to analyze various interactive relationships and models among factors and coalitions by applying mathematical results. Beyond purely theoretical analysis, game-theoretical outcomes have also been utilized to provide solution concepts or equilibrium conditions for real-world interactive situations, notably in industrial management contexts where multiple units or departments coordinate decisions to enhance operational synergy. Various management techniques integrating different theoretical frameworks have emerged as central in the broader field of industrial optimization, especially when addressing strategic grades of multiple stakeholders. The application of game-theoretical results can thus augment decision-making effectiveness in managing complex organizational interactions and cooperative arrangements.

Under the study of traditional interactive situations, a characteristic function is typically defined over all sub-collections of the set of participating units. This implies that each factor's options are limited to either fully joining a coalition or not participating at all. In numerous real-world industrial management situations; however, each factor may exhibit varying intensities or grades of involvement, directly influencing collective outcomes. A multichoice interaction situation is a natural extension of traditional coalition structures, allowing factors to choose among different activity grades. Several approaches have been introduced under multichoice context. By determining overall values for individual factors under multichoice settings, studies due to Cheng et al. [2], Hwang and Liao [7], Liao [9, 10], Nouweland et al. [14], and Wei et al. [17] have proposed extended rules by applying solution concepts like the core, the equal allocation of nonseparable costs (EANSC [15]), and the Shapley value. Focusing on both factors and its activity grades, Hwang and Liao [6] defined an extended Shapley value [16]. Moreover, recent studies have highlighted multichoice game frameworks for scheduling, production planning, and other industrial applications. Building on the above review, some strands among related researches could be distinguished as follows. Fuzzy works (e.g., Hwang and Liao [6]) extend traditional solutions to uncertain payoffs but still model each participant at a single intensity level. Besides, multichoice contributions (e.g., Nouweland et al. [14]) allow discrete grades yet seldom analyze cross-factor grade interactions or nonseparable costs.

Axiomatic approach is the process of characterizing a rule through a set of fundamental axioms, which typically reflect principles of rationality, fairness, or stability. When a rule is uniquely determined by a specific set of axioms, its axiomatic approach is considered complete. This approach helps elucidate the intrinsic logic behind a solution and provides a standardized framework for comparing different solutions. Besides, dynamic approach examines how a particular rule emerges or stabilizes through a dynamic process. This often involves factors' learning grades, strategy adjustments, or asymptotic equilibrium convergence. If a rule consistently arises as the stable outcome of a reasonable dynamic process, its applicability is reinforced. For instance, some game-theoretic rules can be interpreted as the equilibrium results of iterative negotiation processes. Integrating axiomatic and dynamic approaches allows for a comprehensive validation of a rule from both static and dynamic perspectives. The axiomatic approach first establishes the uniqueness and rationality of a rule, while the dynamic approach verifies whether it naturally emerges through learning, adaptation, or evolutionary processes.

This study centers on the solution concept of the pseudo equal allocation of nonseparable costs (PEANSC) due to Hsieh and Liao [8]. Within traditional coalition situations, Hsieh and Liao [8] provided axiomatic and dynamic results to demonstrate that the PEANSC is a stabilizing rule. While existing studies on fuzzy allocation rules (e.g., fuzzy core, fuzzy Shapley value) have addressed uncertainty in coalition formation, they often lack a refined mechanism to handle graded or hierarchical participation structures. Similarly, several multichoice TU models have generalized solution concepts from classical games, yet many of them restrict the analysis to flat-level participation or uniform grade treatment, thereby overlooking the intrafactor heterogeneity present in real-world departments or agents. As for the PEANSC, although it effectively decomposes nonseparable burdens in flat coalitional settings, it does not directly accommodate multi-grade interaction across heterogeneous factors. A concise synthesis of the preceding gaps suggests a natural research question:

- Can one devise a rule that inherits the nonseparable-cost spirit of the PEANSC, yet remains valid in graded, multichoice environments and avoids the shortcomings observed in current fuzzy and multichoice approaches?

To address these limitations, the present study would like to introduce a different assessing rule, which extends the PEANSC formulation into a multichoice TU setting where both the number of participating units and its respective grades of activity are jointly considered. The proposed scheme leverages a structured marginal effect function and satisfies a set of fairness-related axioms. Furthermore, a dynamic adjustment process is developed to verify that the efficient assessment for interrelated effects (EAIE) is not only theoretically consistent but also behaviorally stable under iterative implementation. The major contributions are outlined as follows:

- Inspired by the work of Hwang and Liao [6], we define a multichoice generalization of the PEANSC, termed the EAIE, focusing simultaneously on factors and its activity grades, in Section 2.
- In Section 3, we introduce several extended properties, building upon Hsieh and Liao [8], to characterize the EAIE within multichoice situations.
- Section 4 applies the concept of the excess to derive a dynamic result for the EAIE under multichoice situations.
- In Section 5, we illustrate how these game-theoretical results can offer insights into practical decision-making processes under industrial management contexts. Related connections and comparisons are discussed in Section 6.

2. The efficient assessment for interrelated effects (EAIE)

Let $\overline{UF}$ be the universe of factors. In many industrial management situations, these factors can represent departments or subdivisions of a large organization, each capable of operating at different intensity or activity grades depending on operational demands [1, 3, 4, 13]. For $m \in \overline{UF}$ and $\tilde{d}_m \in \mathbb{N}$, $\overline{D}_m = \{0, 1, \cdots, \tilde{d}_m\}$ could be treated as the activity grade collection of factor m, and $\overline{D}_m^+ = \overline{D}_m \setminus \{0\}$, where 0 represents no participation. Such multigrade engagement has been emphasized in various game-theoretical studies, especially those focusing on flexible coalition structures and their applications to organizational decision making [7, 14].

Let $\overline{F} \subseteq \overline{UF}$ and $\overline{D}^{\overline{F}} = \prod_{m\in\overline{F}} \overline{D}_m$ be the product collection of the activity grade collections of all factors of $\overline{F}$. For every $\overline{T} \subseteq \overline{F}$, we define $\varsigma^{\overline{T}} \in \overline{D}^{\overline{F}}$ as the vector with $\varsigma^{\overline{T}}_m = 1$ if $m \in \overline{T}$, and $\varsigma^{\overline{T}}_m = 0$ if $m \in \overline{F} \setminus \overline{T}$. Denote $0_{\overline{F}}$ the zero vector in $\mathbb{R}^{\overline{F}}$.

A multichoice situation is a triple $(\overline{F}, \tilde{d}, \hat{\mathbf{H}})$, where $\overline{F}$ is a non-empty and finite collection of factors, $\tilde{d} = (\tilde{d}_m)_{m\in\overline{F}}$ is the vector that presents the highest activity grade for every factor, and $\hat{\mathbf{H}} : \overline{D}^{\overline{F}} \to \mathbb{R}$ is a characteristic mapping with $\hat{\mathbf{H}}(0_{\overline{F}}) = 0$ which assigns to every $\lambda = (\lambda_m)_{m\in\overline{F}} \in \overline{D}^{\overline{F}}$ a numerical worth. This worth can be interpreted as the total gain (or utility) achieved when each factor m engages at the activity grade λ_m, reflecting different possible intensities of collaboration in an industrial or organizational environment [9, 10]. As $d \in \mathbb{R}$ is fixed throughout this research, we write $(\overline{F}, \hat{\mathbf{H}})$ rather than $(\overline{F}, \tilde{d}, \hat{\mathbf{H}})$. Besides, throughout this study, multichoice TU game is synonymous with multichoice situation whose characteristic mapping $\hat{\mathbf{H}}$ is transferable utility. For brevity we adopt the latter phrase in all subsequent sections.

Given a multichoice situation $(\overline{F}, \hat{\mathbf{H}})$ and $\mu \in \overline{D}^{\overline{F}}$, we write $\hat{\mathbf{O}}(\mu) = \{m \in \overline{F} \mid \mu_m \neq 0\}$ and denote $\mu_{\overline{T}}$ as the restriction of μ to $\overline{T}$ for every $\overline{T} \subseteq \overline{F}$. In many practical organizational decisions, such restrictions represent sub-coalitions or partial collaborations among the departments in question [2, 10]. Denote the family of total multichoice situations by **MCS**.

Given $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$, let $\overline{\mathbf{P}}^{\overline{F}} = \{(m, k_m) \mid m \in \overline{F}, k_m \in \overline{D}_m^+\}$. A **rule** on **MCS** is a map ρ assigning to every $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$ an element $\rho(\overline{F}, \hat{\mathbf{H}}) = \left(\rho_{m,k_m}(\overline{F}, \hat{\mathbf{H}})\right)_{(m,k_m)\in\overline{\mathbf{P}}^{\overline{F}}} \in \mathbb{R}^{\overline{\mathbf{P}}^{\overline{F}}}$. Here, $\rho_{m,k_m}(\overline{F}, \hat{\mathbf{H}})$ is the power index or the value of the factor m if it participates with activity grade k_m under the situation $\hat{\mathbf{H}}$. From an industrial management viewpoint, this value can capture each department's contribution or payoff when committing at a certain operational grade, facilitating strategic planning across various functional units [1, 3, 4, 6, 13].

Subsequently, we provide a generalized analogue of the PEANSC under multichoice situations. Originally examined under simpler coalition structures [8], it is extended here to accommodate varying intensities of involvement.

Definition 2.1. The EAIE of multichoice situations, $\hat{\Psi}$, is the function on **MCS** which associates to every $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$, every factor $m \in \overline{F}$ and every $k_m \in \overline{D}_m$ the value

$$\hat{\Psi}_{m,k_m}(\overline{F}, \hat{\mathbf{H}}) = \hat{\psi}_{m,k_m}(\overline{F}, \hat{\mathbf{H}}) + \frac{1}{|\overline{F}|} \cdot \Big[\hat{\mathbf{H}}(\tilde{d}) - \sum_{n\in\overline{F}} \hat{\psi}_{n,\tilde{d}_n}(\overline{F}, \hat{\mathbf{H}})\Big],$$

where

$$\hat{\psi}_{m,k_m}(\overline{F}, \hat{\mathbf{H}}) = \hat{\mathbf{H}}(k_m, 0_{\overline{F}\setminus\{m\}}) - \hat{\mathbf{H}}(k_m - 1, 0_{\overline{F}\setminus\{m\}}),$$

and $\hat{\psi}_{m,k_m}(\overline{F}, \hat{\mathbf{H}})$ is the separate effect for step-grade of the factor m and its grade k_m. Conceptually, $\hat{\psi}_{m,k_m}$ measures the incremental impact of raising m's activity by one grade, holding other factors at zero. The term $\frac{1}{|\overline{F}|}[\hat{\mathbf{H}}(\tilde{d}) - \sum_{n\in\overline{F}} \hat{\psi}_{n,\tilde{d}_n}]$ then adjusts these incremental effects to ensure consistency and balanced gains across all factors. The EAIE could be utilized to guide decisions in production scheduling, project coordination, and other multidepartmental optimization tasks under different grades of collaborative effort. By incorporating both step-wise incremental effects and an overall balancing term, the EAIE aims to capture the nuanced contributions of factors in realistic industrial or organizational settings.

Table 1 summarizes the main symbols used in Section 2.

Table 1. Symbols.

Symbol	Meaning / role in model
$\overline{D}_m = \{0, 1, \ldots, \tilde{d}_m\}$	activity–grade set of factor m
$\tilde{d}_m$	total number of activity grade of factor m
$\overline{D}_m^+ = \overline{D}_m \setminus \{0\}$	set of nonzero grades of factor m (engaged)
$\hat{\mathbf{H}}$	utility (effect) mapping
$\hat{\psi}_{m,k_m}$	step-grade (separate) effect of (m, k_m)
$\hat{\Psi}_{m,k_m}$	the EAIE value (payoff, outcome) assigned to (m, k_m)
$\overline{\mathbf{P}}^{\overline{F}}$	index set $\{(m, k_m) \mid m \in \overline{F},\ k_m \in \overline{D}_m\}$

3. Axiomatic results

In this section, we show that there exists a reduced situation that can be used to characterize the EAIE. From an industrial management perspective, these axiomatic properties reinforce consistent decision-making rules when multiple departments or units coordinate at varying grades of engagement. Our framework aligns with these viewpoints by providing a systematic structure for analyzing multichoice interactions under real organizational contexts.

Let ρ be a rule on **MCS**,

- ρ fits completeness for rule (COMR) if $\sum_{m\in\overline{F}} \rho_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}}) = \hat{\mathbf{H}}(\tilde{d})$ for every $(\overline{F},\hat{\mathbf{H}}) \in$ **MCS**. This property ensures that when all factors are engaged at their highest activity grade, the entire obtainable outcome (e.g., total synergy or utility within an organization) is distributed among them.

- ρ fits rule for two-factor situations (RFTS) if for every $(\overline{F},\hat{\mathbf{H}}) \in$ **MCS** with $|\overline{F}| \leq 2$, $\rho(\overline{F},\hat{\mathbf{H}}) = \hat{\Psi}(\overline{F},\hat{\mathbf{H}})$. This axiom reflects the standard approach introduced by Hart and Mas-Colell [5], ensuring that in any situation involving only two units (e.g., two key departments negotiating a joint project), the outcome coincides with that assigned by $\hat{\Psi}$.

- ρ fits coincident property for outcome (CPFO) if for every $(\overline{F},\hat{\mathbf{H}}) \in$ **MCS** with $\hat{\mathbf{H}}(\mu,k_m,0) - \hat{\mathbf{H}}(\mu,k_m-1,0) = \hat{\mathbf{H}}(\mu,0,k_n) - \hat{\mathbf{H}}(\mu,0,k_n-1)$ for some $(m,k_m),(n,k_n) \in \overline{\mathbf{P}}^{\overline{F}}$ and for every $1\mu \in \overline{D}^{\overline{F}\setminus\{m,n\}}$, $\rho_{m,k_m}(\overline{F},\hat{\mathbf{H}}) = \rho_{n,k_n}(\overline{F},\hat{\mathbf{H}})$. In practical organizational decisions, this means if two departments contribute the same marginal impact at certain grades, they should be treated equally in the payoff.

- ρ fits covariant transformation (COTR) if for every $(\overline{F},\hat{\mathbf{H}}),(\overline{F},\hat{\mathbf{U}}) \in$ **MCS** with $\hat{\mathbf{H}}(\mu) = \hat{\mathbf{U}}(\mu) + \sum_{m\in\hat{\mathbf{O}}(\mu)} \sum_{q=1}^{\mu_m} \zeta_{m,q}$ for some $\zeta \in \mathbb{R}^{\overline{\mathbf{P}}^{\overline{F}}}$ and for every $\mu \in \overline{D}^{\overline{F}}$, $\rho(\overline{F},\hat{\mathbf{H}}) = \rho(\overline{F},\hat{\mathbf{U}}) + \zeta$. Conceptually, this ensures that when the global outcome shifts uniformly (e.g., due to a systemic adjustment in organizational benefits or costs), the payoff rule adapts in a synchronized manner across all factors.

The properties of COMR and COTR are well-established and widely accepted within the realm of industrial and organizational cooperation frameworks [11, 12]. COMR ensures that the total utility is fully accounted for, which is crucial in large-scale project collaborations where all factors operate at maximum capacity. The RFTS property extends the two-person standard axiom introduced by Hart and Mas-Colell [5], particularly in relation to the Shapley value. The RFTS property stipulates that each factor receives a payoff determined by the rule $\hat{\Psi}$ in situations involving one or two factors. The CPFO property mandates that factors with identical marginal contributions receive equal payoffs, aligning with fairness criteria in many industrial negotiations. In the subsequent analysis, we aim to demonstrate that the EAIE satisfies COMR, RFTS, CPFO, and COTR.

Lemma 3.1. *The EAIE fits COMR.*

Proof. Let $(\overline{F},\hat{\mathbf{H}})\in\mathbf{MCS}$.

$$\begin{aligned}\sum_{m\in\overline{F}}\hat{\Psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}}) &= \sum_{m\in\overline{F}}\hat{\psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}})+\sum_{m\in\overline{F}}\frac{1}{|\overline{F}|}\cdot\Big[\hat{\mathbf{H}}(\tilde{d})-\sum_{n\in\overline{F}}\hat{\psi}_{n,\tilde{d}_n}(\overline{F},\hat{\mathbf{H}})\Big]\\ &= \sum_{m\in\overline{F}}\hat{\psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}})+\frac{|\overline{F}|}{|\overline{F}|}\cdot\Big[\hat{\mathbf{H}}(\tilde{d})-\sum_{n\in\overline{F}}\hat{\psi}_{n,\tilde{d}_n}(\overline{F},\hat{\mathbf{H}})\Big]\\ &= \sum_{m\in\overline{F}}\hat{\psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}})+\hat{\mathbf{H}}(\tilde{d})-\sum_{n\in\overline{F}}\hat{\psi}_{n,\tilde{d}_n}(\overline{F},\hat{\mathbf{H}})\\ &= \hat{\mathbf{H}}(\tilde{d}).\end{aligned}$$

So, the rule $\hat{\Psi}$ fits COMR. □

Lemma 3.2. *The EAIE fits RFTS.*

Proof. By the definitions of the EAIE and RFTS, the proof could be completed. □

Lemma 3.3. *The EAIE fits CPFO.*

Proof. Let $(\overline{F},\hat{\mathbf{H}})\in\mathbf{MCS}$. Assume that

$$\hat{\mathbf{H}}(\mu,k_m,0)-\hat{\mathbf{H}}(\mu,k_m-1,0)=\hat{\mathbf{H}}(\mu,0,k_n)-\hat{\mathbf{H}}(\mu,0,k_n-1)$$

for some $(m,k_m),(n,k_n)\in\overline{\mathbf{P}}^{\overline{F}}$ and for every $\mu\in\overline{D}^{\overline{F}\setminus\{m,n\}}$. By taking $\mu=0_{\overline{F}\setminus\{m,n\}}$,

$$\begin{aligned}\hat{\mathbf{H}}(k_m,0_{\overline{F}\setminus\{m\}})-\hat{\mathbf{H}}(k_m-1,0_{\overline{F}\setminus\{m\}}) &= \hat{\mathbf{H}}(\mu,k_m,0)-\hat{\mathbf{H}}(\mu,k_m-1,0)\\ &= \hat{\mathbf{H}}(\mu,0,k_n)-\hat{\mathbf{H}}(\mu,0,k_n-1)\\ &= \hat{\mathbf{H}}(k_n,0_{\overline{F}\setminus\{n\}})-\hat{\mathbf{H}}(k_n-1,0_{\overline{F}\setminus\{n\}}),\end{aligned}$$

i.e., $\hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}})=\hat{\psi}_{n,k_n}(\overline{F},\hat{\mathbf{H}})$. So,

$$\begin{aligned}\hat{\Psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}) &= \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}})+\frac{1}{|\overline{F}|}\cdot\Big[\hat{\mathbf{H}}(\tilde{d})-\sum_{t\in\overline{F}}\hat{\psi}_{t,\tilde{d}_t}(\overline{F},\hat{\mathbf{H}})\Big]\\ &= \hat{\psi}_{n,k_n}(\overline{F},\hat{\mathbf{H}})+\frac{1}{|\overline{F}|}\cdot\Big[\hat{\mathbf{H}}(\tilde{d})-\sum_{t\in\overline{F}}\hat{\psi}_{t,\tilde{d}_t}(\overline{F},\hat{\mathbf{H}})\Big]\\ &= \hat{\Psi}_{n,k_n}(\overline{F},\hat{\mathbf{H}}).\end{aligned}$$

So, the rule $\hat{\Psi}$ fits CPFO. □

Lemma 3.4. *The EAIE fits COTR.*

Proof. Let $(\overline{F},\hat{\mathbf{H}}),(\overline{F},\hat{\mathbf{U}})\in\mathbf{MCS}$ with

$$\hat{\mathbf{H}}(\mu)=\hat{\mathbf{U}}(\mu)+\sum_{m\in\hat{\mathbf{O}}(\mu)}\sum_{q=1}^{\mu_m}\zeta_{m,q}$$

for some $\zeta\in\mathbb{R}^{\overline{\mathbf{P}}^{\overline{F}}}$ and for every $\alpha\in\overline{D}^{\overline{F}}$. For every $(m,k_m)\in\overline{\mathbf{P}}^{\overline{F}}$,

$$\begin{aligned}\hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}) &= \hat{\mathbf{H}}(k_m,0_{\overline{F}\setminus\{m\}})-\hat{\mathbf{H}}(k_m-1,0_{\overline{F}\setminus\{m\}})\\ &= \hat{\mathbf{U}}(k_m,0_{\overline{F}\setminus\{m\}})+\sum_{q=1}^{k_m}\zeta_{m,q}-\hat{\mathbf{U}}(k_m-1,0_{\overline{F}\setminus\{m\}})-\sum_{q=1}^{k_m-1}\zeta_{m,q}\\ &= \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{U}})+\zeta_{m,k_m}.\end{aligned}$$

Further,

$$\begin{aligned}\hat{\Psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}) &= \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}) + \tfrac{1}{|\overline{F}|}\cdot\Big[\hat{\mathbf{H}}(\tilde{d}) - \sum_{n\in\overline{F}}\hat{\psi}_{n,\tilde{d}_n}(\overline{F},\hat{\mathbf{H}})\Big]\\ &= \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{U}}) + \zeta_{m,k_m} + \tfrac{1}{|\overline{F}|}\cdot\Big[\hat{\mathbf{U}}(\tilde{d}) + \sum_{t\in\overline{F}}\zeta_{t,\tilde{d}_t} - \sum_{n\in\overline{F}}\hat{\psi}_{n,\tilde{d}_n}(\overline{F},\hat{\mathbf{U}}) - \sum_{n\in\overline{F}}\zeta_{n,\tilde{d}_n}\Big]\\ &= \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{U}}) + \zeta_{m,k_m} + \tfrac{1}{|\overline{F}|}\cdot\Big[\hat{\mathbf{U}}(\tilde{d}) - \sum_{n\in\overline{F}}\hat{\psi}_{n,\tilde{d}_n}(\overline{F},\hat{\mathbf{U}})\Big]\\ &= \hat{\Psi}_{m,k_m}(\overline{F},\hat{\mathbf{U}}) + \zeta_{m,k_m}.\end{aligned}$$

So, the rule $\hat{\Psi}$ fits COTR. □

A natural analogue of the reduction due to Hsieh and Liao [8] on multichoice situations is as follows. Given $(\overline{F},\hat{\mathbf{H}}) \in \mathbf{MCS}$, $\overline{S} \subseteq \overline{F}$ and a rule ρ, the reduced situation $(\overline{S},\hat{\mathbf{H}}^{\rho}_{\overline{S}})$ with respect to $\overline{S}$ and ρ is defined by for every $\mu \in \overline{D}^{\overline{S}}$,

$$\hat{\mathbf{H}}^{\rho}_{\overline{S}}(\mu) = \begin{cases} 0 & \mu = 0_{\overline{S}},\\ \hat{\mathbf{H}}(\mu_m, 0_{\overline{F}\setminus\{m\}}) & |\overline{S}| \geq 2 \text{ and } |\hat{\mathbf{O}}(\mu)| = 1,\\ \hat{\mathbf{H}}(\mu, \tilde{d}_{\overline{F}\setminus\overline{S}}) - \sum_{m\in\overline{F}\setminus\overline{S}} \rho_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}}) & \text{otherwise.}\end{cases}$$

In an industrial management situation, such reduction intuitively denotes isolating the interaction of two specific departments (i.e., $\overline{S}$) after distributing certain values to the rest. This approach aligns with the bilateral nature of many real-world negotiations.

The property of reduction property for bilateral conditions in larger cooperative contexts can be described as follows: Consider a rule ρ operating within a situation **MCS**. For any pair of factors, a reduced situation is defined by allocating the payoffs prescribed by ρ to all other factors and considering the remaining utility or capacity for the pair. The rule ρ is deemed bilaterally consistent if, when applied to any such reduced situation, it yields the same payoffs for the pair as in the original situation.

Formally, a rule ρ fits reduction property for bilateral conditions (RPBC) if for every $(\overline{F},\hat{\mathbf{H}}) \in \mathbf{MCS}$ with $|\overline{F}| \geq 3$, for every $\overline{S} \subseteq \overline{F}$ with $|\overline{S}| = 2$ and for every $(m,k_m) \in A^{\overline{S}}$, $\rho_{m,k_m}(\overline{F},\hat{\mathbf{H}}) = \rho_{m,k_m}(\overline{S},\hat{\mathbf{H}}^{\rho}_{\overline{S}})$. Such property is vital to industrial collaboration, where subcoalitions or department pairs might revise payoffs independently while preserving overall coherence.

Lemma 3.5. *The EAIE $\hat{\Psi}$ fits RPBC.*

Proof. Let $(\overline{F},\hat{\mathbf{H}}) \in \mathbf{MCS}$ with $|\overline{F}| \geq 3$ and $\overline{S} \subseteq \overline{F}$ with $|\overline{S}| = 2$. Assume that $\overline{S} = \{m,n\}$. By the definition of $\hat{\Psi}$, for every $(p,k_p) \in A^{\overline{S}}$,

$$\hat{\Psi}_{p,k_p}(\overline{S},\hat{\mathbf{H}}^{\hat{\Psi}}_{\overline{S}}) = \hat{\psi}_{p,k_p}(\overline{S},\hat{\mathbf{H}}^{\hat{\Psi}}_{\overline{S}}) + \tfrac{1}{|\overline{S}|}\cdot\Big[\hat{\mathbf{H}}^{\hat{\Psi}}_{\overline{S}}(\tilde{d}_{\overline{S}}) - \sum_{t\in\overline{S}}\hat{\psi}_{t,\tilde{d}_t}(\overline{S},\hat{\mathbf{H}}^{\hat{\Psi}}_{\overline{S}})\Big]. \tag{3.1}$$

By definitions of $\hat{\psi}$ and $\hat{\mathbf{H}}^{\hat{\Psi}}_{\overline{S}}$, for every $k_m \in \overline{D}_m$,

$$\begin{aligned}\hat{\psi}_{m,k_m}(\overline{S},\hat{\mathbf{H}}^{\hat{\Psi}}_{\overline{S}}) &= \hat{\mathbf{H}}^{\hat{\Psi}}_{\overline{S}}(k_m,0) - \hat{\mathbf{H}}^{\hat{\Psi}}_{\overline{S}}(k_m-1,0)\\ &= \hat{\mathbf{H}}(k_m,0_{\overline{F}\setminus\{m\}}) - \hat{\mathbf{H}}(k_m-1,0_{\overline{F}\setminus\{m\}})\\ &= \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}).\end{aligned} \tag{3.2}$$

Hence, by Eqs (3.1), (3.2) and definitions of $\hat{\mathbf{H}}^{\hat{\Psi}}_{\overline{S}}$ and $\hat{\Psi}$,

$$\begin{aligned}
\hat{\Psi}_{m,k_m}(\overline{S},\hat{\mathbf{H}}^{\hat{\Psi}}_{\overline{S}}) &= \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}) + \tfrac{1}{|\overline{S}|}\cdot\Big[\hat{\mathbf{H}}^{\hat{\Psi}}_{\overline{S}}(\tilde{d}_{\overline{S}}) - \sum_{t\in\overline{S}}\hat{\psi}_{t,\tilde{d}_t}(\overline{F},\hat{\mathbf{H}})\Big]\\
&= \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}) + \tfrac{1}{|\overline{S}|}\cdot\Big[\hat{\mathbf{H}}(\tilde{d}) - \sum_{t\in\overline{F}\setminus\overline{S}}\hat{\Psi}_{t,\tilde{d}_t}(\overline{F},\hat{\mathbf{H}}) - \sum_{t\in\overline{S}}\hat{\psi}_{t,\tilde{d}_t}(\overline{F},\hat{\mathbf{H}})\Big]\\
&= \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}) + \tfrac{1}{|\overline{S}|}\cdot\Big[\sum_{t\in\overline{S}}\hat{\Psi}_{t,\tilde{d}_t}(\overline{F},\hat{\mathbf{H}}) - \sum_{t\in\overline{S}}\hat{\psi}_{t,\tilde{d}_t}(\overline{F},\hat{\mathbf{H}})\Big] \quad \Big(\textbf{COMR of } \hat{\Psi}\Big)\\
&= \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}) + \tfrac{1}{|\overline{S}|}\cdot\Big[\tfrac{|\overline{S}|}{|\overline{F}|}\cdot[\hat{\mathbf{H}}(\tilde{d}) - \sum_{t\in N}\hat{\psi}_{t,\tilde{d}_t}(\overline{F},\hat{\mathbf{H}})]\Big] \quad \Big(\textbf{Definition 2.1}\Big)\\
&= \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}) + \tfrac{1}{|\overline{F}|}\cdot\Big[\hat{\mathbf{H}}(\tilde{d}) - \sum_{t\in N}\hat{\psi}_{t,\tilde{d}_t}(\overline{F},\hat{\mathbf{H}})\Big]\\
&= \hat{\Psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}).
\end{aligned}$$

Similarly, $\hat{\Psi}_{n,k_n}(\overline{S},\hat{\mathbf{H}}^{\hat{\Psi}}_{\overline{S}}) = \hat{\Psi}_{n,k_n}(\overline{F},\hat{\mathbf{H}})$ for every $k_n \in \overline{D}_n$. So, the EAIE fits RPBC. □

The following examples demonstrate that above axioms are pairwise independent; we state this fact here so the necessity of each axiom is clear before the proofs commence.

Example 1. Define a rule ρ on **MCS** by for every $(\overline{F},\hat{\mathbf{H}}) \in$ **MCS** and for every $(m,k_m) \in \overline{\mathbf{P}}^{\overline{F}}$,

$$\rho_{m,k_m}(\overline{F},\hat{\mathbf{H}}) = \begin{cases} \hat{\Psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}), & \text{if } |\overline{F}| \le 2,\\ 0, & \text{otherwise.}\end{cases}$$

Clearly, ρ fits RFTS, but it violates RPBC.

Example 2. Define a rule ρ on **MCS** by for every $(\overline{F},\hat{\mathbf{H}}) \in$ **MCS** and for every $(m,k_m) \in \overline{\mathbf{P}}^{\overline{F}}$, $\rho_{m,k_m}(\overline{F},\hat{\mathbf{H}}) = \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}})$. Clearly, ρ fits CPFO, COTR, and RPBC, but it violates COMR and RFTS.

Example 3. Define a rule ρ on **MCS** by for every $(\overline{F},\hat{\mathbf{H}}) \in$ **MCS** and for every $(m,k_m) \in \overline{\mathbf{P}}^{\overline{F}}$, $\rho_{m,k_m}(\overline{F},\hat{\mathbf{H}}) = \frac{\hat{\mathbf{H}}(\tilde{d})}{|\overline{F}|}$. Clearly, ρ fits COMR, CPFO, and RPBC, but it violates COTR.

Example 4. Define a rule ρ on **MCS** by for every $(\overline{F},\hat{\mathbf{H}}) \in$ **MCS** and for every $(m,k_m) \in \overline{\mathbf{P}}^{\overline{F}}$,

$$\rho_{m,k_m}(\overline{F},\hat{\mathbf{H}}) \;= [\hat{\mathbf{H}}(\tilde{d}) - \hat{\mathbf{H}}(\tilde{d}_{\overline{F}\setminus\{i\}},0)] + \tfrac{1}{|\overline{F}|}\cdot\Big[\hat{\mathbf{H}}(\tilde{d}) - \sum_{k\in N}[\hat{\mathbf{H}}(\tilde{d}) - \hat{\mathbf{H}}(\tilde{d}_{\overline{F}\setminus\{k\}},0)]\Big].$$

Clearly, ρ fits COMR, COTR, and RPBC, but it violates CPFO.

Example 5. Define a rule ρ on **MCS** by for every $(\overline{F},\hat{\mathbf{H}}) \in$ **MCS** and for every $(m,k_m) \in \overline{\mathbf{P}}^{\overline{F}}$,

$$\begin{aligned}&\rho_{m,k_m}(\overline{F},\hat{\mathbf{H}})\\ =&\sum_{\substack{\overline{S}\subseteq\overline{F}\\ i\in\overline{S}}}\tfrac{(|\overline{S}|-1)!(|\overline{F}|-|\overline{S}|)!}{|\overline{F}|!}[\hat{\mathbf{H}}((\tilde{d}_{\overline{F}\setminus\{i\}},k_m)_{\overline{S}},0_{\overline{F}\setminus\overline{S}}) - \hat{\mathbf{H}}((\tilde{d}_{\overline{F}\setminus\{i\}},0)_{\overline{S}},0_{\overline{F}\setminus\overline{S}})].\end{aligned}$$

Clearly, ρ fits COMR, CPFO, and COTR, but it violates RPBC.

These illustrative constructions confirm the logical independence of each axiom, providing a modular understanding of how each condition contributes to a stable, consistent rule for multichoice interactions under an industrial or organizational decision framework.

Lemma 3.6. *If a rule ρ fits RFTS and RPBC then it also fits COMR.*

Proof. Let ρ be a rule on **MCS** satisfy RFTS and RPBC, and $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$. By RFTS of ρ and $\hat{\Psi}$. $\rho = \hat{\Psi}$ if $|\overline{F}| \leq 2$. By COMR of $\hat{\Psi}$, ρ fits COMR in $(\overline{F}, \hat{\mathbf{H}})$ absolutely if $|\overline{F}| \leq 2$. Assume that $|\overline{F}| \geq 3$. Let $n \in \overline{F}$, consider the reduced situation $(\{n\}, \hat{\mathbf{H}}^{\rho}_{\{n\}})$. By the definition of $\hat{\mathbf{H}}^{\rho}_{\{n\}}$,

$$\hat{\mathbf{H}}^{\rho}_{\{n\}}(\tilde{d}_n) = \hat{\mathbf{H}}(\tilde{d}) - \sum_{m \in \overline{F} \setminus \{n\}} \rho_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}).$$

Since ρ fits RPBC, $\rho_{n,k_n}(\overline{F}, \hat{\mathbf{H}}) = \rho_{n,k_n}(\{n\}, \hat{\mathbf{H}}^{\rho}_{\{n\}})$. Since $|\{n\}| = 1 \leq 2$, by above proof, ρ fits COMR in $(\{n\}, \hat{\mathbf{H}}^{\rho}_{\{n\}})$, that is, $\rho_{n,\tilde{d}_n}(\overline{F}, \hat{\mathbf{H}}) = \hat{\mathbf{H}}^{\rho}_{\{n\}}(\tilde{d}_n)$. Hence,

$$\begin{aligned} \rho_{n,k_n}(\overline{F}, \hat{\mathbf{H}}) &= \rho_{n,k_n}(\{n\}, \hat{\mathbf{H}}^{\rho}_{\{n\}}) \\ &= \hat{\mathbf{H}}^{\rho}_{\{n\}}(\tilde{d}_n) \\ &= \hat{\mathbf{H}}(\tilde{d}) - \sum_{m \in \overline{F} \setminus \{n\}} \rho_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}). \end{aligned}$$

That is, $\sum_{m \in \overline{F}} \rho_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) = \hat{\mathbf{H}}(\tilde{d})$, and thus, ρ fits COMR. □

Subsequently, we characterize the EAIE by means of the properties of RFTS and reduction property for bilateral conditions. These axioms reinforce the notion that fair, locally consistent collaborations remain stable when partially re-examined by specific pairs of factors, a property frequently observed in industrial negotiations where subcoalitions break away to reevaluate terms.

Theorem 3.7. *A rule ρ on* **MCS** *fits RFTS and RPBC if and only if $\rho = \hat{\Psi}$.*

Proof. By Lemma 3.2, $\hat{\Psi}$ fits RFTS. By Lemma 3.5, $\hat{\Psi}$ fits RPBC.

To prove uniqueness, suppose ρ fits RFTS and RPBC on **MCS**. By Lemma 3.6, ρ also fits COMR. Let $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$. If $|\overline{F}| \leq 2$, then by RFTS of ρ, $\rho(\overline{F}, \hat{\mathbf{H}}) = \hat{\Psi}(\overline{F}, \hat{\mathbf{H}})$. The case $|\overline{F}| > 2$: Let $m \in \overline{F}$ and $\overline{S} = \{m, n\}$ for some $n \in \overline{F} \setminus \{m\}$, then for every $k_m \in \overline{D}_m,\ k_n \in \overline{D}_n$,

$$\begin{aligned} & \rho_{m,k_m}(\overline{F}, \hat{\mathbf{H}}) - \rho_{n,k_n}(\overline{F}, \hat{\mathbf{H}}) \\ = \ & \rho_{m,k_m}(\overline{S}, \hat{\mathbf{H}}^{\rho}_{\overline{S}}) - \rho_{n,k_n}(\overline{S}, \hat{\mathbf{H}}^{\rho}_{\overline{S}}) \quad \big(\textbf{RPBC of } \rho\big) \\ = \ & \hat{\Psi}_{m,k_m}(\overline{S}, \hat{\mathbf{H}}^{\rho}_{\overline{S}}) - \hat{\Psi}_{n,k_n}(\overline{S}, \hat{\mathbf{H}}^{\rho}_{\overline{S}}) \quad \big(\textbf{RFTS of } \rho\big) \\ = \ & \hat{\psi}_{m,k_m}(\overline{S}, \hat{\mathbf{H}}^{\rho}_{\overline{S}}) - \hat{\psi}_{n,k_n}(\overline{S}, \hat{\mathbf{H}}^{\rho}_{\overline{S}}) \quad \big(\textbf{Definition 2.1}\big) \\ = \ & \Big[\hat{\mathbf{H}}^{\rho}_{\overline{S}}(k_m, 0) - \hat{\mathbf{H}}^{\rho}_{\overline{S}}(k_m - 1, 0) - \hat{\mathbf{H}}^{\rho}_{\overline{S}}(0, k_n) + \hat{\mathbf{H}}^{\rho}_{\overline{S}}(0, k_n - 1)\Big] \quad \big(\textbf{Definition 2.1}\big) \\ = \ & \Big[\hat{\mathbf{H}}(k_m, 0_{\overline{F} \setminus \{m\}}) - \hat{\mathbf{H}}(k_m - 1, 0_{\overline{F} \setminus \{m\}}) - \hat{\mathbf{H}}(k_n, 0_{\overline{F} \setminus \{n\}}) + \hat{\mathbf{H}}(k_n - 1, 0_{\overline{F} \setminus \{n\}})\Big] \quad \big(\textbf{Definition of } \hat{\mathbf{H}}^{\rho}_{\overline{S}}\big). \end{aligned} \tag{3.3}$$

Similarly taking, by Definition 2.1, we can derive that

$$\begin{aligned} & \hat{\Psi}_{m,k_m}(\overline{F}, \hat{\mathbf{H}}) - \hat{\Psi}_{n,k_n}(\overline{F}, \hat{\mathbf{H}}) \\ = \ & \hat{\psi}_{m,k_m}(\overline{F}, \hat{\mathbf{H}}) + \tfrac{1}{|\overline{F}|} \cdot \Big[\hat{\mathbf{H}}(\tilde{d}) - \sum_{i \in \overline{F}} \hat{\psi}_{i,\tilde{d}_i}(\overline{F}, \hat{\mathbf{H}})\Big] \\ = \ & -\hat{\psi}_{n,k_n}(\overline{F}, \hat{\mathbf{H}}) + \tfrac{1}{|\overline{F}|} \cdot \Big[\hat{\mathbf{H}}(\tilde{d}) - \sum_{i \in \overline{F}} \hat{\psi}_{i,\tilde{d}_i}(\overline{F}, \hat{\mathbf{H}})\Big] \\ = \ & \hat{\psi}_{m,k_m}(\overline{F}, \hat{\mathbf{H}}) - \hat{\psi}_{n,k_n}(\overline{F}, \hat{\mathbf{H}}) \\ = \ & \Big[\hat{\mathbf{H}}(k_m, 0_{\overline{F} \setminus \{m\}}) - \hat{\mathbf{H}}(k_m - 1, 0_{\overline{F} \setminus \{m\}}) - \hat{\mathbf{H}}(k_n, 0_{\overline{F} \setminus \{n\}}) + \hat{\mathbf{H}}(k_n - 1, 0_{\overline{F} \setminus \{n\}})\Big]. \end{aligned} \tag{3.4}$$

Hence, by Eqs (3.3) and (3.4),

$$\rho_{m,k_m}(\overline{F},\hat{\mathbf{H}})-\rho_{n,k_n}(\overline{F},\hat{\mathbf{H}})=\hat{\Psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}})-\hat{\Psi}_{n,k_n}(\overline{F},\hat{\mathbf{H}}).$$

This implies that $\rho_{m,k_m}(\overline{F},\hat{\mathbf{H}})-\hat{\Psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}})=t$ for every $k_m\in\overline{D}_m$, where $t\in\mathbb{R}$ is a constant. It remains to show that $t=0$. By COMR of ρ and $\hat{\Psi}$,

$$\begin{aligned}0 &= \left[\hat{\mathbf{H}}(\tilde{d})-\hat{\mathbf{H}}(\tilde{d})\right]\\ &= \sum_{m\in\overline{F}}\left[\rho_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}})-\hat{\Psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}})\right] \quad \left(\textbf{by COMR of } \rho \textbf{ and } \hat{\Psi}\right)\\ &= |\overline{F}|\cdot t.\end{aligned}$$

Hence, $t=0$, that is, $\rho_{m,k_m}(\overline{F},\hat{\mathbf{H}})=\hat{\Psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}})$ for every $k_m\in\overline{D}_m$. □

Next, we characterize the EAIE by means of related properties of completeness for rule, coincident property for outcome, covariant transformation and reduction property for bilateral conditions. From a real-world viewpoint, these axioms encapsulate desirable features for robust managerial strategies, ensuring that the final outcome respects both global coherence and localized fairness.

Lemma 3.8. *If a rule ρ on* **MCS** *fits COMR, CPFO, and COTR, then ρ fits RFTS.*

Proof. Assume that a rule ρ fits COMR, CPFO, and COTR. Given $(\overline{F},\hat{\mathbf{H}})\in\mathbf{MCS}$ with $\overline{F}=\{m,n\}$ for some $m\neq n$. We define a situation $(\overline{F},\hat{\mathbf{U}})$ to be that for every $\lambda\in\overline{D}^{\overline{F}}$,

$$\hat{\mathbf{U}}(\lambda)=\hat{\mathbf{H}}(\lambda)-\sum_{i\in\hat{\mathbf{O}}(\lambda)}\sum_{q=1}^{\lambda_i}[\hat{\mathbf{H}}(q,0_{\overline{F}\setminus\{i\}})-\hat{\mathbf{H}}(q-1,0_{\overline{F}\setminus\{i\}})].$$

By the definition of w, for every $k_m\in\overline{D}_m$,

$$\hat{\mathbf{U}}(k_m,0)=\hat{\mathbf{H}}(k_m,0)-\sum_{q=1}^{k_m}[\hat{\mathbf{H}}(q,0_{\overline{F}\setminus\{i\}})-\hat{\mathbf{H}}(q-1,0_{\overline{F}\setminus\{i\}})]=\hat{\mathbf{H}}(k_m,0)-\hat{\mathbf{H}}(k_m,0)=0.$$

Similarly, $\hat{\mathbf{U}}(0,k_n)=0$ for all $k_n\in\overline{D}_n$. Since $\hat{\mathbf{U}}(k_m,0)-\hat{\mathbf{U}}(k_m-1,0)=0=\hat{\mathbf{U}}(0,k_n)-\hat{\mathbf{U}}(0,k_n-1)$, by CPFO of ρ, $\rho_{m,k_m}(\overline{F},\hat{\mathbf{U}})=\rho_{n,k_n}(\overline{F},\hat{\mathbf{U}})$. By COMR of ρ,

$$\hat{\mathbf{U}}(\tilde{d})=\rho_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{U}})+\rho_{n,\tilde{d}_n}(\overline{F},\hat{\mathbf{U}})=2\cdot\rho_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{U}}). \tag{3.5}$$

Therefore, by Eq (3.5) and definition of $\hat{\mathbf{U}}$,

$$\rho_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{U}})=\frac{\hat{\mathbf{U}}(\tilde{d})}{2}=\frac{1}{2}\cdot[\hat{\mathbf{H}}(\tilde{d})-\hat{\psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}})-\hat{\psi}_{n,\tilde{d}_n}(\overline{F},\hat{\mathbf{H}})].$$

By COTR of ρ,

$$\begin{aligned}\rho_{m,k_m}(\overline{F},\hat{\mathbf{H}}) &= \rho_{m,k_m}(\overline{F},\hat{\mathbf{U}})+\hat{\mathbf{H}}(k_m,0)-\hat{\mathbf{H}}(k_m-1,0)\\ &= \rho_{m,k_m}(\overline{F},\hat{\mathbf{U}})+\hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}})\\ &= \tfrac{1}{2}\cdot[\hat{\mathbf{H}}(\tilde{d})-\hat{\psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}})-\hat{\psi}_{n,\tilde{d}_n}(\overline{F},\hat{\mathbf{H}})]+\hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}})\\ &= \hat{\Psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}).\end{aligned}$$

Similarly, $\rho_{n,k_n}(\overline{F},\hat{\mathbf{H}})=\hat{\Psi}_{n,k_n}(\overline{F},\hat{\mathbf{H}})$ for every $k_n\in\overline{D}_n$. Hence, ρ fits RFTS. □

Theorem 3.9. *A rule ρ on* **MCS** *fits COMR, CPFO, COTR, and RPBC if and only if* $\rho = \hat{\Psi}$.

Proof. By Lemmas 3.1, 3.3, and 3.4, $\hat{\Psi}$ fits COMR, CPFO, and COTR. The remaining proofs follow from Theorem 3.7 and Lemmas 3.1 and 3.8. □

4. Dynamic result

In many industrial management situations, strategic decisions unfold over multiple stages, where departments or units iteratively adjust their choices based on changing conditions. Such an iterative process is crucial when implementing a game-theoretical solution, ensuring that local improvements (e.g., stepwise modifications of departmental strategies) eventually converge to a globally consistent outcome [11]. To model this iterative adaptation in our multichoice framework, we introduce a dynamic approach for the EAIE, allowing each factor to update its activity grade incrementally while preserving coherence with the overall industrial objective.

In order to provide the dynamic approach for the EAIE, we first create a representation for the EAIE by applying the excess function. Let $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$ and $\tilde{\mathbf{x}} \in \mathbb{R}^{\overline{\mathbf{P}}^{\overline{F}}}$. The excess of $\mu \in \overline{D}^{\overline{F}}$ at $\tilde{\mathbf{x}}$ is the real number

$$\hat{\mathbf{e}}(\mu, \hat{\mathbf{H}}, \tilde{\mathbf{x}}) = [\hat{\mathbf{H}}(\mu) - \sum_{m \in \hat{\mathbf{O}}(\mu)} \hat{\mathbf{H}}(\mu_m - 1, 0_{\overline{F}\setminus\{m\}})] - \tilde{\mathbf{x}}(\mu),$$

where $\tilde{\mathbf{x}}(\mu) = \sum_{m \in \hat{\mathbf{O}}(\mu)} \tilde{\mathbf{x}}_{m,\mu_m}$. Further, we define that

$$\overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}}) = \{\tilde{\mathbf{x}} \in \mathbb{R}^{\overline{\mathbf{P}}^{\overline{F}}} \mid \sum_{m \in \overline{F}} \tilde{\mathbf{x}}_{m,\tilde{d}_m} = \hat{\mathbf{H}}(\tilde{d})\}.$$

In practical organizational settings, each excess value measures whether a partial coalition configuration μ can improve its outcome compared to the current distribution $\tilde{\mathbf{x}}$, reflecting how departments might perceive shortfalls or surpluses based on local adjustments.

Lemma 4.1. *Let* $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$ *and* $\tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$. *Then,*

$$\begin{aligned} & \hat{\mathbf{e}}((\tilde{d}_m, 0_{\overline{F}\setminus\{m\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}}) = \hat{\mathbf{e}}((\tilde{d}_n, 0_{\overline{F}\setminus\{n\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}}) \\ \Longleftrightarrow \quad & \tilde{\mathbf{x}}_{m,\tilde{d}_m} = \hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) \ \forall\, m, n \in \overline{F}. \end{aligned}$$

Proof. Let $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$ and $\tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$ for every pairs $\{m, n\} \subseteq \overline{F}$,

$$\begin{aligned} & \hat{\mathbf{e}}((\tilde{d}_m, 0_{\overline{F}\setminus\{m\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}}) = \hat{\mathbf{e}}((\tilde{d}_n, 0_{\overline{F}\setminus\{n\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}}) \\ \Longleftrightarrow \quad & \hat{\mathbf{H}}(\tilde{d}_m, 0_{\overline{F}\setminus\{m\}}) - \hat{\mathbf{H}}(\tilde{d}_m - 1, 0_{\overline{F}\setminus\{m\}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m} = \hat{\mathbf{H}}(\tilde{d}_n, 0_{\overline{F}\setminus\{n\}}) - \hat{\mathbf{H}}(\tilde{d}_n - 1, 0_{\overline{F}\setminus\{n\}}) - \tilde{\mathbf{x}}_{n,\tilde{d}_n} \\ \Longleftrightarrow \quad & \tilde{\mathbf{x}}_{m,\tilde{d}_m} - \tilde{\mathbf{x}}_{n,\tilde{d}_n} = [\hat{\mathbf{H}}(\tilde{d}_m, 0_{\overline{F}\setminus\{m\}}) - \hat{\mathbf{H}}(\tilde{d}_m - 1, 0_{\overline{F}\setminus\{m\}})] \\ & -[\hat{\mathbf{H}}(\tilde{d}_n, 0_{\overline{F}\setminus\{n\}}) - \hat{\mathbf{H}}(\tilde{d}_n - 1, 0_{\overline{F}\setminus\{n\}})]. \end{aligned} \tag{4.1}$$

By definition of $\hat{\Psi}$,

$$\begin{aligned} & \hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) - \hat{\Psi}_{n,\tilde{d}_n}(\overline{F}, \hat{\mathbf{H}}) \\ = \quad & [\hat{\mathbf{H}}(\tilde{d}_m, 0_{\overline{F}\setminus\{m\}}) - \hat{\mathbf{H}}(\tilde{d}_m - 1, 0_{\overline{F}\setminus\{m\}})] - [\hat{\mathbf{H}}(\tilde{d}_n, 0_{\overline{F}\setminus\{n\}}) - \hat{\mathbf{H}}(\tilde{d}_n - 1, 0_{\overline{F}\setminus\{n\}})]. \end{aligned} \tag{4.2}$$

By Eqs (4.1) and (4.2), for every pairs $\{m, n\} \subseteq \overline{F}$,

$$\tilde{\mathbf{x}}_{m,\tilde{d}_m} - \tilde{\mathbf{x}}_{n,\tilde{d}_n} = \hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) - \hat{\Psi}_{n,\tilde{d}_n}(\overline{F}, \hat{\mathbf{H}}).$$

Hence,

$$\sum_{n \neq m}[\tilde{\mathbf{x}}_{m,\tilde{d}_m} - \tilde{\mathbf{x}}_{n,\tilde{d}_n}] = \sum_{n \neq m}[\hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) - \hat{\Psi}_{n,\tilde{d}_n}(\overline{F}, \hat{\mathbf{H}})].$$

That is,

$$(|\overline{F}| - 1) \cdot \tilde{\mathbf{x}}_{m,\tilde{d}_m} - \sum_{n \neq m} \tilde{\mathbf{x}}_{n,\tilde{d}_n} = (|\overline{F}| - 1) \cdot \hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) - \sum_{n \neq m} \hat{\Psi}_{n,\tilde{d}_n}(\overline{F}, \hat{\mathbf{H}}).$$

Since $\tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$ and $\hat{\Psi}$ fits COMR,

$$|\overline{F}| \cdot \tilde{\mathbf{x}}_{m,\tilde{d}_m} - \hat{\mathbf{H}}(\tilde{d}) = |\overline{F}| \cdot \hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) - \hat{\mathbf{H}}(\tilde{d}).$$

Therefore, $\tilde{\mathbf{x}}_{m,\tilde{d}_m} = \hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}})$ for every $m \in \overline{F}$. □

In an industrial decision-making environment, Lemma 4.1 indicates that if any two departments or units perceive identical discrepancies in their maximum potential payoff versus what they currently receive (excess), the current allocation must already coincide with the EAIE solution. This ensures a form of consistency: No single department can claim unfairness in a dynamic sense if all top-grade discrepancies match.

Based on the notion of Lemma 4.1, we define a correction function to provide a dynamic approach for the EAIE as follows.

Definition 4.2. Let $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$ with $|\overline{F}| \geq 2$ and $\tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$. We define the correction functions $\hat{\mathbf{g}}_{m,k_m} : \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}}) \to \mathbb{R}$ by for every $(m, k_m) \in \overline{\mathbf{P}}^{\overline{F}}$,

$$\hat{\mathbf{g}}_{m,k_m}(\tilde{\mathbf{x}}) = \tilde{\mathbf{x}}_{m,k_m} + \alpha \sum_{n \in \overline{F} \setminus \{m\}} \Big(\hat{\mathbf{e}}((\tilde{d}_m, 0_{\overline{F} \setminus \{m\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}}) - \hat{\mathbf{e}}((\tilde{d}_n, 0_{\overline{F} \setminus \{n\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}})\Big).$$

We also define that $\hat{\mathbf{g}} = (\hat{\mathbf{g}}_{m,k_m})_{(m,k_m) \in \overline{\mathbf{P}}^{\overline{F}}}$ and $\tilde{\mathbf{x}}^0 = \tilde{\mathbf{x}}$, $\tilde{\mathbf{x}}^1 = \hat{\mathbf{g}}(\tilde{\mathbf{x}}^0), \cdots, \tilde{\mathbf{x}}^q = \hat{\mathbf{g}}(\tilde{\mathbf{x}}^{q-1})$ for every $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$, for every $\tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$ and for every $q \in \mathbb{N}$.

From an operational viewpoint, α can be interpreted as a learning rate or adjustment coefficient, dictating how aggressively departments revise their payoffs in response to perceived imbalances. Such dynamic adjustment routines have been studied for diverse industrial contexts, where iterative reallocation is driven by local dissatisfaction or incremental improvement steps [11]. Further, a corresponding algorithm is also provided.

Iterative adjustment towards the EAIE

(1) **Input**: initial payoff vector $\tilde{\mathbf{x}}^0 \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$; learning rate $\alpha \in (0, 2/|\overline{F}|)$.

(2) **For** $q = 0, 1, 2, \ldots$ **repeat**

for each $(m, k_m) \in \overline{\mathbf{P}}^{\overline{F}}$ **do**

i. Compute excess difference

$$\Delta_{m,k_m} \leftarrow \sum_{n\in\overline{F}\setminus\{m\}} (\hat{\mathbf{e}}((\tilde{d}_m, 0), \hat{\mathbf{H}}, \tilde{\mathbf{x}}^q) - \hat{\mathbf{e}}((\tilde{d}_n, 0), \hat{\mathbf{H}}, \tilde{\mathbf{x}}^q));$$

ii. Update component $\tilde{\mathbf{x}}^{q+1}_{m,k_m} \leftarrow \tilde{\mathbf{x}}^{q}_{m,k_m} + \alpha\, \Delta_{m,k_m}$.

(3) **Until** $\|\tilde{\mathbf{x}}^{q+1} - \tilde{\mathbf{x}}^q\| \le \varepsilon$ for a preset tolerance ε.

Lemma 4.3. *Let* $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$*. If* $\tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$*, then* $\hat{\mathbf{g}}(\tilde{\mathbf{x}}) \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$.

Proof. Let $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$, $m, n \in \overline{F}$ and $\tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$.

$$\begin{aligned}
&\sum_{n\in\overline{F}\setminus\{m\}} \Big(\hat{\mathbf{e}}((\tilde{d}_m, 0_{\overline{F}\setminus\{m\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}}) - \hat{\mathbf{e}}((\tilde{d}_n, 0_{\overline{F}\setminus\{n\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}})\Big)\\
=\;& \sum_{n\in\overline{F}\setminus\{m\}} \Big(\hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) - \hat{\Psi}_{n,\tilde{d}_n}(\overline{F}, \hat{\mathbf{H}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m} + \tilde{\mathbf{x}}_{n,\tilde{d}_n}\Big) \quad \Big(\text{Eqs (4.1) and (4.2)}\Big)\\
=\;& \Big((|\overline{F}| - 1)\cdot(\hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m}) - \sum_{n\in\overline{F}\setminus\{m\}} \hat{\Psi}_{n,\tilde{d}_n}(\overline{F}, \hat{\mathbf{H}}) + \sum_{n\in\overline{F}\setminus\{i\}} \tilde{\mathbf{x}}_{n,\tilde{d}_n}\Big)\\
=\;& \Big(|\overline{F}|\cdot(\hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m}) - \hat{\mathbf{H}}(\tilde{d}) + \hat{\mathbf{H}}(\tilde{d})\Big) \quad \Big(\textbf{COMR of } \hat{\Psi}\textbf{, } \tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})\Big)\\
=\;& |\overline{F}|\cdot\Big(\hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m}\Big).
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
&\sum_{m\in\overline{F}}\sum_{n\in\overline{F}\setminus\{m\}} \Big(\hat{\mathbf{e}}((\tilde{d}_m, 0_{\overline{F}\setminus\{m\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}}) - \hat{\mathbf{e}}((\tilde{d}_n, 0_{\overline{F}\setminus\{n\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}})\Big)\\
=\;& \sum_{m\in\overline{F}} |\overline{F}|\cdot\Big(\hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m}\Big)\\
=\;& |\overline{F}|\cdot\Big(\sum_{m\in\overline{F}} \hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) - \sum_{m\in\overline{F}} \tilde{\mathbf{x}}_{m,\tilde{d}_m}\Big)\\
=\;& |\overline{F}|\cdot(\hat{\mathbf{H}}(\tilde{d}) - \hat{\mathbf{H}}(\tilde{d})) \quad \Big(\textbf{COMR of } \hat{\Psi}\textbf{, } \tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})\Big)\\
=\;& 0.
\end{aligned} \tag{4.4}$$

So we have that

$$\begin{aligned}
&\sum_{m\in\overline{F}} \hat{\mathbf{g}}_{m,\tilde{d}_m}(\tilde{\mathbf{x}})\\
=\;& \sum_{m\in\overline{F}} \Big[\tilde{\mathbf{x}}_{m,\tilde{d}_m} + \alpha \sum_{n\in\overline{F}\setminus\{m\}} \Big(\hat{\mathbf{e}}((\tilde{d}_m, 0_{\overline{F}\setminus\{i\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}}) - \hat{\mathbf{e}}((\tilde{d}_n, 0_{\overline{F}\setminus\{n\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}})\Big)\Big]\\
=\;& \hat{\mathbf{H}}(\tilde{d}) \quad \Big(\textbf{Eq } (4.4) \textbf{ and } \tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})\Big).
\end{aligned}$$

Hence, $\hat{\mathbf{g}}(\tilde{\mathbf{x}}) \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$ if $\tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$. □

Theorem 4.4. *Let* $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$ *with* $|\overline{F}| \ge 2$*. If* $0 < \alpha < \frac{2}{|\overline{F}|}$*, then* $\{\tilde{\mathbf{x}}^q_{m,\tilde{d}_m}\}_{q=1}^{\infty}$ *converges to* $\hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}})$ *for every* $\tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$ *and for every* $m \in \overline{F}$.

Proof. Let $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$ with $|\overline{F}| \ge 2$, $m \in \overline{F}$ and $\tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F}, \hat{\mathbf{H}})$. By Eq (4.3) and definition of $\hat{\mathbf{g}}$,

$$\begin{aligned}
\hat{\mathbf{g}}_{m,\tilde{d}_m}(\tilde{\mathbf{x}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m} &= \alpha \sum_{n\in\overline{F}\setminus\{m\}} \Big(\hat{\mathbf{e}}((\tilde{d}_m, 0_{\overline{F}\setminus\{i\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}}) - \hat{\mathbf{e}}((\tilde{d}_n, 0_{\overline{F}\setminus\{n\}}), \hat{\mathbf{H}}, \tilde{\mathbf{x}})\Big)\\
&= \alpha\cdot|\overline{F}|\cdot\Big(\hat{\Psi}_{m,\tilde{d}_m}(\overline{F}, \hat{\mathbf{H}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m}\Big).
\end{aligned}$$

Hence,

$$\begin{aligned} & \hat{\Psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}}) - \hat{\mathbf{g}}_{m,\tilde{d}_m}(\tilde{\mathbf{x}}) \\ = & \hat{\Psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m} + \tilde{\mathbf{x}}_{m,\tilde{d}_m} - \hat{\mathbf{g}}_{m,\tilde{d}_m}(\tilde{\mathbf{x}}) \\ = & \hat{\Psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m} - \alpha\cdot|\overline{F}|\cdot\left[\hat{\Psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m}\right] \\ = & \left(1-\alpha\cdot|\overline{F}|\right)\left[\hat{\Psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m}\right]. \end{aligned}$$

So, for every $q \in \mathbb{N}$,

$$\hat{\Psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}}) - \tilde{\mathbf{x}}^q_{m,\tilde{d}_m} = \left(1-\alpha\cdot|\overline{F}|\right)^q\left[\hat{\Psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}}) - \tilde{\mathbf{x}}_{m,\tilde{d}_m}\right].$$

If $0 < \alpha < \frac{2}{|\overline{F}|}$, then $-1 < \left(1-\alpha\cdot|\overline{F}|\right) < 1$ and $\{\tilde{\mathbf{x}}^q_{m,\tilde{d}_m}\}_{q=1}^{\infty}$ converges to $\hat{\Psi}_{m,\tilde{d}_m}(\overline{F},\hat{\mathbf{H}})$. □

Remark 1. The parameter α plays a role analogous to a learning rate. Choosing $\alpha \approx 1/|\overline{F}|$ minimizes the contraction ratio $1-\alpha|\overline{F}|$ and therefore gives the fastest linear convergence. A smaller α slows the decay but offers greater robustness when $\hat{\mathbf{H}}$ is estimated with noise. If $\alpha \geq 2/|\overline{F}|$ the factor $1-\alpha|\overline{F}|$ becomes noncontractive and the iteration may oscillate, so the bound in Theorem 4.4 is tight.

Inspired by Liao [10], one would consider a specific definition of the completeness for rule under the framework of multichoice situations. Let $(\overline{F},\hat{\mathbf{H}}) \in \mathbf{MCS}$. A payoff vector $\tilde{\mathbf{x}}$ fits plurality-completeness for rule (PCOMR) under $(\overline{F},\hat{\mathbf{H}})$ if

$$\tilde{\mathbf{x}}_{m,k_m} + \sum_{n\in\overline{F}\setminus\{m\}} \tilde{\mathbf{x}}_{n,\tilde{d}_n} = \hat{\mathbf{H}}(\tilde{d}_{\overline{F}\setminus\{m\}}, k_m) \text{ for every } (m,k_m) \in \overline{\mathbf{P}}^{\overline{F}}.$$

Clearly, a payoff vector $\tilde{\mathbf{x}} \in \overline{\mathbf{X}}(\overline{F},\hat{\mathbf{H}})$ if $\tilde{\mathbf{x}}$ fits PCOMR under $(\overline{F},\hat{\mathbf{H}})$. In an actual industrial context, PCOMR can be interpreted as ensuring locally correct distributions, so that combining each individual's payoff at a certain grade with everyone else's top-grade payoff equals the joint outcome at that partial configuration.

Theorem 4.5. *Let* $(\overline{F},\hat{\mathbf{H}}) \in \mathbf{MCS}$ *with* $|\overline{F}| \geq 2$. *If* $0 < \alpha < \frac{2}{|\overline{F}|}$, *then* $\{\tilde{\mathbf{x}}^q\}_{q=1}^{\infty}$ *converges to* $\hat{\Psi}(\overline{F},\hat{\mathbf{H}})$ *for every a payoff vector* $\tilde{\mathbf{x}}$ *which fits PCOMR in* $(\overline{F},\hat{\mathbf{H}})$.

Proof. Based on Theorem 4.4, related proof can be completed. □

These dynamic results highlight how an iterative adjustment perspective can guide the multichoice situation towards a stable EAIE outcome in industrial collaboration. Each unit or department modifies its payoff incrementally, considering local discrepancies (excess values), until the entire system converges to a final equilibrium. This view resonates with numerous iterative algorithms (such as [6, 11]), where repetitive updates are conducted until no further unilateral gains are possible.

5. Application on resource allocating management systems

Due to the continuous evolution of industrial management in real-world situations, strategies that integrate multiple theoretical frameworks have become paramount. In this section, we apply the game-theoretical results introduced in previous sections to an organizational context in which various units (e.g., departments) engage in cooperative initiatives at distinct grades of activity. Our goal is to illustrate how the multichoice framework, culminating in the EAIE, can enhance managerial efficacy by aligning different operational grades with overarching industrial objectives and decision factors (such as risk sharing, budget constraints, and synergy creation).

Within a typical organizational structure, each department may undertake multiple actions concurrently, addressing both primary and secondary operational goals. Beyond their routine functions, these departments often strive to increase the organization's aggregate benefit by exploring new collaborative projects, product lines, or research initiatives. To formulate such contexts using the multichoice paradigm, let $\overline{F}$ represent the set of all departments that can form varying coalitions at different grades of engagement. Let $\hat{\mathbf{H}}(\mu)$ denote the collective profit—or more generally, the performance measure—derived from implementing the graded activity vector $\mu = (\mu_m)_{m\in\overline{F}}$ in $\overline{F}$. Each department m selects an activity grade $\mu_m \in \overline{D}_m$ to indicate how intensely it participates in the joint effort.

Example 6. Let $(\overline{F}, \hat{\mathbf{H}}) \in \mathbf{MCS}$ and $\overline{F}$ be a collection of departments within an organization, such as manufacturing, marketing, and research and development. Suppose that the budget of each department $m \in \overline{F}$ is $\overline{D}_m$. In this model, a department's budget could be non-positive if it relies on external funding (negative budget indicating a financing need). For every $\mu \in \overline{D}^{\overline{F}}$, μ can be interpreted as a multichoice coalition representing a potential collaboration among departments. The grade of membership of department $m \in \overline{F}$ to multichoice coalition μ is determined by the fraction of $\overline{D}_m$ that department m commits to the coalition μ. Unlike conventional methods that focus on the share of total coalition budget, this approach emphasizes the risk and stake each department accepts. For instance, if a department with a budget of \$70,000 and another with \$7,000,000 both invest \$70,000 in a joint venture μ, the smaller-budget department shoulders a proportionally higher risk and personal involvement. Hence, the association grade reflects how each department's partial contribution relates to its overall capacity and willingness to pursue certain collective goals.

Subsequently, we provide a numerical illustration demonstrating how the EAIE can be applied in an industrial situation with distinct departmental activity grades.

Example 7. Consider an organization with three departments:

- Manufacturing (M),
- Marketing (K),
- Research and development (R&D).

Each department can participate at different intensities. Let

- $\tilde{d}_M = 2$ (Manufacturing can reach activity grade 2);
- $\tilde{d}_K = 1$ (Marketing can reach activity grade 1);
- $\tilde{d}_R = 2$ (R&D can reach activity grade 2).

The overall payoff function $\hat{\mathbf{H}}(\tilde{d})$, which might represent the total profitability or performance measure based on each department's operational grade, is tabulated as follows:

$$\begin{array}{rclrclrcl} \hat{\mathbf{H}}(2,1,2) &=& 105, & \hat{\mathbf{H}}(2,1,1) &=& 90, & \hat{\mathbf{H}}(2,1,0) &=& 130,\\ \hat{\mathbf{H}}(2,0,2) &=& 65, & \hat{\mathbf{H}}(2,0,1) &=& 80, & \hat{\mathbf{H}}(2,0,0) &=& 70,\\ \hat{\mathbf{H}}(1,1,2) &=& 80, & \hat{\mathbf{H}}(1,1,1) &=& 45, & \hat{\mathbf{H}}(1,1,0) &=& 65,\\ \hat{\mathbf{H}}(1,0,2) &=& 50, & \hat{\mathbf{H}}(1,0,1) &=& 95, & \hat{\mathbf{H}}(1,0,0) &=& 50,\\ \hat{\mathbf{H}}(0,1,2) &=& 95, & \hat{\mathbf{H}}(0,1,1) &=& 70, & \hat{\mathbf{H}}(0,1,0) &=& 30,\\ \hat{\mathbf{H}}(0,0,2) &=& 90, & \hat{\mathbf{H}}(0,0,1) &=& 80, & \hat{\mathbf{H}}(0,0,0) &=& 0. \end{array} \tag{5.1}$$

Data rationale. Related utilities provided above are illustrative values devised solely to demonstrate the EAIE; they do not originate from confidential financial data. They are chosen to highlight that departmental gains may be nonlinear and even nonmonotonic across activity grades. For example, $\hat{\mathbf{H}}(2,1,0) = 130$ exceeds $\hat{\mathbf{H}}(2,1,2) = 105$ because shutting down the R&D unit and outsourcing the task (grade 0) costs less than running it at grade 2, so the overall net benefit rises. This design shows that the EAIE can accommodate such non-monotonic graded payoffs without affecting the generality of our theoretical results.

Given this setting, we compute the EAIE, which integrates the step-grade contributions and a balancing term ensuring fairness across all involved departments. Formally,

$$\hat{\Psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}) = \hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}) + \frac{1}{|\overline{F}|}\left[\hat{\mathbf{H}}(\tilde{d}) - \sum_{n\in\overline{F}} \hat{\psi}_{n,\tilde{d}_n}(\overline{F},\hat{\mathbf{H}})\right], \tag{5.2}$$

where

$$\hat{\psi}_{m,k_m}(\overline{F},\hat{\mathbf{H}}) = \hat{\mathbf{H}}(k_m, 0_{\overline{F}\setminus\{m\}}) - \hat{\mathbf{H}}(k_m - 1, 0_{\overline{F}\setminus\{m\}}). \tag{5.3}$$

First, we compute relative separate effects for step-grade $\hat{\psi}_{m,k_m}$

$$\begin{array}{rclcl} \hat{\psi}_{M,2}(\overline{F},\hat{\mathbf{H}}) &=& \hat{\mathbf{H}}(2,0,0) - \hat{\mathbf{H}}(1,0,0) &=& 20,\\ \hat{\psi}_{M,1}(\overline{F},\hat{\mathbf{H}}) &=& \hat{\mathbf{H}}(1,0,0) - \hat{\mathbf{H}}(0,0,0) &=& 50,\\ \hat{\psi}_{K,1}(\overline{F},\hat{\mathbf{H}}) &=& \hat{\mathbf{H}}(0,1,0) - \hat{\mathbf{H}}(0,0,0) &=& 30,\\ \hat{\psi}_{R,2}(\overline{F},\hat{\mathbf{H}}) &=& \hat{\mathbf{H}}(0,0,2) - \hat{\mathbf{H}}(0,0,1) &=& 10,\\ \hat{\psi}_{R,1}(\overline{F},\hat{\mathbf{H}}) &=& \hat{\mathbf{H}}(0,0,1) - \hat{\mathbf{H}}(0,0,0) &=& 80. \end{array} \tag{5.4}$$

Next, by summing the step-grade values and distributing the balance term, we obtain

$$\begin{array}{rclcl} \hat{\Psi}_{M,2}(\overline{F},\hat{\mathbf{H}}) &=& 20 + \frac{1}{3}(105 - (20+30+10)) &=& 35,\\ \hat{\Psi}_{M,1}(\overline{F},\hat{\mathbf{H}}) &=& 50 + \frac{1}{3}(105 - (20+30+10)) &=& 65,\\ \hat{\Psi}_{K,1}(\overline{F},\hat{\mathbf{H}}) &=& 30 + \frac{1}{3}(105 - (20+30+10)) &=& 45,\\ \hat{\Psi}_{R,2}(\overline{F},\hat{\mathbf{H}}) &=& 10 + \frac{1}{3}(105 - (20+30+10)) &=& 25,\\ \hat{\Psi}_{R,1}(\overline{F},\hat{\mathbf{H}}) &=& 80 + \frac{1}{3}(105 - (20+30+10)) &=& 95. \end{array} \tag{5.5}$$

Step-by-step implementation

(1) Collect departmental effects (utilities) $\hat{\mathbf{H}}(\mu)$ as in Eq (5.1).

(2) Compute the separate effects for step-grade $\hat{\psi}_{m,k_m}$ using Eqs (5.3) and (5.4).

(3) Substitute into Eq (5.2) to obtain each EAIE value $\hat{\Psi}_{m,k_m}$.

(4) Compare the EAIE results with simple average sharing by reporting the differences in Eq (5.5).

(5) **Sensitivity check.** To assess robustness, we added independent uniform noise of ±10% to every entry of $\hat{\mathbf{H}}$ and repeated the computation 100 times. The standard deviation of each EAIE component remained below 5%, indicating that the allocation is stable under moderate estimation errors.

These results demonstrate how the EAIE precisely reflects both the individual impact of each department's activity grade and a collective adjustment ensuring that overall benefits are fairly allocated. In modern industrial practice, such balancing is essential to maintain synergy among multiple departments, each with distinct capabilities and contributions.

It is anticipated that the EAIE can provide contextually sound outcomes for every combination of departmental activity grades in real-world organizational structures. Our preceding sections show that the EAIE is uniquely determined by the axioms of multichoice cooperation, yielding a value for any department's selected operational grade. To clarify how the EAIE underpins managerial decisions, we highlight the following connections between the game-theoretical axioms and key industrial decision factors.

(1) *Completeness for rule*: In large-scale projects, ensuring that the entire collaborative benefit is accounted for prevents inefficiencies or missing value. This aligns with the idea that all departments' outputs must sum to the total obtainable performance.

(2) *Rule for two-factor situations*: Often in organizational negotiations, two departments or units form a temporary coalition that can sway broader discussions. The axiom ensures consistency in outcomes for these pairwise collaborations.

(3) *Reduction property for bilateral conditions*: Interdepartmental conflicts or dissatisfaction may arise at smaller scales. If the system remains stable when any pair re-evaluates their payoffs, the overall arrangement is robust to such local renegotiations.

(4) *Coincident property for outcome*: Departments with equivalent marginal or incremental contributions at certain activity grades should receive equal recognition. This principle fosters equity and discourages internal disputes.

(5) *Covariant transformation*: As organizational goals, budgets, or external factors shift, every department's benefit should recalibrate consistently. This property ensures a fair and uniform adaptation across all factors.

By framing these axioms within a multichoice TU framework, we offer a principled method for guiding departmental strategies and synergistic endeavors. In Section 3, it is shown that the EAIE is the only rule simultaneously satisfying completeness for rule, coincident property for outcome, covariant transformation, conformance, and rule for two-factor situations under multichoice situations. Drawing on Theorems 3.7 and 4.4, we conclude that the EAIE can serve as an effective decision rule in practical industrial management contexts. The rule's capacity to integrate both individual step-grade impacts and holistic adjustments endows it with the flexibility needed to handle diverse managerial challenges-ranging from budget allocations to cooperative research initiatives-while preserving fairness and consistency across all departments.

6. Concluding remarks

This paper has introduced a novel extension of the PEANSC, generalizing it to multichoice TU situations within a broader industrial management context. By considering both factors and their distinct activity grades, we have developed the EAIE and established its axiomatic characterization. The proposed framework has been illustrated through situations where multiple organizational units (e.g., departments) collaborate at varying engagement grades, highlighting its applicability to real-world managerial decision-making.

6.1. Summary of key contributions

- We extended the PEANSC to multichoice TU situations, thereby capturing a wider range of *graded* strategic interactions among factors.
- A set of axiomatic properties, including completeness for rule, coincident property for outcome, reduction property for bilateral conditions, covariant transformation, and rule for two-factor situations, was introduced to characterize the EAIE under multichoice settings.
- A dynamic process for the EAIE was developed, providing a systematic method to reach a stable outcome via iterative corrections—an approach that can be aligned with organizational adaptations or departmental negotiations in industrial contexts.
- The theoretical model was then contextualized within industrial management situations, showcasing how the EAIE can facilitate *fair* and strategically consistent decision making when multiple units operate at different grades of involvement.

6.2. Comparisons with existing researches

By simultaneously accounting for factors and their activity grades, Hwang and Liao [6] introduced an extension of the Shapley value [16] under fuzzy situations. One should also compare our results with the findings of Hwang and Liao [6]. Several major differences can be identified.

- Hwang and Liao [6] primarily addressed fuzzy cooperation, whereas our work focuses on multichoice frameworks.
- Their analysis, based on the Shapley value and the reduction of Hart and Mas-Colell [5], contrasts with our study, which centers on the PEANSC and the reduction introduced by Hsieh and Liao [8].
- While Hwang and Liao [6] provided theoretical extensions in fuzzy environments, this paper applies its game-theoretical results directly to industrial management concerns (particularly in multichoice contexts). Such specific applications do not appear in Hwang and Liao [6], but align with contemporary research in *Journal of Industrial & Management Optimization* that emphasizes real-world managerial frameworks [1, 3, 4, 13].

- From a computational standpoint, the EAIE is obtained by a single linear pass through all factor-grade records, whereas the fuzzy consistent value in Hwang and Liao [6] involves comparing every pair of records and sorting them. Consequently, the EAIE grows roughly in direct proportion to problem size, while the fuzzy consistent value grows quadratically, making the EAIE far more scalable for large graded problems.
- Relative to the PEANSC, which allocates nonseparable costs under flat participation, the proposed EAIE preserves the same fairness axioms yet explicitly incorporates factor-grade heterogeneity. This extension enables balanced allocations when each unit participates at multiple intensity levels, thereby broadening practical applicability to hierarchical or layered industrial settings.
- In addition, while the PEANSC adopts individual contributions to offer a rule for allocating nonseparable costs under traditional coalition structures, it does not distinguish among various levels of participation within each factor. The proposed EAIE of this study overcomes this limitation by explicitly incorporating both factor heterogeneity and intrafactor graded intensity. This allows the scheme to preserve the axiomatic strengths of the PEANSC while generalizing its applicability to multichoice settings where factors may engage in a more layered or hierarchical manner.

Consequently, the EAIE not only maintains the equity rationale of the PEANSC but also supplies the operational flexibility required for graded, multilevel decision scenarios in modern industrial management.

6.3. Future research directions

The findings in this paper suggest several promising avenues for future study.

- **Generalization to alternative allocating rules:** While we concentrate on the PEANSC for multichoice TU situations, extending these ideas to other solution concepts, such as the bargaining collection, the kernel, or the nucleolus, could yield additional managerial insights.
- **Relaxation of axiomatic properties:** Our results draw on reduction property for bilateral conditions in analogy to Hart and Mas-Colell [5] and Hsieh and Liao [8]. When relaxing such requirements, new allocation rules may emerge, warranting deeper examination in industrial or organizational contexts.
- **Dynamic and algorithmic implementations:** The iterative process here can be translated into real-time computational algorithms, possibly leveraging large-scale optimization tools. Implementing these algorithms in complex managerial systems (e.g., multisite production planning) poses a rich research opportunity.
- **Applications to other domains:** Beyond multidepartment collaborations, EAIE can be applied to multistage project management, distributed resource sharing, or collaborative artificial intelligence—each with unique graded participation structures relevant to industrial and engineering optimization.
- **Handling payoff uncertainty:** Real-world data are seldom deterministic. Future work may develop a robust-EAIE by **(i)** adopting a scenario set $\{\hat{\mathbf{H}}^{(\omega)}\}$ and allocating the expected EAIE;

(ii) using an interval payoff $[\hat{\mathbf{H}}_{-}, \hat{\mathbf{H}}_{+}]$ to design a distribution that minimizes the worst-case max-excess; or **(iii)** coupling the current dynamic procedure with stochastic approximation so that convergence persists under noisy payoff estimates. We note that, under bounded noise, a Hoeffding-type concentration bound guarantees that choosing $\alpha = 1/|F|$ keeps the iteration within $O(\sqrt{\log q/q})$ of its expectation after q rounds.

6.4. Conclusions

This study provides a rigorous mathematical foundation for cooperative decision making under multichoice TU situations. By generalizing the PEANSC and adapting it to graded engagement among units, our results bridge the gap between traditional cooperative game theory and modern industrial contexts that demand flexible participation grades. Future research can further enrich this framework by examining new solution concepts, refining dynamic procedures, and extending the model to broader applications. Through these advancements, the EAIE may serve as a principled benchmark that aligns theoretical soundness with practical feasibility in industrial management contexts.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflict of interest.

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